G. N. BERMAN

A Collection of Problems on a Course of Mathematical Analysis

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FOREWORD TO THE TENTH RUSSIAN EDITION

THE present *Collection of Problems* is intended for students studying mathematical analysis within the framework of a technical college course. In the arrangement of the material, the style of the exposition and basic pedagogical tendencies the *Collection* is most suited to the widely used *Course of Mathematical Analysis* of A. F. Bermant†. At the same time, since the book contains systematically selected problems and exercises on the main branches of a Technical College course of mathematical analysis, it forms a useful adjunct independently of the text-book on which the course is based.

Theoretical information and references to the necessary formulae are omitted in the *Collection of Problems;* it is assumed that the reader can find them in the relevant sections of his text-book. Most of the articles of the *Collection of Problems* are subdivided for convenience of use. A common instruction precedes a group of problems of the same type. Problems with a physics content are preceded by the necessary physical laws. In the case of more or less difficult problems, hints are given in the answers; such problems are marked by an asterisk (*).

The *Collection of Problems* was produced directly for the first edition (1947) by Georgii Nikolayevich Berman. All the subsequent editions, which have twice included substantial revisions, have been brought out without the original author, who died on 9th February 1949 after a long and serious illness, resulting from wounds received at the front in the Second World War. Those who have undertaken the revision — essentially friends and co-workers of Georgii Nikolayevich — always recall him with feelings of great respect; he was a man of wide culture and a talented pedagogue.

Both revisions of the *Collection of Problems* (the first for the second edition of 1950, the second for the sixth edition of 1956) have been carried out by I. G. Aramanovich, B. A.

† This text-book is simply referred to as the *Course* in the text of the *Collection.*

Kordemskii, R. I. Pozoiskii and M. G. Shestopal. A part in this work was taken by A. F. Bermant, the author of the above-mentioned *Course,* who edited the *Collection.*

With the aim of improving the *Collection of Problems* from the methodological point of view, and to take account of criticisms obtained from teachers using the *Collection,* firstly the second, then later the sixth edition were supplied with a substantially increased number of problems in several sections; in addition, the problems were regrouped, the statement of them revised afresh, and the solutions checked. In the present tenth edition certain problems have again been given a fresh statement. The previous numbering has been retained for the unchanged problems. The only deviations from this system occur in two short chapters — the tenth and eleventh, in connection with an additional regrouping of the problems of these chapters and the inclusion of new problems on the theme of "Change of variables." The *Collection* contains a new chapter (XVI) on "Elements of the Theory of Fields" (problems no. 4401–4464) and tables of the values of certain elementary functions as an appendix. (The tables have been borrowed from the book by V. P. Minorskii, *A Collection of Problems on Higher Mathematics* (Sbornik zadach po vysshei matematike), with the consent of their compiler A. T. Tsvetkov.)

Thus certain modifications brought into the tenth edition of the *Collection of Problems* do not hinder, in the vast majority of cases, the simultaneous use of the present and previous editions (as from the sixth).

The work on the tenth edition of the *Collection* has had to proceed in the absence not only of its first author, the late G. N. Berman, but also of one of the co-authors and editor of the work, Professor Anisim Fedorovich Bermant. A. F. Bermant died suddenly on 26 May, 1959. His cherished image will never be erased from our memories — he was a gifted, lively and noble comrade, and a progressive pedagogue.

I. G. Aramanovich
B. A. Kordemskii
R. I. Pozoiskii
M. G. Shestopal

CHAPTER I

FUNCTIONS

1. Functions and Methods of Specifying Them

1. The sum of the interior angles of a plane convex polygon is a function of the number of sides. What sort of numbers can the values of the argument be?

Specify this function analytically.

2. A function is given by the following table:

Independent variable x	0	0·5	1	1·5	2	3
Function y	—1·5	—1	0	3·2	2·6	0

Independent variable x	4	5	6	7	8	9	10
Function y	—1·8	—2·8	0	1·1	1·4	1·9	2·4

Draw its graph by joining the points with a "smooth" curve, and enlarge the table by using the graph to find the values of the function at $x = 2{\cdot}5;\ 3{\cdot}5;\ 4{\cdot}5;\ 5{\cdot}5;\ 6{\cdot}5;\ 7{\cdot}5;\ 8{\cdot}5;\ 9{\cdot}5$.

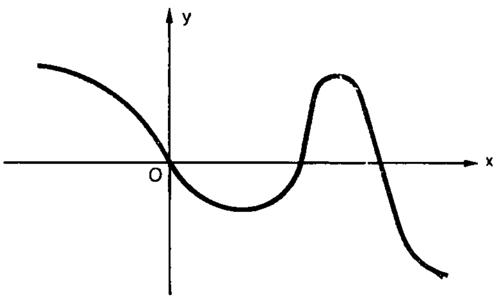

Fig. 1.

3. A function is given by the graph illustrated in Fig. 1. Transfer the figure to millimetre paper by choosing a scale and several values of the independent variable. Take from the figure the values of the function corresponding to the chosen values of the independent variable and form a table of these values.

4. A function is given by the graph illustrated in Fig. 2. Use the graph to answer the following questions:

(a) For what values of the independent variable does the function vanish?

(b) For what values of the independent variable is the function positive?

(c) For what values of the independent variable is the function negative?

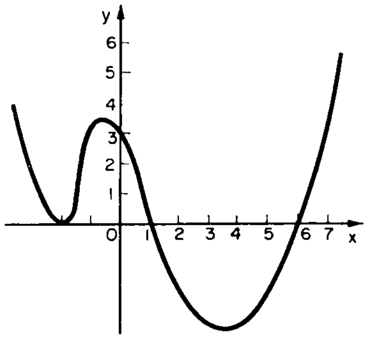

FIG. 2.

5. The force F of interaction of two electric charges e_1 and e_2 depends by Coulomb's law on the distance r between them:

$$F = \frac{e_1 e_2}{\varepsilon r^2}.$$

Putting $e_1 = e_2 = 1$ and $\varepsilon = 1$, form a table of values of this function for $r = 1, 2, 3, \ldots, 10$ and draw its graph by joining the points obtained with a "smooth" curve.

6. Form the function expressing the dependence of the radius r of a cylinder on its height h for a given volume V $(= 1)$. Work out the values of r for the following values of h: 0·5; 1; 1·5; 2; 2·5; 3; 3·5; 4; 4·5; 5. Draw the graph of the function.

7. Write down the expression for the area of the isosceles trapezium with bases a and b as a function of the angle α at base a. Draw the graph of the function for $a = 2$, $b = 1$.

8. Express the length b of one adjacent side of a right-angled triangle as a function of the length a of the other with constant hypotenuse c ($= 5$). Verify that the graph of the function is a quadrant of a circle.

2. Notation for and Classification of Functions

9. Given the functions

$$\text{(a) } f(x) = \frac{x-2}{x+1}; \qquad \text{(b) } \varphi(x) = \frac{|x-2|}{x+1}.$$

Find: $f(0)$; $f(1)$; $f(2)$; $f(-2)$; $f\left(-\frac{1}{2}\right)$; $f(\sqrt{2})$; $\left|f\left(\frac{1}{2}\right)\right|$; $\varphi(0)$; $\varphi(1)$; $\varphi(2)$; $\varphi(-2)$; $\varphi(4)$. Do $f(-1)$, $\varphi(-1)$ exist?

10. Given the function

$$f(u) = u^3 - 1,$$

find: $f(1)$; $f(a)$; $f(a+1)$; $f(a-1)$; $2f(2a)$.

11. Given the functions

$$F(z) = 2^{z-2} \quad \text{and} \quad \varphi(z) = 2^{|z|-2},$$

find: $F(0)$; $F(2)$; $F(3)$; $F(-1)$; $F(2{\cdot}5)$; $F(-1{\cdot}5)$ and $\varphi(0)$; $\varphi(2)$; $\varphi(-1)$; $\varphi(x)$; $\varphi(-1) + F(1)$.

12. Given the function

$$\psi(t) = ta^t,$$

find: $\psi(0)$; $\psi(1)$; $\psi(-1)$; $\psi\left(\frac{1}{a}\right)$; $\psi(a)$; $\psi(-a)$.

13. $\varphi(t) = t^3 + 1$. Find $\varphi(t^2)$ and $[\varphi(t)]^2$.

14. $F(x) = x^4 - 2x^2 + 5$. Show that $F(a) = F(-a)$.

15. $\Phi(z) = z^3 - 5z$. Show that $\Phi(-z) = -\Phi(z)$.

16. $f(t) = 2t^2 + \frac{2}{t^2} + \frac{5}{t} + 5t$. Show that $f(t) = f\left(\frac{1}{t}\right)$.

17. $f(x) = \sin x - \cos x$. Show that $f(1) > 0$.

18. $\psi(x) = \log x$. Show that $\psi(x) + \psi(x + 1) = \psi[x(x + 1)]$.

19. $F(z) = a^z$. (1) Show that, for any z,

$$F(-z)\,F(z) - 1 = 0.$$

(2) Show that

$$F(x)\,F(y) = F(x + y).$$

20. We are given the graph of function $y = f(x)$ and the values a and b of the independent variable x (Fig. 3). Construct $f(a)$ and $f(b)$ on the figure. What is the geometric meaning of the ratio $\frac{f(b) - f(a)}{(b - a)}$?

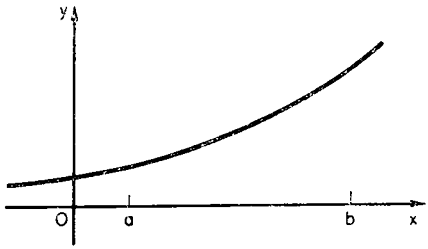

FIG. 3.

21. Show that, if any chord of the graph of the function $y = f(x)$ lies above the arc subtending it, we must have

$$\frac{f(x_1) + f(x_2)}{2} > f\left(\frac{x_1 + x_2}{2}\right) \text{ for all } x_1 \neq x_2.$$

22. Given: $f(x) = x^2 - 2x + 3$. Find all the roots of the equations (a) $f(x) = f(0)$; (b) $f(x) = f(-1)$.

23. Given: $f(x) = 2x^3 - 5x^2 - 23x$. Find all the roots of the equation $f(x) = f(-2)$.

24. Given a function $f(x)$, indicate a root of the equation $f(x) = f(a)$.

25. Indicate the roots of the equation

$$f(x) = f\left(\frac{x+8}{x-1}\right),$$

if it is known that $f(x)$ is defined in the interval $[-5, 5]$. Find all the roots of the equation for the case when

$$f(x) = x^2 - 12x + 3.$$

26. $F(x) = x^2 + 6$; $\varphi(x) = 5x$. Find all the roots of the equation $F(x) = |\varphi(x)|$.

27. $f(x) = x + 1$; $\varphi(x) = x - 2$. Solve the equation

$$|f(x) + \varphi(x)| = |f(x)| + |\varphi(x)|.$$

28. Find the values of a and b in the expression for the function $f(x) = ax^2 + bx + 5$ for which we have identically $f(x + 1) - f(x) \equiv 8x + 3$.

29. Let $f(x) = a \cos (bx + c)$. Find constants a, b and c from the condition that $f(x + 1) - f(x) \equiv \sin x$.

Functions of a Function

30. Given: $y = z^2$, $z = x + 1$, express y as a function of x.

31. Given: $y = \sqrt{z+1}$, $z = \tan^2 x$, express y as a function of x.

32. Given: $y = z^2$, $z = \sqrt[3]{x+1}$, $x = a^t$, express y as a function of t.

33. Given: $y = \sin x$; $v = \log y$; $u = \sqrt{1+v^2}$, express u as a function of x.

34. Given: $y = 1 + x$; $z = \cos y$; $v = \sqrt{1 - z^2}$, express v as a function of x.

35. Represent the following functions of a function with the aid of chains made up of basic elementary functions:

(1) $y = \sin^3 x$; (2) $y = \sqrt[3]{(1+x)^2}$; (3) $y = \log \tan x$;
(4) $y = \sin^3 (2x + 1)$; (5) $y = 5^{(3x+1)^2}$.

36. $f(x) = x^3 - x$; $\varphi(x) = \sin 2x$. Find:

(a) $f\left[\varphi\left(\frac{\pi}{12}\right)\right]$; (b) $\varphi[f(1)]$; (c) $\varphi[f(2)]$; (d) $f[\varphi(x)]$;
(e) $f[f(x)]$; (f) $f\{f[f(1)]\}$; (g) $\varphi[\varphi(x)]$.

37. Check the validity of the following method for drawing the graph of the function of a function $y = f[\varphi(x)] = F(x)$ when the graphs of the component functions $y = f(x)$, $y = \varphi(x)$ are known. From a point A of the graph of $\varphi(x)$ (Fig. 4) corresponding to a given value of independent variable x a straight line is drawn parallel to Ox to its intersection at B with the bisector of the first and third quadrants; from B a straight line is drawn parallel to Oy to its intersection with the graph of $f(x)$ at C. If a straight line is drawn from C parallel to Ox, its point of intersection D with the straight line NN' will be the point of the graph of $F(x)$ corresponding to the given value of x.

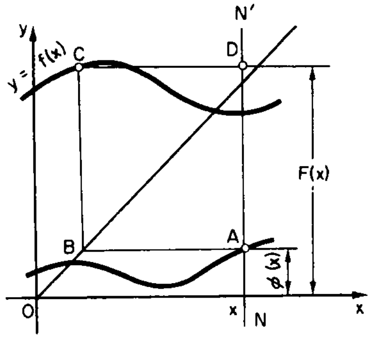

FIG. 4.

Implicit Functions

38. Write down the explicit expressions for y, given implicitly by the following equations:

(1) $x^2 + y^2 = 1$; (2) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$;

(3) $x^3 + y^3 = a^3$;

(4) $xy = C$; (5) $2^{xy} = 5$;

(6) $\log x + \log (y + 1) = 4$;

(7) $2^{x+y} (x^2 - 2) = x^3 + 7$;

(8) $(1 + x) \cos y - x^2 = 0$.

39*. Show that, for $x > 0$, the graph of function y given by the equation $y + |y| - x - |x| = 0$ is the bisector of the first quadrant, whilst for $x \leqq 0$ the function is many-valued and its "graph" is the set of points of the third quadrant (including its boundary points).

3. Elementary Investigation of Functions

Domain of Definition of a Function

40. Form a table of the values of the function of an integral argument $y = \dfrac{1}{x!}$ for $1 \leqq x \leqq 6$.

41. The value of the function of an integral argument $u = \varphi(n)$ is equal to the number of positive integers not exceeding n. Make up a table of the values of u for $1 \leqq n \leqq 20$.

42. The value of the function of an integral argument $u = f(n)$ is equal to the number of integral divisors of the argument differing from 1 and n itself. Form a table of the values of u for $1 \leqq n \leqq 20$.

43. A girder consists of three sections of 1, 2, 1 units of length weighing 2, 3, 1 units of weight respectively (Fig. 5). The weight of a variable section AM of length x is a function of x. For what values of x is this function defined? Form an analytic expression for the function and draw its graph.

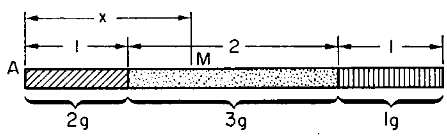

FIG. 5.

44. A tower has the following shape: a cylinder of radius R and height $2R$ is mounted on a right circular truncated cone with base radii $2R$ (lower) and R (upper); a hemisphere of radius R is mounted on the cylinder. Express the area

S of the cross-section of the tower as a function of the distance x of the cross-section from the lower base of the cone. Draw the graph of $S = f(x)$.

45. A cylinder is inscribed in a sphere of radius R. Find the volume V of the cylinder as a function of its height x. State the domain of definition of this function and the domain of definiteness of the corresponding analytic expression.

46. A right circular cone is inscribed in a sphere of radius R. Find the area S of the lateral surface of the cone as a function of its generator x. State the domain of definition of this function and the domain of definiteness of the corresponding analytic expression.

Give the domains of definition of the functions of problems 47–48.

47. (1) $y = 1 - \log x$; (2) $y = \log (x + 3)$;
(3) $y = \sqrt{5 - 2x}$; (4) $y = \sqrt{-px}\ (p > 0)$;
(5) $y = \dfrac{1}{x^2 - 1}$; (6) $y = \dfrac{1}{x^2 + 1}$; (7) $y = \dfrac{1}{x^3 - x}$;
(8) $y = \dfrac{2x}{x^2 - 3x + 2}$; (9) $y = 1 - \sqrt{1 - x^2}$;
(10) $y = \dfrac{1}{\sqrt{x^2 - 4x}}$; (11) $y = \sqrt{x^2 - 4x + 3}$;
(12) $y = \dfrac{x}{\sqrt{x^2 - 3x + 2}}$; (13) $y = \arcsin \dfrac{x}{4}$;
(14) $y = \arcsin (x - 2)$; (15) $y = \arccos (1 - 2x)$;
(16) $y = \arccos \dfrac{1 - 2x}{4}$; (17) $y = \arcsin \sqrt{2x}$;
(18) $y = \sqrt{1 - |x|}$; (19) $y = \dfrac{1}{\sqrt{|x| - x}}$;
(20) $y = \dfrac{1}{\sqrt{x - |x|}}$; (21) $y = \sqrt{\log \dfrac{5x - x^2}{4}}$;
(22) $y = \log \sin x$; (23) $y = \arccos \dfrac{2}{2 + \sin x}$;
(24) $y = \log_x 2$.

48. (1) $y = \dfrac{1}{\log(1-x)} + \sqrt{x+2}$;

(2) $y = \sqrt{3-x} + \arcsin\dfrac{3-2x}{5}$;

(3) $y = \arcsin\dfrac{x-3}{2} - \log(4-x)$;

(4) $y = \sqrt{x} + \sqrt[3]{\dfrac{1}{x-2}} - \log(2x-3)$;

(5) $y = \sqrt{x-1} + 2\sqrt{1-x} + \sqrt{x^2+1}$;

(6) $y = \dfrac{3}{4-x^2} + \log(x^3-x)$;

(7) $y = \log\sin(x-3) + \sqrt{16-x^2}$;

(8) $y = \sqrt{\sin x} + \sqrt{16-x^2}$;

(9) $y = \dfrac{1}{\sqrt{\sin x}} + \sqrt[3]{\sin x}$;

(10) $y = \log\dfrac{x-5}{x^2-10x+24} - \sqrt[3]{x+5}$;

(11) $y = \sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{1-x}{\sqrt{1+x}}}$;

(12) $y = \sqrt{x^2-3x+2} + \dfrac{1}{\sqrt{3+2x-x^2}}$;

(13) $y = (x^2+x+1)^{-\frac{3}{2}}$;

(14) $y = \log(\sqrt{x-4} + \sqrt{6-x})$;

(15) $y = \log[1 - \log(x^2-5x+16)]$.

49. Are the following functions identical?

(1) $f(x) = \dfrac{x}{x^2}$ and $\varphi(x) = \dfrac{1}{x}$;

(2) $f(x) = \dfrac{x^2}{x}$ and $\varphi(x) = x$;

(3) $f(x) = x$ and $\varphi(x) = \sqrt{x^2}$;

(4) $f(x) = \log x^2$ and $\varphi(x) = 2\log x$?

50. Suggest an example of a function specified analytically which is

(1) defined only in the interval $-2 \leqq x \leqq 2$;

(2) defined only in the interval $-2 < x < 2$ and is not defined at $x = 0$;

(3) defined everywhere except for $x = 2$, $x = 3$, $x = 4$.

51. Find the domains of definition of the single-valued branches of the function $y = \varphi(x)$ given by
(1) $y^2 - 1 + \log_2 (x - 1) = 0$; (2) $y^4 - 2xy^2 + x^2 - x = 0$

Elements of the Behaviour of Functions

52. $f(x) = \dfrac{x^2}{1 + x^2}$; state the domain of definition of $f(x)$ and verify that the function is non-negative.

53. Find the intervals in which the following functions are of constant sign or zero:

(1) $y = 3x - 6$; (2) $y = x^2 - 5x + 6$; (3) $y = 2^{x-1}$;
(4) $y = x^3 - 3x^2 + 2x$; (5) $y = |x|$.

54. Which of the functions below are even, which are odd, and which are neither even nor odd?

(1) $y = x^4 - 2x^2$; (2) $y = x - x^2$;
(3) $y = \cos x$; (4) $y = 2^x$;
(5) $y = x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$; (6) $y = \sin x$;
(7) $y = \sin x - \cos x$; (8) $y = 1 - x^2$;
(9) $y = \tan x$; (10) $y = 2^{-x^2}$;
(11) $y = \dfrac{a^x + a^{-x}}{2}$; (12) $y = \dfrac{a^x - a^{-x}}{2}$;
(13) $y = \dfrac{x}{a^x - 1}$; (14) $y = \dfrac{a^x + 1}{a^x - 1}$;
(15) $y = x \cdot \dfrac{a^x - 1}{a^x + 1}$; (16) $y = 2^{x - x^4}$;
(17) $y = \ln \dfrac{1 - x}{1 + x}$.

55. Write each of the following functions as the sum of an even and an odd function:

$$(1)\ y = x^2 + 3x + 2;$$
$$(2)\ y = 1 - x^3 - x^4 - 2x^5;$$
$$(3)\ y = \sin 2x + \cos \frac{x}{2} + \tan x.$$

56. Show that $f(x) + f(-x)$ is an even function, and $f(x) - f(-x)$ odd.

57. Write the following as the sums of even and odd functions:

$$(1)\ y = a^x; \quad (2)\ y = (1 + x)^{100} \quad \text{(see problem 56).}$$

58. Show that the product of two even functions is an even function, the product of two odd functions is even, and the product of an even and an odd function is odd.

59. Which of the following functions are periodic?

$$(1)\ y = \sin^2 x; \quad (2)\ y = \sin x^2; \quad (3)\ y = x \cos x;$$
$$(4)\ y = \sin \frac{1}{x}; \quad (5)\ y = 1 + \tan x; \quad (6)\ y = 5;$$
$$(7)\ y = E(x); \quad (8)\ y = x - E(x).$$

(The function $E(x)$ is defined thus: if x is an integer, $E(x) = x$; if x is not an integer, $E(x)$ is the greatest integer less than x. Thus $E(2) = 2$; $E(3{\cdot}25) = 3$; $E(-1{\cdot}37) = -2$.)

60. Draw the graph of the periodic function of period $T = 1$ which is given in the semi-open interval $[0, 1)$ by

$$(1)\ y = x; \quad (2)\ y = x^2.$$

61. State the intervals in which the following functions are increasing, decreasing and constant:

$$(1)\ y = |x|; \quad (2)\ y = |x| - x.$$

62. Give the maxima and minima of the functions

$$(1)\ y = \sin x^2; \quad (2)\ y = \cos x^3; \quad (3)\ y = 1 - \sin x; \quad (4)\ y = 2^{x^2}.$$

63. Use graphical addition to draw the graph of the function

$$y = f(x) + \varphi(x):$$

(1) with the graphs shown in Fig. 6;

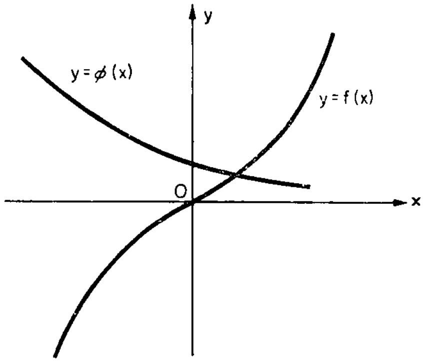

FIG. 6.

(2) with the graphs shown in Fig. 7.

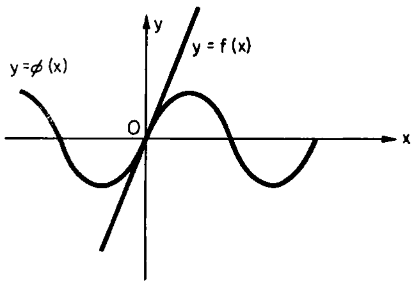

FIG. 7.

64. Knowing the graph of $y = f(x)$, draw the graphs of:

$$(1)\ y = |f(x)|; \quad (2)\ y = \frac{1}{2}[|f(x)| + f(x)];$$

$$(3)\ y = \frac{1}{2}[|f(x)| - f(x)].$$

4. Elementary Functions

Linear Functions

65. Given that the current $I = 0{\cdot}8a$ when the voltage $E = 2{\cdot}4$ V, use Ohm's law to express analytically the relationship between the current and voltage; draw the graph of the function obtained.

66. A vessel of arbitrary shape is filled with liquid. At a depth $h = 25{\cdot}3$ cm the pressure of the liquid is $p = 18{\cdot}4$ g/cm^2.

(a) Obtain the function expressing the relationship between the pressure and the depth;

(b) find the pressure at depth $h = 14{\cdot}5$ cm;

(c) at what depth does the pressure become $26{\cdot}5$ g/cm^2?

67. Starting from Newton's law, obtain the function giving the relationship between the force F and acceleration w if the force performs work $A = 32$ ergs at an acceleration of 12 cm/sec^2 over a path $s = 15$ cm.

68. Find the linear function $y = ax + b$ from the following data:

(1)

x	y
0	4
3	6

(2)

x	y
2	4·3
−1·6	0

(3)

x	y
2·5	7·2
3·2	6·8.

69. A certain quantity of gas occupies a volume 107 cm^3 at 20° C, and 114 cm^3 at 40° C.

(a) Starting from the Gay-Lussac law, obtain the function giving the relationship between the volume V of the gas and its temperature t.

(b) What is the volume at 0° C?

70. A particle in uniform motion is at a distance of 32·7 cm from the initial point after 12 sec; after 20 sec from the initial instant the distance is 43·4 cm. Find the distance s as a function of time t.

71. The voltage in a circuit falls uniformly (with a linear law). The voltage at the start is 12 V, and falls to 6·4 V at the end of the experiment lasting 8 sec. Express the voltage V as a function of time t and draw the graph of the function.

72. Find the increment of the linear function $y = 2x - 7$ when the independent variable x passes from the value $x_1 = 3$ to $x_2 = 6$.

73. Find the increment of the linear function $y = -3x + 1$ corresponding to an increment $\Delta x = 2$ of the independent variable.

74. The function $y = 2{\cdot}5x + 4$ has received the increment $\Delta y = 10$. Find the increment of the argument.

75. Given the function $y = \dfrac{x - a}{a^2 - b^2}$ and the initial value of the independent variable $x_1 = a - b$, for what finite value x_2 of the independent variable x is the increment $\Delta y = \dfrac{1}{a - b}$?

76. Function $\varphi(x)$ is given by: $\varphi(x) = \frac{1}{2}x + 2$ for $-\infty < x \leqq 2$; $\varphi(x) = 5 - x$ for $2 \leqq x < +\infty$. Find the roots of $\varphi(x) = 2x - 4$ analytically and graphically.

77. Draw the graphs of the functions

(1) $y = |x + 1| + |x - 1|$;

(2) $y = |x + 1| - |x - 1|$;

(3) $y = |x - 3| - 2\,|x + 1| + 2\,|x| - x + 1$.

78*. For what values of x does the inequality hold:

$$|f(x) + \varphi(x)| < |f(x)| + |\varphi(x)|,$$

if $f(x) = x - 3$ and $\varphi(x) = 4 - x$.

79*. For what values of x does the inequality hold:

$$|f(x) - \varphi(x)| > |f(x)| - |\varphi(x)|,$$

if $f(x) = x$ and $\varphi(x) = x - 2$.

80. A function is defined thus: $f(x)$ varies linearly in each of the intervals $n \leqq x < n + 1$ where n is a positive integer, whilst $f(n) = -1$, $f\left(n + \frac{1}{2}\right) = 0$. Draw the graph of the function.

Quadratic Functions

81. Draw the graphs and indicate the intervals of increase and decrease of the functions:

$$(1)\ y = \frac{1}{2}x^2;\quad (2)\ y = x^2 - 1;\quad (3)\ y = |x^2 - 1|;$$
$$(4)\ y = 1 - x^2;\quad (5)\ y = x^2 - x + 4;$$
$$(6)\ y = x - x^2;\quad (7)\ y = |x - x^2|;$$
$$(8)\ y = 2x^2 + 3;\quad (9)\ y = 2x^2 - 6x + 4;$$
$$(10)\ y = -3x^2 + 6x - 1;$$
$$(11)\ y = |-3x^2 + 6x - 1|;\quad (12)\ y = -x\,|x|.$$

82. The graph of a single-valued function defined in the interval $(-\infty, 6]$ consists of:

points of Ox with abscissae less than -3;

points of a parabola symmetric about Oy and passing through the points $A\,(-3,0)$, $B\,(0,5)$;

points of the straight line CD, with $C\,(3,0)$ and $D\,(6,2)$. Form the analytic expression for the function.

83. Find the maxima of:

$$(1)\ y = -2x^2 + x - 1;\quad (2)\ y = -x^2 - 3x + 2;$$
$$(3)\ y = 5 - x^2;\quad (4)\ y = -2x^2 + ax - a^2;$$
$$(5)\ y = a^2x - b^2x^2.$$

84. Find the minima of:

$$(1)\ y = x^2 + 4x - 2;\quad (2)\ y = 2x^2 - 1{\cdot}5x + 0{\cdot}6;$$
$$(3)\ y = 1 - 3x + 6x^2;\quad (4)\ y = a^2x^2 + a^4;$$
$$(5)\ y = (ax + b)(ax - 2b).$$

85. Express the number a as the sum of two terms such that their product is a maximum.

86. Express the number a as a sum of two terms such that the sum of their squares is a minimum.

87. We want to build a wooden fence so as to enclose a rectangular piece of ground next to a stone wall. The total length of the fence is 8 m. What must be the length of the part of the fence parallel to the wall for the enclosed area to be a maximum?

88. In a triangle ABC angle $A = 30°$ and the sum of the sides including this angle is 100 cm. What must be the length of side AB for the area of the triangle to be a maximum?

89. Which of the cylinders with axial section of given perimeter $P = 100$ cm has the greatest lateral surface?

90. Which of the cones with axial sections of perimeter P has the greatest lateral surface?

91. A body consists of a right circular cylinder with a cone (of the same base) mounted on it. The angle at the vertex of the cone is 60°. The perimeter of the axial section of the body is 100 cm. What must be the radius of the cylinder for the lateral surface of the body to be a maximum?

92. A rectangle is inscribed in an isosceles triangle of base a and height h, as shown in Fig. 8. What must be the height of the rectangle for its area to be a maximum?

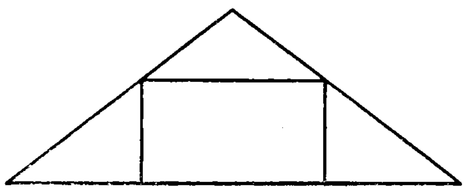

FIG. 8.

93. A cylinder is inscribed in a right circular cone such that the planes and centres of the circular bases of the cylinder and cone are the same. What is the ratio of the base radii of cylinder and cone for the lateral surface of the cylinder to be a maximum?

94. A cylinder is inscribed in a given right circular cone of base radius R and height H, such that the planes and centres of the circular bases of cone and cylinder coincide. What must be the radius of the cylinder for its total surface area to be a maximum? Consider the cases $H > 2R$, $H \leqq 2R$.

