Recommended Problems 2 – Solutions

2.1 For a vertex $v = a_1a_2\ldots a_k$ $(a_i \in \{0,1\})$, let $p(v)$ denote the parity of $a_1+\ldots+a_k$, that is

$$p(v) := (a_1+a_2+\ldots+a_k) \bmod 2$$

If $v \leftrightarrow w$, then $p(v)=p(w)$, because by flipping v at two coordinates the sum ~~changes~~ $a_1+\ldots+a_n$ changes by $0, 2$ or -2:

$$\begin{array}{c|c|c|c} v = \ldots 0 \ldots 0 \ldots & v = \ldots 0 \ldots 1 \ldots & v = \ldots 1 \ldots 0 \ldots & v = \ldots 1 \ldots 1 \ldots \\ w = \ldots 1 \ldots 1 \ldots & w = \ldots 1 \ldots 0 \ldots & w = \ldots 0 \ldots 1 \ldots & w = \ldots 0 \ldots 0 \ldots \end{array}$$

~~p(v)~~

By induction, if $u_1, u_2, \ldots, u_n$ is a path (or a walk) then

$$p(u_1) = p(u_2) = \ldots = p(u_n)$$

Therefore, if $p(u) \neq p(v)$, then u and v are not connected, so G has at least 2 components. We will show that any two vertices u, v with $p(u) = p(v)$ are connected.

By induction on $j = 0, 1, \ldots, k$ we show that

Claim: $\boxed{a_1a_2\ldots a_k \text{ and } b_1b_2\ldots b_k \text{ are connected whenever } a_1=b_1, a_2=b_2, \ldots, a_{k-j}=b_{k-j} \text{ and } p(a_1\ldots a_k) = p(b_1\ldots b_k)}$

- for $j=0$ the tuples are equal
- for $j=1$ the tuples are equal as well, because $a_1=b_1, a_2=b_2, \ldots a_{k-1}=b_{k-1}$ but also $a_1+a_2+\ldots+a_k \equiv b_1+b_2+\ldots+b_k \pmod 2$, so $a_k=b_k$
- assume $k \geq j > 1$ and that the Claim is true for smaller j's.

  if $a_{k-j+1} = b_{k-j+1}$ we can use the induction hypothesis

  otherwise $a_1a_2\ldots a_{k-j}\, a_{k-j+1}\, a_{k-j+2} \ldots$

  $$\updownarrow$$

  $$a_1a_2\ldots a_{k-j}\,(1-a_{k-j+1})(1-a_{k-j+2})\,a_{k-j+3}\ldots$$

  and this new tuple agrees with $b_1\ldots b_k$ on the coordinates $1, 2, \ldots, k-j+1$, therefore ~~so~~ it is connected to $b_1\ldots b_k$ by the induction hypothesis

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$\rightarrow$ G has two components $\{v : p(v)=0\}$ and $\{v : p(v)=1\}$

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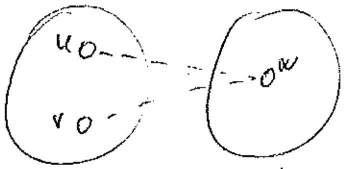
(2.2) Claim: The complement of a simple disconnected graph G is connected.

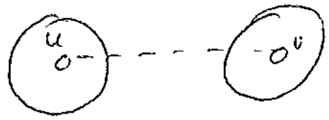
Proof: If $u, v \in V(G)$ are in the same component of G, then there exists a vertex $w \in V(G)$ in a different component (since we assume G is disconnected). Then $uw \notin E(G)$ and

(A) $wv \notin E(G)$ (since w is in a different component than u and v) so $uw, wv \in E(\overline{G})$, so u, v are connected in $\overline{G}$.

If $u, v \in V(G)$ are in different components of G,

(B) then $uv \notin E(G)$, hence $uv \in E(\overline{G})$ and, again, u, v are connected (even adjacent) in $\overline{G}$.

