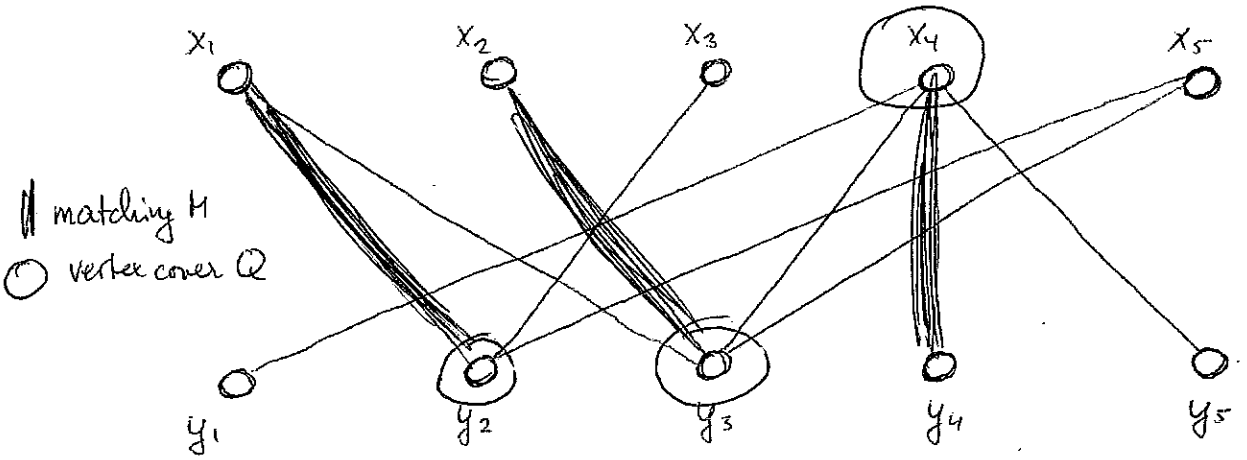
1)[8 marks]

a) Find a maximum matching in the graph below. Prove that it is a maximum matching by exhibiting an optimal solution to the dual problem (minimum vertex cover).



$M = \{x_1y_2, x_2y_3, x_4y_4\}$

$Q = \{x_4, y_2, y_3\}$

From the duality theorem (max. matching has the same size as min. vertex cover) it follows that M is a maximum matching (and Q is a minimum vertex cover)

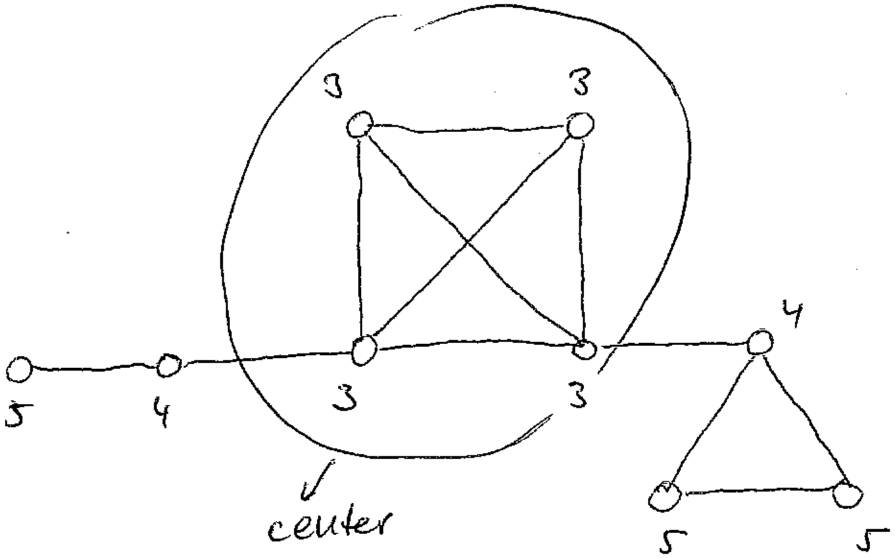
b) The graph in part a) does not have a matching which saturates the upper partite set X. Find a subset S of X which violates Hall's condition.

$S = \{x_1, x_2, x_3\}$

$N(S) = \{y_2, y_3\}$

$|S| > |N(S)|$ so S violates Hall's condition

2)[8 marks]
a) Find the diameter, radius, center and eccentricities of all vertices in the graph below (no justification needed).