95. What must be the radius of a circle for the area of a sector of given perimeter P to be a maximum?

96. A window is in the form of a rectangle with an equilateral triangle on top. The perimeter of the window is P. What must be the base a of the rectangle for the window to have maximum area?

97. A window is in the form of a rectangle with a semicircle on top. What must be the base of the rectangle for the window to have maximum area when its perimeter is 2 m?

98. We want to cut out the corners from a rectangular piece of card-board of 30×50 cm^2 so that, on bending along the dotted lines (Fig. 9), a box is obtained with the greatest lateral surface. Find the side of the squares cut out.

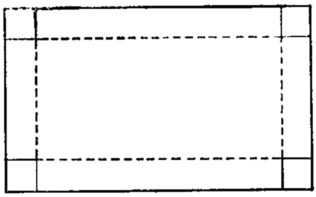

FIG. 9.

99. Using a piece of wire of length 120 cm, we want to make a model of a rectangular parallelepiped with a square base. What must be the side of the base for the total area of the parallelepiped to be a maximum?

100. A piece of wire of length a cm is to be cut in two; a square is made from one piece and an equilateral triangle from the other. How must the wire be cut for the sum of the areas of the figures thus obtained to be a minimum?

101. Find the point on the straight line $y = x$ such that the sum of the squares of its distances from the points $(-a, 0)$, $(a, 0)$ and $(0, b)$ is a minimum.

102. Find the point on the straight line $y = x + 2$ such that the sum of the squares of its distances from the straight lines $3x - 4y + 8 = 0$ and $3x - y - 1 = 0$ is a minimum.

103. An electrical current J divides into two branches with resistances r_1 and r_2 (Fig. 10). Show that the least loss of energy passing into heat per unit time corresponds to a distribution of the currents in inverse proportion to the resistances of the branches. (Start from the law: the heat given out $Q = 0{\cdot}24J^2Rt$.)

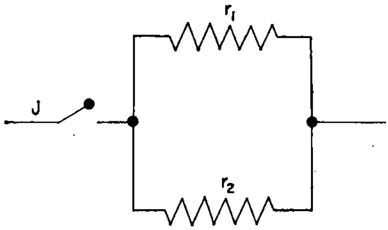

FIG. 10.

104. Trace the parabola $y = x^2$ and use it for graphical solution of the following equations:

(1) $x^2 - x - 2{\cdot}25 = 0$; (2) $2x^2 - 3x - 5 = 0$;
(3) $3{\cdot}1x^2 - 14x + 5{\cdot}8 = 0$; (4) $4x^2 - 12x + 9 = 0$;
(5) $3x^2 - 8x + 7 = 0$.

105. Function $\varphi(x)$ is given by: $\varphi(x) = \frac{1}{2}x - \frac{1}{2}$ for $-\infty < x \leqq \frac{11}{3}$; $\varphi(x) = 1 + x$ for $\frac{11}{3} \leqq x < +\infty$. Find analytically and graphically all the real roots of the equation $[\varphi(x)]^2 = 7x + 25$.

106. Give the domain of definition of the function

$$y = \log(ax^2 + bx + c).$$

107. Find $f(x + 1)$, given that $f(x - 1) = 2x^2 - 3x + 1$.

108*. Show that the function $f(x) = \dfrac{x^2 + 2x + c}{x^2 + 4x + 3c}$ takes any real value if $0 < c \leqq 1$.

Linear Rational Functions

109. Starting from the Boyle–Mariotte law, find the function showing how the volume of a gas depends on the pressure at $t = \text{const}$, if it is known that the volume is 2·3 l. at 760 mm pressure. Trace the graph of the function.

110. The variable x is inversely proportional to y, y is inversely proportional to z, and z in turn is inversely proportional to v. What is the relationship between x and v?

111. Variable x is inversely proportional to y, y is directly proportional to z, z is directly proportional to u, u is inversely proportional to v. What is the relationship between x and v?

112. During electrolysis the quantity of material separated at the electrode is proportional to the current, the current is proportional to the conductivity of the electrolyte, the conductivity is proportional to the concentration of electrolyte, the concentration for a given quantity of material is inversely proportional to the volume of solvent. How does the quantity of material separated at the electrode depend on the volume of solvent?

113. Draw the graphs of the linear rational functions:

$$(1)\ y = \frac{x-1}{x-2};\quad (2)\ y = \frac{2x}{3-x};\quad (3)\ y = \frac{2x-5}{3x-7{\cdot}5};$$

$$(4)\ y = \frac{x}{1-\frac{1}{2}x};\quad (5)\ y = \frac{4-3x}{3-2{\cdot}25x}.$$

114. Find graphically the maxima and minima of the following linear rational functions in the stated intervals:

$$(1)\ y = \frac{4}{x}\quad [1, 5];\quad (2)\ y = \frac{x}{2x-5}\quad [-1, 2];$$

$$(3)\ y = \frac{1-x}{1+x}\quad [0, 4].$$

115. Prove: (1) if the abscissae of the four points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $M_3(x_3, y_2)$, $M_4(x_4, y_4)$ of the graph of $y = \frac{k}{x}$ (Fig. 11) form the proportion $\frac{x_1}{x_2} = \frac{x_3}{x_4}$, the rectangular trapezia $M_1M_2N_2N_1$ and $M_3M_4N_4N_3$ are of equal area;

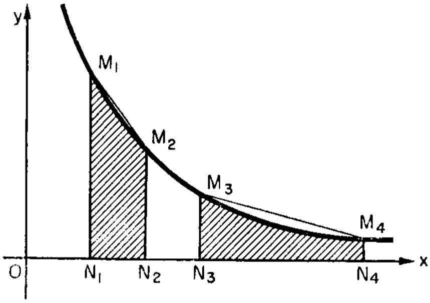

FIG. 11.

(2) if points M_1 and M_2 lie on the graph of $y = \frac{k}{x}$ (Fig. 12), the area of figures $A_1M_1M_2A_2$ and $B_1M_1M_2B_2$ are equal.

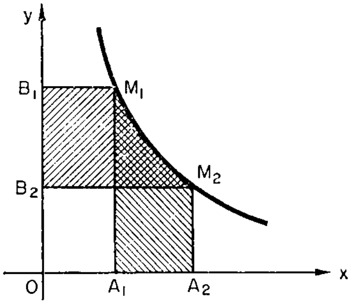

FIG. 12.

116. Use graphical addition to draw the graph of $y = \frac{x^2+1}{x}$.

5. The Inverse Functions. Power, Exponential and Logarithmic Functions

117. Find the inverses of the following functions:

(1) $y = x$; (2) $y = 2x$; (3) $y = 1 - 3x$;

(4) $y = x^2 + 1$; (5) $y = \frac{1}{x}$; (6) $y = \frac{1}{1-x}$;

(7) $y = x^2 - 2x$; (8) $y = \sqrt[3]{x^2 + 1}$; (9) $y = 10^{x+1}$;

(10) $y = 1 + \log(x + 2)$; (11) $y = \log_x 2$;

(12) $y = \frac{2^x}{1 + 2^x}$; (13) $y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}} + 1$;

(14) $y = 2 \sin 3x$; (15) $y = 1 + 2 \sin \frac{x - 1}{x + 1}$;

(16) $y = 4 \arcsin \sqrt{1 - x^2}$.

118. Show that the function $y = \frac{1 - x}{1 + x}$ is the inverse of itself. Give further examples of such functions.

119. Show that the function $f(x) = \frac{ax - b}{cx - a}$ is the same as its inverse.

120. Show that, if $f(x) = \sqrt[n]{a - x^n}$, $x > 0$, then $f[f(x)] = x$. Find the inverse of $f(x)$.

121. What is the special feature of the graph of a function which is the same as its inverse?

122. A function y of x is given by the equation $y^2 - 1 + \log_2(x - 1) = 0$. Find the domain of definition of the function and write down its inverse.

123. A function y of x is given by the equation $y^2 + \sin^3 x - y + 2 = 0$. Find the inverse of the function.

Power Functions

124. Draw the graphs of the functions:

(1) $y = \frac{1}{3} x^3$; (2) $y = -\frac{1}{2} x^3$; (3) $y = x^3 + 3x^2$;

(4) $y = x^3 - x + 1$; (5) $y = -x^3 + 2x - 2$;

(6) $y = 2x^{\frac{3}{2}}$; (7) $y = \frac{1}{2} x^{\frac{5}{4}}$; (8) $y = x^{0 \cdot 3}$;

(9) $y = x^{2 \cdot 1}$; (10) $y = x^{0 \cdot 62}$; (11) $y = \frac{1}{2} x^{-0 \cdot 2}$;

(12) $y = 5x^{-2 \cdot 5}$; (13) $y = 1 - \sqrt{|x|}$.

125. Find graphically the approximate values of the real roots of the equation $x + 3 = 4\sqrt[3]{x^2}$.

126*. Draw the graph of the cubical parabola $y = x^3$ and use the graph to find graphically the solutions of the equations:

$$(1)\ x^3 + x - 4 = 0;\quad (2)\ x^3 - 3x^2 - x + 3 = 0;$$
$$(3)\ x^3 - 6x^2 + 9x - 4 = 0;$$
$$(4)\ x^3 + 3x^2 + 6x + 4 = 0.$$

127. Given the following data, form the corresponding equation and solve it graphically:

(1) The square of a number is equal to the sum of the number and its reciprocal.

(2) A wooden sphere of radius 10 cm and density 0·8 g/cm² floats on water. Find the height of the segment submerged in the water.

(3) A wooden cube and a pyramid with square base together weigh 0·8 kg. The side of the cube is equal to the side of the base of the pyramid, the height of which is 45 cm. Find the side of the cube. The specific gravity of wood is 0·8.

128. For what values of x has the function $y = x^n$, $x > 0$, values greater than those of its inverse, and for what x has it smaller values?

Exponential and Hyperbolic Functions

129. Draw the graphs of the functions:

$$(1)\ y = -2^x;\quad (2)\ y = 2^{x+3};\quad (3)\ y = \frac{1}{3}\,3^x;$$
$$(4)\ y = 1 - 3^{x-3};\quad (5)\ y = \left(\frac{1}{2}\right)^{|x|};\quad (6)\ y = 2^{-x^2}.$$

130. Draw the graph of $y = 2^x$. Obtain on the same figure without further calculations the graphs of the functions:

$$(1)\ y = 2^{x-1};\quad (2)\ y = \frac{1}{12}2^{\frac{x}{2}};\quad (3)\ y = \frac{1}{3}2^{\frac{x-1}{2}} + 1.$$

131. Show that the graph of $y = ka^x$ $(k > 0)$ is the same curve as for $y = a^x$, but displaced with respect to the axes.

132. Use graphical addition to draw on millimetre paper the graphs of:

$$(1)\ y = x^2 + 2^x; \quad (2)\ y = x^2 - 2^x.$$

133. Solve graphically: $2^x - 2x = 0$.

134. Draw on millimetre paper the figure bounded by the curves $y = 2^x$, $y = \dfrac{1+x}{x}$, and $x = 3$. Find approximately from the graph the coordinates of the points of intersection of these curves.

135. Find the greatest possible value of n for which $2^x > x^n$ for all $x \geqq 100$ (n is an integer).

136. Show that $y = \sinh x$ and $y = \tanh x$ are odd functions, and $y = \cosh x$ even (see the *Course*, sec. 22). Are these functions periodic?

137. Prove the relationships:

$$(1)\ \cosh^2 x - \sinh^2 x = 1; \quad (2)\ \cosh^2 x + \sinh^2 x = \cosh 2x; \quad (3)\ 2 \sinh x \cosh x = \sinh 2x;$$

$$(4)\ \sinh(\alpha \pm \beta) = \sinh \alpha \cosh \beta \pm \sinh \beta \cosh \alpha;$$

$$(5)\ \cosh(\alpha \pm \beta) = \cosh \alpha \cosh \beta \pm \sinh \alpha \sinh \beta;$$

$$(6)\ 1 - \tanh^2 x = \frac{1}{\cosh^2 x}; \quad (7)\ 1 - \coth^2 x = -\frac{1}{\sinh^2 x}.$$

Logarithmic Functions

138. Draw the graphs of:

$$(1)\ y = -\log_2 x; \quad (2)\ y = \log \frac{10}{x}; \quad (3)\ y = |\log x|;$$

$$(4)\ y = \log_2 |x|; \quad (5)\ y = 1 + \log(x + 2);$$

$$(6)\ y = \log_2 |1 - x|; \quad (7)\ y = a^{\log_a x}; \quad (8)\ y = \log_x 2.$$

139. Draw the graph of $y = \log x$. Obtain on the same figure without further calculations the graphs of:

$$(1)\ y = \frac{1}{2} \log(x + 1); \quad (2)\ y = 2 \log\left(\frac{x+1}{2}\right).$$

140. Use graphical addition to draw the graph of the function $y = x + \log\frac{1}{x}$ and use the graph to find the minima of the function in the interval $(0, 2]$.

141. Show that the graph of $y = \log_a(x + \sqrt{x^2 + 1})$ is symmetric about the origin. Find the inverse function.

142. Show that the ordinate of the graph of $y = \log_a x$ is equal to the ordinate of the graph of $y = \log_{a^n} x$ multiplied by n.

6. The Trigonometric and Inverse Trigonometric Functions

Trigonometric Functions

143. Give the amplitude and period of the following harmonic oscillations:

$$(1)\ y = \sin 3x; \quad (2)\ y = 5\cos 2x;$$

$$(3)\ y = 4\sin \pi x; \quad (4)\ y = 2\sin\frac{x}{2};$$

$$(5)\ y = \sin\frac{3\pi x}{4}; \quad (6)\ y = 3\sin\frac{5x}{8}.$$

144. Give the amplitude, period, frequency and initial phase of the harmonic oscillations:

$$(1)\ y = 2\sin(3x + 5); \quad (2)\ y = -\cos\frac{x-1}{2};$$

$$(3)\ y = \frac{1}{3}\sin 2\pi\left(\omega - \frac{1}{6}\right); \quad (4)\ y = \sin\frac{2t+3}{6\pi}.$$

145. Draw the graphs of:

$$(1)\ y = -\sin x; \quad (2)\ y = 1 - \sin x;$$

$$(3)\ y = 1 - \cos x; \quad (4)\ y = \sin 2x; \quad (5)\ y = \sin\frac{x}{2};$$

$$(6)\ y = -2\sin\frac{x}{3}; \quad (7)\ y = \cos 2x;$$

$$(8)\ y = 2\sin\left(x - \frac{\pi}{3}\right); \quad (9)\ y = 2\sin\left(3x + \frac{3\pi}{4}\right);$$

$$(10)\ y = \frac{1}{2}\sin(2\pi x - 1{\cdot}2);$$

(11) $y = 2 + 2\sin\left(\frac{\pi x}{2} + \frac{\pi}{6}\right)$; (12) $y = 2\cos\frac{x-\pi}{3}$;

(13) $y = |\sin x|$; (14) $y = |\cos x|$; (15) $y = |\tan x|$;

(16) $y = |\cot x|$; (17) $y = \sec x$; (18) $y = \operatorname{cosec} x$.

$$(19)\quad y = \begin{cases} \cos x \text{ for } -\pi \leqq x \leqq 0, \\ 1 \quad \text{for} \quad 0 < x < 1, \\ \frac{1}{x} \quad \text{for} \quad 1 \leqq x \leqq 2. \end{cases}$$

146. A triangle has sides of 1 cm and 2 cm. Draw the graph of the area of the triangle as a function of the angle x between these two sides. Find the domain of definition of this function and the value of argument x for which the area is a maximum.

147. A particle moves uniformly on a circle of radius R with centre at the origin anticlockwise with linear velocity v cm/sec. The abscissa of the particle at the initial instant was a. Form the equation of the harmonic oscillation of the abscissa (see *Course*, sec. 25).

148. A point moves uniformly along the circle $x^2 + y^2 = 1$. At time t_0 its ordinate is y_0, at time t_1 the ordinate is y_1. Find the ordinate of the point as a function of the period and initial phase of the vibration and time.

149. Figure 13 illustrates a crank mechanism. The radius of the fly-wheel is R, the length of the connecting-rod a. The fly-wheel rotation is clockwise and uniform, at a rate of n revolutions per second. At the instant $t = 0$, when the connecting-rod and crank form a straight line (the "dead" position), the cross-head (A) is at point 0. Find the displacement x of the cross-head (A) as a function of time t.

150. Use graphical addition to draw the graphs of:

(1) $y = \sin x + \cos x$; (2) $y = \sin 2\pi x + \sin 3\pi x$;

(3) $y = 2\sin\frac{x}{2} + 3\sin\frac{x}{3}$; (4) $y = x + \sin x$;

(5) $y = x - \sin x$; (6) $y = -2^x + \cos x$.

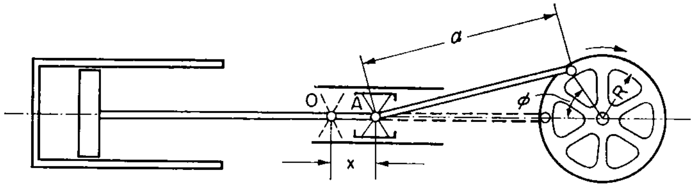

FIG. 13.

151. Solve graphically the equations:

(1) $x = 2\sin x$; (2) $x = \tan x$; (3) $x - \cos x = 0$;
(4) $4\sin x = 4 - x$; (5) $2^{-x} = \cos x$.

152. Find the periods of the compound harmonic vibrations:

(1) $y = 2\sin 3x + 3\sin 2x$; (2) $y = \sin t + \cos 2t$;

(3) $y = \sin\frac{\pi t}{3} + \sin\frac{\pi t}{4}$;

(4) $y = \sin\left(2\pi t + \frac{\pi}{3}\right) + 2\sin\left(3\pi t + \frac{\pi}{4}\right) + 3\sin 5\pi t$.

153. Express as a simple harmonic vibration:

(1) $y = \sin x + \cos x$; (2) $y = \sin x + 2\sin\left(x + \frac{\pi}{6}\right)$.

154. Give a proof of the following graphical method for adding harmonic vibrations. Given the vibrations

$$A_1 \sin(\omega x + \varphi_1) \text{ and } A_2 \sin(\omega x + \varphi_2).$$

we draw vectors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ of lengths A_1 and A_2 respectively at angles φ_1 and φ_2 to a horizontal axis (Fig. 14). On adding vectors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ we obtain the vector $\boldsymbol{A}$ of length A at an angle φ to the horizontal axis; A and φ are the amplitude and initial phase respectively of the sum

$$A_1 \sin(\omega x + \varphi_1) + A_2 \sin(\omega x + \varphi_2) = A\sin(\omega x + \varphi).$$

155*. Give the periods and draw the graphs of:

(1) $y = |\sin x| + |\cos x|$;

(2) $y = \frac{1}{2}\left(\frac{|\sin x|}{\cos x} + \frac{\sin x}{|\cos x|}\right)$.

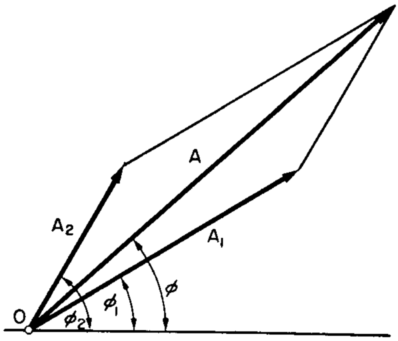

FIG. 14.

156. Find the domain of definition and indicate the shape of the graph of:

(1) $y = \log \sin x$; (2) $y = \sqrt{\log \sin x}$;

(3) $y = \sqrt{\log \dfrac{1}{|\sin x|}}$.

Inverse Trigonometric Functions

157. Draw the graphs of:

(1) $y = \operatorname{arc\,cot} x$; (2) $y = 2 \arcsin \dfrac{x}{2}$;

(3) $y = 1 + \arctan 2x$; (4) $y = \dfrac{\pi}{2} - \arccos 2x$;

(5) $y = \arcsin \dfrac{1-x}{4}$.

158. A circular sector with central angle α is folded into a cone. Find the angle ω at the vertex of the cone as a function of angle α and draw the graph of the function.

159. A painting of height a m hangs at a slope against a wall so that the dihedral angle between them is φ. The lower edge of the painting is at b m above the eye-level of the observer, who stands at a distance l m from the wall. Find the relationship between the angle γ at which the observer sees the painting and the angle φ.

160. Give the relationship between the angle α of rotation of the crank and the displacement x of the cross-head for the crank mechanism (see Fig. 13, problem 149).

161. Indicate the domains of x in which the identities hold:

(1) $\arcsin x + \arccos x = \dfrac{\pi}{2}$;

(2) $\arcsin \sqrt{x} + \arccos \sqrt{x} = \dfrac{\pi}{2}$;

(3) $\arccos \sqrt{1 - x^2} = \arcsin x$;

(4) $\arccos \sqrt{1 - x^2} = -\arcsin x$;

(5) $\arctan x = \operatorname{arc\,cot} \dfrac{1}{x}$;

(6) $\arctan x = \operatorname{arc\,cot} \dfrac{1}{x} - \pi$;

(7) $\arccos \dfrac{1 - x^2}{1 + x^2} = 2 \arctan x$;

(8) $\arccos \dfrac{1 - x^2}{1 + x^2} = -2 \arctan x$;

(9) $\arctan x + \arctan 1 = \arctan \dfrac{1 + x}{1 - x}$;

(10) $\arctan x + \arctan 1 = \pi + \arctan \dfrac{1 + x}{1 - x}$.

162. Using the identities of problem 161, find the domain of definition and draw the graph of:

(1) $y = \arccos \sqrt{1 - x^2}$;

(2) $y = \arcsin \sqrt{1 - x} + \arcsin \sqrt{x}$;

(3) $y = \arccos \dfrac{1 - x^2}{1 + x^2}$;

(4) $y = \arctan x - \operatorname{arc\,cot} \dfrac{1}{x}$.

163*. Draw the graph of $y = \arcsin (\sin x)$. Show that this is a periodic function.

164. Draw the graph of $y = \arccos (\cos x)$.

165. Draw the graph of $y = \arctan(\tan x)$.

166. Draw the graph of:

(1) $y = x - \arctan(\tan x)$;

(2) $y = x - \arcsin(\sin x)$; (3) $y = x \arcsin(\sin x)$;

(4) $y = \arccos(\cos x) - \arcsin(\sin x)$.

7. Numerical Problems

167. Draw the graph of $y = x^3 + 2x^2 - 4x + 7$ in the interval $[-4, 2]$ for values of x at intervals of 0·2; use an ordinate scale 20 times smaller than the abscissa scale. Find from the graph the maxima and minima of the function in the interval $[-3, 2]$. What is the point of transition from increase to decrease of the function? Find the zero of the function in $[-4, 2]$. The accuracy of the evaluation to be 0·1.

168. When studying the dispersion of shrapnel in artillery theory it is required to draw the graph of $y = e^{A\cos^2\alpha}$; $e \approx 2{\cdot}718$. Carry out the construction for $A = 2$, giving α values from 0 to 90° every 5°. The accuracy required is 0·01.

169. Draw the parabola $y = ax^2 + bx + c$ through three given points $M_1(1, 8)$, $M_2(5, 6)$, $M_3(9, 3)$. Find the zeros of $ax^2 + bx + c$. The required accuracy is 0·01.

170. We require to cut out equal squares from the corners of a square sheet of tin 30×30 cm² so that a box of capacity 1600 cm³ can be made by bending the remainder. What must be the length of side x of the squares cut out? The required accuracy is 0·01.

171. Show that, if we put $x^2 = y$ in the equation

$$x^4 + px^2 + qx + s = 0,$$

this can be replaced by the system

$$\left\{ \begin{aligned} x^2 &= y, \\ (y - y_0)^2 + (x - x_0)^2 &= r^2, \end{aligned} \right.$$

$$y_0 = \frac{1-p}{2}, \quad x_0 = -\frac{q}{2} \quad \text{and} \quad r^2 = y_0^2 + x_0^2 - s.$$

Using this method, solve graphically the equation

$$x^4 - 3x^2 - 8x - 29 = 0.$$

The required accuracy is 0·1.

172*. Using the method indicated in problem 171, show that, with the aid of the further substitution $x = x' + \alpha$, every fourth degree equation $x^4 + ax^3 + bx^2 + cx + d = 0$ can be solved graphically by drawing a circle and the parabola $y = x^2$.

Using this method, solve graphically the equation

$$x^4 + 1{\cdot}2x^3 - 22x^2 - 39x + 31 = 0.$$

The required accuracy is 0·1.

173. Find graphically the roots of the equation

$$\mathrm{e}^x \sin x = 1, \quad \mathrm{e} \approx 2{\cdot}718,$$

lying between 0 and 10; give an approximate general formula for the remaining roots. The accuracy required is 0·01.

174. Solve graphically the system:

$$x + y^2 = 1; \quad 16x^2 + y = 4.$$

The required accuracy is 0·01.

175. Draw the graphs of the following functions (in the polar system of coordinates) for values of φ every $\frac{\pi}{12}$:

(1) $\varrho = a\varphi$ (spiral of Archimedes)

(2) $\varrho = \frac{a}{\varphi}$ (hyperbolic spiral)

(3) $\varrho = \mathrm{e}^{a\varphi}$ $(e \approx 2{\cdot}718)$ (logarithmic spiral)

(4) $\varrho = a \sin 3\varphi$ (three-petal rose)

(5) $\varrho = a \cos 2\varphi$ (four-petal rose)

(6) $\varrho = a(1 - \cos \varphi)$ (cardioid)

The required accuracy is 0·01. Choose an arbitrary constant $a > 0$.

CHAPTER II

LIMITS

1. Basic Definitions

Functions of an Integral Argument

176. A function of an integral argument takes the values

$$u_1 = 0{\cdot}9;\ u_2 = 0{\cdot}99;\ u_3 = 0{\cdot}999;\ldots,\ u_n = \underbrace{0{\cdot}999\ldots9}_{n \text{ times}};\ldots$$

What is the value of $\lim\limits_{n\to\infty} u_n$? What must be the value of n for the absolute value of the difference between u_n and its limit not to exceed $0{\cdot}0001$?

177. The function u_n takes the values

$$u_1 = 1;\quad u_2 = \frac{1}{4};\quad u_3 = \frac{1}{9};\quad \ldots;\quad u_n = \frac{1}{n^2};\ldots$$

Find $\lim\limits_{n\to\infty} u_n$. What must n be for the difference between u_n and its limit to be less than a given positive ε?

178. Show that $u_n = \dfrac{n-1}{n+1}$ tends to unity as n increases indefinitely. As from what n is the absolute value of the difference between u_n and unity not greater than 10^{-4}?

179. The function v_n takes the values

$$v_1 = \frac{\cos\frac{\pi}{2}}{1};\quad v_2 = \frac{\cos\pi}{2};\quad v_3 = \frac{\cos\frac{3\pi}{2}}{3};\ldots;$$

$$v_n = \frac{\cos\frac{n\pi}{2}}{n};\ldots$$

Find $\lim\limits_{n\to\infty} v_n$. What must n be for the absolute value of the difference between v_n and its limit not to exceed $0{\cdot}001$?

Does v_n take the value of its limit?

180. The general term of the sequence $u_1 = \frac{1}{2}$, $u_2 = \frac{5}{4}$, $u_3 = \frac{7}{8}$, $u_4 = \frac{17}{16}, \ldots$ has the form $\frac{2^n - 1}{2^n}$, if n is odd, and $\frac{2^n + 1}{2^n}$ if n is even.

Find $\lim\limits_{n \to \infty} u_n$. What must n be for the absolute value of the difference between u_n and its limit not to exceed (i) 10^{-4}; (ii) a given ε?

181. Show that the sequence $u_n = \frac{4n^2 + 1}{3n^2 + 2}$ tends to a limit equal to $\frac{3}{4}$ whilst increasing monotonically on indefinite increase of n. As from what n is $\frac{3}{4} - u_n$ not greater than a given positive ε?

182. Show that $u_n = \frac{\sqrt{n^2 + a^2}}{n}$ has a limit equal to unity as n increases indefinitely. As from what n is $|1 - u_n|$ not greater than a given positive ε?

What is the nature of the variation of variable u_n in the limit?

183. A function v_n takes the values ("binomial coefficients")

$$v_1 = m, \quad v_2 = \frac{m(m-1)}{1\,.\,2}, \quad v_3 = \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}, \ldots,$$

$$v_n = \frac{m(m-1)(m-2)\ldots[m-(n-1)]}{1\,.\,2\,.\,3\ldots n}, \ldots,$$

where m is a positive integer. Find $\lim\limits_{n \to \infty} v_n$.

184. Show that the sequence $u_n = 1 + (-1)^n$ has no limit as n increases indefinitely.

185. Show that the sequence $u_n = \frac{2^n + (-2)^n}{2^n}$ has no limit as n increases indefinitely, whilst the sequence $v_n = \frac{2^n + (-2)^n}{3^n}$ has a limit.

What is it?

186. Do the following sequences have limits?

$$(1)\ u_n = n \sin\frac{n\pi}{2};\quad (2)\ u_n = \frac{\sin\frac{n\pi}{2}}{\log n}\quad (n > 1)?$$

187. Prove the theorem: if sequences $u_1, u_2, \ldots, u_n, \ldots$ and $v_1, v_2, \ldots, v_n, \ldots$ tend to a common limit a, the sequence $u_1, v_1, u_2, v_2, \ldots, u_n, v_n, \ldots$ tends to the same limit.

188. Prove the theorem: if a sequence $u_1, u_2, \ldots, u_n, \ldots$ tends to a limit a, any infinite subsequence of it (say $u_1, u_3, u_5, \ldots$) tends to the same limit.

189. The sequence $u_1, u_2, \ldots, u_n, \ldots$ has limit $a \neq 0$. Show that $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$. What can be said about this limit if $a = 0$? (Give examples.)

Functions of a Continuous Argument

190. Given $y = x^2$, when $x \to 2$, $y \to 4$. What must δ be for $|x - 2| < \delta$ to imply $|y - 4| < \varepsilon = 0{\cdot}001$?

191. Let $y = \frac{x^2 - 1}{x^2 + 1}$. When $x \to 2$, we have $y \to \frac{3}{5}$. What must δ be for $|x - 2| < \delta$ to imply $\left|y - \frac{3}{5}\right| < 0{\cdot}1$?

192. Let $y = \frac{x - 1}{2(x + 1)}$. When $x \to 3$, we have $y \to \frac{1}{4}$. What must δ be for $|x - 3| < \delta$ to imply $\left|\frac{1}{4} - y\right| < 0{\cdot}01$?

193. Show that $\sin x$ tends to unity as $x \to \frac{\pi}{2}$. What condition must x satisfy in the neighbourhood of the point $x = \frac{\pi}{2}$ for $1 - \sin x < 0{\cdot}01$?

194. When x increases indefinitely the function $y = \frac{1}{x^2+1}$ tends to zero: $\lim\limits_{x\to\infty} \frac{1}{x^2+1} = 0$. What must N be for $|x| > N$ to imply $y < \varepsilon$?

