NMAG 405 - Universal Algebra 1 - fall semester 2022/23

Homework 1

Deadline 27.10.2022, 17:20

1. (10 points) A *latin square* $(A, *)$ is an algebra of type (2), such that for each $a, b \in A$ there exists a unique $x \in A$ with $x * a = b$ and a unique $y \in A$ with $a * y = b$; we then denote x by b/a and y by $a \backslash b$. (For finite A each row and each column of the multiplication table of $*$ contains every element of A exactly once, hence the name.)
   A *quasigroup* is an algebra $(A, *, \backslash, /)$ of type $(2, 2, 2)$, which satisfies the identities:
   $$y \approx x * (x \backslash y) \approx x \backslash (x * y) \approx (y/x) * x \approx (y * x)/x.$$
   Let A be a fixed set. Prove that the map Φ that assigns to every latin square $(A, *)$ the algebra $(A, *, \backslash, /)$ as above, and the map Ψ that forgets the operations $\backslash, /$ are mutually inverse bijections between the set of latin squares and the quasigroups (with universe A).

2. (10 points) Let $\mathbb{R}^n$ be the n-dimensional euclidean space and $\mathcal{C}$ be the set of all its (topologically) closed subsets. Show that $(\mathcal{C}, \cap, \cup)$ is a complete lattice and describe $\bigwedge$ and $\bigvee$. What are the compact elements of this lattice? Is it an algebraic lattice?

3. (10 points) A map $f\colon L_1 \to L_2$ between two lattices is called *monotone* if $x \leq y$ implies $f(x) \leq f(y)$. Let L be a complete lattice, and $f\colon L \to L$ an monotone map. Prove that there is a fixpoint a of f, i.e. a point $a \in L$ such that $f(a) = a$.