Preliminaries 2

Definition 1.

- An *algebra* is a tuple $\mathbf{A} = (A; f_1, \dots, f_k)$, where A is a set (the *universe* of $\mathbf{A}$) and $f_1, \dots, f_k$ are operations on A (called the *basic operations*).
- A *clone* is a pair $\mathbf{A} = (A; \mathcal{F})$, where $\mathcal{F}$ is a set of operations on A such that
  - For every n and every $i \leq n$ the operation π_i^n, defined by $\pi_i^n(x_1, \dots, x_n) = x_i$, belongs to $\mathcal{F}$
  - For any m, n, if f is an n-ary operation from $\mathcal{F}$ and $g_1, \dots, g_n$ are m-ary operations from $\mathcal{F}$, then the (m-ary) operation h, defined by

    $$h(x_1, \dots, x_m) = f(g_1(x_1, \dots, x_m), g_2(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m))$$

    also belongs to $\mathcal{F}$.

  The set of all n-ary members of $\mathcal{F}$ will be denoted by $\mathbf{A}^{(n)}$ or $\mathcal{F}^{(n)}$.
- We say that an algebra $\mathbf{A} = (A; f_1, \dots, f_k)$ generates a clone $(A; \mathcal{F})$, if $f_1, \dots, f_k \in \mathcal{F}$ and $(A; \mathcal{F})$ is the smallest clone with this property (more precisely, whenever a clone $(A; \mathcal{G})$ contains $f_1, \dots, f_k$, then $\mathcal{F} \subseteq \mathcal{G}$). Such a clone is uniquely determined by $\mathbf{A}$ (why?), we call it the *clone generated by* $\mathbf{A}$.
- Let $\mathbb{A} = (A, \dots)$ be a relational structure and let $\mathcal{F}$ be the set of all polymorphisms of $\mathbb{A}$. Then $(A, \mathcal{F})$ is a clone (see Problem 1 (a),(b) in Problem Set 1). We will denote this clone by $\mathrm{Pol}(\mathbb{A})$.

Remark 2.

- Let $(A, \mathcal{F})$ be a clone. Whenever an operation h can be defined by a formula using members of $\mathcal{F}$, then h is a member of $\mathcal{F}$. For instance, if $f \in \mathcal{F}^{(2)}$ and $g \in \mathcal{F}^{(3)}$ then the following operation h belongs to $\mathcal{F}^{(4)}$:

  $$h(x, y, z, v) = f(x, y, (g(g(y, z), f(z, x, y))))$$

  Prove it!
- The clone $(A, \mathcal{F})$ generated by an algebra $\mathbf{A}$ can be described alternatively as follows. The elements of $\mathcal{F}$ are precisely those operation which can be defined by a formula using only operations of the algebra $\mathbf{A}$.

  When we want to find the clone generated by an algebra we may proceed as follows. First we try to find as many operations from the basic operations (by means of composition) as we can. We keep doing it until we are unable to find new operations. In this way we obtain a candidate for the clone. Then we check whether our set is really a clone.
- It is sometimes convenient to describe a clone as $\mathrm{Pol}(\mathbb{A})$ for some relational structure $\mathbb{A}$.

Some important operations on the set $\{0, 1\}$

- $\neg$: unary operation defined by $\neg x = 1 - x$ (negation)
- $\wedge$: binary operation defined by $x \wedge y = \min\{x, y\} = xy$, i.e. $x \wedge y = 1$ iff $x = y = 1$ (AND)
- $\vee$: binary operation defined by $x \vee y = \max\{x, y\}$, i.e. $x \vee y = 0$ iff $x = y = 0$ (OR)
- $\rightarrow$: binary operation defined by $x \rightarrow y = 0$ iff $x = 1$ and $y = 0$. (implication)
- $+$: binary addition modulo 2 (XOR)