195. As $x \to \infty$, $y = \frac{x^2-1}{x^2+3} \to 1$. What must N be for $|x| > N$ to imply $|y - 1| < \varepsilon$?

2. Orders of Magnitude. Tests for the Existence of a Limit

Magnitudes of Large and Small Orders

196. A function u_n takes the values

$$u_1 = 3, \quad u_2 = 5, \quad u_3 = 7, \ldots, \quad u_n = 2n + 1, \ldots$$

Prove that u_n is a large order magnitude as $n \to \infty$. As from what n is u_n greater than N?

197. Show that the general term u_n of any arithmetic progression is a large order magnitude as $n \to \infty$. (When is it positive and when negative?)

Does this statement hold for any geometric progression?

198. As $x \to 0$, we have $y = \frac{1+2x}{x} \to \infty$. What condition must x satisfy for the inequality $|y| > 10^4$ to hold?

199. Show that the function $y = \frac{x}{x-3}$ is of large order as $x \to 3$. What must x be for $|y|$ to be greater than 1000?

200. When x tends to unity, the function $y = \frac{1}{(x-1)^2}$ increases indefinitely. What must δ be for $|x - 1| < \delta$ to imply $\frac{1}{(x-1)^2} > N = 10^4$?

201. The function $y = \frac{1}{2^x - 1}$ is infinitely large as $x \to 0$. What inequality must x satisfy for $|y|$ to be greater than 100?

202. As $x \to \infty$, we have $y = \log x \to \infty$. What must M be for $x > M$ to imply $y > N = 100$?

203. Which of the basic elementary functions are bounded throughout their domain of definition?

204. Show that the function $y = \dfrac{x^2}{1 + x^4}$ is bounded throughout the real axis.

205. Is the function $y = \dfrac{x^2}{1 + x^5}$ bounded throughout the real axis? Is it bounded in the interval $(0, \infty)$?

206. Is the function $y = \log \sin x$ bounded throughout its domain of existence?

Answer the same question for $y = \log \cos x$.

207. Show that the functions $y = x \sin x$ and $y = x \cos x$ are not bounded as $x \to \infty$ (indicate for each of them at least one sequence of x_n such that $y_n \to \infty$).

Do the functions become infinitely large?

Sketch the graphs of the functions.

208. Sketch the graphs of $f(x) = 2^{x \sin x}$ and $f(x) = 2^{-x \sin x}$.

Indicate two sequences x_n and x_n' of values of x for each of these functions such that $\lim\limits_{n \to \infty} f(x_n) = \infty$ and $\lim\limits_{n \to \infty} f(x_n') = 0$

209. For what values of a is the function $y = a^x \sin x$ unbounded as $x \to +\infty$ $(x \to -\infty)$?

210. Are the following functions unbounded?

(1) $f(x) = \dfrac{1}{x} \cos \dfrac{1}{x}$ as $x \to 0$;

(2) $f(x) = x \arctan x$ as $x \to \infty$;

(3) $f(x) = 2^x \arcsin (\sin x)$ as $x \to +\infty$;

(4) $f(x) = (2 + \sin x) \log x$ as $x \to +\infty$;

(5) $f(x) = (1 + \sin x) \log x$ as $x \to +\infty$.

211. The function u_n takes the values

$$u_1 = 2, \quad u_2 = \frac{3}{4}, \quad u_3 = \frac{4}{9}, \ldots, \quad u = \frac{n+1}{n^2}, \ldots$$

Show that u_n is an infinitesimal as $n \to \infty$.

212. Function u_n takes the values

$$u_1 = -7, \quad u_2 = -\frac{1}{2}, \quad u_3 = \frac{1}{27}, \quad u_4 = \frac{1}{8}, \ldots, \quad u_n = \frac{n^2 - 8}{n^3}, \ldots$$

Show that u_n is an infinitesimal as $n \to \infty$.

213. Show that

$$y = \frac{x}{x+1} \to 0$$

as $x \to 0$. What condition must x satisfy for the inequality $|y| < 10^{-4}$ to hold?

214. Prove that the function

$$y = \sqrt{x+1} - \sqrt{x}$$

tends to zero as $x \to \infty$. What must N be for $x > N$ to imply $y < \varepsilon$?

215. Write each of the following functions, which has a limit as $x \to \infty$, as the sum of a constant (equal to the limit) and a function; prove that the latter function is an infinitesimal as $x \to \infty$:

$$(1)\ y = \frac{x^3}{x^3 - 1}; \quad (2)\ y = \frac{x^2}{2x^2 + 1}; \quad (3)\ y = \frac{1 - x^2}{1 + x^2}.$$

Tests for the Existence of Limits

216*. u_n takes the values

$$u_1 = \frac{1}{4}, \quad u_2 = \frac{1}{4} + \frac{1}{10}, \quad u_3 = \frac{1}{4} + \frac{1}{10} + \frac{1}{28}, \ldots, \quad u_n = \frac{1}{3+1} + \frac{1}{3^2+1} + \ldots + \frac{1}{3^n+1}, \ldots$$

Show that u_n tends to a limit as $n \to \infty$.

217. u_n takes the values

$$u_1 = \frac{1}{2}, \quad u_2 = \frac{1}{2} + \frac{1}{2\,.\,4}, \quad u_3 = \frac{1}{2} + \frac{1}{2\,.\,4} + \frac{1}{2\,.\,4\,.\,6}, \ldots, \quad u_n = \frac{1}{2} + \frac{1}{2\,.\,4} + \ldots + \frac{1}{2\,.\,4 \ldots (2n)}, \ldots$$

Show that u_n tends to a limit as $n \to \infty$.

218. Prove the theorem:

If, given the same variation of the independent variable, the difference between two functions is an infinitesimal, one function being increasing and the other decreasing, they both tend to the same limit.

219. The terms of two sequences u_n and v_n are given by

$$u_1 = \frac{u_0 + v_0}{2}, \quad v_1 = \frac{u_0 + 2v_0}{3};$$

$$u_2 = \frac{u_1 + v_1}{2}, \quad v_2 = \frac{u_1 + 2v_1}{3};$$

and in general

$$u_n = \frac{u_{n-1} + v_{n-1}}{2}, \quad v_n = \frac{u_{n-1} + 2v_{n-1}}{3}.$$

where u_0 and v_0 are given numbers ($u_0 < v_0$). Use the theorem of the previous problem to show that the sequences both tend to the same limit, lying between u_0 and v_0.

220. Show that the sequence u_n:

$$u_1 = \sqrt{6}, \quad u_2 = \sqrt{6 + u_1}, \ldots, u_n = \sqrt{6 + u_{n-1}}, \ldots$$

has a limit and find the limit.

3. Continuous Functions

221. A function is defined as follows:

$$\begin{array}{ll} y = 0 & \text{for } x < 0; \\ y = x & \text{for } 0 \leqq x < 1; \\ y = -x^2 + 4x - 2 & \text{for } 1 \leqq x < 3; \\ y = 4 - x & \text{for } x \geqq 3. \end{array}$$

Is this function continuous?

222. Three cylinders of the same height 5 m and base radii 3, 2 and 1 m respectively are set up end to end. Express the cross-sectional area of the figure obtained as a function

of the distance of the section from the lower base of the bottom cylinder. Is this function continuous? Draw its graph.

223. Let

$$f(x) = \begin{cases} x + 1, & \text{if } x \leqq 1; \\ 3 - ax^2, & \text{if } x > 1. \end{cases}$$

For what choice of number a is $f(x)$ continuous? (Draw its graph.)

224. Let

$$f(x) = \begin{cases} -2\sin x, & \text{if} \quad x \leqq -\frac{\pi}{2}; \\ A\sin x + B, & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}; \\ \cos x, & \text{if} \quad x \geqq \frac{\pi}{2}. \end{cases}$$

Choose the numbers A and B so that $f(x)$ is continuous; draw its graph.

225. At what points have the functions $y = \frac{1}{x-2}$ and $y = \frac{1}{(x+2)^2}$ discontinuities? Draw the graphs of the functions. Describe the difference in the behaviour of the functions close to their discontinuities.

226. The function $f(x) = \frac{x^2 - 1}{x^3 - 1}$ is not defined at $x = 1$. What value must we give $f(1)$ in order to make $f(x)$ continuous at $x = 1$?

227. What sort of discontinuities do the functions $y = \frac{\sin x}{x}$ and $y = \frac{\cos x}{x}$ have at $x = 0$?

Show the nature of the graphs of the functions in the neighbourhood of $x = 0$.

228. Investigate the continuity of the function given by:

$$y = \frac{|x|}{x} \text{ at } x \neq 0, \quad y = 0 \text{ at } x = 0.$$

Draw the graph of the function.

229. How many discontinuities (and of what kind) has the function $y = \dfrac{1}{\log|x|}$?

Sketch its graph.

230. The function $y = \arctan\dfrac{1}{x}$ is not defined at $x = 0$. Is it possible further to define $f(x)$ at $x = 0$ in such a way that the function is continuous at this point? Sketch the graph of the function.

231. Investigate the continuity of the function given by

$$f(x) = \sin\frac{\pi}{2x} \text{ at } x \neq 0,\ f(0) = 1.$$

Sketch the graph of the function.

232. Sketch the graph of $f(x) = x\sin\dfrac{\pi}{x}$. What value must we give $f(0)$ in order to make the function continuous everywhere?

233. Show that the function $y = \dfrac{1}{1 + 2^{\frac{1}{x}}}$ has a discontinuity of the first kind at $x = 0$. Sketch the graph of the function in the neighbourhood of $x = 0$ (see *Course*, sec. 36).

234. Investigate the character of the discontinuity of the function $y = 2^{-2^{\frac{1}{1-x}}}$ at $x = 1$. Could y be defined at $x = 1$ in such a way that the function would become continuous at $x = 1$?

235. Investigate the nature of the discontinuity of the function $y = \dfrac{2^{\frac{1}{x}} - 1}{2^{\frac{1}{x}} + 1}$ at the point $x = 0$.

236. A function $f(x)$ is defined as follows: $f(x) = (x + 1)2^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)}$ for $x \neq 0$ and $f(0) = 0$. Verify that the values of $f(x)$ lie exclusively between $f(-2)$ and $f(2)$ in the interval $-2 \leqq x \leqq 2$ and that $f(x)$ is nevertheless discontinuous (at what point?). Sketch its graph.

237. Investigate the continuity of the function $y = \dfrac{1}{1 + 2^{\tan x}}$. What sort of graph has the function?

238. A function is defined thus: if x is a rational number, $f(x) = 0$; if x is irrational, $f(x) = x$. For what value of x is the function continuous?

239. Investigate the continuity and draw the graphs of:

(1) $y = x - E(x)$; (2) $y = \dfrac{1}{x - E(x)}$;

(3) $y = (-1)^{E(x)}$.

Function $E(x)$ is equal to the greatest integer not greater than x (see also problem 59).

240. Use the properties of continuous functions to show that the equation $x^5 - 3x = 1$ has at least one root lying between 1 and 2.

241*. Prove that: (a) a polynomial of odd degree has at least one real root; (b) a polynomial of even degree has at least two real roots if it takes at least one value of opposite sign to its first coefficient.

242. Prove that the equation $x2^x = 1$ has at least one positive root less than unity.

243. Show that the equation $x = a \sin x + b$, where $0 < a < 1$, $b > 0$, has at least one positive root, which does not exceed $b + a$.

244*. Show that the equation $\dfrac{a_1}{x - \lambda_1} + \dfrac{a_2}{x - \lambda_2} + \dfrac{a_3}{x - \lambda_3} = 0$, where $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $\lambda_1 < \lambda_2 < \lambda_3$, has two real roots in the intervals (λ_1, λ_2) and (λ_2, λ_3).

4. Finding Limits. Comparison of Infinitesimals

Functions of an Integral Argument

Find the limits in problems 245–267:

245. $\lim\limits_{n \to \infty} \dfrac{n + 1}{n}$.

246. $\lim\limits_{n \to \infty} \dfrac{(n + 1)^2}{2n^2}$.

247. $\lim\limits_{n\to\infty} \frac{(n+1)^3-(n-1)^3}{(n+1)^2+(n-1)^2}$. **248.** $\lim\limits_{n\to\infty} \frac{n^3-100n^2+1}{100n^2+15n}$.

249. $\lim\limits_{n\to\infty} \frac{1000n^3+3n^2}{0{\cdot}001n^4-100n^3+1}$. **250.** $\lim\limits_{n\to\infty} \frac{(n+1)^4-(n-1)^4}{(n+1)^4+(n-1)^4}$.

251. $\lim\limits_{n\to\infty} \frac{(2n+1)^4-(n-1)^4}{(2n+1)^4+(n-1)^4}$. **252.** $\lim\limits_{n\to\infty} \frac{\sqrt[3]{n^3+2n-1}}{n+2}$.

253. $\lim\limits_{n\to\infty} \frac{\sqrt[3]{n^2+n}}{n+1}$. **254.** $\lim\limits_{n\to\infty} \frac{(\sqrt{n^2+1}+n)^2}{\sqrt[3]{n^6+1}}$.

255. $\lim\limits_{n\to\infty} \frac{\sqrt{n^3-2n^2+1}+\sqrt[3]{n^4+1}}{\sqrt[4]{n^6+6n^5+2}-\sqrt[5]{n^7+3n^3+1}}$.

256. $\lim\limits_{n\to\infty} \frac{\sqrt[4]{n^5+2}-\sqrt[3]{n^2+1}}{\sqrt[5]{n^4+2}-\sqrt{n^3+1}}$.

257. $\lim\limits_{n\to\infty} \frac{n!}{(n+1)!-n!}$. **258.** $\lim\limits_{n\to\infty} \frac{(n+2)!+(n+1)!}{(n+3)!}$.

259. $\lim\limits_{n\to\infty} \frac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!}$.

260. $\lim\limits_{n\to\infty} \frac{1+\frac{1}{2}+\frac{1}{4}+\ \dots\ +\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{9}+\ \dots\ +\frac{1}{3^n}}$.

261. $\lim\limits_{n\to\infty} \frac{1}{n^2}(1+2+3+\dots+n)$.

262. $\lim\limits_{n\to\infty} \left(\frac{1+2+3+\dots+n}{n+2}-\frac{n}{2}\right)$.

263. $\lim\limits_{n\to\infty} \left(\frac{1-2+3-4+\dots-2n}{\sqrt{n^2+1}}\right)$.

264*. $\lim\limits_{n\to\infty} \left(\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\dots+\frac{1}{(n-1)n}\right)$.

265. $\lim\limits_{n\to\infty} \left(\frac{1}{1\,.\,3}+\frac{1}{3\,.\,5}+\dots+\frac{1}{(2n-1)(2n+1)}\right)$.

266. $\lim\limits_{n\to\infty} \dfrac{2^n - 1}{2^n + 1}$.

267. $\lim\limits_{n\to\infty} \dfrac{2^{\frac{1}{n}} - 1}{2^{\frac{1}{n}} + 1}$.

Functions of a Continuous Argument

Find the limits in problems 268–304:

268. $\lim\limits_{x\to 2} \dfrac{x^2 + 5}{x^2 - 3}$.

269. $\lim\limits_{x\to 0} \left(\dfrac{x^3 - 3x + 1}{x - 4} + 1\right)$.

270. $\lim\limits_{x\to 1} \dfrac{x}{1 - x}$.

271. $\lim\limits_{x\to \sqrt{3}} \dfrac{x^2 - 3}{x^4 + x^2 + 1}$.

272. $\lim\limits_{x\to 1} \dfrac{x^2 - 2x + 1}{x^3 - x}$.

273. $\lim\limits_{x\to -2} \dfrac{x^3 + 3x^2 + 2x}{x^2 - x - 6}$.

274. $\lim\limits_{x\to 1} \dfrac{(x - 1)\sqrt{2 - x}}{x^2 - 1}$.

275. $\lim\limits_{x\to \frac{1}{2}} \dfrac{8x^3 - 1}{6x^2 - 5x + 1}$.

276. $\lim\limits_{x\to 1} \dfrac{x^3 + x - 2}{x^3 - x^2 - x + 1}$.

277. $\lim\limits_{x\to 1} \left(\dfrac{1}{1 - x} - \dfrac{3}{1 - x^3}\right)$.

278. $\lim\limits_{x\to 2} \left[\dfrac{1}{x(x - 2)^2} - \dfrac{1}{x^2 - 3x + 2}\right]$.

279. $\lim\limits_{x\to 1} \left[\dfrac{x + 2}{x^2 - 5x + 4} + \dfrac{x - 4}{3(x^2 - 3x + 2)}\right]$.

280. $\lim\limits_{x\to 1} \dfrac{x^m - 1}{x^n - 1}$ (m and n are integers).

281. $\lim\limits_{x\to \infty} \dfrac{x^3 + x}{x^4 - 3x^2 + 1}$.

282. $\lim\limits_{x\to \infty} \dfrac{x^4 - 5x}{x^2 - 3x + 1}$.

283. $\lim\limits_{x\to \infty} \dfrac{x^2 - 1}{2x^2 + 1}$.

284. $\lim\limits_{x\to \infty} \dfrac{1 + x - 3x^3}{1 + x^2 + 3x^3}$.

285. $\lim\limits_{x\to \infty} \left(\dfrac{x^3}{x^2 + 1} - x\right)$.

286. $\lim\limits_{x\to \infty} \left(\dfrac{x^3}{2x^2 - 1} - \dfrac{x^2}{2x + 1}\right)$.

287. $\lim\limits_{x\to \infty} \left[\dfrac{3x^2}{2x + 1} - \dfrac{(2x - 1)(3x^2 + x + 2)}{4x^2}\right]$.

288. $\lim\limits_{x\to \infty} \dfrac{(x + 1)^{10} + (x + 2)^{10} + \ldots + (x + 100)^{10}}{x^{10} + 10^{10}}$.

289. $\lim\limits_{x\to+\infty} \dfrac{\sqrt{x^2+1}+\sqrt{x}}{\sqrt[4]{x^3+x}-x}$.

290. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+1}-\sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$.

291. $\lim\limits_{x\to+\infty} \dfrac{\sqrt[5]{x^7+3}+\sqrt[4]{2x^3-1}}{\sqrt[6]{x^8+x^7+1}-x}$.

292. $\lim\limits_{x\to\infty} \dfrac{\sqrt[3]{x^4+3}-\sqrt[5]{x^3+4}}{\sqrt[3]{x^7+1}}$.

293. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x^2}-1}{x}$.

294. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1}{x^2}$.

295. $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$.

296. $\lim\limits_{x\to 5} \dfrac{\sqrt{x-1}-2}{x-5}$.

297. $\lim\limits_{x\to 1} \dfrac{x^2-\sqrt{x}}{\sqrt{x}-1}$.

298. $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$.

299. $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+x^2}-1}{x^2}$.

300. $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}$.

301. $\lim\limits_{x\to a} \dfrac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$ $(a > b)$.

302. $\lim\limits_{x\to 1} \dfrac{\sqrt[n]{x}-1}{\sqrt[m]{x}-1}$ (n and m are integers).

303*. $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+x^2}-\sqrt[4]{1-2x}}{x+x^2}$.

304. $\lim\limits_{x\to 1} \dfrac{\sqrt[3]{7+x^3}-\sqrt{3+x^2}}{x-1}$.

305. How do the roots of the quadratic equation $ax^2 + bx + c = 0$ vary when b and c remain constant $(b \neq 0)$ and a tends to zero?

Find the limits in problems 306–378:

306. $\lim\limits_{x\to\infty} (\sqrt{x+a}-\sqrt{x})$.

307. $\lim\limits_{x\to\infty} (\sqrt{x^2+1}-\sqrt{x^2-1})$.

308. $\lim\limits_{x \to \pm\infty} (\sqrt{x^2 + 1} - x)$†. **309.** $\lim\limits_{x \to \pm\infty} x(\sqrt{x^2 + 1} - x)$.

310. $\lim\limits_{x \to \pm\infty} (\sqrt{(x + a)(x + b)} - x)$.

311. $\lim\limits_{x \to \pm\infty} (\sqrt{x^2 - 2x - 1} - \sqrt{x^2 - 7x + 3})$.

312. $\lim\limits_{x \to \infty} (\sqrt[3]{(x + 1)^2} - \sqrt[3]{(x - 1)^2})$.

313. $\lim\limits_{x \to \infty} x^{\frac{3}{2}} (\sqrt{x^3 + 1} - \sqrt{x^3 - 1})$.

314. $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x}$. **315.** $\lim\limits_{x \to 0} \dfrac{\tan kx}{x}$.

316. $\lim\limits_{x \to 0} \dfrac{\sin \alpha x}{\sin \beta x}$. **317.** $\lim\limits_{x \to 0} \dfrac{\tan 2x}{\sin 5x}$.

318. $\lim\limits_{\alpha \to 0} \dfrac{\sin (\alpha^n)}{(\sin \alpha)^m}$ (n and m are positive integers).

319. $\lim\limits_{x \to 0} \dfrac{2 \arcsin x}{3x}$. **320.** $\lim\limits_{x \to 0} \dfrac{2x - \arcsin x}{2x + \arctan x}$.

321. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$. **322.** $\lim\limits_{x \to 0} \dfrac{1 - \cos^3 x}{x \sin 2x}$.

323. $\lim\limits_{\alpha \to 0} \dfrac{\tan \alpha}{\sqrt[3]{(1 - \cos \alpha)^2}}$. **324.** $\lim\limits_{x \to 0} \dfrac{1 + \sin x - \cos x}{1 - \sin x - \cos x}$.

325. $\lim\limits_{\alpha \to 0} \dfrac{\tan \alpha - \sin \alpha}{\alpha^3}$. **326.** $\lim\limits_{\alpha \to 0} \dfrac{(1 - \cos \alpha)^2}{\tan^3 \alpha - \sin^3 \alpha}$.

327. $\lim\limits_{x \to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{\tan x}\right)$. **328.** $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2}$.

329. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos x}{\sqrt[3]{(1 - \sin x)^2}}$. **330.** $\lim\limits_{x \to \pi} \dfrac{\sin 3x}{\sin 2x}$.

331. $\lim\limits_{x \to \frac{\pi}{2}} \left(\dfrac{\pi}{2} - x\right) \tan x$. **332.** $\lim\limits_{\alpha \to \pi} \dfrac{\sin \alpha}{1 - \dfrac{\alpha^2}{\pi^2}}$.

† In problems where we indicate $x \to \pm\infty$, the cases $x \to +\infty$ and $x \to -\infty$ have to be considered separately.

333. $\lim\limits_{z\to 1} (1-z)\tan\frac{\pi z}{2}$.

334. $\lim\limits_{y\to a} \left(\sin\frac{y-a}{2}\tan\frac{\pi y}{2a}\right)$.

335. $\lim\limits_{x\to\frac{\pi}{4}} \frac{\cos x - \sin x}{\cos 2x}$.

336. $\lim\limits_{x\to\frac{\pi}{6}} \frac{\sin\left(x-\frac{\pi}{6}\right)}{\frac{\sqrt{3}}{2}-\cos x}$.

337. $\lim\limits_{x\to\pi} \frac{1-\sin\frac{x}{2}}{\cos\frac{x}{2}\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)}$.

338. $\lim\limits_{x\to\frac{\pi}{2}} \left(2x\tan x - \frac{\pi}{\cos x}\right)$.

339. $\lim\limits_{x\to 0} \frac{\cos(a+x)-\cos(a-x)}{x}$.

340. $\lim\limits_{x\to 0} \frac{\cos\alpha x - \cos\beta x}{x^2}$.

341. $\lim\limits_{x\to 0} \frac{\sin(a+x)-\sin(a-x)}{\tan(a+x)-\tan(a-x)}$.

342. $\lim\limits_{\alpha\to\beta} \frac{\sin^2\alpha - \sin^2\beta}{\alpha^2-\beta^2}$.

343. $\lim\limits_{h\to 0} \frac{\sin(a+2h)-2\sin(a+h)+\sin a}{h^2}$.

344. $\lim\limits_{h\to 0} \frac{\tan(a+2h)-2\tan(a+h)+\tan a}{h^2}$.

345. $\lim\limits_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$.

346. $\lim\limits_{x\to 0} \frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan x}$.

347. $\lim\limits_{x\to 0} \frac{\sqrt{1+x\sin x}-\sqrt{\cos 2x}}{\tan^2\frac{x}{2}}$.

348. $\lim\limits_{x\to 0} \frac{1-\cos x\sqrt{\cos 2x}}{x^2}$.

349. $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+\arctan 3x}-\sqrt[3]{1-\arcsin 3x}}{\sqrt{1-\arcsin 2x}-\sqrt{1+\arctan 2x}}$.

350*. $\lim\limits_{x\to -1} \dfrac{\sqrt{\pi}-\sqrt{\arccos x}}{\sqrt{x+1}}$.

351. $\lim\limits_{x\to\infty} \left(\dfrac{x}{1+x}\right)^x$.

352. $\lim\limits_{x\to\infty} \left(1-\dfrac{1}{t}\right)^t$.

353. $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{x}\right)^{\frac{x+1}{x}}$.

354. $\lim\limits_{x\to\infty} \left(1+\dfrac{k}{x}\right)^{mx}$.

355. $\lim\limits_{x\to\infty} \left(\dfrac{x+1}{x-2}\right)^{2x-1}$.

356. $\lim\limits_{x\to\infty} \left(\dfrac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}$.

357. $\lim\limits_{x\to\infty} \left(\dfrac{x^2+1}{x^2-1}\right)^{x^2}$.

358. $\lim\limits_{x\to\pm\infty} \left(\dfrac{x+1}{2x-1}\right)^x$.

359. $\lim\limits_{x\to\pm\infty} \left(\dfrac{2x+1}{x-1}\right)^x$.

360. $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{x^2}\right)^x$.

361. $\lim\limits_{x\to\pm\infty} \left(1+\dfrac{1}{x}\right)^{x^2}$.

362. $\lim\limits_{x\to\infty} \left(\dfrac{x^2-2x+1}{x^2-4x+2}\right)^x$.

363. $\lim\limits_{x\to 0} (1+\sin x)^{\operatorname{cosec} x}$.

364. $\lim\limits_{x\to 0} (1+\tan^2\sqrt{x})^{\frac{1}{2x}}$.

365. $\lim\limits_{x\to 0} \dfrac{\ln(1+kx)}{x}$.

366. $\lim\limits_{x\to 0} \dfrac{\ln(a+x)-\ln a}{x}$.

367. $\lim\limits_{x\to\infty} \{x[\ln(x+a)-\ln x]\}$.

368. $\lim\limits_{x\to e} \dfrac{\ln x-1}{x-e}$.

369. $\lim\limits_{h\to 0} \dfrac{a^h-1}{h}$.

370. $\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{3x}$.

371. $\lim\limits_{x\to 1} \dfrac{e^x-e}{x-1}$.

372*. $\lim\limits_{x\to 0} \dfrac{e^{x^2}-\cos x}{x^2}$.

373. $\lim\limits_{x\to 0} \dfrac{e^x-e^{-x}}{\sin x}$.

374. $\lim\limits_{x\to 0} \dfrac{e^{\sin 2x}-e^{\sin x}}{x}$.

375. $\lim\limits_{x\to 0} \dfrac{e^{ax}-e^{bx}}{x}$.

376. $\lim\limits_{x\to\infty} x\,(e^{\frac{1}{x}} - 1)$.

377. $\lim\limits_{x\to\pm\infty} (\cosh x - \sinh x)$.

378. $\lim\limits_{x\to\pm\infty} \tanh x$.

Miscellaneous Limits

Find the limits in problems 379–401:

379. $\lim\limits_{x\to\infty} \dfrac{(ax+1)^n}{x^n + A}$. Consider the cases separately when (1) n is a positive integer, (2) n is a negative integer, (3) n is zero.

380. $\lim\limits_{x\to\pm\infty} x\left(\sqrt{x^2 + \sqrt{x^4+1}} - x\sqrt{2}\right)$.

381. $\lim\limits_{x\to\pm\infty} \dfrac{a^x}{a^x+1}$ $(a > 0)$.

382. $\lim\limits_{x\to\pm\infty} \dfrac{a^x - a^{-x}}{a^x + a^{-x}}$ $(a > 0)$.

383. $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}$.

384. $\lim\limits_{x\to\infty} \dfrac{\arctan x}{x}$.

385. $\lim\limits_{x\to\infty} \dfrac{x + \sin x}{x + \cos x}$.

386. $\lim\limits_{x\to 1} \dfrac{\arcsin x}{\tan \dfrac{\pi x}{2}}$.

387. $\lim\limits_{h\to 0} \dfrac{\sin(a+3h) - 3\sin(a+2h) + 3\sin(a+h) - \sin a}{h^3}$.

388. $\lim\limits_{x\to\frac{\pi}{2}} \tan^2 x\left(\sqrt{2\sin^2 x + 3\sin x + 4} - \sqrt{\sin^2 x + 6\sin x + 2}\right)$.

389. $\lim\limits_{x\to 0} \dfrac{1 - \cos(1-\cos x)}{x^4}$.

390*. $\lim\limits_{n\to\infty} \left(\cos\dfrac{x}{2}\cos\dfrac{x}{4}\dots\cos\dfrac{x}{2^n}\right)$.

391. $\lim\limits_{x\to\infty} x^2\left(1 - \cos\dfrac{1}{x}\right)$.

392. $\lim\limits_{x\to\infty} \left(\cos\sqrt{x+1} - \cos\sqrt{x}\right)$.

393*. $\lim\limits_{x\to\infty} x\left(\arctan\dfrac{x+1}{x+2} - \dfrac{\pi}{4}\right)$.

394. $\lim\limits_{x\to\infty} x\left(\arctan\dfrac{x+1}{x+2} - \arctan\dfrac{x}{x+2}\right)$.

395*. $\lim\limits_{x\to 0}\dfrac{\arcsin x - \arctan x}{x^3}$.

396. $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x^n}\right)^x$ $(n > 0)$.

397*. $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\sin x}}$.

398. $\lim\limits_{x\to 0}\dfrac{\ln\cos x}{x^2}$.

399. $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$.

400. $\lim\limits_{x\to 0}(\cos x + \sin x)^{\frac{1}{x}}$.

401. $\lim\limits_{x\to 0}(\cos x + a\sin bx)^{\frac{1}{x}}$.

Comparison of Infinitesimals

402. An infinitesimal u_n takes the values

$$u_1 = 1,\quad u_2 = \frac{1}{2},\quad u_3 = \frac{1}{3},\ldots,\quad u_n = \frac{1}{n},\ldots,$$

whilst the corresponding values of infinitesimal v_n are

$$v_1 = 1,\quad v_2 = \frac{1}{2!},\ u_3 = \frac{1}{3!},\ldots,\quad v_n = \frac{1}{n!},\ldots$$

Compare u_n and v_n; which is of the higher order of smallness?

403. The function u_n takes the values

$$u_1 = 0,\quad u_2 = \frac{3}{8},\quad u_3 = \frac{8}{27},\ldots,\quad u_n = \frac{n^2-1}{n^3},\ldots,$$

whilst the corresponding values of v_n are

$$v_1 = 2,\quad v_2 = \frac{5}{8},\quad v_3 = \frac{10}{27},\ldots,\quad v_n = \frac{n^2+1}{n^3},\ldots$$

Compare these infinitesimals.

