Universal Algebra 1 – Exercises 8

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Exercise 1. Let ρ be the similarity type consisting of a single unary operation symbol f, and let $\mathcal{V}$ be the subvariety of the variety of algebras of type ρ defined by the single identity $f^6(x) \approx f^2(x)$. Determine and draw the graph of the free algebras $\mathbf{F}_{\mathcal{V}}(\{x\})$ and $\mathbf{F}_{\mathcal{V}}(\{x, y\})$.

Exercise 2. Let $\mathcal{V}$ be the variety of distributive lattices. Determine and draw the Hasse diagram of the free algebras $\mathbf{F}_{\mathcal{V}}(\{x\}), \mathbf{F}_{\mathcal{V}}(\{x, y\}), \mathbf{F}_{\mathcal{V}}(\{x, y, z\})$.

Exercise 3. Let $\mathcal{V}$ be the variety of commutative semigroups satisfying the identity $x^2 \approx x^3$. Show that $|\mathbf{F}_{\mathcal{V}}(\{x_1, \dots, x_n\})| = 3^n - 1$.

Exercise 4. Let $\mathcal{V}$ be the variety of binars satisfying the identities

$$x \cdot x \approx x \tag{1}$$
$$(x \cdot y) \cdot z \approx (z \cdot y) \cdot x \tag{2}$$

(i) Show that every member of $\mathcal{V}$ satisfies the following identities.

$$(x \cdot y) \cdot (z \cdot w) \approx (x \cdot z) \cdot (y \cdot w) \tag{3}$$
$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot (x \cdot z) \tag{4}$$
$$(y \cdot z) \cdot x \approx (y \cdot x) \cdot (z \cdot x) \tag{5}$$
$$y \cdot (x \cdot y) \approx (y \cdot x) \cdot y \tag{6}$$
$$(y \cdot x) \cdot x \approx x \cdot y \tag{7}$$

(ii) Let $\mathcal{W}$ be the subvariety of $\mathcal{V}$ defined by the additional identity

$$y \cdot (x \cdot y) \approx x \tag{8}$$

Determine and write the Cayley table of the free algebra $\mathbf{F}_{\mathcal{W}}(\{x, y\})$.