Recommended Problems 6 – Solutions

6.1 For an edge e of K_n let $b(e)$ denote the number of spanning trees of K_n containing e.

K_n has n^{n-2} spanning trees, each has $n-1$ edges, so

$$\sum_{e \in E(K_n)} b(e) = n^{n-2} \cdot (n-1). \qquad (1)$$

By symmetry, $b(e) = b(f)$ for every $e, f \in E(K_n)$. Therefore the left hand side is a sum of equal numbers, the number of summands is $|E(K_n)| = \frac{n(n-1)}{2}$. Thus, for every $g \in E(K_n)$ we have

$$\sum_{e \in E(K_n)} b(e) = \frac{n(n-1)}{2} b(g) \qquad (2)$$

By comparing (1) & (2) we get

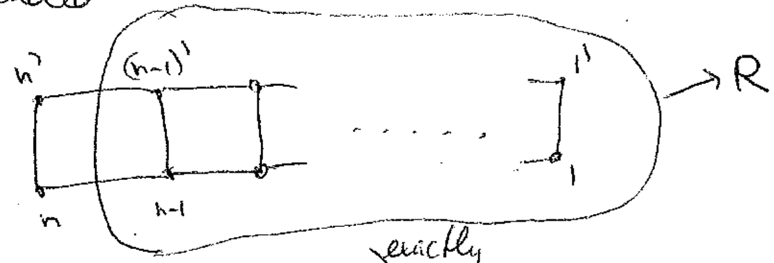
$$b(g) = 2n^{n-3}.$$

# of spanning trees of $K_n - g$ is equal to the # of spanning trees of K_n which do not contain g. This is equal to

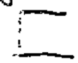
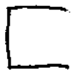
$$\underbrace{n^{n-2}}_{\text{\# of all spanning trees of } K_n} - \underbrace{b(g)}_{\text{\# of spanning trees containing } g} = n^{n-2} - 2n^{n-3} = n^{n-3}(n-2)$$

Recommended Problems 6 – Solutions

6.2

$G_n =$ 

Every spanning tree T is of exactly one of the following 4 types (this follows from ~~acyclicity and~~ connectivity)

1) $n'(n-1)'$, $n(n-1) \in E(T)$, $nn' \notin E(T)$ 
2) nn', $n'(n-1)' \in E(T)$, $n(n-1) \notin E(T)$ 
3) nn', $n(n-1) \in E(T)$, $n'(n-1)' \notin E(T)$ 
4) nn', $n(n-1)$, $n'(n-1)' \in E(T)$ 

\# of trees of type 1) is $\tau(G_{n-1})$, since ~~every~~ for every such a tree T, the subgraph of T induced by R is a tree (correct number of edges, acyclic), so trees of type 1) are in bijective correspondence with spanning trees of G_{n-1}.

Similarly, # of trees of type 2) (as well as 3)) is $\tau(G_{n-1})$.

It remains to count trees of type 4). If we remove $n \leftrightarrow n'$ from such a tree and add $(n-1) \leftrightarrow (n-1)'$ we get a tree (correct # of edges, connected ~~acyclic~~) and, as above, the subgraph of this new tree induced by R is a tree.

It follows that trees of type 4) are in bijection with spanning trees of G_{n-1} containing the edge $(n-1)(n-1)'$. The number of the latter trees can be computed as

$$\underset{\text{all spanning trees}}{\tau(G_{n-1})} - \underbrace{\#\text{ of sp. trees of } G_{n-1} \text{ which do not contain } (n-1)(n-1)'}_{k}$$

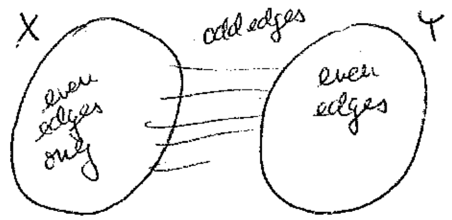
$= \tau(G_{n-1}) - \tau(G_{n-2})$ (since k = # sp. trees of the graph G_{n-1} of type 1))

Altogether we have $\tau(G_{n-1}) + \tau(G_{n-1}) + \tau(G_{n-1}) + (\tau(G_{n-1}) - \tau(G_{n-2}))$

$= 4\tau(G_{n-1}) - \tau(G_{n-2})$ spanning trees of G_n.

Recommended Problems 6 – Solutions

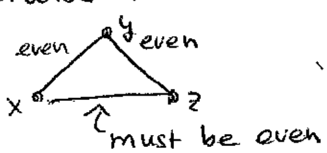
(6.3) $\Leftarrow$



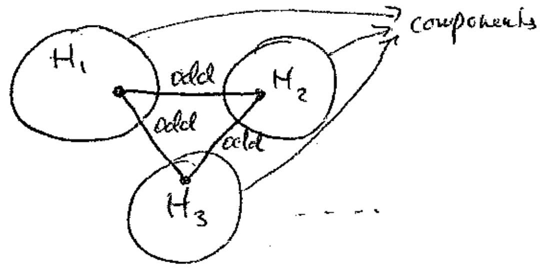
even edge = edge of even weight
odd edge = edge of odd weight

Every cycle has even number of odd edges (if, say, the cycle starts in X, then every time we exit X we have to return to X), therefore every cycle has even weight.

$\Rightarrow$ Assume that every cycle has even weight and let H be the subgraph (spanning) formed by even edges*. Observe that if $xy, yz \in E(H)$ then $xz \in E(H)$ otherwise we would have a triangle of odd weight in K_n:



It follows that every component of H is a complete graph. The number of components of H is at most 2, otherwise we would again get a cycle of odd weight:



So either $H = K_n$, ~~therefore~~ then $\overline{H}$ has only isolated vertices (which can be viewed, stretching the definitions a bit, as $K_{0,n}$)

or $H = K_m + K_{n-m}$, then $\overline{H} = \overline{K_m + K_{n-m}} = K_{m,n-m}$ as required.

(6.4) The proof is similar to the correctness proof of Kruskal's algorithm. See "Prim's algorithm" on Wikipedia for details.

---

* Note that the subgraph we are interested in is the complement of H