404. An infinitesimal u_n takes the values

$$u_1 = 0,\ u_2 = \frac{1}{4},\quad u_3 = \frac{2}{9},\ldots,\quad u_n = \frac{n-1}{n^2},\ldots,$$

whilst the corresponding values of infinitesimal v_n are

$$v_1 = 3,\ v_2 = \frac{5}{4},\quad v_3 = \frac{7}{9}, \ldots,\quad v_n = \frac{2n+1}{n^2}, \ldots$$

Show that u_n and v_n are infinitesimals of the same order but are non-equivalent.

405. Functions $y = \dfrac{1-x}{1+x}$ and $y = 1 - \sqrt{x}$ are infinitesimals as $x \to 1$. Which has the higher order?

406. Given the function $y = x^3$, show that Δy and Δx are in general infinitesimals of the same order as $\Delta x \to 0$.

For what value of x will the order of smallness of the increments be different?

For what values of x are increments Δx and Δy equivalent?

407. Show that $1 - x$ and $1 - \sqrt[3]{x}$ are infinitesimals of the same order as $x \to 1$. Are they equivalent?

408. Let $x \to 0$. Then $\sqrt{a + x^2} - \sqrt{a} (a > 0)$ is an infinitesimal. Find its order with respect to x.

409. Find the order with respect to x of the following infinitesimals as $x \to 0$:

(1) $x^3 + 1000\,x^2$; (2) $\sqrt[3]{x^2} - \sqrt{x}$;

(3) $\dfrac{x(x+1)}{1+\sqrt{x}}$; (4) $\dfrac{7x^{10}}{x^3+1}$.

410. Show that the increments of functions $y = a\sqrt{x}$ and $v = bx^2$ are of the same order of smallness for $x > 0$ and the common increment $\Delta x \to 0$. For what value of x are they equivalent (a and b differ from zero)?

411. Show that, as $x \to 1$, the infinitesimals $1 - x$ and $a(1 - \sqrt[k]{x})$, where $a \neq 0$ and k is a positive integer, are of the same order.

For what value of a are they equivalent?

412. Prove that $\sec x - \tan x$ and $\pi - 2x$ are infinitesimals of the same order as $x \to \dfrac{\pi}{2}$.

Are they equivalent?

413. Prove that the infinitesimals $e^{2x} - e^x$ and $\sin 2x - \sin x$ are equivalent as $x \to 0$.

414. Find the order with respect to x of the following functions, infinitesimal as $x \to 0$:

(1) $\sqrt[3]{1 + \sqrt[3]{x}} - 1$; (2) $\sqrt{1 + 2x} - 1 - \sqrt{x}$;

(3) $e^{\sqrt{x}} - 1$; (4) $e^{\sin x} - 1$; (5) $\ln(1 + \sqrt{x \sin x})$;

(6) $\sqrt{1 + x^2} \tan \frac{\pi x}{2}$; (7) $e^x - \cos x$; (8) $e^{x^2} - \cos x$;

(9) $\cos x - \sqrt[3]{\cos x}$; (10) $\sin(\sqrt{1 + x} - 1)$;

(11) $\ln(1 + x^2) - 2\sqrt[3]{(e^x - 1)^2}$;

(12) $\arcsin(\sqrt{4 + x^2} - 2)$.

Some Geometrical Problems

415. Starting from an equilateral triangle of side a, a new triangle is constructed from the three heights of the first triangle, and so on n times; find the limit of the sum of the areas of all the triangles as $n \to \infty$.

416. A square is inscribed in a circle of radius R, a circle is inscribed in the square, then a square in this circle, and so on n times. Find the limit of the sum of the areas of all the circles and the limit of the sum of the areas of all the squares as $n \to \infty$.

417. A step figure is inscribed as shown in Fig. 15 in a right-angled isosceles triangle, the base of which is divided

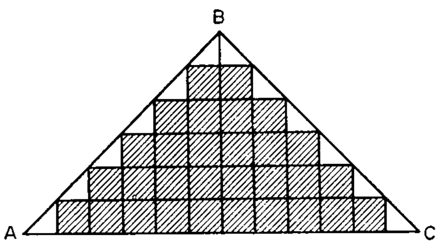

FIG. 15.

into $2n$ equal parts. Show that the difference between the area of the triangle and that of the step figure is an infinitesimal as n increases indefinitely.

418. The hypotenuse of a right-angled isosceles triangle of adjacent side a is divided into n equal parts and straight lines drawn from the points of subdivision parallel to the adjacent sides. The step line $AKLMNOPQRTB$ (Fig. 16)

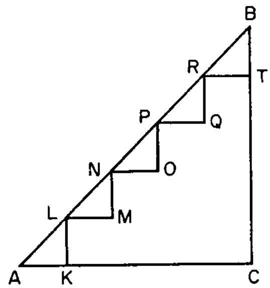

FIG. 16.

is thus obtained. The length of this step line is equal to $2a$ for any n, i.e. the limit of its length is equal to $2a$. But on the other hand, as n increases indefinitely the step line approaches indefinitely the hypotenuse. Consequently the length of the hypotenuse is equal to the sum of the lengths of the adjacent sides. Find the error in this argument.

419. The straight line AB of length a is divided into equal parts by n points, and lines are drawn from these points at angles $\frac{\pi}{2n}$ (Fig. 17). Find the limit of the length of the step line obtained as n increases indefinitely. Compare with the result of the previous problem.

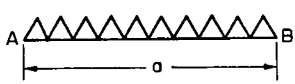

FIG. 17.

420. The straight line AB of length a is divided into n equal parts. An arc of a circle equal to $\frac{\pi}{n}$ radians is erected on each part of AB (Fig. 18). Find the limit of the length of the resulting curve as $n \to \infty$. How does the result change if semicircles are erected on each subdivision?

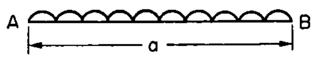

FIG. 18.

421. A circle of radius R is divided into equal parts by n points. Taking the points as centres, arcs of circles of radius r are drawn to their intersections with neighbouring arcs (Fig. 19). Find the limit of the length of the resulting closed curve when n increases indefinitely.

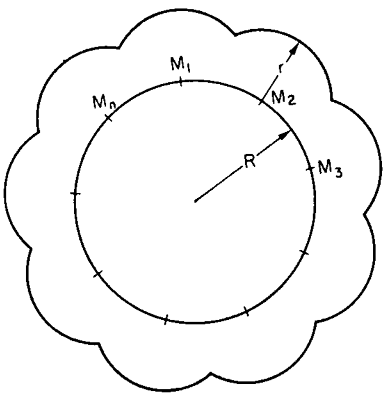

FIG. 19.

422. Two circles of radii R and r $(R > r)$ are located to the right of OY and touch it at the origin (Fig. 20). As $x \to 0$, of what order with respect to x are the infinitesimal segment MM' and the infinitesimal angle α?

423. The straight line OP joins the centre of a circle to a point P lying outside the circle. A tangent PT is drawn to the circle from P and a perpendicular TN dropped from T to OP. Prove that segments AP and AN, where A is the point of intersection of OP with the circle, are equivalent infinitesimals as $P \to A$.

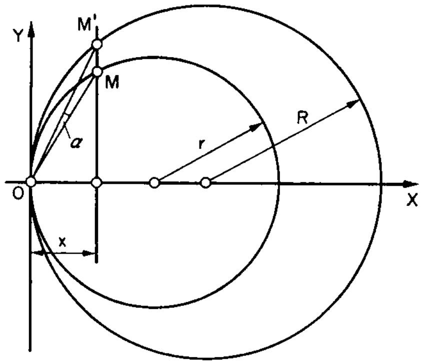

FIG. 20.

424. Tangents are drawn at the ends and mid-point of the arc AB of a circle and points A and B are joined by a chord. Show that the ratio of the areas of the two triangles thus formed tends to 4 as arc AB diminishes indefinitely.

Numerical Problems

425. Starting from the equivalence of functions $\sqrt{1+x}-1$ and $\frac{1}{2}x$ (see *Course,* sec. 40) as $x \to 0$, evaluate approximately:

(1) $\sqrt{105}$; (2) $\sqrt{912}$; (3) $\sqrt{260}$; (4) $\sqrt{1632}$;
(5) $\sqrt{0{\cdot}31}$; (6) $\sqrt{0{\cdot}021}$.

426. Prove that $\sqrt[n]{1+x}-1$ and $\frac{x}{n}$ are equivalent infinitesimals as $x \to 0$. Use this fact to find approximately the roots

(1) $\sqrt[3]{1047}$; (2) $\sqrt[3]{8144}$; (3) $\sqrt[5]{1{\cdot}1}$; (4) $\sqrt[5]{1080}$.

Find the same roots from logarithmic tables. Compare the results.

427. Use the equivalence of $\ln(1+x)$ and x as $x \to 0$ for approximate evaluation of the natural logarithms of the following numbers: 1·01; 1·02; 1·1; 1·2. Find the logarithms to base ten of the same numbers and compare with the tables.

CHAPTER III

DERIVATIVES AND DIFFERENTIALS. DIFFERENTIAL CALCULUS

1. Derivatives. The Rate of Change of a Function

Some Physical Concepts

428. A particle moves in a straight line according to the law

$$s = 5t + 6.$$

Find the average velocity: (a) during the first six seconds, (b) during the interval from the end of the third to the end of the sixth second.

429. A particle M moves away from a fixed point A so that the distance AM increases proportionally to the square of time. After 2 min from the initial instant distance AM is equal to 12 m. Find the average velocity: (a) during the first 5 min, (b) during the interval from $t = 4$ min to $t = 7$ min, (c) during the interval from $t = t_1$ to $t = t_2$.

430. The equation of a rectilinear motion is

$$s = t^3 + \frac{3}{t}.$$

Find the average velocity during the interval from $t =$ $= 4$ to $t = 4 + \Delta t$, putting $\Delta t = 2, 1, 0{\cdot}1, 0{\cdot}03$.

431. A freely falling body moves according to the law $s = \dfrac{gt^2}{2}$, where g ($= 980$ cm/sec^2) is the acceleration due to gravity. Find the average velocity during the interval from $t = 5$ sec to $(t + \Delta t)$ sec, putting $\Delta t = 1$ sec, $0{\cdot}1$ sec, $0{\cdot}05$ sec, $0{\cdot}001$ sec. Find the velocity at the end of the fifth second and at the end of the tenth second. Obtain the formula for the velocity of the falling body at any instant t.

432. AB is a thin non-homogeneous rod of length $L = 20$ cm. The mass of a piece AM increases proportionally to the square of the distance of point M from point A, and we know that the mass of $AM = 2$ cm is equal to 8 g. Find: (a) the average linear density of the piece of rod $AM = 2$ cm, (b) of the whole rod, (c) the density of the rod at point M.

433. The mass (in g) of a thin non-homogeneous rod AB of length 30 cm is distributed according to the law

$$m = 3l^2 + 5l,$$

where l is the length of a piece of rod measured from A. Find: (1) the average linear density of the rod, (2) the linear density: (a) at the point distant $l = 5$ cm from A, (b) at point A itself, (c) at the end of the rod.

434. The amount of heat Q required to raise unit mass of water from 0 to $t°$ C is given by

$$Q = t + 0{\cdot}00002t^2 + 0{\cdot}0000003t^3 \text{ (cal/g)}.$$

Find the specific heat of water at $t = 30°$, $t = 100°$.

435*. The angular velocity of a uniform rotation is defined as the ratio of the angle of rotation to the corresponding time interval. Give the definition of the angular velocity of a non-uniform rotation.

436. If the process of radioactive decay were uniform, the rate of decay would be reckoned as the amount of material disintegrating in unit time. The process is in fact non-uniform. Given the definition of the rate of radioactive decay.

437. A constant current is defined as the quantity of electricity flowing through the conductor cross-section in unit time. Define a variable current.

438. The thermal coefficient of linear expansion of a rod is the increase in unit length per 1° C rise in temperature if we assume uniform expansion. The process is in fact non-uniform. Let $l = f(t)$, where l is the length of the rod, t the temperature. Define the coefficient of linear expansion.

439. The coefficient of extension of a spring is defined as the increase in unit length of the spring under the action of unit force acting per square centimetre of the spring cross-section. It is assumed here that the extension is proportional to the force (Hooke's law). Define the coefficient of extension k when there is a deviation from Hooke's law. (Let l be the spring length, S the cross-sectional area, P the extending force and $l = \varphi(P)$.)

Derivatives

440. Find the increment of the function $y = x^3$ at the point $x_1 = 2$ when the increment Δx of the independent variable is (1) 2, (2) 1, (3) 0·5, (4) 0·1.

441. Find the ratio $\frac{\Delta y}{\Delta x}$ for the functions:

(1) $y = 2x^3 - x^2 + 1$ for $x = 1$; $\Delta x = 0{\cdot}1$;

(2) $y = \frac{1}{x}$ for $x = 2$; $\Delta x = 0{\cdot}01$;

(3) $y = \sqrt{x}$ for $x = 4$; $\Delta x = 0{\cdot}4$.

Show that, as $\Delta x \to 0$, the ratio tends in the first case to 4, in the second to $-\frac{1}{4}$, in the third to $\frac{1}{4}$.

442. Given the function $y = x^2$, find approximate numerical values for the derivative at $x = 3$ when Δx is equal to (a) 0·5, (b) 0·1, (c) 0·01, (d) 0·001.

443. $f(x) = x^2$; find $f'(5)$; $f'(-2)$; $f'\left(-\frac{3}{2}\right)$.

444. $f(x) = x^3$; find $f'(1)$; $f'(0)$; $f'(-\sqrt{2})$; $f'\left(\frac{1}{3}\right)$.

445. $f(x) = x^2$. At what point does $f(x) = f'(x)$?

446. Given $f(x) = x^2$, show that $f'(a + b) = f'(a) + f'(b)$. Does the same equation hold for $f(x) = x^3$?

447. Find the numerical value of the derivative of $y = \sin x$ at $x = 0$.

448. Find the numerical value of the derivative of $y = \log x$ at $x = 1$.

449. Find the numerical value of the derivative of $y = 10^x$ at $x = 0$.

450. What is the limit of $\frac{f(x)}{x}$ as $x \to 0$ if $f(0) = 0$?

451. Prove the theorem: if $f(x)$ and $\varphi(x)$ vanish at $x = 0$: $f(0) = 0$, $\varphi(0) = 0$, and their derivatives exist at $x = 0$, whilst $\varphi'(0) \neq 0$, we have

$$\lim_{x \to 0} \frac{f(x)}{\varphi(x)} = \frac{f'(0)}{\varphi'(0)}\,.$$

452. Prove that, if $f(x)$ has a derivative at $x = a$, then

$$\lim_{x \to a} \frac{xf(a) - af(x)}{x - a} = f(a) - af'(a).$$

453. Find the derivatives of the functions:

(1) x^5; (2) x^{10}; (3) $x^{\frac{3}{7}}$; (4) $\sqrt[3]{x^2}$; (5) $\sqrt{x}$;

(6) x^{-3}; (7) $\frac{1}{x}$; (8) $\sqrt[5]{\frac{1}{x^3}}$; (9) $x\sqrt[4]{x}$; (10) $0{\cdot}7x^5$;

(11) $\frac{1}{12}x^{12}$; (12) ax^{-7}; (13) $\sqrt[n]{x}$; (14) $\frac{p}{x}$;

(15) $ax^{-\frac{2}{3}}$.

Geometrical Meaning of the Derivative

454. Find the slope of the tangent to the parabola $y = x^2$: (1) at the origin, (2) at the point (3, 9), (3) at the point $(-2, 4)$, (4) at its points of intersection with the straight line $y = 3x - 2$.

455. At what points is the slope of the tangent to the cubical parabola $y = x^3$ equal to 3?

456. At what point is the tangent to the parabola $y = x^2$: (1) parallel to Ox, (2) at an angle of 45° to Ox?

457. Can the tangent to the cubical parabola $y = x^3$ form an obtuse angle with Ox?

458. At what angle does the parabola $y = x^2$ cut the straight line $3x - y - 2 = 0$?

459. At what angles do the parabolas $y = x^2$ and $y^2 = x$ intersect?

460. At what angles do the hyperbola $y = \frac{1}{x}$ and the parabola $y = \sqrt{x}$ intersect?

461. Write down the equations of the tangent and normal to $y = x^3$ at the point with abscissa 2. Find the subtangent and subnormal.

462. For what values of the independent variable are the tangents to $y = x^2$ and $y = x^3$ parallel?

463. At what point is the tangent to the parabola $y = x^2$: (1) parallel to the straight line $y = 4x - 5$; (2) perpendicular to the straight line $2x - 6y + 5 = 0$; (3) at an angle of 45° to the straight line $3x - y + 1 = 0$?

464. Show that the subtangent corresponding to any point of the parabola $y = ax^2$ is equal to half the abscissa of the point of contact. Using this fact, give a method of drawing the tangent to the parabola at any given point.

465. Show that the normal to a parabola at any point is the bisector of the angle between the line joining the point to the focus and the line through the point parallel to the parabola's axis.

2. Differentiation of Functions

Sums, Products and Quotients of Power Functions

466. Differentiate the following functions (x, y, z, t, u, v are independent variables; a, b, c, m, n, p, q are constants):

(1) $3x^2 - 5x + 1$; (2) $x^4 - \frac{1}{3}x^3 + 2{\cdot}5x^2 - 0{\cdot}3x + 0{\cdot}1$;

(3) $ax^2 + bx + c$; (4) $\sqrt[3]{x} + \sqrt[3]{2}$;

(5) $2\sqrt{x} - \frac{1}{x} + \sqrt[4]{3}$; (6) $0{\cdot}8\sqrt[4]{y} - \frac{y^3}{0{\cdot}3} + \frac{1}{5y^2}$;

(7) $\frac{x}{n} + \frac{n}{x} + \frac{x^2}{m^2} + \frac{m^2}{x^2}$; (8) $\frac{mx^2}{\sqrt{x}} + \frac{nx\sqrt{x}}{\sqrt[3]{x}} - \frac{p\sqrt{x}}{x}$;

(9) $\frac{mz^2 + nz + 4p}{p + q}$; (10) $0{\cdot}1t^{-\frac{2}{3}} - \frac{5{\cdot}2}{t^{1{\cdot}4}} + \frac{2{\cdot}5}{\sqrt[5]{t}}$;

(11) $(x - 0{\cdot}5)^2$; (12) $\sqrt{x}\,(x^3 - \sqrt{x} + 1)$;

(13) $(v + 1)^2 (v - 1)$; (14) $0{\cdot}5 - 3(a - x)^2$;

(15) $\frac{ax^3 + bx^2 + c}{(a + b)x}$; (16) $\left(\frac{mu + n}{p}\right)^3$.

467. $f(x) = 3x - 2\sqrt{x}$. Find: $f(1)$; $f'(1)$; $f(4)$; $f'(4)$; $f(a^2)$; $f'(a^2)$.

468. $f(t) = \frac{t^2 - 5t - 1}{t^3}$. Find: $f(-1)$; $f'(-1)$; $f'(2)$; $f'\left(\frac{1}{a}\right)$.

469. $f(z) = \frac{2z^3 - 3z + \sqrt{z} - 1}{z}$. Find: $f'\left(\frac{1}{4}\right)$.

470. $f(x) = 4 - 5x + 2x^3 - x^5$. Show that

$$f'(a) = f'(-a).$$

Differentiate the functions of problems 471–489 (x, y, z, t, u, v, s are variables; a, b, c, d, m are constants).

471. (1) $y = (x^2 - 3x + 3)(x^2 + 2x - 1)$;

(2) $y = (x^3 - 3x + 2)(x^4 + x^2 - 1)$;

(3) $y = (\sqrt{x} + 1)\left(\frac{1}{\sqrt{x}} - 1\right)$;

(4) $y = \left(\frac{2}{\sqrt{x}} - \sqrt{3}\right)\left(4x\sqrt[3]{x} + \frac{\sqrt[3]{x^2}}{3x}\right)$;

(5) $y = (\sqrt[3]{x} + 2x)(1 + \sqrt[3]{x^2} + 3x)$;

(6) $y = (x^2 - 1)(x^2 - 4)(x^2 - 9)$;

(7) $y = (1 + \sqrt{x})(1 + \sqrt{2x})(1 + \sqrt{3x})$.

472. $y = \dfrac{x+1}{x-1}$.

473. $y = \dfrac{x}{x^2+1}$.

474. $s = \dfrac{3t^2+1}{t-1}$.

475. $u = \dfrac{v^3-2v}{v^2+v+1}$.

476. $y = \dfrac{ax+b}{cx+d}$.

477. $z = \dfrac{x^2+1}{3(x^2-1)} + (x^2-1)(1-x)$.

478. $u = \dfrac{v^5}{v^3-2}$.

479. $y = \dfrac{1-x^3}{1+x^3}$.

480. $y = \dfrac{2}{x^3-1}$.

481. $u = \dfrac{v^2-v+1}{a^2-3}$.

482. $y = \dfrac{1-x^3}{\sqrt{\pi}}$.

483. $z = \dfrac{1}{t^2+t+1}$.

484. $s = \dfrac{1}{t^2-3t+6}$.

485. $y = \dfrac{2x^4}{b^2-x^2}$.

486. $y = \dfrac{x^2+x-1}{x^3+1}$.

487. $y = \dfrac{3}{(1-x^2)(1-2x^3)}$.

488. $y = \dfrac{ax+bx^2}{am+bm^2}$.

489. $y = \dfrac{a^2b^2c^2}{(x-a)(x-b)\,x-c)}$.

490. $f(x) = (x^2+x+1)(x^2-x+1)$; find $f'(0)$ and $f'(1)$.

491. $F(x) = (x-1)(x-2)(x-3)$; find $F'(0)$; $F'(1)$ and $F'(2)$.

492. $F(x) = \dfrac{1}{x+2} + \dfrac{3}{x^2+1}$; find $F'(0)$ and $F'(-1)$.

493. $s(t) = \dfrac{3}{5-t} + \dfrac{t^2}{5}$; find $s'(0)$ and $s'(2)$.

494. $y(x) = (1+x^3)\left(5-\dfrac{1}{x^2}\right)$; find $y'(1)$ and $y'(a)$.

495. $\varrho(\varphi) = \dfrac{\varphi}{1-\varphi^2}$; find $\varrho'(2)$ and $\varrho'(0)$.

496. $\varphi(z) = \dfrac{a - z}{1 + z}$; find $\varphi'(1)$.

497. $z(t) = (\sqrt{t^3} + 1)\,t$; find $z'(0)$.

Powers of Functions

Differentiate the functions of problems 498–515:

498. (1) $(x - a)(x - b)(x - c)(x - d)$;

(2) $(x^2 + 1)^4$; (3) $(1 - x)^{20}$; (4) $(1 + 2x)^{30}$;

(5) $(1 - x^2)^{10}$; (6) $(5x^3 + x^2 - 4)^5$; (7) $(x^3 - x)^6$;

(8) $\left(7x^2 - \dfrac{4}{x} + 6\right)^6$; (9) $s = \left(t^3 - \dfrac{1}{t^3} + 3\right)^4$;

(10) $y = \left(\dfrac{x + 1}{x - 1}\right)^2$; (11) $y = \left(\dfrac{1 + x^2}{1 + x}\right)^5$;

(12) $y = (2x^3 + 3x^2 + 6x + 1)^4$.

499. $v = \dfrac{(s + 4)^2}{s + 3}$.

500. $s = \dfrac{t^3}{(1 - t)^2}$.

501. $y = \dfrac{1 + \sqrt{x}}{1 + \sqrt{2x}}$.

502. $y = \dfrac{1 - \sqrt[3]{2x}}{1 + \sqrt[3]{2x}}$.

503. $y = \sqrt{1 - x^2}$.

504. $y = (1 - 2x^{\frac{1}{2}})^4$.

505. $u = \left(\dfrac{v}{1 - v}\right)^m$.

506. $y = \dfrac{2}{(x^2 - x + 1)^2}$.

507. $y = \dfrac{1}{\sqrt{a^2 - x^2}}$.

508. $y = \sqrt[3]{\dfrac{1}{1 + x^2}}$.

509. $y = \dfrac{1}{\sqrt{1 - x^4 - x^3}}$.

510. $y = \dfrac{1 + x}{\sqrt{1 - x}}$.

511. $y = \dfrac{x^2}{\sqrt{x^2 + a^2}}$.

512. $u = \dfrac{1}{v - \sqrt{a^2 + v^2}}$.

513. $y = \dfrac{1}{\sqrt[3]{2x - 1}} + \dfrac{5}{\sqrt[4]{(x^2 + 2)^3}}$.

514. $u(v) = (v^2 + v + 2)^{\frac{3}{2}}$; find $u'(1)$.

515. $y(x) = \sqrt{\dfrac{x+1}{x-1}}$; find $y'(2)$.

516. Show that the derivative of an even function is an odd function, and the derivative of an odd function is even.

Trigonometric Functions

Differentiate the functions in problems 517–546:

517. $y = \sin x + \cos x$.

518. $y = \dfrac{x}{1 - \cos x}$.

519. $y = \dfrac{\tan x}{x}$.

520. $\varrho = \varphi \sin \varphi + \cos \varphi$.

521. $z = \dfrac{\sin \alpha}{\alpha} + \dfrac{\alpha}{\sin \alpha}$.

522. $s = \dfrac{\sin t}{1 + \cos t}$.

523. $y = \dfrac{x}{\sin x + \cos x}$.

524. $y = \dfrac{x \sin x}{1 + \tan x}$.

525. $y = \cos^2 x$.

526. $y = \dfrac{1}{4} \tan^4 x$.

527. $y = \cos x - \dfrac{1}{3} \cos^3 x$.

528. $y = 3 \sin^2 x - \sin^2 x$.

529. $y = \dfrac{1}{3} \tan^3 x - \tan x + x$.

530. $y = x \sec^2 x - \tan x$.

531. $y = \sec^2 x + \operatorname{cosec}^2 x$.

532. $y = \sin 3x$.

533. $y = a \cos \dfrac{x}{3}$.

534. $y = 3 \sin (3x + 5)$.

535. $y = \tan \dfrac{x+1}{2}$.

536. $y = \sqrt{1 + 2 \tan x}$.

537. $y = \sin \dfrac{1}{x}$.

538. $y = \sin (\sin x)$.

539. $y = \cos^3 4x$.

540. $y = \sqrt{\tan \dfrac{x}{2}}$.

541. $y = \sin \sqrt{1 + x^2}$.

542. $y = \cot \sqrt[3]{1 + x^2}$.

543. $y = (1 + \sin^2 x)^4$.

544. $y = \sqrt{1 + \tan \left(x + \dfrac{1}{x}\right)}$.

545. $y = \cos^2 \dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$.

546. $y = \sin^2 (\cos 3x)$.

547. Deduce the formulae:

$$(\sin^n x \cos nx)' = n \sin^{n-1} x \cos (n+1) x;$$
$$(\sin^n x \sin nx)' = n \sin^{n-1} x \sin (n+1) x;$$
$$(\cos^n x \sin nx)' = n \cos^{n-1} x \cos (n+1) x;$$
$$(\cos^n x \cos nx)' = -n \cos^{n-1} x \sin (n+1) x.$$

Inverse Trigonometric Formulae

Differentiate the functions in problems 548–572:

548. $y = x \arcsin x.$

549. $y = \dfrac{\arcsin x}{\arccos x}.$

550. $y = (\arcsin x)^2.$

551. $y = x \arcsin x + \sqrt{1 - x^2}.$

552. $y = \dfrac{1}{\arcsin x}.$

553. $y = x \sin x \arctan x.$

554. $y = \dfrac{\arccos x}{x}.$

555. $y = \sqrt{x} \arctan x.$

556. $y = (\arccos x + \arcsin x)^n.$

557. $y = \operatorname{arc\,sec} x.$

558. $y = \dfrac{x}{1 + x^2} - \arctan x.$

559. $y = \dfrac{\arcsin x}{\sqrt{1 - x^2}}.$

560. $y = \dfrac{x^2}{\arctan x}.$

561. $y = \arcsin (x - 1).$

562. $y = \arccos \dfrac{2x - 1}{\sqrt{3}}.$

563. $y = \arctan x^2.$

564. $y = \arcsin \dfrac{2}{x}.$

565. $y = \arcsin (\sin x).$

566. $y = \arctan^2 \dfrac{1}{x}.$

567. $y = \sqrt{1 - (\arccos x)^2}.$

568. $y = \arcsin \sqrt{\dfrac{1 - x}{1 + x}}.$

569. $y = \dfrac{1}{2} \sqrt[4]{\arcsin \sqrt{x^2 + 2x}}.$

570. $y = \arcsin \dfrac{\sin \alpha \sin x}{1 - \cos \alpha \cos x}.$

571. $y = \arccos \dfrac{b + a \cos x}{a + b \cos x}$.

572. $y = \arctan(x - \sqrt{1 + x^2})$.

Logarithmic Functions

Differentiate the functions of problems 573–597:

573. $y = x^2 \log_3 x$.

574. $y = \ln^2 x$.

575. $y = x \log x$.

576. $y = \sqrt{\ln x}$.

577. $y = \dfrac{x - 1}{\log_2 x}$.

578. $y = x \sin x \ln x$.

579. $y = \dfrac{1}{\ln x}$.

580. $y = \dfrac{\ln x}{x^n}$.

581. $y = \dfrac{1 - \ln x}{1 + \ln x}$.

582. $y = \dfrac{\ln x}{1 + x^2}$.

583. $y = x^n \ln x$.

584. $y = \sqrt{1 + \ln^2 x}$.

585. $y = \ln (1 - 2x)$.

586. $y = \ln (x^2 - 4x)$.

587. $y = \ln \sin x$.

588. $y = \log_3 (x^2 - 1)$.

589. $y = \ln \tan x$.

590. $y = \ln \arccos 2x$.

591. $y = \ln^4 \sin x$.

592. $y = \arctan [\ln(ax + b)]$.

593. $y = (1 + \ln \sin x)^n$.

594. $y = \log_2 [\log_3 (\log^5 x)]$.

595. $y = \ln \arctan \sqrt{1 + x^2}$.

596. $y = \arcsin^2 [\ln (a^3 + x^3)]$.

597. $y = \sqrt[3]{\ln \sin \dfrac{x + 3}{4}}$.

Exponential Functions

Differentiate the functions of problems 598–633:

598. $y = 2^x$.

599. $y = 10^x$.

600. $y = \dfrac{1}{3^x}$.

601. $y = \dfrac{x}{4^x}$.

602. $y = x10^x$.

603. $y = xe^x$.

604. $y = \dfrac{x}{e^x}$.

605. $y = \dfrac{x^3 + 2^x}{e^x}$.

606. $y = e^x \cos x$.

607. $y = \dfrac{e^x}{\sin x}$.

608. $y = \dfrac{\cos x}{e^x}$.

609. $y = 2^{\frac{x}{\ln x}}$.

610. $y = x^3 - 3^x$.

611. $y = \sqrt{1 + e^x}$.

612. $y = (x^2 - 2x + 3)\, e^x$.