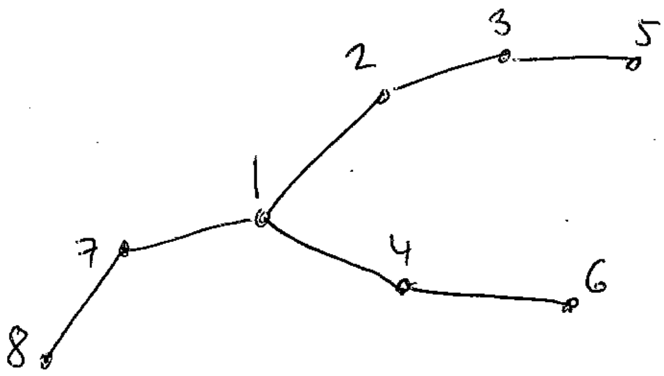
eccentricities are labels
diameter is 5
radius is 3

b) State Jordan's theorem about centers of trees.

The center of a tree is a vertex or an edge.

3)[8 marks]

a) Find the tree with vertex set $\{1, 2, \ldots, 8\}$ and Prüfer code $(3, 2, 1, 4, 1, 7)$ (no justification needed).



b) Count the trees with vertex set $\{1, 2, \ldots, 6\}$ and degrees $d(1) = d(5) = 2, d(2) = 3, d(3) = d(4) = d(6) = 1$.

The Prüfer code of such a tree is a permutation of (1, 5, 2, 2). # of such permutations is

$$\frac{4! \leftarrow \text{all permutations on a 4 element set}}{2! \leftarrow \text{counting each twice}}$$

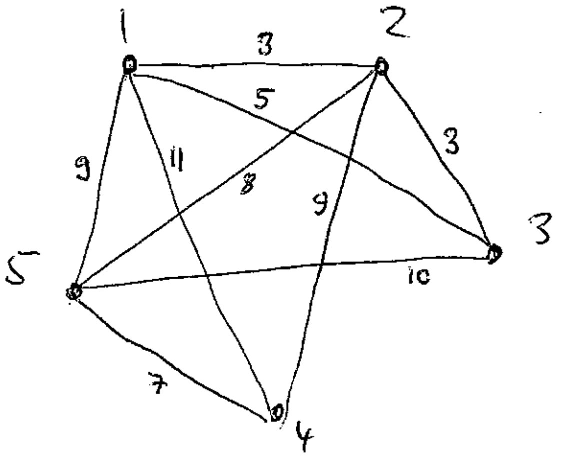
$$\frac{4!}{2!} = \frac{24}{2} = \boxed{12}$$

(or we can use the formula $\frac{(n-2)!}{\prod(d_i-1)!} = \frac{4!}{0!0!1!1!2!} = 12$)

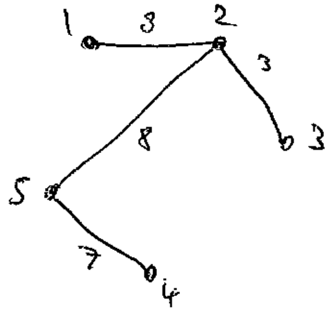
4)[6 marks] There are five cities in a network. The cost of building a road directly between i and j is the entry $a_{i,j}$ in the matrix below. An infinite entry indicates that there is a mountain in the way and the road cannot be build. Determine the least cost of making all the cities reachable from each other. (For full credit, it is enough to rephrase this problem in the language of graph theory and solve it. You do not need to explain the algorithm you are using.)

$$\begin{pmatrix} 0 & 3 & 5 & 11 & 9 \\ 3 & 0 & 3 & 9 & 8 \\ 5 & 3 & 0 & \infty & 10 \\ 11 & 9 & \infty & 0 & 7 \\ 9 & 8 & 10 & 7 & 0 \end{pmatrix}$$

We need to find a minimum weight spanning tree of the weighted graph



Kruskal's algorithm gives us the tree



The least cost is $3+3+7+8 = 21$ //

5)[8 marks]

a) G is a forest with 123 vertices and 111 edges. How many components does G have?

A tree with n vertices has n-1 edges

It follows that a forest with k components has n-k edges

(discussed in detail in class)

In our situation

$$123 - k = 111$$

$$k = 12 /\!/$$

b) G is a tree which has exactly four non-leaf vertices, their degrees are $5, 6, 3, 8$. How many leaves does G have?

Let k denote the number of leaves.

Then $\sum_{v \in V(G)} d(v) = 5+6+3+8+k\cdot 1 = 22+k$

By the Degree-Sum Formula $|E(G)| = \frac{22+k}{2}$ (*)

Since G is a tree we have $|E(G)| = |V(G)| - 1 = 4+k-1$ (**)

By comparing (*) and (**) we get

$$\frac{22+k}{2} = 3+k \Rightarrow k = 16 /\!/$$