

## Homework 4

Deadline 15.12.2022, 17:20

1. (10 points) Let  $\mathbf{R}$  be a fixed ring (with 1), and let  $\mathcal{V}$  be the variety of (left)-modules over  $\mathbf{R}$ . Prove that the free  $\mathbf{R}$ -module is isomorphic to  $R^{(X)} = \{(u_x)_{x \in X} \in R^X \mid \text{only finitely many } u_x \text{ are not } 0\}$ . Use the universal algebraic characterization of free algebras as “terms modulo identities”, do not use the categorical/module-theoretical definition of freeness.
2. (10 points) Let  $\mathcal{V}$  be the variety of algebras  $(A, \cdot, l, r)$  of type  $(2, 1, 1)$  that satisfy the identities

$$l(x \cdot y) \approx x, \quad r(x \cdot y) \approx y, \quad l(x) \cdot r(x) \approx x.$$

- (a) Show that every non-trivial member of  $\mathcal{V}$  is infinite.
  - (b) Prove that, if  $\mathbf{A} \in \mathcal{V}$  is generated by  $\{a_1, a_2, \dots, a_n\}$ , then it is already generated by  $\{(a_1 \cdot a_2), a_3, \dots, a_n\}$
  - (c) Prove that  $\mathbf{F}_{\mathcal{V}}(n) = \mathbf{F}_{\mathcal{V}}(m)$  for all positive integers  $n, m$ .
3. (10 points) Let  $\mathbf{A}$  be the semigroup given by the following multiplication table:

·	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
3	0	1	2	3

Prove that the variety generated by  $\mathbf{A}$  is exactly the variety of commutative semigroups satisfying  $x^3 \approx x^4$ .