613. $y = \dfrac{1 + e^x}{1 - e^x}$.

614. $y = \dfrac{1 - 10^x}{1 + 10^x}$.

615. $y = \dfrac{e^x}{1 + x^2}$.

616. $y = xe^x (\cos x + \sin x)$.

617. $y = e^{-x}$.

618. $y = 10^{2x-3}$.

619. $y = e^{\sqrt{x+1}}$.

620. $y = \sin (2^x)$.

621. $y = 3^{\sin x}$.

622. $y = a^{\sin^3 x}$.

623. $y = e^{\arcsin 2x}$.

624. $y = 2^{3x}$.

625. $y = e^{\sqrt{\ln x}}$.

626. $y = \sin (e^{x^2 + 3x - 2})$.

627. $y = 10^{1 - \sin^4 3x}$.

628. $y = e^{\sqrt{\ln (ax^2+bx+c)}}$

629. $y = \ln \sin \sqrt[3]{\arctan e^{3x}}$.

630. $y = ae^{-b^2x^2}$.

631. $y = x^2 e^{-\frac{x^2}{a^2}}$.

632. $y = Ae^{-k^2x} \sin (\omega x + \alpha)$.

633. $y = a^x x^a$.

Hyperbolic Functions

Differentiate the functions of problems 634–649:

634. $y = \sinh^3 x$.

635. $y = \ln \cosh x$.

636. $y = \arctan (\tanh x)$.

637. $y = \tanh (1 - x^2)$.

638. $y = \sinh^2 x + \cosh^2 x$.

639. $y = \cosh (\sinh x)$.

640. $y = \sqrt{\cosh x}$.

641. $y = e^{\cosh^2 x}$.

642. $y = \tanh (\ln x)$.

643. $y = x \sinh x - \cosh x$.

644. $y = \sqrt[4]{(1 + \tanh^2 x)^3}$.

645. $y = \dfrac{1}{2} \tanh \dfrac{x}{2} - \dfrac{1}{6} \tanh^3 \dfrac{x}{2}$.

646. $y = \sqrt[4]{\dfrac{1 + \tanh x}{1 - \tanh x}}$.

647. $y = \dfrac{1}{2} \tanh x + \dfrac{\sqrt{2}}{8} \ln \dfrac{1 + \sqrt{2} \tanh x}{1 - \sqrt{2} \tanh x}$.

648. $y = \frac{1}{x}\cosh 2x + \sqrt{x}\sinh 2x.$ **649.** $y = x^2 e^{3x} \operatorname{cosech} x.$

Logarithmic Differentiation

Differentiate the functions of problems 650–666 by using the rule for logarithmic differentiation:

650. $y = x^{x^2}.$

651. $y = x^{x^x}.$

652. $y = (\sin x)^{\cos x}.$

653. $y = (\ln x)^x.$

654. $y = \sqrt[x]{(x+1)^2}.$

655. $y = x^3 e^{x^2} \sin 2x.$

656. $y = \dfrac{(x-2)^2 \sqrt[3]{x+1}}{(x-5)^3}.$

657. $y = x^{\ln x}.$

658. $y = \dfrac{(x+1)^3 \sqrt[4]{x-2}}{\sqrt[5]{(x-3)^2}}.$

659. $y = \sqrt{x \sin x \sqrt{1 - e^x}}.$

660. $y = \sqrt{\dfrac{1 - \arcsin x}{1 + \arcsin x}}.$

661. $y = x^{\frac{1}{x}}.$

662. $y = x^{\sin x}.$

663. $y = \left(\dfrac{x}{1+x}\right)^x.$

664. $y = 2x^{\sqrt{x}}.$

665. $y = (x^2 + 1)^{\sin x}.$

666. $y = \sqrt[3]{\dfrac{x(x^2+1)}{(x^2-1)^2}}.$

Various Functions

Differentiate the functions of problems 667–770:

667. $y = (1 + \sqrt[3]{x})^3.$

668. $y = a \tan\left(\dfrac{x}{k} + b\right).$

669. $y = \sqrt{1 + \sqrt{2px}}.$

670. $y = \arctan(x^2 - 3x + 2).$

671. $y = \log(x - \cos x).$

672. $y = 3\cos^2 x - \cos^3 x.$

673. $y = 5 \tan \dfrac{x}{5} + \tan \dfrac{\pi}{8}.$

674. $y = \dfrac{1}{\sqrt[3]{x + \sqrt{x}}}.$

675. $y = \sin\frac{x}{2}\sin 2x.$

676. $y = \sin x e^{\cos x}.$

677. $y = x^5\sqrt[3]{x^6 - 8}.$

678. $y = e^{-x^2}\ln x.$

679. $y = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^{10}.$

680. $y = \arctan\frac{x+1}{x-1}.$

681. $y = e^{2x+3}\left(x^2 - x + \frac{1}{2}\right).$

682. $y = \frac{2\sin^2 x}{\cos 2x}.$

683. $y = \frac{1}{\sqrt{3}}\arctan\frac{x\sqrt{3}}{1-x^2}.$

684. $y = \frac{\tan\frac{x}{2} + \cot\frac{x}{2}}{x}.$

685. $y = \sin^2\frac{x}{3}\cot\frac{x}{2}.$

686. $y = \frac{\sqrt[9]{4x^5 + 2}}{3x^4}.$

687. $y = \ln\left(x + \sqrt{a^2 + x^2}\right).$

688. $y = x\arctan\sqrt{x}.$

689. $y = \sqrt{1 + \tan^2 x + \tan^4 x}.$

690. $y = \cos 2x\ln x.$

691. $y = \frac{2}{3}\arctan x + \frac{1}{3}\arctan\frac{x}{1-x^2}.$

692. $y = \arcsin(n\sin x).$

693. $y = \arcsin\sqrt{\sin x}.$

694. $y = \frac{1}{18}\sin^6 3x - \frac{1}{24}\sin^8 3x.$

695. $y = x - \sqrt{1-x^2}\arcsin x.$

696. $y = \cos\frac{\arcsin x}{2}.$

697. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}.$

698. $y = \arccos\sqrt{1-3x}.$

699. $y = \sin^2\left(\frac{1-\ln x}{x}\right).$

700. $y = \log_3(x^2 - \sin x).$

701. $y = \arctan\sqrt{\frac{1-x}{1+x}}.$

702. $y = \ln\frac{x + \sqrt{1-x^2}}{x}.$

703. $y = x\arcsin(\ln x).$

704. $y = \tan \dfrac{1 - e^x}{1 + e^x}$.

705. $y = \cos x \sqrt{1 + \sin^2 x}$.

706. $y = 0{\cdot}4 \left(\cos \dfrac{2x + 1}{2} - \sin 0{\cdot}8\, x\right)^2$.

707. $y = x10^{\sqrt{x}}$.

708. $y = \dfrac{1}{\tan^2 2x}$.

709. $y = \ln \arctan \dfrac{1}{1 + x}$.

710. $y = \ln \dfrac{1}{x + \sqrt{x^2 - 1}}$.

711. $y = \sqrt[3]{1 + x\sqrt{x + 3}}$.

712. $y = x^2 \sqrt{1 + \sqrt{x}}$.

713. $y = \dfrac{1}{\sqrt{1 + \sin^2 x}}$.

714. $y = x^3 \arctan x^3$.

715. $y = \dfrac{\ln \sin x}{\ln \cos x}$.

716. $y = \arcsin x + \sqrt{1 - x^2}$.

717. $y = \dfrac{\arcsin 4x}{1 - 4x}$.

718. $y = e^{\frac{1}{\ln x}}$.

719. $y = \ln \dfrac{1 - e^x}{e^x}$.

720. $y = 10^{x \tan x}$.

721. $y = \sin^2 x \sin x^2$.

722. $y = \dfrac{2 \cos x}{\sqrt{\cos 2x}}$.

723. $y = x \sqrt{\dfrac{1 - x}{1 + x^2}}$.

724. $y = \dfrac{1}{4} \ln \dfrac{1 + x}{1 - x} - \dfrac{1}{2} \arctan x$.

725. $y = 2^{\frac{x}{\ln x}}$.

726. $y = \sqrt{(a - x)(x - b)} - (a - b) \arctan \sqrt{\dfrac{a - x}{x - b}}$.

727. $y = \dfrac{\sin 3x}{2 \sin^2 x \cos x}$.

728. $y = e^{\sqrt{\frac{1-x}{1+x}}}$.

729. $y = \sqrt{a^2 - x^2} - a \arccos \dfrac{x}{a}$.

730. $y = \sqrt{x^2 + 1} - \ln \left(\dfrac{1}{x} + \sqrt{1 + \dfrac{1}{x^2}}\right)$.

731. $y = \dfrac{\sin^2 x}{1 + \cot x} + \dfrac{\cos^2 x}{1 + \tan x}$.

732. $y = \ln\left(x + \sqrt{x^2 - 1}\right) - \dfrac{x}{\sqrt{x^2 - 1}}$.

733. $y = e^{ax}(a \sin x - \cos x)$. **734.** $y = xe^{1 - \cos x}$.

735. $y = \dfrac{1}{\arctan e^{-2x}}$. **736.** $y = e^x(\sin 3x - 3\cos 3x)$.

737. $y = 3x^3 \arcsin x + (x^2 + 2)\sqrt{1 - x^2}$.

738. $y = \dfrac{1}{\sqrt{1 + e^{-\sqrt{x}}}}$.

739. $y = 2 \arcsin \dfrac{x - 2}{\sqrt{6}} - \sqrt{2 + 4x - x^2}$.

740. $y = \ln(e^x \cos x + e^{-x} \sin x)$.

741. $y = \dfrac{1 + x \arctan x}{\sqrt{1 + x^2}}$. **742.** $y = \dfrac{1}{\cos(x - \cos x)}$.

743. $y = e^x \sin x \cos^3 x$. **744.** $y = \sqrt[11]{9 + 6\sqrt[5]{x^9}}$.

745. $y = x - \ln\left(2e^x + 1 + \sqrt{e^{2x} + 4e^x + 1}\right)$.

746. $y = e^{\arctan\sqrt{1 + \ln(2x + 3)}}$. **747.** $y = \dfrac{e^{x^2}}{e^x + e^{-x}}$.

748. $y = \ln \tan \dfrac{x}{2} - \cot x \ln(1 + \sin x) - x$.

749. $y = 2\ln\left(2x - 3\sqrt{1 - 4x^2}\right) - 6 \arcsin 2x$.

750. $y = \dfrac{3x^2 - 1}{3x^3} + \ln\sqrt{1 + x^2} + \arctan x$.

751. $y = \dfrac{1}{2}(3 - x)\sqrt{1 - 2x - x^2} + 2 \arcsin \dfrac{x + 1}{\sqrt{2}}$.

752. $y = \ln\left(x \sin x \sqrt{1 - x^2}\right)$. **753.** $y = x\sqrt{1 + x^2} \sin x$.

754. $y = \dfrac{\sqrt{x + 2}\,(3 - x)^4}{(x + 1)^5}$. **755.** $y = \sqrt[5]{\left(1 + xe^{\sqrt{x}}\right)^3}$.

756. $y = \frac{1}{\sqrt{x}} e^{x^2 - \arctan x + \frac{1}{2}\ln x + 1}$.

757. $y = \frac{\sin x}{4 \cos^4 x} + \frac{3 \sin x}{8 \cos^2 x} + \frac{3}{8} \ln \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}}$.

758. $y = \frac{x e^x \arctan x}{\ln^5 x}$.

759. $y = \frac{(1 - x^2) e^{3x-1} \cos x}{(\arccos x)^3}$.

760. $y = x \sqrt{(x^2 + a^2)^3} + \frac{3a^2 x}{2} \sqrt{x^2 + a^2} + \frac{3a^4}{2} \ln (x +$
$+ \sqrt{x^2 + a^2})$.

761. $y = x (\arcsin x)^2 - 2x + 2 \sqrt{1 - x^2} \arcsin x$.

762. $y = \ln \cos \arctan \frac{e^x - e^{-x}}{2}$.

763. $y = \frac{1}{m \sqrt{ab}} \arctan \left(e^{mx} \sqrt{\frac{a}{b}} \right)$.

764. $y = \frac{1}{3} \ln \frac{x + 1}{\sqrt{x^2 - x + 1}} + \frac{1}{\sqrt{3}} \arctan \frac{2x - 1}{\sqrt{3}}$.

765. $y = \ln \frac{\sqrt{1 + x} - \sqrt{1 - x}}{\sqrt{1 + x} + \sqrt{1 - x}} + 2 \arctan \sqrt{\frac{1 - x}{1 + x}}$.

766. $y = (\tan 2x)^{\cot \frac{x}{2}}$.

767. $y = \sqrt[3]{\frac{x - 5}{\sqrt[5]{x^2 + 4}}}$.

768. $y = \ln \sqrt[4]{\frac{x^2 + x + 1}{x^2 - x + 1}} + \frac{1}{2\sqrt{3}} \left(\arctan \frac{2x + 1}{\sqrt{3}} + \right.$
$\left. + \arctan \frac{2x - 1}{\sqrt{3}} \right)$.

769. $y = \arccos \frac{x^{2n} - 1}{x^{2n} + 1}$.

770. $y = - \frac{x}{1 + 8x^3} + \frac{1}{12} \ln \frac{(1 + 2x)^2}{1 - 2x + 4x^2} +$
$+ \frac{\sqrt{3}}{6} \arctan \frac{4x - 1}{3}$.

771. Show that the function $y = \ln \frac{1}{1+x}$ satisfies the relationship

$$x \frac{dy}{dx} + 1 = e^y.$$

772. Show that the function

$$y = \frac{x^2}{2} + \frac{1}{2} x \sqrt{x^2+1} + \ln \sqrt{x + \sqrt{x^2+1}}$$

satisfies the relationship

$$2y = xy' + \ln y'.$$

773. Show that the function

$$y = \frac{\arcsin x}{\sqrt{1-x^2}}$$

satisfies the relationship

$$(1-x^2)\, y' - xy = 1.$$

774*. Evaluate the sums:

$$1 + 2x + 3x^2 + \ldots + nx^{n-1};$$
$$2 + 2 \,.\, 3x + 3 \,.\, 4x^2 + \ldots + n\,(n-1)\, x^{n-2}.$$

Inverse Functions

775. Suppose that the rule for differentiating power functions is only proved for positive integral powers. Deduce the formula for differentiating a root by using the rule for differentiating an inverse function.

776. $x = e^{\arcsin y}$; find $\frac{dy}{dx}$ in terms of y, in terms of x.

777. $t = 2 - 3s + s^3$; express $\frac{ds}{dt}$ in terms of s.

778. $u = \frac{1}{2} \ln \frac{1+v}{1-v}$; prove the relationship

$$\frac{du}{dv} \cdot \frac{dv}{du} = 1.$$

779. Knowing that $\arcsin\sqrt{x}$ and $\sin^2 x$ are the inverse of each other, and that $(\sin^2 x)' = \sin 2x$, find:

$$(\arcsin\sqrt{x})'.$$

780. We use the symbol $\alpha(x)$ to denote the inverse of the exponential function $y = x^x$, i.e. $y = x^x$ implies $x = \alpha(y)$. Find the formula for the derivative of the function $y = \alpha(x)$.

781. The inverse hyperbolic functions are written as $\operatorname{Arc\,sinh} x$ $\operatorname{Arc\,cosh} x$, $\operatorname{Arc\,tanh} x$. Find the derivatives of these functions.

782. $s = te^{-t}$; find $\dfrac{dt}{ds}$.

783. $y = \dfrac{1 - x^4}{1 + x^4}$. Express $\dfrac{dx}{dy}$ in terms of x, in terms of y. Show that $\dfrac{dy}{dx}\cdot\dfrac{dx}{dy} = 1$.

784. $x = y^3 - 4y + 1$. Find $\dfrac{dy}{dx}$.

785. $t = \arcsin 2^s$. Find $\dfrac{ds}{dt}$ in terms of s, in terms of t.

786. Show that $\dfrac{dy}{dx}\cdot\dfrac{dx}{dy} = 1$ if x and y are connected by the relationships:

(1) $y = x^2 + ax + b$; (2) $y = x^{-n}$;
(3) $y = \ln(x^2 - 1)$.

Functions Given Implicitly

787. Show that the derivatives of both sides of the equation

$$\sin^2 x = 1 - \cos^2 x$$

are identically equal, i.e. that the equation can be differentiated term by term. Is it "possible" to differentiate term by term the equation $\sin x = 1 - \cos x$?

788. Show that the equation

$$\frac{2\sin^2 x - 1}{\cos x} + \frac{\cos x\,(2\sin x + 1)}{1 + \sin x} = \tan x$$

can be differentiated term by term.

789. What must the function $y = f(x)$ be for the equation

$$\cos^4 x + 2\sin^2 x \cos^2 x + y^2 = 1$$

to be differentiable term by term (i.e. for the derivatives of both sides to be identical)?

790. What must y be for the derivatives of both sides of

$$x^2 + y^2 = 1$$

to be equal, i.e. for the equation to be differentiable term by term?

791. What is the slope of the tangent to the circle

$$(x - 1)^2 + (y + 3)^2 = 17,$$

at the point (2, 1)?

Find the derivatives of functions y given implicitly in problems 792–812:

792. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ **793.** $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$

794. $x^3 + y^3 - 3axy = 0.$ **795.** $y^2 \cos x = a^2 \sin 3x.$

796. $y^3 - 3y + 2ax = 0.$ **797.** $y^2 - 2xy + b^2 = 0.$

798. $x^4 + y^4 = x^2y^2.$

799. $x^3 + ax^2y + bxy^2 + y^3 = 0.$

800. $\sin(xy) + \cos(xy) = \tan(x + y).$

801. $2^x + 2^y = 2^{x+y}.$ **802.** $2y \ln y = x.$

803. $x - y = \arcsin x - \arcsin y.$

804. $x^y = y^x.$ **805.** $y = \cos(x + y).$

806. $\cos(xy) = x.$ **807.** $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$

808. $y = 1 + xe^y.$

809. $x \sin y - \cos y + \cos 2y = 0.$

810. $\tan \frac{y}{2} = \sqrt{\frac{1-k}{1+k}} \tan \frac{x}{2}.$

811. $y \sin x - \cos(x - y) = 0.$

812. $y = x + \arctan y.$

813. Show that the function y defined by the equation $xy - \ln y = 1$ satisfies also the relationship

$$y^2 + (xy - 1)\frac{dy}{dx} = 0.$$

Geometrical and Mechanical Applications of the Derivative

814. Two points with abscissae $x_1 = 1$, $x_2 = 3$ are taken on the parabola $y = x^2$. A secant is drawn through these points. At what point of the parabola is the tangent to it parallel to this secant?

815. A chord is drawn through the focus of a parabola perpendicular to its axis. Tangents are drawn to the parabola at its points of intersection with the chord. Show that the tangents cut at a right angle.

816. Form the equations of the tangent and normal to the hyperbola $y = \frac{1}{x}$ at the point with abscissa $x = -\frac{1}{2}$. Find the subtangent and subnormal.

817. Show that the intercept of the tangent to the hyperbola $y = \frac{a}{x}$ between the coordinate axes is bisected at the point of contact.

818. Show that the area of the triangle formed by any tangent to the hyperbola $xy = a$ and the coordinate axes is equal to the area of the square constructed on the semi-transverse axis.

819. A particle moves along a straight line so that its distance s from the initial point after t seconds is equal to

$$s = \frac{1}{4}t^4 - 4t^3 + 16t^2.$$

(a) At what instant was the particle at the initial point? (b) At what instant does its velocity vanish?

820. A body of mass 3 kg moves along a straight line according to the law

$$s = 1 + t + t^2;$$

s is given in centimetres, t in seconds. Find the kinetic energy $\frac{mv^2}{2}$ of the body 5 sec after the initial instant.

821. The angle α of rotation of pulley is given as a function of time t by $\alpha = t^2 + 3t - 5$. Find the angular velocity at $t = 5$ sec.

822. The angle of rotation of a wheel is proportional to the square of the time. The first revolution is accomplished in 8 sec. Find the angular velocity ω, 32 sec from the start of the motion.

823. The angle θ through which a wheel rotates in t sec is equal to

$$\theta = at^2 - bt + c,$$

where a, b, c are positive constants. Find the angular velocity ω of the wheel. After how long will the angular velocity be zero?

824. The quantity of electricity flowing through a conductor, starting from the instant $t = 0$, is given by

$$Q = 2t^2 + 3t + 1 \quad \text{(coulombs)}$$

Find the current at the end of the fifth second.

825. Find the points of the curve $y = x^2 (x - 2)^2$ at which the tangents are parallel to the axis of abscissae.

826. Show that the curve $y = x^5 + 5x - 12$ is inclined to Ox at every point at an acute angle.

827. At what points of the curve $y = x^3 + x - 2$ is the tangent parallel to the straight line $y = 4x - 1$.

828. Form the equations of the tangents to the curve $y = x - \frac{1}{x}$ at its points of intersection with the axis of abscissae.

829. Find the equation of the tangent to the curve $y = x^3 + 3x^2 - 5$ perpendicular to the straight line $2x - 6y + 1 = 0$.

Find the equations of the tangents and normals to the curves of problems 830–833:

830. $y = \sin x$ at the point $M(x_0, y_0)$.

831. $y = \ln x$ at the point $M(x_0, y_0)$.

832. $y = \dfrac{8a^3}{4a^2 + x^2}$ at the point with abscissa $x = 2a$.

833. $y^2 = \dfrac{x^3}{2a - x}$ (cissoid) at the point $M(x_0, y_0)$.

834. Show that the subtangent to the nth order parabola $y = x^n$ is equal to $\dfrac{1}{n}$ times the abscissa of the point of contact.

Give a method of drawing the tangent to the curve $y = x^n$.

835. Find the subtangents and subnormals to the curves $y = x^3$; $y^2 = x^3$; $xy^2 = 1$. Give methods of drawing the tangents to these curves.

836. Find the equations of the tangent and normal to the parabola $x^2 = 4ay$ at the point (x_0, y_0); show that the equation of the tangent at the point with abscissa $x_0 = 2am$ is $x = \dfrac{y}{m} + am$.

837. A chord of the parabola $y = x^2 - 2x + 5$ joins the points with abscissae $x_1 = 1$, $x_2 = 3$. Find the equation of the tangent to the parabola parallel to the chord.

838. Find the equation of the normal to the curve

$$y = \frac{x^2 - 3x + 6}{x^2}$$

at the point with abscissa $x = 3$.

839. Find the equation of the normal to the curve $y = -\sqrt{x} + 2$ at its point of intersection with the bisector of the first quadrant.

840. Find the equation of the normal to the parabola $y = x^2 - 6x + 6$ perpendicular to the straight line joining the origin to the vertex.

841. Show that the normals to the curve $y = x^2 - x + 1$ through the points with abscissae $x_1 = 0$, $x_2 = -1$ and $x_3 = \dfrac{5}{2}$ intersect in a single point.

842. Normals are drawn to the parabola $y = x^2 - 4x + 5$ at its intersections with the straight line $x - y + 1 = 0$. Find the area of the triangle formed by the normals and the chord joining the points of intersection.

843. Show that the tangents to the hyperbola $y = \frac{x-4}{x-2}$ at its points of intersection with the coordinate axes are parallel.

844. Find the tangent to the hyperbola $y = \frac{x+9}{x+5}$ such that it passes through the origin.

845. Find the point of the curve $y = \frac{1}{1+x^2}$ at which the tangent is parallel to the axis of abscissae.

846. Find the equation of the tangent to the curve $x^2(x+y) = a^2(x-y)$ at the origin.

847. Show that the tangents to the curve $y = \frac{1+3x^2}{3+x^2}$ through the points for which $y = 1$ intersect at the origin.

848. Draw the normal to the curve $y = x \ln x$ parallel to the straight line $2x - 2y + 3 = 0$.

849. Find the distance from the origin to the normal to the curve $y = e^{2x} + x^2$ at the point $x = 0$.

850. Draw the graph of the function $y = \sin\left(2x - \frac{\pi}{3}\right)$ and find the point of intersection of the tangents to the graph when one is drawn through the point of intersection of the graph with Oy and the other through the point $\left(\frac{5\pi}{12}, 1\right)$.

851. Show that the subtangent at any point of the curve $y = ae^{bx}$ (a and b are constants) is of constant length.

852. Show that the subnormal at any point of the curve $y = x \ln(cx)$ (c is an arbitrary constant) is the fourth proportional of the abscissa, ordinate and sum of abscissa and ordinate of the point.

853. Show that any tangent to the curve $y = x\frac{1}{2}\sqrt{x - 4x^2}$ cuts the axis of ordinates at a point equidistant from the point of contact and the origin.

854. Show that the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $M(x_0, y_0)$ has the equation $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$.

855. Show that the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} =$ $= 1$ at the point $M(x_0, y_0)$ has the equation $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$.

856. Prove that the normal at any point of an ellipse bisects the angle between the focal radii of the point (Fig. 21).

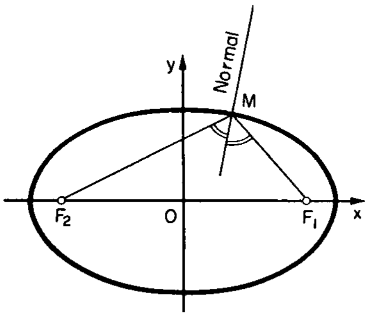

FIG. 21.

Deduce from this a method of drawing the tangent and normal to the ellipse; solve the corresponding problem for the hyperbola.

857. Find the equations of the tangents to the hyperbola $\frac{x^2}{2} - \frac{y^2}{7} = 1$ perpendicular to the straight line $2x + 4y -$ $- 3 = 0$.

858. A straight line is drawn through the origin parallel to the tangent to a curve at a given point M. Find the locus of point P of intersection of this straight line with the straight line through M parallel to the axis of ordinates.

Find these loci for (a) the parabola $y^2 = 2px$, (b) for the logarithmic curve $y = \log_b x$, (c) for the circle $x^2 + y^2 = a^2$, (d) for the tractrix $y = \sqrt{a^2 - x^2} - a \ln \dfrac{a + \sqrt{a^2 - x^2}}{x}$.

Find the angles at which the curves of problems 859–864 intersect:

859. (1) $y = \dfrac{x+1}{x+2}$ and $y = \dfrac{x^2 + 4x + 8}{16}$.

(2) $y = (x-2)^2$ and $y = 4x - x^2 + 4$.

860. (1) $x^2 + y^2 = 8$ and $y^2 = 2x$.

(2) $x^2 + y^2 - 4x = 1$ and $x^2 + y^2 + 2y = 9$.

861. $x^2 - y^2 = 5$ and $\dfrac{x^2}{18} + \dfrac{y^2}{8} = 1$.

862. $x^2 + y^2 = 8ax$ and $y^2 = \dfrac{x^3}{2a - x}$.

863. $x^2 = 4ay$ and $y = \dfrac{8a^3}{x^2 + 4a^2}$.

864. $y = \sin x$ and $y = \cos x$ $(0 \leqq x \leqq \pi)$.

865. Find the equations of the tangent and normal to the curve

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$$

at the point with abscissa a.

866. Prove that the sum of the intercepts cut from the coordinate axes by the tangent at any point of the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ is equal to a.

867. Prove that the segment of the tangent to the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ lying between the coordinate axes has a constant length a.

868. Prove that the segment of the tangent to the tractrix

$$y = \frac{a}{2} \ln \frac{a + \sqrt{a^2 - x^2}}{a - \sqrt{a^2 - x^2}} - \sqrt{a^2 - x^2},$$

lying between the axis of ordinates and the point of contact has a constant length.

869. Prove that the length of the normal at any point $M(x_0, y_0)$ of the equilateral hyperbola $x^2 - y^2 = a^2$, measured from M to the point at which it cuts the axis of abscissae, is equal to the radius vector of point M.

870. Show that the intercept cut off on the axis of abscissae by the tangent at any point of the curve $\frac{a}{x^2} + \frac{b}{y^2} = 1$ is proportional to the cube of the abscissa of the point of contact.

871. Prove that the ordinate of any point of the curve $2x^2y^2 - x^4 = c$ (c is an arbitrary constant) is the mean proportional between the abscissa and sum of abscissa and subnormal to the curve at that point.

872. Show that the tangents at points with the same abscissa to the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where the axis $2a$ is common and axes $2b$ are different (Fig. 22), have a common point of intersection on the axis of abscissae. Use this fact to indicate a simple method of drawing the tangent to an ellipse.

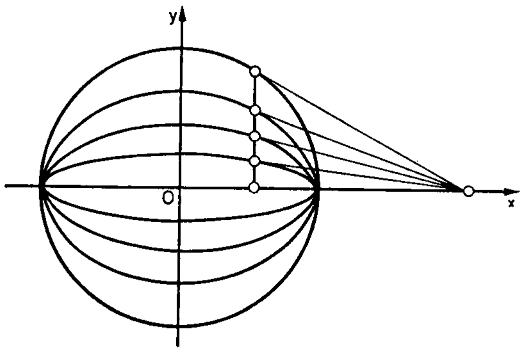

FIG. 22.

873. Prove that the curve $y = e^{kx} \sin mx$ is touched by each of the curves $y = e^{kx}$, $y = -e^{kx}$ at every point common to them.

874. The following method is used for drawing the tangent to the catenary $y = \frac{1}{2}a\,(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$: a semi-circle is drawn on the ordinate MN of point M as diameter (Fig. 23) and the chord $NP = a$ obtained; the line MP is the required tangent. Prove this.

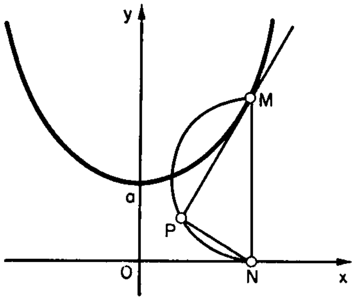

FIG. 23.

Graphical Differentiation

875. The following results were obtained by measuring the winding temperature of an electric motor when current passed:

Time t min	0	5	10	15	20	25
Temperature θ °C	20	26	32·5	41	46	49
Time t min	30	35	40	45	50	55
Temperature θ °C	52·5	54·5	56·5	58	59·5	61

Draw a rough graph of the continuous dependence of temperature on time. Use graphical differentiation to draw the graph of the rate of change of temperature with time.

876. Figure 24 illustrates the curve of the rise of a steam engine (low pressure) inlet valve. Draw the velocity curve by graphical differentiation.

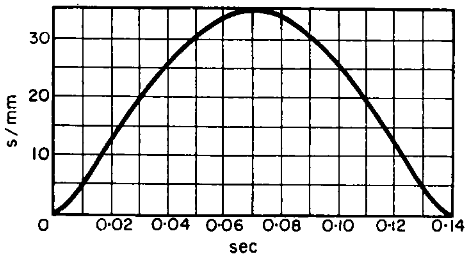

FIG. 24.

3. Differentials. Differentiability of a Function

Differentials

877. Find the increment of the function $y = x^2$ corresponding to the increment Δx of the independent variable. Evaluate Δy if $x = 1$ and $\Delta x = 0{\cdot}1$; $0{\cdot}01$. What will be the error (absolute and relative) in the value of Δy if we confine ourselves to the term linear in Δx?

