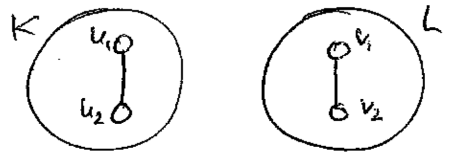
Recommenden Problems 3 – Solutions

3.1

- <u>G is connected</u>: If K, L are different components, then take $u_1u_2 \in E(K)$, $v_1v_2 \in E(L)$ (such edges exist as there are no isolated vertices). But then the subgraph of G induced by $\{u_1, u_2, v_1, v_2\}$ has exactly to edges (it is $\simeq P_2 + P_2$)



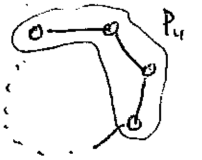
- <u>G is complete</u>. Take any $u, v \in V(G)$, $u \neq v$. Since G is connected, there exists a u,v-path. Take a shortest u,v-path. It is induced (see homework problem 3.3a). If its length is > 1 then the first three vertices induce a subgraph isomorphic to P_3 ↯. So the length is 1, i.e. $u \leftrightarrow v$.

3.2

- <u>G is bipartite</u>: ~~Assume otherwise~~ Assume it is not, i.e. G has an odd cycle
    - <u>The shortest odd cycle is an induced subgraph.</u>

      If not  we have two shorter cycles, one of them is odd (think it over!)

    - <u>The shortest odd cycle cannot have length 3</u>. The graph would otherwise contain C_3 as an induced subgraph
    - <u>The shortest odd cycle cannot have length ≥ 5.</u>

      otherwise ~~the~~ four consecutive vertices induce a subgraph isomorphic to P_4

Recommended Problems 3 – Solutions

3.2 contd

Let X, Y be the partite sets of a bipartition.
Let $u \in X$, $v \in Y$ be arbitrary.

- There is a u,v-path (as G is connected)
- Every u,v-path has an odd length. (discussed in class)
- The shortest u,v-path is an induced subgraph (see homework 3.3a)
- The shortest u,v-path cannot have length ≥ 3 (otherwise the first 4 vertices induce P_4)

It follows that the shortest u,v-path has length 1 — $uv \in E(G)$.

3.3

By induction n.

basic step: $n = 1$ ~~...~~ obvious

induction step: we assume that the claim is true for all graphs with at most $n-1$ vertices and ~~pro~~ we prove it for an n-vertex graph G.

- case 1 All vertices of G have degree at least 2. In this case we can use a lemma from class.
- case 2 G contains a vertex v of degree at most 1.
  Then we consider G' obtained by deleting v and the incident edge (if there is one).
  G' has $n-1$ vertices and at least $n-1$ edges (since we deleted at most 1 edge), so, by induction hypothesis, G' contains a cycle and this cycle is of course also contained in G.

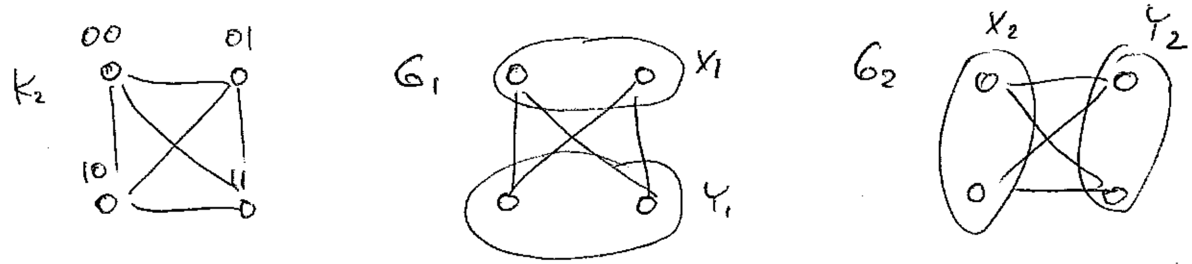
Recommended Problems 3 – Solutions

3.4 a) Let $c(v)$ denote the code of a vertex $v \in K_n$. We define bipartite subgraphs $G_1, \dots, G_k$ of K_n as follows.

G_i is the complete bipartite subgraph of K_n with bipartition X_i, Y_i, where

$$X_i = \{v;\ c(v) \text{ has } 0 \text{ at } i\text{-th coordinate}\},$$
$$Y_i = \{v;\ c(v) \text{ has } 1 \text{ at } i\text{-th coordinate}\}.$$

We have to show that each edge $u \leftrightarrow v$ in K_n is in G_i for some i. But this is easy: $c(u) \neq c(v)$, differ at some coordinate, say i, and so uv is an edge of G_i



b) Let X_i, Y_i be the partite set of a bipartition of G_i, $i = 1, 2, \dots, k$.

We define an encoding of vertices:

$c(v)$ has 0 at i-th coordinate iff $v \in X_i$

We need to show that distinct vertices have different codes. So let $u, v \in K_n$, $u \neq v$. The edge uv is in some of the graphs $G_1, \dots, G_k$, say G_i. One of the vertices u, v is in X_i and the other one in Y_i (which is disjoint from X_i) so $c(u)$ and $c(v)$ differ at the i-th coordinate.

We have shown that c is an injective (= one-to-one) mapping

$c: V(K_n) \longrightarrow$ Set of ~~k-tuples~~ binary k-tuples.

Therefore $\underset{\overset{\|}{n}}{|V(K_n)|} \leq |\text{Set of all binary } k\text{-tuples}| = 2^k$