Homework 5 Solutions

5.1 Let G be the graph whose vertices are atoms and edges are bonds. We know that

- G is connected (molecules are)
- G is acyclic

$\Big\}$ G is a tree

- G has $\underline{k}$ vertices of degree 4 and $\underline{\ell}$ vertices of degree 1

We have $\sum d(v) = 4k + \ell$, therefore, by the Degree-Sum Formula

$$(*)\quad 2\,|E(G)| = 4k + \ell$$

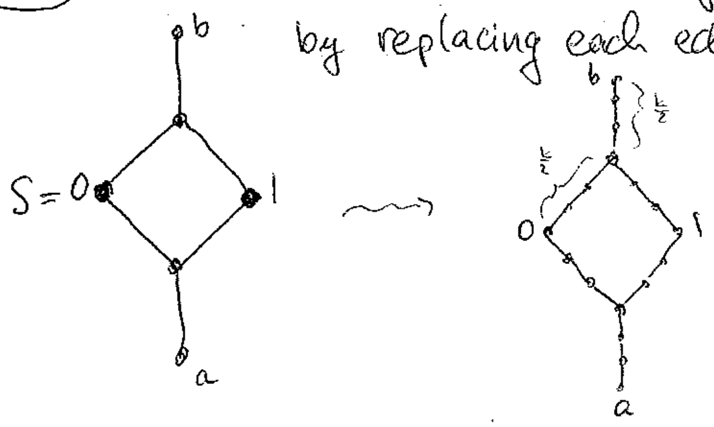
Moreover, $|V(G)| = k + \ell$ and, since G is a tree, $|E(G)| = k + \ell - 1$ $(**)$

By comparing $(*)$ and $(**)$ we get

$$4k + \ell = 2(k + \ell - 1)$$

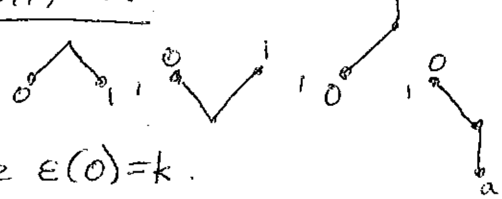
$$\underline{\ell = 2k + 2}$$

5.2 For even k consider the graph obtained from S by replacing each edge by a path of length $\frac{k}{2}$.



There are 2 $0,1$-paths: and , both have length k, so $\underline{d(0,1) = k}$

The paths



have length k and contain all vertices, therefore $\varepsilon(0) = k$.

Similarly $\varepsilon(1) = k$. Every vertex v above $0,1$ has distance at least $k+1$ from a (because every path to a contains either the path or), hence the eccentricity of v is at least $k+1$.

Similarly (by considering distances to b), the eccentricity of every vertex below $0,1$ is at least $k+1$. It follows that the center of our graph is formed by vertices 0 and 1. Their distance is k as observed above

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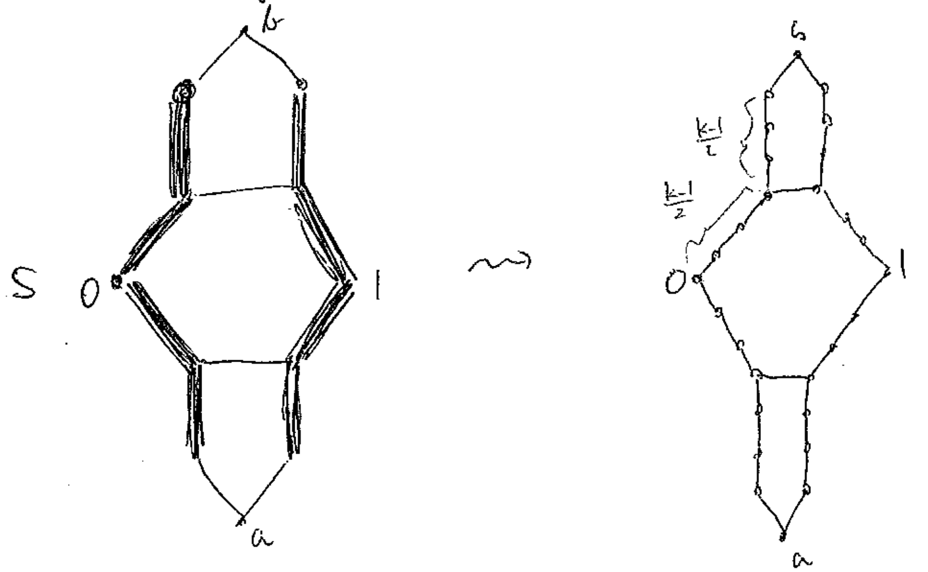
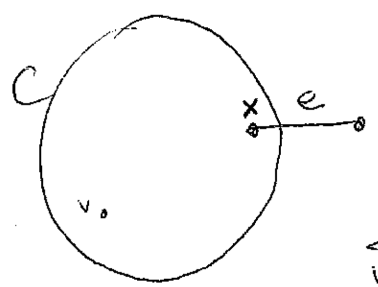
5.2 contd. For odd $k > 1$ consider the graph obtained from S by replacing each thick edge by a path of length $\frac{k-1}{2}$



In a similar way as in the first case we observe that $d(0,1) = k$, the center consists of 0 and 1.

The case $k=1$ is left for the reader ☺

5.3 "$\Rightarrow$" Assume $d(v)$ is odd for every $v \in V(T)$. Take any $e \in E(T)$, and let C be a component of $T-e$ and let x be the endpoint of e contained in C. The degree of every vertex $v \in V(C)$, $v \neq x_i$ is the same as in T and $d_C(x) = d_T(x) - 1$. Then



$\sum_{v \in V(C)} d_C(v)$ is a sum of $|V(C)|-1$ odd numbers and one even number ($d_C(x)$). The sum must be even (by the Degree-Sum Formula), so $|V(C)|$ must be even.

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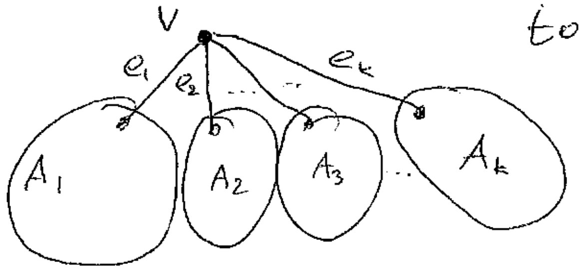
5.3 contd. "$\Leftarrow$" Assume $|V(C)|$ is odd for every $e \in E(T)$ and every component C of $T-e$. Let $v \in V(T)$ be arbitrary, ~~and~~ let $A_1, \dots, A_k$ be components of $T-v$ (observe that $k = d(v)$) and let e_i be the edge joining v to A_i.



A_i is a component of $T-e_i$, so $|V(A_i)|$ is odd for every i.
The component of $T-e_k$ different ~~of~~ from A_k (contains $A_1, \dots, A_{k-1}$ and v) has $1 + |V(A_1)| + \dots + |V(A_{k-1})|$ vertices, therefore this sum must be odd as well. Hence $|V(A_1)| + \dots + |V(A_{k-1})|$ is even.
A sum of $k-1$ odd numbers is even iff $k-1$ is even. Thus $k-1$ is even and $k = d(v)$ is odd.