878. Find the increment Δv of the volume v of a sphere when the radius $R = 2$ changes by ΔR. Evaluate Δv if $\Delta R = 0{\cdot}5$; $0{\cdot}1$; $0{\cdot}01$. What will be the error in the value of Δv if we confine ourselves to the term which is linear in ΔR?

879. Given the function $y = x^3 + 2x$, find the value of the increment and its linear principal part corresponding to variation of x from $x = 2$ to $x = 2{\cdot}1$.

880. What is the increment of the function $y = 3x^2 - x$ when the independent variable passes from the value $x = 1$ to $x = 1{\cdot}02$. What is the value of the corresponding principal part? Find the ratio of the second quantity to the first.

881. We know that the increment $\Delta x = 0{\cdot}2$ for a given function $y = f(x)$ at the point x. The corresponding principal part of the increment of the function is known to be $0{\cdot}8$. Find the numerical value of the derivative at x.

882. We know that the principal part of the increment $\mathrm{d}f(x) = -0{\cdot}8$ of the function $f(x) = x^2$ corresponds to the increment $\Delta x = 0{\cdot}2$ of the independent variable at a certain point. Find the initial value of the independent variable.

883. Find the increment and differential of the function $y = x^2 - x$ for $x = 10$ and $\Delta x = 0{\cdot}1$. Find the absolute and relative error on replacing the increment by the differential. Draw a figure.

884. Find the increment and differential of the function $y = \sqrt{x}$ for $x = 4$ and $\Delta x = 0{\cdot}41$. Find the absolute and relative errors. Draw a figure.

885. $y = x^3 - x$. Evaluate Δy and dy at $x = 2$, giving Δx the values $\Delta x = 1$, $\Delta x = 0{\cdot}1$, $\Delta x = 0{\cdot}01$. Find the corresponding value of the relative error $\delta = \dfrac{|\Delta y - dy|}{|\Delta y|}$.

886. Find graphically (by drawing a large scale figure on millimetre paper) the increment and differential, and evaluate the absolute and relative errors, on replacing the increment by the differential for the function $y = 2^x$ with $x = 2$ and $\Delta x = 0{\cdot}4$.

887. The side of a square is 8 cm. How much is its area increased if each side is increased by (a) 1 cm, (b) 0·5 cm, (c) 0·1 cm. Find the linear principal part of the increment of the area of the square and estimate the relative error (in per cent) on replacing the increment by its principal part.

888. We know that, when the side of a given square is increased by 0·3 cm, the linear principal part of the increment in the area amounts to 2·4 cm^2. Find the linear part of the increment of the area corresponding to an increment in each side of (a) 0·6 cm, (b) 0·75 cm, (c) 1·2 cm.

889. Find the differentials of the functions:

$$(1)\ 0{\cdot}25\sqrt{x};\quad (2)\ \frac{\sqrt[3]{x}}{0{\cdot}2};\quad (3)\ \frac{1}{0{\cdot}5x^2};\quad (4)\ \frac{1}{4x^4};\quad (5)\ \frac{1}{2\sqrt{x}};$$

$$(6)\ \frac{1}{n\sqrt[3]{x}};\quad (7)\ \frac{\sqrt{x}}{a+b};\quad (8)\ \frac{p}{q^x};\quad (9)\ \frac{m-n}{x^{0{\cdot}2}};$$

(10) $\dfrac{m+n}{\sqrt{x}}$; (11) $(x^2+4x+1)(x^2-\sqrt{x})$;

(12) $\dfrac{x^3+1}{x^3-1}$; (13) $\dfrac{1}{1-t^2}$; (14) $(1+x-x^2)^3$;

(15) $\tan^2 x$; (16) $5^{\ln\tan x}$; (17) $2^{-\frac{1}{\cos x}}$;

(18) $\ln\tan\left(\dfrac{\pi}{2}-\dfrac{x}{4}\right)$; (19) $\dfrac{\cos x}{1-x^2}$;

(20) $\sqrt{\arcsin x}+(\arctan x)^2$;

(21) $3\arcsin x-4\arctan x+\dfrac{1}{2}\operatorname{arc\,cos} x-3\dfrac{1}{2}\operatorname{arc\,cot} x$;

(22) $3^{-\frac{1}{x^2}}+3x^3-4\sqrt{x}$.

890. Find the value of the differential of the function: (1) $y=\dfrac{1}{(\tan x+1)^2}$ when the independent variable changes from $x=\dfrac{\pi}{6}$ to $x=\dfrac{61\pi}{360}$; (2) $y=\cos^2\varphi$ when φ varies from $60°$ to $60°30'$; (3) $y=\sin 2\varphi$ when φ varies from $\dfrac{\pi}{6}$ to $\dfrac{61\pi}{360}$; (4) $y=\sin 3\varphi$ when φ varies from $\dfrac{\pi}{6}$ to $\dfrac{61\pi}{360}$; (5) $y=\sin\dfrac{\theta}{3}$ when θ varies from $\dfrac{\pi}{6}$ to $\dfrac{61\pi}{360}$.

891. Find the approximate value of the increment of the function $y=\sin x$ when x varies from $30°$ to $30°1'$. What does $\sin 30°1'$ equal?

892. Find the approximate value of the increment of the function $y=\tan x$ when x varies from $45°$ to $45°10'$.

893. Find the approximate value of the increment of $y=\dfrac{1+\cos x}{1-\cos x}$ when x varies from $\dfrac{\pi}{3}$ to $\dfrac{\pi}{3}+\dfrac{1}{100}$.

894. $\varrho=k\sqrt{\cos 2\varphi}$; find $d\varrho$.

895. $y=3^{\frac{1}{x}}+\dfrac{1}{2^{2x}}+6^{\sqrt{x}}$. Evaluate dy for $x=1$ and $dx=0{\cdot}2$.

896. Evaluate approximately sin 60°3′, sin 60°18′. Compare the results with the tabulated figures.

897. Show that the function $y = \dfrac{1 + \ln x}{x - x \ln x}$ satisfies the relationship

$$2x^2\,\mathrm{d}y = (x^2y^2 + 1)\,\mathrm{d}x.$$

898. Show that function y given by

$$\operatorname{arc\,tan}\frac{y}{x} = \ln\sqrt{x^2 + y^2}$$

satisfies the relationship $x(\mathrm{d}y - \mathrm{d}x) = y(\mathrm{d}y + \mathrm{d}x)$.

899. $f(x) = \mathrm{e}^{0{\cdot}1x(1-x)}$. Work out approximately $f(1{\cdot}05)$.

900. Evaluate arc tan 1·02, arc tan 0·97.

901. Evaluate approximately $\sqrt{\dfrac{(2{\cdot}037)^2 - 1}{(2{\cdot}037)^2 + 1}}$.

902. Evaluate approximately arc sin 0·4983.

903. If the length of a heavy cord (cable, chain) (Fig. 25) is $2s$, the half-span l, and the sag f, the approximate equation holds:

$$s = l\left(1 + \frac{2}{3}\frac{f^2}{l^2}\right).$$

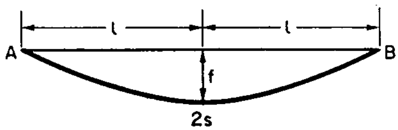

FIG. 25.

(a) Work out the change in the length of the cord when its sag f increases by $\mathrm{d}f$.

(b) If we assume a change in the length of the cable $\mathrm{d}x$ (say due to a change in temperature or load), how does the sag vary?

904. Compare the errors in finding an angle from its tangent and from its sine with the aid of logarithmic tables, i.e. compare the accuracy of finding x from the formula $\log\sin x = y$ and $\log\tan x = z$, if y and z are given with the same errors.

905. In engineering calculations π and $\sqrt{g}$ (g is the acceleration due to gravity) are often cancelled when one occurs in the numerator and the other in the denominator. Find the relative error resulting from this.

906. Express the differentials of the following functions of a function in terms of the independent variable and its differential:

(1) $y = \sqrt[3]{x^2 + 5x}$; $x = t^3 + 2t + 1$;

(2) $s = \cos^2 z$, $z = \dfrac{t^2 - 1}{4}$;

(3) $z = \arc\tan v$, $v = \dfrac{1}{\tan s}$;

(4) $v = 3^{-\frac{1}{x}}$, $x = \ln\tan s$;

(5) $s = e^z$, $z = \dfrac{1}{2}\ln t$, $t = 2u^2 - 3u + 1$;

(6) $y = \ln\tan\dfrac{u}{2}$; $u = \arc\sin v$, $v = \cos 2s$.

Differentiability of Functions

907. The function $y = |x|$ is continuous for any x. Show that it is not differentiable at $x = 0$.

908. Investigate the continuity and differentiability of the function $y = |x^3|$ at $x = 0$.

909. A function $f(x)$ is defined as follows: $f(x) = 1 + x$ for $x \leqq 0$; $f(x) = x$ for $0 < x < 1$; $f(x) = 2 - x$ for $1 \leqq x \leqq 2$ and $f(x) = 3x - x^2$ for $x > 2$. Investigate the continuity of $f(x)$ and examine the existence and continuity of $f'(x)$.

910. The function $y = |\sin x|$ is continuous for any x. Show that it is not differentiable at $x = 0$. Are there any other values of the independent variable at which the function is not differentiable?

911. Investigate the continuity and differentiability of the function $y = e^{-|x|}$ at $x = 0$.

912. $f(x) = x^2 \sin\frac{1}{x}$ for $x \neq 0$, $f(0) = 0$. Is $f(x)$ differentiable at $x = 0$?

913. $f(x) = \frac{\sqrt{x+1}-1}{\sqrt{x}}$ for $x \neq 0$, $f(0) = 0$. Is $f(x)$ continuous and differentiable at $x = 0$?

914. Given $f(x) = 1 + \sqrt[3]{(x-1)^2}$, show that it is impossible to extract from the increment of $f(x)$ at $x = 1$ a linear principal part, so that $f(x)$ has no derivative at $x = 0$. Interpret the result geometrically.

915. $f(x) = x \arctan 1/x$ for $x = 0$, $f(0) = 0$. If $f(x)$ continuous and differentiable at $x = 0$? Interpret the result geometrically.

916. $f(x) = \frac{x}{1 + e^{\frac{1}{x}}}$ for $x \neq 0$ and $f(0) = 0$. Is $f(x)$ continuous or differentiable at $x = 0$?

4. Derivative as Rate of Change (Further Examples)

Relative Velocity

917. A particle moves along the spiral of Archimedes $\varrho = a\varphi$. Find the rate of change of the radius vector ϱ relative to the polar angle φ.

918. A particle moves along the logarithmic spiral $\varrho = e^{a\varphi}$. Find the rate of change of the radius vector if it is known to rotate with angular velocity ω.

919. A particle moves along the circle $\varrho = 2r\cos\varphi$. Find the rate of change of the abscissa and ordinate of the particle if the radius vector rotates with angular velocity ω. The polar axis is the axis of abscissae and the pole the origin in the system of Cartesian coordinates.

920. A circle of radius R rolls along a straight line without slip. The centre of the circle moves with constant velocity v. Find the rate of change of the abscissa x and ordinate y of a point on the circumference of the circle.

921. The barometric pressure p varies with the height h in accordance with the function

$$\ln \frac{p}{p_0} = ch,$$

where p_0 denotes the standard pressure. The pressure at a height of 5540 m is half the standard; find the rate of change of the pressure with the height.

922. The relationship $y^2 = 12x$ connects y and x. The argument x increases uniformly with a velocity of 2 units per second. What is the rate of increase of y at $x = 3$?

923. The ordinate of a point describing the circle $x^2 + y^2 = 25$ decreases at a rate of 1·5 cm/sec. What is the rate of change of the abscissa of the point when the ordinate becomes 4 cm?

924. At what point of the ellipse $16x^2 + 9y^2 = 400$ does the ordinate decrease at the same rate as the abscissa increases?

925. The side of a square increases at a rate of v cm/sec. What are the rates of change of the perimeter and area of the square at the instant when the side is a cm.

926. The radius of a circle varies with velocity v. At what rates do the area and circumference of the circle change?

927. The radius of a sphere varies with a speed v. What are the rates of change of the volume and surface of the sphere?

928. For what angles does the sine vary twice as slowly as the argument?

929. At what angles are the rates of change of the sine and tangent of the same angle equal?

930. The rate of increase of a sine is increased n times. How many times will the rate of increase of the tangent be increased?

931. Assuming that the volume of a wooden cask is proportional to the cube of its diameter and that the latter increases uniformly from year to year, show that the rate of increase of the volume when the diameter is 90 cm is 25 times greater than the rate when the diameter is 18 cm.

Functions given Parametrically

932. How do you prove whether or not a point given by Cartesian coordinates lies on a curve whose equation is given in the parametric form? (a) Does the point (5, 1) lie on the circle $x = 2 + 5\cos t$, $y = -3 + 5\sin t$? (b) Does the point $(2, \sqrt{3})$ lie on the circle $x = 2\cos t$, $y = 2\sin t$?

933. Plot the graphs of the functions given parametrically:

(a) $x = 3\cos t$, $\quad y = 4\sin t$;
(b) $x = t^2 - 2t$, $\quad y = t^2 + 2t$;
(c) $x = \cos t$, $\quad y = t + 2\sin t$;
(d) $x = 2^{t-1}$, $\quad y = \frac{1}{4}(t^3 + 1)$.

934. Eliminate the parameter from the parametric equations of the functions:

(1) $x = 3t$, $y = 6t - t^2$; (2) $x = \cos t$, $y = \sin 2t$;
(3) $x = t^3 + 1$, $y = t^2$; (4) $x = \varphi - \sin\varphi$, $y = 1 - \cos\varphi$;
(5) $x = \tan t$; $y = \sin 2t + 2\cos 2t$.

935. Given the following curves, specified by parametric equations, find the values of the parameter corresponding to the points with given coordinates on the curves:

(1) $x = 3(2\cos t - \cos 2t)$, $y = 3(2\sin t - \sin 2t)$; $(-9, 0)$;
(2) $x = t^2 + 2t$, $y = t^3 + t$; $(3, 2)$;
(3) $x = 2\tan t$, $y = 2\sin^2 t + \sin 2t$; $(2, 2)$;
(4) $x = t^2 - 1$, $y = t^3 - t$; $(0, 0)$.

Find the derivatives of y with respect to x in problems 936–945:

936. $x = a\cos\varphi$, $\quad y = b\sin\varphi$.

937. $x = a\cos^3\varphi$, $\quad y = b\sin^3\varphi$.

938. $x = a(\varphi - \sin\varphi)$, $\quad y = a(1 - \cos\varphi)$.

939. $x = 1 - t^2$, $\quad y = t - t^3$.

940. $x = \dfrac{t+1}{t}$, $\quad y = \dfrac{t-1}{t}$.

941. $x = \ln(1 + t^2)$, $\quad y = t - \arctan t$.

942. $x = \varphi(1 - \sin\varphi)$, $\quad y = \varphi\cos\varphi$.

943. $x = \dfrac{1+t^3}{t^2-1}$, $\quad y = \dfrac{t}{t^2-1}$.

944. $x = e^t \sin t$, $\quad y = e^t \cos t$.

945. $x = \dfrac{3at}{1+t^3}$, $\quad y = \dfrac{3at^2}{1+t^3}$.

Find the slope of the tangents to the curves of problems 946–949:

946. $x = 3\cos t$, $y = 4\sin t$ at the point $\left(\dfrac{3\sqrt{2}}{2}, 2\sqrt{2}\right)$.

947. $x = t - t^4$, $y = t^2 - t^3$ at the point $(0, 0)$.

948. $x = t^3 + 1$, $y = t^2 + t + 1$ at the point $(1, 1)$.

949. $x = 2\cos t$, $y = \sin t$ at the point $\left(1, -\dfrac{\sqrt{3}}{2}\right)$.

950. Give the simplest geometrical significance of parameter t for the following curves specified parametrically:

$$(1)\quad \begin{cases} x = \cos t + t\sin t - \dfrac{t^2}{2}\cos t, \\ y = \sin t - t\cos t - \dfrac{t^2}{2}\sin t; \end{cases}$$

(2) $\quad x = a\cos^3 t$, $y = a\sin^3 t$;

(3) $\quad x = a\cos t\sqrt{2\cos 2t}$, $y = a\sin t\sqrt{2\cos 2t}$.

951. Show that the function given by the parametric equations

$$x = 2t + 3t^2, \quad y = t^2 + 2t^3,$$

satisfies the relationship $y = y'^2 + 2y'^3$ (the prime denotes differentiation with respect to x, i.e. $y' = \dfrac{dy}{dx}\Big)$.

952. Show that the function given by the parametric equations

$$x = \frac{1+t}{t^3}, \quad y = \frac{3}{2t^2} + \frac{2}{t},$$

satisfies the relationship

$$xy'^3 = 1 + y' \quad \left(y' = \frac{dy}{dx}\right).$$

953. Show that the function given by the parametric equations

$$x = \frac{at}{1+t^3}, \quad y = \frac{a^2}{6}\,\frac{4t^3+1}{(1+t^3)^2},$$

satisfies the relationship

$$x^3 + y^3 = axy' \quad \left(y' = \frac{dy}{dx}\right).$$

954. Show that the function given by the parametric equations

$$x = \frac{1}{\sqrt{1+t^2}} - \ln\frac{1+\sqrt{1+t^2}}{t}, \quad y = \frac{t}{\sqrt{1+t^2}},$$

satisfies the relationship

$$y\sqrt{1+y'^2} = y' \quad \left(y' = \frac{dy}{dx}\right).$$

955. Show that the function given by the parametric equations

$$x = \frac{1+\ln t}{t^2}, \quad y = \frac{3+2\ln t}{t},$$

satisfies the relationship

$$yy' = 2xy'^2 + 1 \quad \left(y' = \frac{dy}{dx}\right).$$

956. Find the angles at which the following curves intersect:

$$(1) \begin{cases} y = x^2 \\ x = \frac{5}{3}\cos t, \quad y = \frac{5}{4}\sin t; \end{cases}$$

$$(2) \begin{cases} x = a\cos\varphi, \quad y = a\sin\varphi \quad \text{and} \\ x = \frac{at^2}{1+t^2}, \quad y = \frac{at\sqrt{3}}{1+t^2}. \end{cases}$$

957. Show that, whatever the position of the generating circle of the cycloid, the tangent and normal at the corresponding point of the cycloid pass through its highest and lowest points (see *Course*, sec. 55).

958. Find the lengths of the tangent, normal, subtangent and subnormal at any point of the curve (cardioid)

$$x = a(2 \cos t - \cos 2t),\ y = a(2 \sin t - \sin 2t)$$

959. Find the lengths of the tangent, normal, subtangent and subnormal at any point of the curve (astroid)

$$x = a \sin^3 t, \quad y = a \cos^3 t$$

960. Prove by evaluation that the tangent to the circle $x^2 + y^2 = a^2$ is normal to the curve (involute of circle)

$$x = a(\cos t + t \sin t),\ y = a(\sin t - t \cos t).$$

961. Find the lengths of the tangent, normal, subtangent and subnormal of the involute of the circle (see the equations of the latter in the previous problem).

962. Prove that the segment of the normal to the curve

$$x = 2a \sin t + a \sin t \cos^2 t, \quad y = -a \cos^3 t,$$

lying between the coordinate axes is equal to $2a$.

Find the equations of the tangent and normal to each of the curves of problems 963–966 at the point in question:

963. $x = 2\mathrm{e}^t;\ y = \mathrm{e}^{-t}$ at $t = 0$.

964. $x = \sin t, \quad y = \cos 2t$ at $t = \dfrac{\pi}{6}$.

965. $x = 2 \ln \cot t + 1, \quad y = \tan t + \cot t$ at $t = \dfrac{\pi}{4}$.

966. (1) $x = \dfrac{3at}{1 + t^2}, \quad y = \dfrac{3at^2}{1 + t^2}$ at $t = 2$;

(2) $\begin{cases} x = t(t \cos t - 2 \sin t) \\ y = t(t \sin t + 2 \cos t) \end{cases}$ at $t = \dfrac{\pi}{4}$;

(3) $x = \sin t,\ y = a^t$ at $t = 0$.

967. Show that, for the cardioid (see problem 958), the tangents are parallel at any two points corresponding to values of the parameter differing by $\frac{2}{3}\pi$.

968. If OT and ON are the perpendiculars dropped from the origin to the tangent and normal at any point of the astroid (see problem 959), we have

$$4OT^2 + ON^2 = a^2.$$

Prove this.

969. Find the length of the perpendicular dropped from the origin on to the tangent to the curve

$$2x = a(3 \cos t + \cos 3t),\ 2y = a(3 \sin t + \sin 3t).$$

Prove that

$$4\varrho^2 = 3p^2 + 4a^2;$$

ϱ is the radius vector of the given point, and p the length of the perpendicular.

Rate of Change of Radius Vector

970. Find the angle θ between the radius vector and tangent, and the angle α between the polar axis and tangent, for the circle $\varrho = 2r \sin \varphi$.

971. Show that the sum of the angles formed by the tangent with the radius vector and with the polar axis is equal to two right angles for the parabola $\varrho = a \sec^2 \frac{\varphi}{2}$. Use this property for drawing the tangent to the parabola.

972. Given the curve $\varrho = a \sin^3 \frac{\varphi}{3}$ (conchoid), show that $\alpha = 4\theta$ (the notation is the same as in problem 970).

973. Prove that the two parabolas $\varrho = a \sec^2 \frac{\varphi}{2}$ and $\varrho = b \operatorname{cosec}^2 \frac{\varphi}{2}$ intersect at right angles.

974. Find the tangent of the angle between the polar axis and tangent to the curve $\varrho = a \sec^2 \varphi$ at the point for which $\varrho = 2a$.

975. Find the tangent of the angle between the polar axis and the tangent at the origin for (1) the curve $\varrho = \sin^3 \varphi$, (2) for the curve $\varrho = \sin 3\varphi$.

976. Show that the two cardioids $\varrho = a(1 + \cos \varphi)$ and $\varrho = a(1 - \cos \varphi)$ intersect at right angles.

977. The equation of a curve in polar coordinates is given parametrically: $\varrho = f_1(t)$, $\varphi = f_2(t)$. Express the tangent of the angle θ between the tangent and radius vector as a function of t.

978. A curve is given by the equations $\varrho = at^3$, $\varphi = bt^2$. Find the angle between the radius vector and tangent.

979. Express the radius vector ϱ and polar angle φ as functions of parameter t for the ellipse $x = a \cos t$, $y = b \sin t$. Use this form of writing the ellipse to evaluate the angle between the tangent and radius vector.

The polar subtangent is defined as the projection of the piece of tangent, measured from its point of contact to its intersection with the perpendicular erected on the radius vector at the pole, on to this perpendicular. The polar subnormal is similarly defined. Use the definitions to solve problems 980–984.

980. Deduce the expression for the polar subtangent and polar subnormal of the curve $\varrho = f(\varphi)$.

981. Show that the length of the polar subtangent of the hyperbolic spiral $\varrho = \dfrac{a}{\varphi}$ is constant.

982. Show that the length of the polar subnormal of the spiral of Archimedes $\varrho = a\varphi$ is constant.

983. Find the length of the polar subtangent of the logarithmic spiral $\varrho = a^\varphi$.

984. Find the length of the polar subnormal of the logarithmic spiral $\varrho = a^\varphi$.

Rate of Change of Arc Length

In problems 985–999, s denotes the length of arc of the curve.

985. The straight line is $y = ax + b$. Find $\frac{\mathrm{d}s}{\mathrm{d}x}$.

986. The circle is $x^2 + y^2 = r^2$; $\frac{\mathrm{d}s}{\mathrm{d}x} = ?$

987. The ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $\frac{\mathrm{d}s}{\mathrm{d}y} = ?$

988. The parabola is $y^2 = 2px$; $\mathrm{d}s = ?$

989. The semi-cubical parabola is $y^2 = ax^3$; $\frac{\mathrm{d}s}{\mathrm{d}y} = ?$

990. The sine wave is $y = \sin x$; $\mathrm{d}s = ?$

991. The catenary is $y = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}$ $(y = \cosh x)$; $\frac{\mathrm{d}s}{\mathrm{d}x} = ?$

992. The circle is $x = r\cos t$, $y = r\sin t$; $\frac{\mathrm{d}s}{\mathrm{d}t} = ?$

993. The cycloid is $x = a(t - \sin t)$, $y = a(1 - \cos t)$; $\frac{\mathrm{d}s}{\mathrm{d}t} = ?$

994. The astroid is $x = a\cos^3 t$, $y = a\sin^3 t$; $\mathrm{d}s = ?$

995. The spiral of Archimedes is $x = at\sin t$, $y = at\cos t$; $\mathrm{d}s = ?$

996. The cardioid is $\begin{cases} x = a(2\cos t - \cos 2t), \\ y = a(2\sin t - \sin 2t); \end{cases}$ $\mathrm{d}s = ?$

997. The tractrix is

$$x = a\left(\cos t + \ln\tan\frac{t}{2}\right), \quad y = a\sin t; \quad \mathrm{d}s = ?$$

998. The involute of a circle is

$$x = a(\cos t + t\sin t), \quad y = a(\sin t - t\cos t); \quad \frac{\mathrm{d}s}{\mathrm{d}t} = ?$$

999. The hyperbola is $x = a\cosh t$; $y = a\sinh t$; $\mathrm{d}s = ?$

Velocity of Motion

1000. A ladder of length 10 m has one end resting against a vertical wall and the other on the ground. The lower end moves away from the wall at a speed of 2 m/min. At what speed is the upper end falling when the bottom is 6 m from the wall? What is the direction of the velocity vector?

1001. A train and a balloon leave the same point at the same instant. The train travels uniformly at a speed of 50 km/hr, and the sphere rises (also uniformly) at a speed of 10 km/hr. At what speed are they leaving each other? What is the direction of the velocity vector?

1002. A man of height 1·7 m moves away from a light source at a height of 3 m at a speed of 6·34 km/hr. What is the speed of the shadow of his head?

1003. A horse runs round a circle at a speed of 20 km/hr. A lamp is located at the centre of the circle, whilst there is a fence tangential to the circle at the point at which the horse starts. At what speed is the shadow of the horse moving along the fence when the horse has travelled $\frac{1}{8}$ of the circle?

1004. Figure 26 illustrates schematically the crank mechanism of a steam engine: A is the cross-head, BB' the guide,

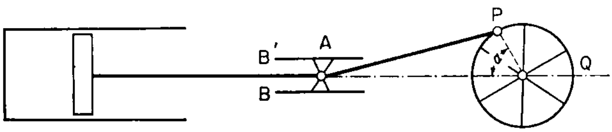

FIG. 26.

AP the connecting-rod, Q the fly-wheel. The fly-wheel rotates uniformly with angular velocity ω, its radius is R, and the length of the connecting-rod is l. At what speed does the cross-head move when the fly-wheel has rotated through the angle α?

1005. A fly-wheel disrupts when rotating at 80 rev/min. The radius of the wheel is 90 cm, the centre is 1 m above the ground. What is the speed of the piece indicated by A in Fig. 27 when it strikes the ground?

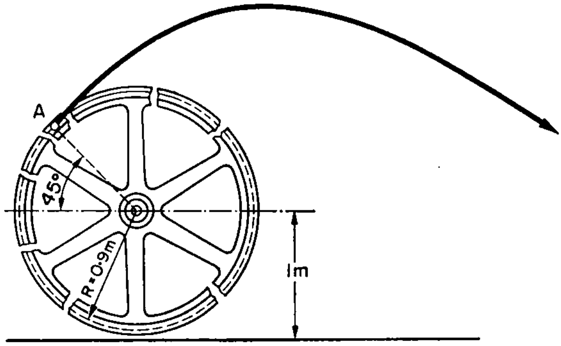

FIG. 27.

5. Repeated Differentiation

Functions given Explicitly

1006. $y = x^2 - 3x + 2$; $y'' = ?$

1007. $y = 1 - x^2 - x^4$; $y''' = ?$

1008. $f(x) = (x + 10)^6$; $f'''(2) = ?$

1009. $f(x) = x^6 - 4x^3 + 4$; $f^{(IV)}(1) = ?$

1010. $y = (x^2 + 1)^3$; $y'' = ?$ 1011. $y = \cos^2 x$; $y''' = ?$

1012. $f(x) = e^{2x-1}$; $f''(0) = ?$

1013. $f(x) = \arctan x$; $f''(1) = ?$

1014. $f(x) = \dfrac{1}{1 - x}$; $f^{(V)}(x) = ?$

1015. $y = x^3 \ln x$; $y^{(IV)} = ?$ 1016. $f(x) = \dfrac{a}{x^n}$; $y''(x) = ?$

1017. $\varrho = a \sin 2\varphi$; $\dfrac{d^4\varrho}{d\varphi^4} = ?$ 1018. $y = \dfrac{1 - x}{1 + x}$; $y^{(n)} = ?$

Find the second derivatives of the functions of problems 1019–1028:

1019. $y = xe^{x^2}$. 1020. $y = \dfrac{1}{1 + x^3}$.

1021. $y = (1 + x^2) \arctan x$. 1022. $y = \sqrt{a^2 - x^2}$.

1023. $y = \ln (x + \sqrt{1 + x^2})$. 1024. $y = \dfrac{1}{a + \sqrt{x}}$.

1025. $y = e^{\sqrt{x}}$.

1026. $y = \sqrt{1 - x^2} \arcsin x$.

1027. $y = \arcsin (a \sin x)$.

1028. $y = x^x$.

Find general expressions for the nth order derivatives of the functions of problems 1029–1040:

1029. $y = e^{ax}$.

1030. $y = e^{-x}$.

1031. $y = \sin ax + \cos bx$.

1032. $y = \sin^2 x$.

1033. $y = xe^x$.

1034. $y = x \ln x$.

1035. $y = \dfrac{1}{ax + b}$.

1036. $y = \ln (ax + b)$.

1037. $y = \log_a x$.

1038. $y = \dfrac{x}{x^2 - 1}$.

1039. $y = \dfrac{1}{x^2 - 3x + 2}$.

1040. $y = \sin^4 x + \cos^4 x$.

1041. Prove that the function $y = (x^2 - 1)^n$ satisfies the relationship

$$(x^2 - 1)\, y^{(n+2)} - 2xy^{(n+1)} - n(n + 1)\, y^{(n)} = 0.$$

1042. Prove that the function $y = e^x \sin x$ satisfies the relationship $y'' - 2y' + 2y = 0$, whilst $y = e^{-x} \sin x$ satisfies $y'' + 2y' + 2y = 0$.

1043. Show that the function $y = \dfrac{x - 3}{x + 4}$ satisfies the relationship $2y'^2 = (y - 1)\, y''$.

1044. Show that the function $y = \sqrt{2x - x^2}$ satisfies the relationship $y^3y'' + 1 = 0$.

1045. Show that the function $y = e^{4x} + 2e^{-x}$ satisfies the relationship $y''' - 13y' - 12y = 0$.

1046. Show that $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$ satisfies the relationship $xy'' + \dfrac{1}{2} y' - \dfrac{1}{4} y = 0$.

