Recommended Problems 4 - Solutions

4.1 Consider one division and games between them. This defines a graph with 13 vertices (the teams) and such that every vertex has degree 9 (edges are the games). Sum of the degrees is then $9 \cdot 13$ which is odd - this is <u>impossible</u>.

4.2 a) True.

average degree of G is $\dfrac{2|E(G)|}{n}$ $\left(\text{because } \sum_{v \in V(G)} d(v) = 2|E(G)|\right)$

average degree after removing a vertex of degree $\Delta(G)$ is $\dfrac{2(|E(G)| - \Delta(G))}{n-1}$

(we have removed $\Delta(G)$ edges and 1 vertex)

we want to show

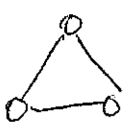
$$\frac{2|E(G)|}{n} \geq \frac{2(|E(G)| - \Delta(G))}{n-1}$$

$$\Longleftrightarrow \quad (n-1)|E(G)| \geq n(|E(G)| - \Delta(G))$$

$$\Longleftrightarrow \quad n\Delta(G) \geq E(G)$$

This is true since the left hand side is greater or equal to the sum of the degrees $\left(\sum_{v \in V(G)} d(v) \leq \sum_{v \in V(G)} \Delta(G) = n \cdot \Delta(G)\right)$ and this sum is equal to $2|E(G)|$, which is greater than or equal to $|E(G)|$.

b) False

G  average degree 2

 average degree 1

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4.3 "$\Rightarrow$" If $Q'_k \cong Q_k$ then k is even.

This doesn't work for $k=1$.
But in this case it's true as well

If k is odd, Q'_k contains an odd cycle:

$$00\ldots0 \xleftrightarrow{1} 11\ldots10 \xleftrightarrow{2} 0\ldots011 \xleftrightarrow{3} 11\ldots1000 \xleftrightarrow{4} 00\ldots01111 \leftrightarrow$$

$$\leftrightarrow \ \ldots\ \xleftrightarrow{k-1} 011\ldots1 \xleftrightarrow{k} 00\ldots0$$

So Q'_k is not bipartite while Q_k is so $Q_k \not\cong Q'_k$

"$\Leftarrow$" If k is even then $Q'_k \cong Q_k$

We will define an isomorphism $f: Q'_k \to Q_k$.

For a tuple $u = a_1 \ldots a_k$ let $\bar u = (1-a_1)(1-a_2)\ldots(1-a_k)$

and $p(u) = (a_1 + \ldots + a_k) \bmod 2$

As k is even we have $p(u) = p(\bar u)$. We define f by

$$f(u) = \begin{cases} u & \text{if } p(u) = 0 \\ \bar u & \text{if } p(u) = 1 \end{cases}$$

- <u>f is onto</u>: we have to check that for each $v \in V(Q_k)$ there exists $u \in V(Q'_k)$ such that $f(u) = v$.
  If $p(v) = 0$ we can take $u := v$.
  If $p(v) = 1$ then $p(\bar v) = 1$ and we can take $u := \bar v$
  (then $f(u) = f(\bar v) = \bar{\bar v} = v$)
- <u>f is a bijection</u> since f is onto and $|V(Q_k)| = |V(Q'_k)|$
- <u>f is an isomorphism</u>: we have to check that $u \leftrightarrow v$ in Q'_k iff $f(u) \leftrightarrow f(v)$ in Q_k. Observe that $u \leftrightarrow v$ in Q'_k iff $u \leftrightarrow \bar v$ in Q_k iff $\bar u \leftrightarrow v$ in Q_k.
  Also observe that if $u \leftrightarrow v$ in Q'_k or $f(u) \leftrightarrow f(v)$ in Q_k then $p(u) = 1 - p(v)$.
- <u>If $u \leftrightarrow v$</u> in Q'_k then <u>$f(u)f(v) = u\bar v$ or $f(u)f(v) = \bar u v$</u>, therefore <u>$f(u) \leftrightarrow f(v)$ in Q_k</u>
- <u>If $f(u) \leftrightarrow f(v)$ in Q_k</u> then $u \leftrightarrow \bar v$ in Q_k or $\bar u \leftrightarrow v$ in Q_k, so <u>$u \leftrightarrow v$ in Q'_k</u>

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(4.4) We define a graph with vertex set $\{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ by $a_i \leftrightarrow b_j$ iff $i+j > k$

and $\{a_1, \ldots, a_k\}$, $\{b_1, \ldots, b_k\}$ are independent.

- $d(a_i) = i$ as a_i is adjacent to vertices $b_k, b_{k-1}, \ldots, b_{k-i+1}$
- $d(b_i) = i$ as b_i is adjacent to vertices $a_k, \ldots, a_{k-i+1}$

So the degree sequence of this simple graph is $(1,1,2,2,\ldots,k,k)$ as required.

Picture for $k=4$