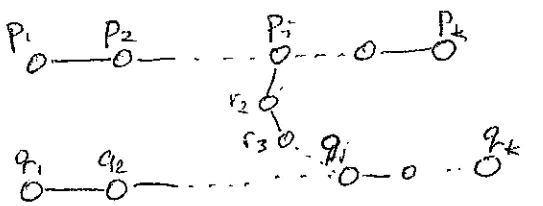
(A) 



(2.3) Assume the contrary and let $P = p_1, p_2, \ldots, p_k$ and $Q = q_1, \ldots q_k$ be disjoint paths of maximum length. Let

~~$R = p_i, \ldots, q_j$ be a shortest path~~

$R = p_i = r_1, r_2, \ldots, r_\ell = q_j$ be a shortest path from a vertex of P to a vertex of Q. Since P and Q are disjoint we have $\ell \geq 2$.

P 

Now we take the longer of the paths $p_1 p_2 \ldots p_i$ and $p_k p_{k-1} \cdots p_i$, join it with R and join it with the longer of the paths $q_1, \ldots, q_j$ and $q_k q_{k-1} \cdots q_j$.

It is easy to see that this is a path and its length is at least $\frac{k-1}{2} + 1 + \frac{k-1}{2} = k$ which is strictly greater than the length of P $(k-1)$. A contradiction!

Recommended Problems 2 - Solutions

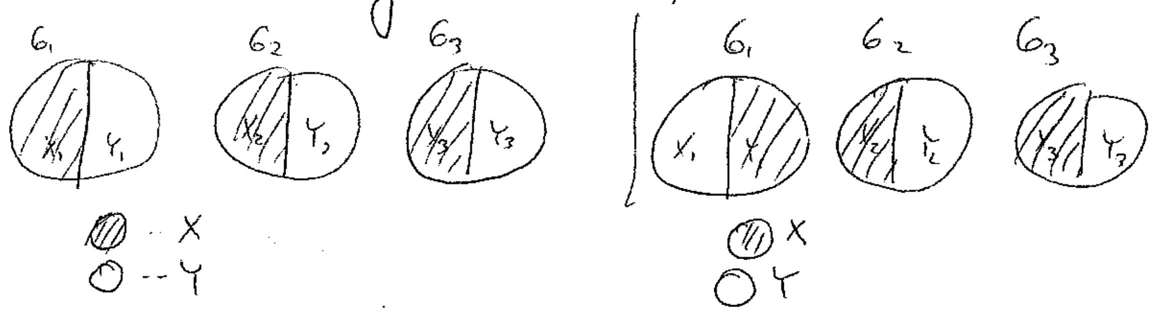
2.4

($\Rightarrow$) ~~If G is not connected~~
Assume G is not connected and let $G_1, \dots G_k$ be its components. For each component G_i take its partite sets X_i, Y_i. Now

$$X = X_1 \cup X_2 \cup \dots \cup X_k, \quad Y = Y_1 \cup Y_2 \cup \dots \cup Y_k$$

and $X = Y_1 \cup X_2 \cup \dots X_k$, $\quad Y = X_1 \cup Y_2 \cup \dots Y_k$

are two ~~do~~ essentially different bipartitions



($\Leftarrow$) Let G be connected and let X, Y be any bipartition.
Let u be any vertex of G.
If $u \in X$, then every vertex with an even walk from u must be in X and every vertex with an odd walk from u must be in Y (the reader saw this argument many times...). But every vertex has a walk from u (as G is connected), so in fact

$$X = \{v \in V(G);\ \text{there is an even } u,v\text{-walk}\}$$
$$Y = \{v \in V(G);\ \text{---''--- odd } u,v\text{-walk}\}$$

(Similarly, if $u \notin X$ then $X = \{v;\ \text{---''--- odd } u,v\text{-walk}\}$, $Y = \{v;\ \text{even}\}$)

So, bipartition X, Y is uniquely determined by choosing one vertex in X.