1047. Show that $y = \cos e^x + \sin e^x$ satisfies the equation $y'' - y' + ye^{2x} = 0$.

1048. Show that the function

$$y = A \sin (\omega t + \omega_0) + B \cos (\omega t + \omega_0)$$

(A, B, ω, ω_0 constant) satisfies the equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0.$$

1049. Show that the function

$$a_1 e^{nx} + a_2 e^{-nx} + a_3 \cos nx + a_4 \sin nx$$

(a_1, a_2, a_3, a_4, n constant) satisfies the equation $\frac{d^4y}{dx^4} = n^4 y$.

1050. Show that the function

$$y = \sin (n \arcsin x)$$

satisfies the equation

$$(1 - x^2)y'' - xy' + n^2 y = 0.$$

1051. Show that the function $e^{\alpha \arcsin x}$ satisfies the equation

$$(1 - x^2)\, y'' - xy' - \alpha^2 y = 0.$$

1052. Prove that the function $y = (x + \sqrt{x^2 + 1})^k$ satisfies the relationship

$$(1 + x^2)\, y'' + xy' - k^2 y = 0.$$

1053. Prove that the expression

$$S = \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2$$

is unchanged if y is replaced by $\frac{1}{y}$, i.e. if we put $y = \frac{1}{y_1}$, we have

$$\frac{y_1'''}{y_1'} - \frac{3}{2}\left(\frac{y_1''}{y_1'}\right)^2 = S.$$

1054. Given $y = f(x)$, express $\frac{d^2x}{dy^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Show that the formula $R = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''}$ can be transformed to

$$R^{\frac{2}{3}} = \frac{1}{\left(\frac{d^2y}{dx^2}\right)^{\frac{2}{3}}} + \frac{1}{\left(\frac{d^2x}{dy^2}\right)^{\frac{2}{3}}}.$$

1055. Given $F(x) = f(x)\varphi(x)$, where $f'(x)\varphi'(x) = C$, show that

$$\frac{F''}{F} = \frac{f''}{f} + \frac{\varphi''}{\varphi} + \frac{2C}{f\varphi} \text{ and } \frac{F'''}{F} = \frac{f'''}{f} + \frac{\varphi'''}{\varphi}.$$

Functions given Implicitly

1056. $b^2x^2 + a^2y^2 = a^2b^2$; $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = ?$

1057. $x^2 + y^2 = r^2$; $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} = ?$

1058. $y = \tan(x + y)$; $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} = ?$

1059. $s = 1 + t\mathrm{e}^s$; $\dfrac{\mathrm{d}^2s}{\mathrm{d}t^2} = ?$

1060. $y^3 + x^3 - 3axy = 0$; $y'' = ?$

1061. $y = \sin(x + y)$; $y'' = ?$

1062. $\mathrm{e}^{x+y} = xy$; $y'' = ?$

1063. Deduce the formula for the second derivative of the inverse of $y = f(x)$.

1064. $\mathrm{e}^y + xy = \mathrm{e}$; find $y''(x)$ for $x = 0$.

1065. $y^2 = 2px$; find $k = \dfrac{y''}{\sqrt{(1 + y'^2)^3}}$.

1066. Show that $y^2 + x^2 = R^2$ implies $k = \dfrac{1}{R}$, where $k = \dfrac{|y''|}{\sqrt{(1 + y'^2)^3}}$.

1067. Prove that, if

$$ax^2 + 2bxy + cy^2 + 2gx + 2fy + h = 0,$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{ax + by + g}{bx + cy + f} \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{A}{(bx + cy + f)^3},$$

where A is constant (independent of x and y).

1068. Prove that, if

$$(a + bx)\, e^{\frac{y}{x}} = x,$$

then

$$x^2 \frac{d^2y}{dx^2} = \left(x \frac{dy}{dx} - y\right)^2.$$

Functions given Parametrically

1069. $x = at^2,\quad y = bt^3,\quad \frac{d^2x}{dy^2} = ?$

1070. $x = a \cos t,\quad y = a \sin t;\quad \frac{d^2y}{dx^2} = ?$

1071. $x = a \cos t,\quad y = b \sin t;\quad \frac{d^3y}{dx^3} = ?$

1072. $x = a(\varphi - \sin \varphi),\quad y = a(1 - \cos \varphi);\quad \frac{d^2y}{dx^2} = ?$

1073. (1) $x = a \cos^3 t,\quad y = a \sin^3 t;\quad \frac{d^3y}{dx^3} = ?$

(2) $x = a \cos^2 t,\quad y = a \sin^2 t;\quad \frac{d^2y}{dx^2} = ?$

1074. (1) $x = \ln t,\quad y = t^2 - 1;\quad \frac{d^2y}{dx^2} = ?$

(2) $x = \arcsin t,\quad y = \ln (1 - t^2);\quad \frac{d^2y}{dx^2} = ?$

1075. $x = at \cos t,\quad y = at \sin t;\quad \frac{d^2y}{dx^2} = ?$

1076. Show that $y = f(x)$ given by the parametric equations $y = e^t \cos t$, $x = e^t \sin t$, satisfies $y''(x + y)^2 = 2(xy' - y)$.

1077. Show that $y = f(x)$ given by the parametric equations $y = 3t - t^3$, $x = 3t^2$, satisfies

$$36y''\left(y - \sqrt{3x}\right) = x + 3.$$

1078. Show that the function given by the parametric equations

$$x = \sin t,\quad y = \sin kt,$$

satisfies the relationship

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + k^2y = 0.$$

1079. Prove that, if

$$x = f(t)\cos t - f'(t)\sin t,\ y = f(t)\sin t + f'(t)\cos t,$$

we have

$$ds^2 = dx^2 + dy^2 = [f(t) + f''(t)]^2\,dt^2.$$

Acceleration

1080. A particle moves along a straight line such that $s = \frac{4}{3}t^3 - t + 5$. Find the acceleration a at the end of the second second (s is given in metres, t in seconds).

1081. A rectilinear motion is given by

$$s = t^2 - 4t + 1.$$

Find the speed and acceleration.

1082. A particle moves on a straight line such that $s = \frac{2}{9}\sin\frac{1}{2}\pi t + s_0$. Find the acceleration at the end of the first second (s in cm, t in sec).

1083. A particle moves in a straight line, and $s = \sqrt{t}$. Show that the motion is slowing down and that the acceleration a is proportional to the cube of the velocity v.

1084. A heavy beam of length 13 m is being lowered to the ground, its lower end being fixed to a trolley (Fig. 28) and its upper end attached to a rope wound round a windlass. The rope unwinds at a speed of 2 m/min. What is the acceleration of the trolley away from point O when its distance from O is 5 m?

1085. A barge, the deck of which is 4 m below the level of the wharf, is drawn towards the latter by a rope wound up on a windlass at a speed of 2 m/sec.

What is the acceleration of the barge at the instant when it is 8 m away from the wharf (measured horizontally)?

1086. A particle is moving along a straight line in such a way that its speed is changing proportionally to the square root of the path traversed. Show that the motion occurs under the action of a constant force.

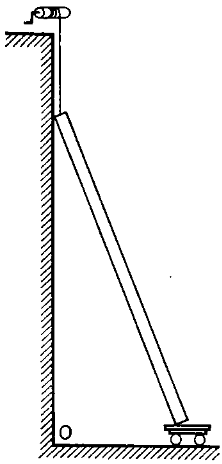

FIG. 28.

1087. If the force acting on a particle is inversely proportional to the speed, show that the kinetic energy of the particle is a linear function of time.

Leibniz's Formula

1088. Use Leibniz's formula to evaluate the following derivatives:

(1) $[(x^2+1)\sin x]^{(20)}$; (2) $(e^x \sin x)^{(n)}$;

(3) $(x^3 \sin \alpha x)^{(n)}$.

1089. Show that, if $y = (1-x)^{-\alpha} e^{-\alpha x}$, then

$$(1-x)\frac{dy}{dx} = \alpha x y.$$

Prove by using Leibniz's formula that

$$(1-x)\, y^{(n+1)} - (n+\alpha x)\, y^{(n)} - n\alpha y^{(n-1)} = 0.$$

1090. The function $y = e^{\alpha \arcsin x}$ satisfies the equation

$$(1 - x^2)\, y'' - xy' - \alpha^2 y = 0.$$

(see problem 1051). By applying Leibniz's theorem and differentiating this equation n times, show that

$$(1 - x^2)\, y^{(n+2)} - (2n + 1)\, xy^{(n+1)} - (n^2 + \alpha^2)\, y^{(n)} = 0.$$

1091. Prove that

$$(e^{ax} \cos bx)^{(n)} = r^n e^{ax} \cos (bx + n\varphi),$$

where

$$r = \sqrt{a^2 + b^2}, \ \tan \varphi = \frac{b}{a}.$$

Obtain the following formulae by using Leibniz's theorem:

$$r^n \cos n\varphi = a^n - C_n^2 a^{n-2} b^2 + C_n^4 a^{n-4} b^4 - \ldots,$$

$$r^n \sin n\varphi = C_n^1 a^{n-1} b - C_n^3 a^{n-3} b^3 + C_n^5 a^{n-5} b^5 - \ldots$$

1092. Prove that

$$\left(x^{n-1}\, e^{\frac{1}{x}}\right)^{(n)} = (-1)^n \frac{e^{\frac{1}{x}}}{x^{n+1}}.$$

1093. Prove that $y = \arcsin x$ satisfies the equation

$$(1 - x^2)\, y'' = xy'.$$

Apply Leibniz's formula to both sides of this equation to find $y^{(n)}(0)$ $(n \geqq 2)$.

1094. By applying Leibniz's formula n times, show that

$$y = \cos (m \arcsin x)$$

satisfies the relationship

$$(1 - x^2)\, y^{(n+2)} - (2n + 1)\, xy^{(n+1)} - (m^2 - n^2)\, y^{(n)} = 0.$$

1095. If $y = (\arcsin x)^2$, we have

$$(1 - x^2)\, y^{(n+1)} - (2n - 1)\, xy^{(n)} - (n - 1)^2\, y^{(n-1)} = 0.$$

Find $y'(0)$, $y''(0) \ldots, y^{(n)}(0)$.

Differentials of Higher Orders

1096. $y = \sqrt[3]{x^2}$; $d^2y = ?$

1097. $y = x^m$; $d^3y = ?$

1098. $y = (x + 1)^3 (x - 1)^2$; $d^2y = ?$

1099. $y = 4^{-x^2}$; $d^2y = ?$

1100. $y = \operatorname{arc} \tan \left(\frac{b}{a} \tan x\right)$; $d^2y = ?$

1101. $y = \sqrt{\ln^2 x - 4}$; $d^2y = ?$

1102. $y = \sin^2 x$; $d^3y = ?$

1103. $\varrho^2 \cos^3 \varphi - a^2 \sin^3 \varphi = 0$; $d^2\varrho = ?$

1104. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$; $d^2y = ?$

1105. $y = \ln \frac{1 - x^2}{1 + x^2}$; $x = \tan t$; express d^2y in terms of: (1) x and dx, (2) t and dt.

1106. $y = \sin z$; $z = a^x$; $x = t^3$; express d^2y in terms of: (1) z and dz, (2) x and dx, (3) t and dt.

CHAPTER IV

THE INVESTIGATION OF FUNCTIONS AND CURVES

1. The Behaviour of a Function "at a Point"

1107. Plot the graph of the function

$$y = 3x^4 - 4x^3 + 12x^2 + 1.$$

1108. Starting directly from the definition, show that the function $y = x^3 - 3x + 2$ is increasing at $x_1 = 2$, is decreasing at $x_2 = 0$, has a maximum at $x_3 = -1$ and a minimum at $x_4 = 1$.

1109. Starting directly from the definition, show that the function $y = \cos 2x$ is increasing at $x_1 = \frac{3\pi}{4}$, is decreasing at $x_2 = \frac{\pi}{6}$, has a maximum at $x_3 = 0$ and a minimum at $x_4 = \frac{\pi}{2}$.

1110. Starting directly from the definition, describe the behaviour of the following functions at $x = 0$:

(1) $y = 1 - x^4$; (2) $y = x^5 - x^3$; (3) $y = \sqrt[3]{x}$;
(4) $y = \sqrt[3]{x^2}$; (5) $y = 1 - \sqrt[5]{x^4}$; (6) $y = |\tan x|$;
(7) $y = |\ln (x + 1)|$; (8) $y = e^{-|x|}$;
(9) $y = \sqrt{x^3 + x^2}$.

1111. Show by using the tests for the behaviour of a function at a point that $y = \ln (x^2 + 2x - 3)$ is increasing at $x_1 = 2$, is decreasing at $x_2 = -4$ and has no stationary points.

1112. Examine the behaviour of the function

$$y = \sin x + \cos x$$

at $x_1 = 0$, $x_2 = 1$, $x_3 = -\frac{\pi}{3}$ and $x_4 = 2$.

1113. Examine the behaviour of the function

$$y = x - \ln x$$

at $x_1 = \frac{1}{2}$, $x_2 = 2$, $x_3 = e$ and $x_4 = 1$, and show that, if the function is increasing at $x = a > 0$, it is decreasing at $x' = \frac{1}{a}$.

1114. How does the function

$$y = x \operatorname{arc} \tan x$$

behave at $x_1 = 1$, $x_2 = -1$ and $x_3 = 0$?

1115. Examine the behaviour of the function given by

$$y = \begin{cases} \dfrac{\sin x}{x} & \text{at } x \neq 0, \\ 1 & \text{at } x = 0, \end{cases}$$

at $x_1 = \frac{1}{2}$, $x_2 = -\frac{1}{2}$ and $x_3 = 0$.

2. Applications of the First Derivative

Theorems of Rolle and Lagrange

1116. Prove that Rolle's theorem holds for $y = x^3 + 4x^2 - 7x - 10$ in the interval $[-1, 2]$.

1117. Prove that Rolle's theorem holds for $y = \ln \sin x$ in the interval $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$.

1118. Prove that Rolle's theorem holds for $y = 4^{\sin x}$ in the interval $[0, \pi]$.

1119. Prove that Rolle's theorem holds for the function $y = \sqrt[3]{x^2 - 3x + 2}$ in the interval $[1, 2]$.

1120. The function $y = \frac{2 - x^2}{x^4}$ takes equal values at the ends of the interval $[-1, 1]$. Show that there is no point in the interval at which the derivative vanishes, and explain this deviation from Rolle's theorem.

1121. The function $y = |x|$ takes equal values at the ends of the interval $[-a, a]$. Verify that there is no point of the interval at which the derivative of the function vanishes, and explain this deviation from Rolle's theorem.

1122. Prove the theorem: if the equation

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x = 0$$

has a positive root $x = x_0$, the equation

$$na_0x^{n-1} + (n - 1)\, a_1x^{n-2} + \ldots + a_{n-1} = 0$$

also has a positive root, which is less than x_0.

1123. Given the function $f(x) = 1 + x^m(x - 1)^n$, where m and n are positive integers, show without working out the derivative that the equation $f'(x) = 0$ has at least one root in the interval $[0, 1]$.

1124. Show that the equation $x^3 - 3x + c = 0$ cannot have two different roots in the interval $[0, 1]$.

1125. Given

$$f(x) = (x - 1)\,(x - 2)\,(x - 3)\,(x - 4),$$

discover how many roots the equation $f'(x) = 0$ has, and indicate the intervals in which they lie.

1126. Show that the function $f(x) = x^n + px + q$ cannot have more than two real roots for n even or more than three for n odd.

1127. Obtain Lagrange's formula for the function $y =$ $= \sin 3x$ in the interval $[x_1, x_2]$.

1128. Obtain Lagrange's formula for $y = x(1 - \ln x)$ in the interval $[a, b]$.

1129. Obtain Lagrange's formula for $y = \arcsin 2x$ in the interval $[x_0, x_0 + \Delta x]$.

1130. Prove that Lagrange's theorem holds for the function $y = x^n$ in the interval $[0, a]$; $n > 0$, $a > 0$.

1131. Prove that Lagrange's theorem holds for the function $y = \ln x$ in the interval $[1, \mathrm{e}]$.

1132. Use Lagrange's formula to prove the inequality

$$\frac{a-b}{a} \leqq \ln\frac{a}{b} \leqq \frac{a-b}{b},$$

when $0 < b \leqq a$.

1133. Use Lagrange's formula to prove the inequality

$$\frac{\alpha-\beta}{\cos^2\beta} \leqq \tan\alpha - \tan\beta \leqq \frac{\alpha-\beta}{\cos^2\alpha}, \quad \text{if} \quad 0 < \beta \leqq \alpha < \frac{1}{2}\pi.$$

1134. Prove with the aid of Lagrange's formula that the inequalities

$$nb^{n-1}(a-b) < a^n - b^n < na^{n-1}(a-b),$$

hold with $a > b$ if $n > 1$, whilst they hold in the opposite sense if $n < 1$.

1135. The function $f(x) = x^2 \sin\frac{1}{x}$ for $x \neq 0$, $f(0) = 0$ is differentiable for any x. Lagrange's formula for it gives in the interval $[0, x]$:

$$f(x) - f(0) = xf'(\xi)\ (0 < \xi < x).$$

We have:

$$x^2 \sin\frac{1}{x} = x\left(2\xi\sin\frac{1}{\xi} - \cos\frac{1}{\xi}\right),$$

whence $\cos\frac{1}{\xi} = 2\xi\sin\frac{1}{\xi} - x\sin\frac{1}{x}$. If we now let x tend to zero, ξ will also tend to zero, and we get: $\lim\limits_{\xi\to 0}\cos\frac{1}{\xi} = 0$.

Explain this paradoxical result.

1136. By applying to the function $f(x) = \operatorname{arc}\tan x$ in the interval $[1, 1{\cdot}1]$ the formula

$$f(x_0 + \Delta x) \approx f(x_0) + f'\left(x_0 + \frac{\Delta x}{2}\right)\Delta x,$$

find the approximate value of $\operatorname{arc}\tan 1{\cdot}1$.

In problems 1137–1141, use the formula

$$f(x_0 + \Delta x) \approx f(x_0) + f'\left(x_0 + \frac{\Delta x}{2}\right)\Delta x,$$

to find approximate values of the given expressions.

1137. arc sin 0·54.

1138. log 11. Compare with tables.

1139. $\ln\left(x + \sqrt{1 + x^2}\right)$ for $x = 0{\cdot}2$.

1140. log 7, knowing that $\log 2 = 0{\cdot}3010$ and $\log 3 = 0{\cdot}4771$. Compare the result with tables.

1141. log 61. Compare the result with tables.

1142. Show that, if we use the formula

$$f(b) = f(a) + (b - a)\, f'\left(\frac{a + b}{2}\right)$$

to work out the logarithm of $N + 0{\cdot}01N$, i.e. if we put

$$\log(N + 0{\cdot}01\,N) = \log N + \frac{0{\cdot}43429}{N + \frac{0{\cdot}01}{2}N}\,0{\cdot}01\,N$$

$$= \log N + \frac{0{\cdot}43429}{100{\cdot}5},$$

the error involved is less than 0·00001, i.e. the answer is correct to five figures after the decimal point, provided log N is correct to five figures.

The Behaviour of a Function in an Interval

1143. Prove that the function

$$y = 2x^3 + 3x^2 - 12x + 1$$

is decreasing in the interval $(-2, 1)$.

1144. Prove that the function

$$y = \sqrt{2x - x^2}$$

is increasing in the interval $(0, 1)$ and decreasing in $(1, 2)$. Draw the graph of the function.

1145. Prove that the function $y = x^3 + x$ is always increasing.

1146. Prove that the function $y = \arctan x - x$ is always decreasing.

1147. Prove that the function $y = \frac{x^2 - 1}{x}$ is increasing in any interval not containing the point $x = 0$.

1148. Prove that the function $y = \frac{\sin(x + a)}{\sin(x + b)}$ varies monotonically in any interval not containing a discontinuity.

1149*. Prove that $\frac{\tan x_2}{\tan x_1} > \frac{x_2}{x_1}$ if

$$0 < x_1 < x_2 < \frac{\pi}{2}.$$

1150. Find the interval in which

$$y = x^3 - 3x^2 - 9x + 14$$

is monotonic, and plot its graph in the interval $(-2, 4)$.

1151. Do the same for the function

$$y = x^4 - 2x^2 - 5$$

in the interval $(-2, 2)$.

Find the intervals in which the functions of problems 1152–1164 are monotonic:

1152. $y = (x - 2)^5 (2x + 1)^4$.

1153. $y = \sqrt[3]{(2x - a)(a - x)^2}$ $(a > 0)$.

1154. $y = \frac{1 - x + x^2}{1 + x + x^2}$.

1155. $y = \frac{10}{4x^3 - 9x^2 + 6x}$.

1156. $y = x - e^x$.

1157. $y = x^2 e^{-x}$.

1158. $y = \frac{x}{\ln x}$.

1159. $y = 2x^2 - \ln x$.

1160. $y = x - 2 \sin x$ $(0 \leqq x \leqq 2\pi)$.

1161. $y = 2 \sin x + \cos 2x$ $(0 \leqq x \leqq 2\pi)$.

1162. $y = x + \cos x$.

1163. $y = \ln(x + \sqrt{1 + x^2})$.

1164. $y = x\sqrt{ax - x^2}$ $(a > 0)$.

Find the extrema of the functions of problems 1165–1184:

1165. $y = 2x^3 - 3x^2$.

1166. $y = 2x^3 - 6x^2 - 18x + 7$.

1167. $y = \dfrac{3x^2 + 4x + 4}{x^2 + x + 1}$. **1168.** $y = \sqrt[3]{x^3 - 3x^2 + 8}$.

1169. $y = \dfrac{1}{\ln(x^4 + 4x^3 + 30)}$.

1170. $y = -x^2\sqrt{x^2 + 2}$. **1171.** $y = \dfrac{2}{3}x^2\sqrt[3]{6x - 7}$.

1172. $y = \dfrac{4\sqrt{3}}{9x\sqrt{1 - x}}$. **1173.** $y = \dfrac{1 + 3x}{\sqrt{4 + 5x^2}}$.

1174. $y = \sqrt[3]{(x^2 - a^2)^2}$. **1175.** $y = x - \ln(1 + x)$.

1176. $y = x - \ln(1 + x^2)$. **1177.** $y = (x - 5)^2\sqrt[3]{(x + 1)^2}$.

1178. $y = (x^2 - 2x)\ln x - \dfrac{3}{2}x^2 + 4x$.

1179. $y = \dfrac{1}{2}(x^2 + 1)\arctan x - \dfrac{\pi}{8}x^2 - \dfrac{x - 1}{2}$.

1180. $y = \dfrac{1}{2}\left(x^2 - \dfrac{1}{2}\right)\arcsin x + \dfrac{1}{4}x\sqrt{1 - x^2} - \dfrac{\pi}{12}x^2$.

1181. $y = x\sin x + \cos x - \dfrac{1}{4}x^2$ $\left(-\dfrac{\pi}{2} \leqq x \leqq \dfrac{\pi}{2}\right)$.

1182. $y = \left(\dfrac{1}{2} - x\right)\cos x + \sin x - \dfrac{x^2 - x}{4}$ $\left(0 \leqq x \leqq \dfrac{\pi}{2}\right)$.

1183. $y = \dfrac{2 - x}{\pi}\cos\pi(x + 3) + \dfrac{1}{\pi^2}\sin\pi(x + 3)$ $(0 < x < 4)$.

1184. $y = ae^{px} + be^{-px}$.

Find the maxima and minima of the functions of problems 1185–1197 in the intervals quoted:

1185. $y = x^4 - 2x^2 + 5$; $[-2, 2]$.

1186. $y = x + 2\sqrt{x}$; $[0, 4]$.

1187. $y = x^5 - 5x^4 + 5x^3 + 1$; $[-1, 2]$.

1188. $y = x^3 - 3x^2 + 6x - 2;\quad [-1, 1]$.

1189. $y = \sqrt{100 - x^2}\quad (-6 \leqq x \leqq 8)$.

1190. $y = \dfrac{1 - x + x^2}{1 + x - x^2}\quad (0 \leqq x \leqq 1)$.

1191. $y = \dfrac{x - 1}{x + 1}\quad (0 \leqq x \leqq 4)$.

1192. $y = \dfrac{a^2}{x} + \dfrac{b^2}{1 - x}\quad (0 < x < 1)\quad (a > 0, b > 0)$.

1193. $y = \sin 2x - x\quad \left(-\dfrac{\pi}{2} \leqq x \leqq \dfrac{\pi}{2}\right)$.

1194. $y = 2\tan x - \tan^2 x\quad \left(0 \leqq x < \dfrac{\pi}{2}\right)$.

1195. $y = x^x\quad (0{\cdot}1 \leqq x < \infty)$.

1196. $y = \sqrt[3]{(x^2 - 2x)^2}\quad (0 \leqq x \leqq 3)$.

1197. $y = \arctan \dfrac{1 - x}{1 + x}\quad (0 \leqq x \leqq 1)$.

Inequalities

Prove the inequalities of problems 1198–1207:

1198. $2\sqrt{x} > 3 - \dfrac{1}{x}\quad (x > 1)$.

1199. $e^x > 1 + x\quad (x \neq 0)$.

1200. $x > \ln(1 + x)\quad (x > 0)$.

1201. $\ln x > \dfrac{2(x - 1)}{x + 1}\quad (x > 1)$.

1202. $2x \arctan x \geqq \ln(1 + x^2)$.

1203. $1 + x \ln(x + \sqrt{1 + x^2}) \geqq \sqrt{1 + x^2}$.

1204. $\ln(1 + x) > \dfrac{\arctan x}{1 + x}\quad (x > 0)$.

1205. $\sin x < x - \dfrac{x^3}{6} + \dfrac{x^5}{120}\quad (x > 0)$.

1206. $\sin x + \tan x > 2x \quad \left(0 < x < \frac{\pi}{2}\right).$

1207. $\frac{e^x + e^{-x}}{2} \geqq 1 + \frac{x^2}{2}.$

Problems on Finding Maxima and Minima

1208. Divide the number 8 into two parts such that the sum of their cubes is a minimum.

1209. What positive number gives the least sum when added to its reciprocal?

1210. Divide the number 36 into two factors such that the sum of their squares is a minimum.

1211. We want to make a box with a lid, its volume being 72 cm^3, and the sides of the base in the ratio 1:2. What must be the dimensions of all the sides for the total surface area to be a minimum?

1212. We want to cut out equal squares from the corners of a square piece of paper measuring 18×18 cm^2 so that the box made by folding the paper along the dotted lines (Fig. 29) has the maximum capacity. What is the side of the squares cut out?

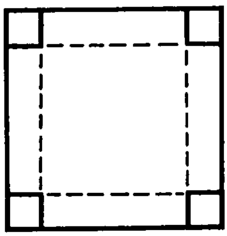

FIG. 29.

1213. Solve the previous problem for a rectangular sheet measuring 8×5 cm^2.

1214. The volume of a regular triangular prism is v. What must the side of the base be for the total surface area of the prism to be a minimum?

1215. An open vat has a cylindrical shape. Given the volume v, what must be the radius of the base and the height for its surface area to be a minimum?

1216. Find the ratio of the radius R and height H of a cylinder of given volume when its surface area is a minimum.

1217. We want to make a conical funnel with a generator 20 cm long. What must be the height of the funnel for its volume to be a maximum?

1218. A sector with central angle α is cut out of a circle. The sector is folded to form a conical shape. What is the value of α for the cone to have maximum volume?

1219. The perimeter of an isosceles triangle is $2p$. What are its sides for the volume of the solid formed by revolving the triangle about its base to be a maximum?

1220. The perimeter of an isosceles triangle is $2p$. What are its sides if the volume of the cone formed by revolving the triangle about its height is a maximum?

1221. Find the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R.

1222. Find the height of the cone of maximum volume that can be inscribed in a sphere of radius R.

1223. A rain drop of initial mass m_0 evaporates uniformly whilst falling under the action of gravity, so that its mass decreases proportionally to time (coefficient of proportionality k). How many seconds after the drop starts falling is its kinetic energy a maximum, and what is the maximum? (Air resistance is neglected.)

1224. A lever of the second kind has support point A; the load P is suspended at point B ($AB = a$). The weight of unit length of the lever is k. What is the length of the lever for load P to be balanced by the least force? (The moment of the balancing force must be taken equal to the sum of the moments of load P and the lever.)

1225. The cost of fuel for the furnace of a steamer is proportional to the cube of its speed. We know that the fuel costs at a speed of 10 km/hr amount to 30 roubles/hour, the remaining costs (independent of the speed) being 480 roubles/hour. What is the speed for the total cost per km travelled to be a minimum? What is the total cost per hour in this case?

1226. A, B and C are three non-collinear points, and $\angle ABC = 60°$. A car leaves from A, and a train from B, at the same instant. The car travels towards B at a speed of 80 km/hr and the train towards C at 50 km/hr. If $AB = 200$ km, how long after the start are the car and train a minimum distance apart?

1227. Given the point A on a circle, draw the chord BC parallel to the tangent at A such that the area of the triangle ABC is a maximum.

1228. Find the sides of the rectangle of maximum perimeter inscribed in a semi-circle of radius R.

1229. Inscribe the rectangle of maximum area in a given segment of a circle.

1230. Circumscribe the cone of maximum volume about a given cylinder (the planes of the bases of cylinder and cone must coincide).

1231. Find the height of the right circular cone of least volume circumscribed about a sphere of radius R.

1232. Find the vertex angle of the axial section of the cone of least lateral surface area circumscribed about a given sphere.

1233. What is the angle at the vertex of an isosceles triangle of given area for maximum radius of the inscribed circle?

1234. Find the height of the cone of least volume circumscribed about a hemisphere of radius R (the centre of the base of the cone lies at the centre of the sphere).

1235. What must be the height of a cone inscribed in a sphere of radius R if its lateral surface is a maximum?

1236. Prove that a conical tent of given capacity requires a minimum amount of material when its height is $\sqrt{2}$ times the base radius.

1237. Draw a straight line through the given point P (1,4) such that the sum of the lengths of the positive intercepts that it cuts off the coordinate axes is a minimum.

1238. Find the sides of the rectangle of greatest area inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

1239. Find the ellipse of least area circumscribed about a given rectangle (the area of an ellipse with semi-axes a and b is πab).

1240. If the area of the triangle formed by a tangent to the ellipse $\frac{x^2}{8} + \frac{y^2}{18} = 1$ and the coordinate axes is a minimum, what is the point of contact?

1241. Two points A (1, 4) and B(3, 0) are given on the ellipse $2x^2 + y^2 = 18$. Find the third point C of the ellipse such that the area of triangle ABC is a maximum, a minimum.

1242. Given a point on the axis of the parabola $y^2 = 2px$ at a distance a from the vertex, find the abscissa x of the point of the parabola closest to it.

1243. An iron strip of width a has to be bent to form an open cylindrical gutter (the gutter section is the arc of a circle). Find the angle subtended by the arc at the centre of the circle for maximum capacity of the gutter.

1244. A log 20 m long is in the form of a frustum of a cone, the base diameters of which are 2 m and 1 m respectively. We want to cut out a beam from the log with a square cross-section and the same axis as the log, the volume of the beam being a maximum. What will be the dimensions of the beam?

