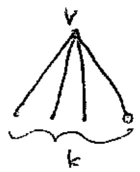
Homework 4 - Solutions

(4.1) Every vertex v is incident to k edges (as G is k-regular),



so the number of copies of P_3 in G, with its middle vertex equal to v, is $\binom{k}{2}$. There are n vertices which gives altogether $n\binom{k}{2}$ copies of P_3.

(4.2) a) k-regular graph has at least $k+1$ vertices (for every vertex there are k different adjacent vertices), so the smallest n is at least $k+1$. On the other hand, K_{k+1} has $k+1$ vertices and is k-regular. Therefore the smallest n is $n=k+1$

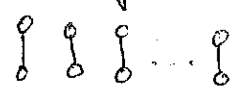
b) • $n \geq k+1$ (as above)

• $n \neq k+1$ the only k-regular graph with $k+1$ vertices is K_{k+1} (as every vertex must be adjacent to all other vertices)

• $n \neq k+2$ If $n=k+2$ then the complement of a k-regular graph is 1-regular graph

  • if ~~k~~ n is odd no 1-regular graph exists (as sum of the degrees must be even)

  • if n is even then the unique 1-regular graph is clearly a disjoint union of $\frac{n}{2}$ edges ($\simeq P_2+P_2+\dots+P_2$)



Going back to complements

  • if ~~n is odd~~ $n=k+2$ is odd then no k-regular graph with $|V(G)|=n$ exists

  • if $n=k+2$ is even then the only k-regular graph with n vertices is $\overline{P_2+P_2+\dots+P_2}$.

Homework 4 - Solutions

4.2 contd

So far we have shown that the smallest n satisfies $n \geq k+3$

- $\boxed{n = k+3}$

$\overline{C_3 + C_k}$ and $\overline{C_{k+3}}$ are for $k \geq 3$ both k-regular and they are not isomorphic (as the complement of the first graph is disconnected while the complement of the second graph is connected).

4.3

$\boxed{1.}$ If G is loopless, with degree sequence $d_1, \dots, d_n$, $d_1 \geq \dots \geq d_n \geq 0$, then $\sum_1 d_i$ is even and $d_1 \leq d_2 + \dots + d_n$.

Proof: By the Degree-Sum Formula $\sum d_i$ is even and equal to $2|E(G)|$. As G is loopless, a vertex of degree d_1 is incident to d_1 edges,

so $d_1 \leq |E(G)| \overset{\text{above}}{=} \dfrac{d_1 + \dots + d_n}{2}$

or $d_1 \leq d_2 + \dots + d_n$

$\boxed{2.}$ If $d_1 \geq \dots \geq d_n \geq 0$ are integers such that $\sum d_i$ is even and $d_1 \leq d_2 + \dots + d_n$, then there exists a loopless graph with degree sequence $d_1, d_2, \dots, d_n$.

By induction on $k = d_1 + \dots + d_n$

For $k = 0$ we take $\overbrace{o \; o \cdots o}^{n}$

Assume $k > 0$.

Observe that the sequence $d_1, \dots, d_n$ has at least 2 nonzero entries, otherwise the condition $d_1 \leq d_2 + \dots + d_n$ is violated

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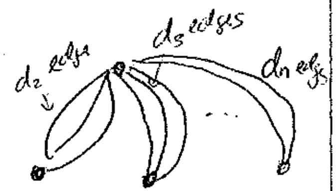
4.3 contd

Case 1 $d_1 = d_2 + \dots + d_n$. Then we can take $G =$ 

Case 2 $d_1 < d_2 + \dots + d_n$. We cannot have $d_1 = d_2 + \dots + d_n - 1$ since the degree sum $d_1 + \dots + d_n = 2(d_2 + \dots + d_n) - 1$ would be odd. Therefore

$$\underline{d_1 \leq d_2 + \dots + d_n - 2}$$

Form a new sequence $d_1', \dots, d_n'$ by subtracting 1 from two smallest nonzero members of the sequence $d_1, \dots, d_n$ (there are such entries by the remark above)

For this new sequence we have

- $d_1' \geq d_2' \geq \dots \geq d_n' \geq 0$ (by construction)
- $\sum d_i' = \sum d_i - 2$ is even
- $\underbrace{d_1'} \leq d_1 \leq d_2 + \dots + d_n - 2 \underbrace{\leq d_2' + \dots + d_n'}$

Also $\sum d_i' < k$, so, by induction hypothesis, there exists a loopless graph G with degree sequence $d_1', \dots, d_n'$. Now we just add one edge between the vertices corresponding to degrees we ~~subtracted~~ decreased.

This graph realizes $d_1, d_2, \dots, d_n$.