1245. A series of experiments leads to n different values $x_1, x_2, \ldots, x_n$ for the required quantity A. The value of A is often taken as the x such that the sum of the squares of its deviations from $x_1, x_2, \ldots, x_n$ is a minimum. Find the x satisfying this requirement.

1246. A torpedo-boat stands at anchor 9 km from the nearest point of the coast; an express messenger has to be sent from the boat to a camp 15 km along the coast, measured from the point nearest the boat. If the messenger can walk at 5 km/hr and row at 4 km/hr, at what point should he land on the coast in order to reach the camp in the shortest time?

1247. A lamp has to be suspended directly above the centre of a circular area of radius R. What should its height be in order to get the best illumination of a path round the area? (The degree of illumination of an area is directly proportional to the cosine of the angle of incidence of the ray and inversely proportional to the square of the distance from the source.)

1248. Find the least illuminated point on a straight line of length l joining two light sources of intensities I_1 and I_2.

1249. A picture 1·4 m high hangs on a wall so that its lower edge is 1·8 m above the eye of an observer. At what distance should the observer stand from the wall for his position to be the most favourable for viewing the picture? (i.e. for maximum angle of vision).

1250. A load of weight P on a horizontal plane has to be shifted by applying a force F to it. The friction force is proportional to the force pressing the body to the plane and is directed in opposition to the displacing force. The coefficient of proportionality (coefficient of friction) is k. At what angle φ to the horizontal should the force be applied in order for it to be a minimum? What is the value of this minimum?

1251. The speed of flow of water along a circular pipe is directly proportional to the so-called hydraulic radius R, calculated from the formula $R = \frac{S}{p}$, where S is the cross-sectional area of the water in the pipe and p is the wetted perimeter of the pipe section. The degree to which the water fills the pipe is characterized by the angle subtended at the centre by the horizontal surface of the water. At what degree of filling is the speed of flow of the water a maximum? (Find graphically the roots of the transcendental equation encountered in this problem.)

1252. The printed text has to occupy S square centimetres on the page of a book. The upper and lower margins must be

a cm, the left and right margins b cm. If we are concerned only with paper economy, what are the best dimensions of a page?

1253*. A conical funnel of base radius R and height H is filled with water. A sphere is submerged in the water. What should the radius of the sphere be for the maximum volume of water to be displaced from the funnel by the submerged part of the sphere?

1254. The vertex of a parabola lies on a circle of radius R, the parabola axis being along a diameter. What should the parabola parameters be for maximum area of segment bounded by the parabola and its common chord with the circle? (The area of a symmetrical parabolic segment is equal to two-thirds the product of its base and height.)

1255. A cone of base radius R and height H is cut by a plane parallel to a generator. What is the distance between the line of intersection of this plane with the base plane and the centre of the base for a maximum area of intersection (See the previous problem.)

1256. At what point P of the parabola $y^2 = 2px$ is the segment of the normal at P, contained inside the curve, a minimum?

1257. Prove that the tangent to an ellipse, the intercept of which between its axes has minimum length, is divided by the point of contact into two pieces respectively equal to the semi-axes.

1258. Prove that the distance from the centre of an ellipse to any normal does not exceed the difference between the semi-axes. (It is advisable to use the parametric equations of the ellipse.)

1259. A point (a, b) and a curve $y = f(x)$ are given in a rectangular system of coordinates xOy. Show that the distance between the fixed point (a, b) and the variable point $(x, f(x))$ can only have an extremum in the direction of the normal to the curve $y = f(x)$.

A Property of the Primitive

1260. Prove (by two methods) that the functions $y = \ln ax$ and $y = \ln x$ are primitives of the same function.

1261. The same for functions $y = 2\sin^2 x$ and $y = -\cos 2x$.

1262. The same for $y = (e^x + e^{-x})^2$ and $y = (e^x - e^{-x})^2$.

1263*. Prove that the function

$$y = \cos^2 x + \cos^2\left(\frac{\pi}{3} + x\right) - \cos x \cos\left(\frac{\pi}{3} + x\right)$$

is constant (independent of x). Find the value of this constant.

1264. Prove that the function

$$y = 2 \arctan x + \arcsin \frac{2x}{1 + x^2}$$

is constant for $x \geqq 1$. Find the value of the constant.

1265. Prove that the function

$$y = \arccos \frac{a \cos x + b}{a + b \cos x} - 2 \arctan\left(\sqrt{\frac{a - b}{a + b}} \tan \frac{x}{2}\right),$$

where $0 < b \leqq a$, is constant for $x \geqq 0$. Find the value of the constant.

1266. Show that the functions $\frac{1}{2} e^{2x}$, $e^x \sinh x$ and $e^x \cosh x$ differ by a constant. Show that each function is a primitive of $\dfrac{e^x}{\cosh x - \sinh x}$.

3. Applications of the Second Derivative

Extrema

Find the extrema of the functions of problems 1267–1275 by using the second derivative:

1267. $y = x^3 - 2ax^2 + a^2 x \quad (a > 0)$.

1268. $p = x^2 (a - x)^2$. **1269.** $y = x + \dfrac{a^2}{x} \quad (a > 0)$.

1270. $y = x + \sqrt{1-x}$.

1271. $y = x\sqrt{2-x^2}$.

1272. $y = e^{\frac{x}{a}} + e^{-\frac{x}{a}}$.

1273. $y = x^2 e^{-x}$.

1274. $y = \frac{x}{\ln x}$.

1275. $y = x^{\frac{1}{x}}$.

1276. For what value of a has the function

$$f(x) = a \sin x + \frac{1}{3} \sin 3x$$

an extremum at $x = \frac{\pi}{3}$? Is this a maximum or minimum?

1277. Find the values of a and b for which the function

$$y = a \ln x + bx^2 + x$$

has an extremum at points $x_1 = 1$ and $x_2 = 2$. Show that, with these values of a and b, the function has a minimum at x_1 and a maximum at x_2.

Convexity, Concavity, Points of Inflexion

1278. Find whether $y = x^5 - 5x^3 - 15x^2 + 30$ is convex or concave in the neighbourhood of points (1, 11) and (3, 3).

1279. Find whether the curve $y = \arctan x$ is convex or concave in the neighbourhood of points $\left(1, \frac{\pi}{4}\right)$ and $\left(-1, -\frac{\pi}{4}\right)$.

1280. Find whether the curve $y = x^2 \ln x$ is convex or concave in the neighbourhood of points (1, 0) and $\left(\frac{1}{e^2}, -\frac{2}{e^4}\right)$.

1281. Prove that the graph of $y = x \arctan x$ is concave everywhere.

1282. Prove that the graph of $y = \ln(x^2 - 1)$ is convex everywhere.

1283. Prove that, if the graph of a function is everywhere concave or convex, the function cannot have more than one extremum.

1284. Let $P(x)$ be a polynomial with positive coefficients and even powers. Show that the graph of $y = P(x) + ax + b$ is everywhere concave.

1285. The curves $y = \varphi(x)$ and $y = \psi(x)$ are concave in the interval (a, b). Show that, in this interval: (a) the curve $y = \varphi(x) + \psi(x)$ is concave; (b) if $\varphi(x)$ and $\psi(x)$ are positive and have a common minimum, the curve $y = \varphi(x)\,\psi(x)$ is concave.

1286. Find the shape of the graph of the function when we know that, in the interval $[a, b]$:

(1) $y > 0,\ y' > 0,\ y'' < 0$; (2) $y > 0,\ y' < 0,\ y'' > 0$;
(3) $y < 0,\ y' > 0,\ y'' > 0$; (4) $y > 0,\ y' < 0,\ y'' < 0$.

Find the points of inflexion and the intervals of convexity and concavity of the graphs of the functions of problems 1287–1300:

1287. $y = x^3 - 5x^2 + 3x - 5$. **1288.** $y = (x + 1)^4 + e^x$.

1289. $y = x^4 - 12x^3 + 48x^2 - 50$.

1290. $y = x + 36x^2 - 2x^3 - x^4$.

1291. $y = 3x^5 - 5x^4 + 3x - 2$.

1292. $y = (x + 2)^6 + 2x + 2$. **1293.** $y = \dfrac{x^3}{x^2 + 3a^2}$ $(a > 0)$.

1294. $y = a - \sqrt[3]{x - b}$.

1295. $y = e^{\sin x}$ $\left(-\dfrac{\pi}{2} \leqq x \leqq \dfrac{\pi}{2}\right)$.

1296. $y = \ln(1 + x^2)$. **1297.** $y = \dfrac{a}{x} \ln \dfrac{x}{a}$ $(a > 0)$.

1298. $y = a - \sqrt[5]{(x - b)^2}$. **1299.** $y = e^{\arctan x}$.

1300. $y = x^4(12 \ln x - 7)$.

1301. Show that the curve $y = \dfrac{x + 1}{x^2 + 1}$ has three points of inflexion which are collinear.

1302. Prove that the points of inflexion of $y = x \sin x$ lie on the curve $y^2 (4 + x^2) = 4x^2$.

1303. Prove that the points of inflexion of the curve $y = \frac{\sin x}{x}$ lie on the curve $y^2(4 + x^4) = 4$.

1304. Show that the graphs of $y = \pm e^{-x}$ and $y = e^{-x} \sin x$ (the curve of a damped vibration) have common tangents at the points of inflexion of the curve $y = e^{-x} \sin x$.

1305. For what values of a and b is the point (1, 3) a point of inflexion of the curve $y = ax^3 + bx^2$?

1306. Choose α and β such that the curve $x^2y + \alpha x + \beta y = 0$ has a point of inflexion at $A(2, 2{\cdot}5)$. How many further points of inflexion are there?

1307. For what values of a has the graph of $y = e^x + ax^3$ a point of inflexion?

1308. Prove that the abscissa of the point of inflexion of the graph of a function cannot coincide with an extremal point of the function.

1309. Prove that, for any twice differentiable function, at least one abscissa of a point of inflexion of the graph lies between two extremal points.

1310. Prove by taking $y = x^4 + 8x^3 + 18x^2 + 8$ as an example that there can be no extremal points between the abscissae of points of inflexion of a function (cf. the previous problem).

1311. Find the shapes of the graphs of the first and second derivatives from the graph of the function of Fig. 30.

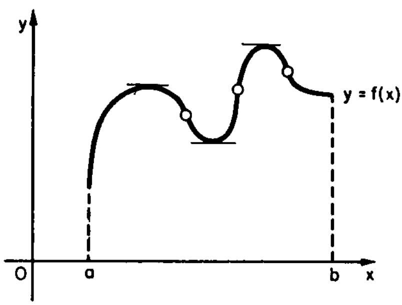

FIG. 30.

1312. Do the same from the graph of the function of Fig. 31.

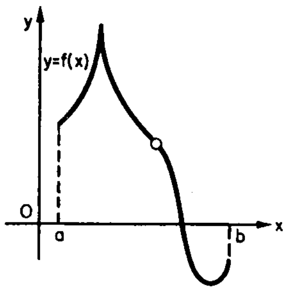

FIG. 31.

1313. Find the shape of the graph of a function, given the graph of its derivative (Fig. 32).

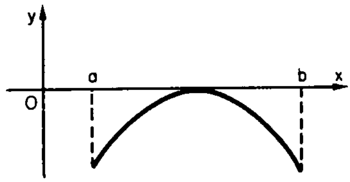

FIG. 32.

1314. Find the shape of the graph of a function, given the graph of its derivative (Fig. 33).

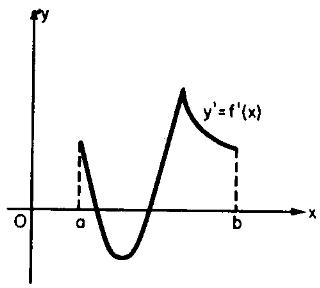

FIG. 33.

1315. A curve is given by the parametric equations $x = \varphi(t)$, $y = \psi(t)$. Show that the curve has points of inflexion at the values of t for which the expression $\frac{(\varphi'\psi'' - \psi'\varphi'')}{\varphi'}$ changes sign (the primes denote differentiations with respect to t), and $\varphi'(t) \neq 0$.

1316. Find the points of inflexion of the curve $x = t^2$, $y = 3t + t^3$.

1317. Find the points of inflexion of the curve $x = e^t$, $y = \sin t$.

4. Auxiliary Problems. Solution of Equations

Cauchy's Formula and l'Hôpital's Rule

1318. Write Cauchy's formula for the functions $f(x) = \sin x$ and $\varphi(x) = \ln x$ in the interval $[a, b]$, $0 < a < b$.

1319. Write Cauchy's formula for the functions $f(x) = e^{2x}$ and $\varphi(x) = 1 + e^x$ in the interval $[a, b]$.

1320. Prove that Cauchy's formula holds for $f(x) = x^3$ and $\varphi(x) = x^2 + 1$ in the interval $[1, 2]$.

1321. Prove that Cauchy's formula holds for $f(x) = \sin x$ and $\varphi(x) = x + \cos x$ in the interval $\left[0, \frac{\pi}{2}\right]$.

1322. Prove that, if $|f'(x)| \geqq |\varphi'(x)|$ in the interval $[a, b]$, we also have $|\Delta f(x)| \geqq |\Delta\varphi(x)|$, where $\Delta f(x) = f(x + \Delta x) - f(x)$, $\Delta\varphi(x) = \varphi(x + \Delta x) - \varphi(x)$, and x and $x + \Delta x$ are arbitrary points of interval $[a, b]$.

1323. Prove that the increment of $y = \ln(1 + x^2)$ is less than the increment of $y = \arctan x$ in the interval $\left[x, \frac{1}{2}\right]$ $(x \geqq 0)$, whilst the converse holds in $\left[\frac{1}{2}, x\right]$: $\Delta \arctan x < \Delta \ln(1 + x^2)$. Use the latter relationship to show that, in the interval $\left[\frac{1}{2}, 1\right]$,

$$\arctan x - \ln(1 + x^2) \geqq \frac{\pi}{4} - \ln 2.$$

Find the limits of the functions of problems 1324–1364:

1324. $\lim\limits_{x\to a} \dfrac{\sqrt[3]{x}-\sqrt[3]{a}}{\sqrt{x}-\sqrt{a}}$.

1325. $\lim\limits_{x\to 0} \dfrac{\ln\cos x}{x}$.

1326. $\lim\limits_{x\to 0} \dfrac{e^x-1}{\sin x}$.

1327. $\lim\limits_{x\to 0} \dfrac{e^{\alpha x}-\cos\alpha x}{e^{\beta x}-\cos\beta x}$.

1328. $\lim\limits_{x\to 0} \dfrac{x-\arctan x}{x^3}$.

1329. $\lim\limits_{x\to 0} \dfrac{e^{a\sqrt{x}}-1}{\sqrt{\sin bx}}$.

1330. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x-\tan x}$.

1331. $\lim\limits_{x\to \infty} \dfrac{\pi-2\arctan x}{\ln\left(1+\dfrac{1}{x}\right)}$.

1332. $\lim\limits_{x\to a} \dfrac{x^m-a^m}{x^n-a^n}$.

1333. $\lim\limits_{x\to 0} \dfrac{a^x-b^x}{c^x-d^x}$.

1334. $\lim\limits_{x\to 0} \dfrac{e^{x^2}-1}{\cos x-1}$.

1335. $\lim\limits_{x\to 0} \dfrac{e^x-e^{-x}}{\sin x\cos x}$.

1336. $\lim\limits_{x\to 0} \dfrac{a^x-b^x}{x\sqrt{1-x^2}}$.

1337. $\lim\limits_{x\to a} \dfrac{\cos x\ln(x-a)}{\ln(e^x-e^a)}$.

1338. $\lim\limits_{x\to 0} \dfrac{e^x-e^{-x}-2x}{x-\sin x}$.

1339. $\lim\limits_{x\to 0} \dfrac{e^{\tan x}-e^x}{\tan x-x}$.

1340. $\lim\limits_{x\to 0} \dfrac{e^x-\dfrac{x^3}{6}-\dfrac{x^2}{2}-x-1}{\cos x+\dfrac{x^2}{2}-1.}$.

1341. $\lim\limits_{x\to 0} \dfrac{e^{x^3}-1-x^3}{\sin^6 2x}$.

1342. $\lim\limits_{x\to 0} \dfrac{\ln(1+x)^4-4x+2x^2-\dfrac{4}{3}x^3+x^4}{6\sin x-6x+x^3}$.

1343. $\lim\limits_{x\to 0} \dfrac{\ln\sin 2x}{\ln\sin x}$.

1344. $\lim\limits_{x\to 0} \dfrac{\ln x}{\ln\sin x}$.

1345. $\lim\limits_{x\to 1} \dfrac{\ln(1-x)+\tan\dfrac{\pi x}{2}}{\cot\pi x}$.

1346. $\lim\limits_{x\to +\infty} (x^n e^{-x})$.

1347. $\lim\limits_{x\to\infty}[(\pi - 2\arctan x)\ln x]$.

1348. $\lim\limits_{x\to\infty}\left[x\sin\frac{a}{x}\right]$.

1349. $\lim\limits_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{\ln x}\right]$.

1350. $\lim\limits_{\varphi\to a}\left[(a^2-\varphi^2)\tan\frac{\pi\varphi}{2a}\right]$.

1351. $\lim\limits_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{\ln x}\right)$.

1352. $\lim\limits_{x\to 0}\left(\cot x-\frac{1}{x}\right)$.

1353. $\lim\limits_{x\to 1}\frac{1}{\cos\frac{\pi x}{2}\ln(1-x)}$.

1354. $\lim\limits_{x\to\infty}[\sqrt[3]{(a+x)(b+x)(c+x)}-x]$.

1355. $\lim\limits_{x\to\infty}\left[x\left(e^{\frac{1}{x}}-1\right)\right]$.

1356. $\lim\limits_{x\to 0}\left[x^2 e^{\frac{1}{x^2}}\right]$.

1357. $\lim\limits_{x\to\frac{\pi}{2}}(\tan x)^{2x-\pi}$.

1358. $\lim\limits_{x\to 0}x^{\sin x}$.

1359. $\lim\limits_{x\to 0}x^{\frac{1}{\ln(e^x-1)}}$.

1360. $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$.

1361. $\lim\limits_{x\to 0}(e^x+x)^{\frac{1}{x}}$.

1362. $\lim\limits_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$.

1363. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x$.

1364. $\lim\limits_{x\to 0}\left[\frac{\ln(1+x)^{1+x}}{x^2}-\frac{1}{x}\right]$.

1365. Prove that $\lim\limits_{x\to\infty}\frac{x-\sin x}{x+\sin x}$ exists but cannot be evaluated by l'Hôpital's rule.

1366. For sufficiently large values of x, which is the bigger: $a^x x^a$ or x^x?

1367. Assuming that $f(x)\to\infty$ as $x\to\infty$, which is the greater, $f(x)$ or $\ln f(x)$, for sufficiently large values of x?

1368. Let $x\to 0$. Prove that $e-(1+x)^{\frac{1}{x}}$ is a first-order infinitesimal with respect to x.

1369. Let $x\to 0$. Prove that $\ln(1+x)-e(\ln\ln(e+x))$ is a second-order infinitesimal with respect to x.

1370. A length AN is marked off along the tangent to a circle of radius r at point A (Fig. 34) equal to the length of arc AM. NM produced cuts diameter AO produced at B. Prove that

$$OB = \frac{r\,(\alpha\cos\alpha - \sin\alpha)}{\sin\alpha - \alpha},$$

where α is the angle in radians subtended at the centre by arc AM, and show that $\lim\limits_{\alpha\to 0} OB = 2r$.

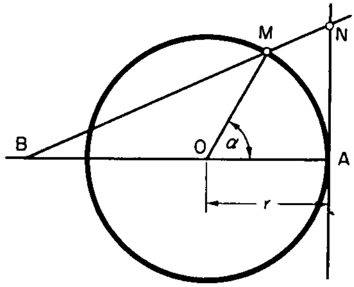

FIG. 34.

Asymptotic Variation of Functions and Asymptotes to Curves

1371. Prove directly from the definition that $y = 2x + 1$ is the asymptote to the curve

$$y = \frac{2x^4 + x^3 + 1}{x^3}.$$

1372. Prove directly from the definition that $x + y = 0$ is an asymptote to the curve $x^2y + xy^2 = 1$.

1373. Prove that the curves $y = \sqrt[3]{x^3 + 3x^2}$ and $y = \dfrac{x^2}{x - 1}$ are asymptotic to each other as $x \to \pm\infty$.

1374. Prove that the functions

$$f(x) = \sqrt{x^6 + 2x^4 + 7x^2 + 1} \text{ and } \varphi(x) = x^3 + x$$

are asymptotically equal as $x \to \infty$. Use this fact to evaluate approximately $f(115)$ and $f(120)$. What is the error if we put $f(100) = \varphi(100)$?

Find the asymptotes of the curves of problems 1375–1391:

1375. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$ **1376.** $xy = a.$

1377. $y = \frac{1}{x^2 - 4x + 5}.$ **1378.** $y = c + \frac{a^3}{(x-b)^2}.$

1379. $2y(x+1)^2 = x^3.$ **1380.** $y^3 = a^3 - x^3.$

1381. $y^3 = 6x^2 + x^3.$ **1382.** $y^2(x^2+1) = x^2(x^2-1).$

1383. $xy^2 + x^2y = a^3.$

1384. $y(x^2 - 3bx + 2b^2) = x^3 - 3ax^2 + a^3.$

1385. $(y + x + 1)^2 = x^2 + 1.$

1386. $y = x \ln\left(e + \frac{1}{x}\right).$ **1387.** $y = xe^{\frac{1}{x^2}}.$

1388. $y = xe^{\frac{2}{x}} + 1.$ **1389.** $y = x \operatorname{arc\,sec} x.$

1390. $y = 2x + \arctan\frac{x}{2}.$

1391. $y = \frac{xf(x) + a}{f(x)}$, where $f(x)$ is a polynomial $(a \neq 0)$.

1392. A curve is given by the parametric equations $x = \varphi(t)$, $y = \psi(t)$. Show that there can only be asymptotes, not parallel to either axis, for the values $t = t_0$ at which simultaneously

$$\lim_{t \to t_0} \varphi(t) = \infty \quad \text{and} \quad \lim_{t \to t_0} \psi(t) = \infty.$$

In this case, if the equation of the asymptote is $y = ax + b$, we have

$$a = \lim_{t \to t_0} \frac{\psi(t)}{\varphi(t)}, \quad b = \lim_{t \to t_0} [\psi(t) - a\varphi(t)].$$

How do we find the asymptotes parallel to an axis?

1393. Find the asymptotes of the curve

$$x = \frac{1}{t}, \quad y = \frac{t}{t+1}.$$

1394. Find the asymptotes of the curve

$$x = \frac{2e^t}{t-1}, \quad y = \frac{te^t}{t-1}.$$

1395. Find the asymptotes of the curve

$$x = \frac{2t}{1-t^2}, \quad y = \frac{t^2}{1-t^2}.$$

1396. Find the asymptotes of the folium of Descartes:

$$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}.$$

1397. Find the asymptotes of the curve

$$x = \frac{t-8}{t^2-4}, \quad y = \frac{3}{t(t^2-4)}.$$

General Investigation of Functions and Curves

Carry out a full investigation and draw the graphs of the functions of problems 1398–1464:

1398. $y = \dfrac{x}{1+x^2}$.

1399. $y = \dfrac{1}{1-x^2}$.

1400. $y = \dfrac{x}{x^2-1}$.

1401. $y(x-1)(x-2)(x-3) = 1$

1402. $y = \dfrac{x^2}{x^2-1}$.

1403. $y = (x^2-1)^3$.

1404. $y = 32x^2(x^2-1)^3$.

1405. $y = \dfrac{1}{x} + 4x^2$ (Newton's "trident").

1406. $y = x^2 + \dfrac{1}{x^2}$.

1407. $y = \dfrac{2x-1}{(x-1)^2}$.

1408. $y = \dfrac{x^3}{3-x^2}$.

1409. $y = \dfrac{x^3}{2(x+1)^2}$.

1410. $y(x-1) = x^3$.

1411. $y(x^3-1) = x^4$.

1412. $y = \frac{(x-1)^2}{(x+1)^3}$.

1413. $y = \frac{x^3+2x^2+7x-3}{2x^2}$.

1414. $xy = (x^2-1)(x-2)$.

1415. $(y-x)x^4+8=0$.

1416. $y = \frac{x}{e^x}$.

1417. $y = x^2e^{-x}$.

1418. $y = \frac{e^x}{x}$.

1419. $y = x - \ln(x+1)$.

1420. $y = \ln(x^2+1)$.

1421. $y = x^2e^{-x^2}$.

1422. $y = x^3e^{-x}$.

1423. $y = xe^{-\frac{x^2}{2}}$

1424. $y = \frac{1}{e^x-1}$.

1425. $y = x + \frac{\ln x}{x}$.

1426. $y = \left(1+\frac{1}{x}\right)^x$.

1427. $y = x + \sin x$.

1428. $y = x \sin x$.

1429. $y = \ln \cos x$.

1430. $y = \cos x - \ln \cos x$.

1431. $y = x - 2 \arctan x$.

1432. $y = e^{\frac{1}{x^2-4x+3}}$.

1433. $y = e^{\sin x} - \sin x$ (without seeking the points of inflexion).

1434. $y = \sqrt[3]{x^2} - x$.

1435. $y^3 = x^2(x^2-4)^3$.

1436. $(3y+x)^3 = 27x$.

1437. $y = \sqrt[3]{(x+1)^2} - \sqrt[3]{x^2} + 1$.

1438. $y = (x-1)^{\frac{2}{3}}(x+1)^3$.

1439. $y^3 = 6x^2 - x^3$.

1440. $(y-x)^2 = x^5$.

1441. $(y-x^2)^2 = x^5$.

1442. $y^2 = x^3+1$.

1443. $y^2 = x^3 - x$.

1444. $y^2 = x(x-1)^2$.

1445. $y^2 = x^2(x-1)$.

1446. $y^2 = \frac{x^3-2}{3x}$.

1447. $x^2y + xy^2 = 2$.

1448. $y^2 = x^2\frac{a+x}{a-x}$ (strophoid)

1449. $9y^2 = 4x^3 - x^4$. **1450.** $25y^2 = x^2(4 - x^2)^3$.

1451. $y^2 = x^2 - x^4$. **1452.** $x^2y^2 = 4(x - 1)$.

1453. $y^2(2a - x) = x^3$ (cissoid)

1454. $x^2y^2 = (x - 1)(x - 2)$.

1455. $x^2y^2 = (a + x)^3 (a - x)$ (conchoid)

1456. $16y^2 = (x^2 - 4)^2 (1 - x^2)$

1457. $y^2 = (1 - x^2)^3$. **1458.** $y^2x^4 = (x^2 - 1)^3$.

1459. $y^2 = 2e\,xe^{-2x}$. **1460.** $y = e^{\frac{1}{x}} - x$.

1461. $y = e^{\tan x}$. **1462.** $f(x) = \dfrac{\sin x}{x}$, $f(0) = 1$.

1463. $y = 1 - xe^{-\frac{1}{|x|} - \frac{1}{x}}$ for $x \neq 0$, $y = 1$ for $x = 0$.

1464. $y = x^2 - 4|x| + 3$.

Investigate the functions given parametrically in problems 1465–1469 and sketch their graphs:

1465. $x = t^3 + 3t + 1$, $y = t^3 - 3t + 1$.

1466. $x = t^3 - 3\pi$, $y = t^3 - 6 \arctan t$.

1467. $x = \dfrac{3t}{1 + t^3}$, $y = \dfrac{3t^2}{1 + t^3}$.

1468. $x = te^t$, $y = te^{-t}$.

1469. $x = 2a \cos t - a \cos 2t$, $y = 2a \sin t - a \sin 2t$ (cardioid).

Investigate the curves whose equations are given in polar coordinates in problems 1470–1477:

1470. $\varrho = a \sin 3\varphi$ (three-petal rose).

1471. $\varrho = a \tan \varphi$. **1472.** $\varrho = a(1 + \tan \varphi)$.

1473. $\varrho = a(1 + \cos \varphi)$ (cardioid).

1474. $\varrho = a(1 + b \cos \varphi)$ $(a > 0, b > 1)$ (limaçon).

1475. $\varrho = \sqrt{\dfrac{\pi}{\varphi}}$ (litus). **1476.** $\varrho = \dfrac{2}{\pi} \arctan \dfrac{\varphi}{\pi}$.

1477. $\varrho = \sqrt{1 - t^2}$, $\varrho = \arcsin t + \sqrt{1 - t^2}$.

First reduce the equations of the curves of problems 1478–1481 to polar coordinates, then investigate and plot the curves:

1478. $(x^2 + y^2)^3 = 4a^2x^2y^2$. **1479.** $(x^2 + y^2)\, x = a^2y$.

1480. $x^4 + y^4 = a^2(x^2 + y^2)$.

1481. $(x^2 + y^2)\,(x^2 - y^2)^2 = 4x^2y^2$.

Solution of Equations

1482. Prove that the equation

$$x^3 - x^2 - 8x + 12 = 0$$

has one simple root $x_1 = -3$ and one double root $x_2 = 2$.

1483. Prove that the equation

$$x^4 + 2x^3 - 3x^2 - 4x + 4 = 0$$

has two double roots $x_1 = 1$ and $x_2 = -2$.

1484. Show that the equation $x \arcsin x = 0$ has only one real root, this being double.

1485. Show that the roots of the equation $x \sin x = 0$ have the form $x = k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$, where $k = 0$ corresponds to a double root. What is the multiplicity of the remaining roots?

1486. Show that the equation $x^3 - 3x^2 + 6x - 1 = 0$ has a unique real simple root in the interval $(0, 1)$, and find the root, by using trial and error, to an accuracy of 0·1.

1487. Show that the equation $x^4 + 3x^2 - x - 2 = 0$ has two (and only two) real simple roots, lying in the intervals $(-1, 0)$ and $(0, 1)$ respectively. Use trial and error to find these roots to an accuracy of 0·1.

1488. Show that the equation $f(x) = a \neq 0$, where $f(x)$ is a polynomial with positive coefficients and odd powers only, has one and only one real root (which may be multiple). Consider the case when $a = 0$. Find to an accuracy of 0·01 the root of the equation

$$x^3 + 3x - 1 = 0,$$

by combining trial and error and the chord method.

1489. Prove the theorem: the necessary and sufficient condition for the equation $x^3 + px + 1 = 0$ to have three simple real roots is that coefficients p and q satisfy the inequality $4p^3 + 27q^2 < 0$. Find to an accuracy of 0·01 all the roots of the equation

$$x^3 - 9x + 2 = 0,$$

by combining trial and error and the chord method.

1490. Prove that the equation

$$x^4 + 2x^2 - 6x + 2 = 0$$

has two (and only two) real simple roots, lying in the intervals (0, 1) and (1, 2) respectively. Find these roots to an accuracy of 0·01 by combining the tangent and chord methods.

1491. Prove that the equation

$$x^5 - 5x + 1 = 0$$

has a unique real simple root lying in the interval $(-1, 0)$, and find the root to an accuracy of 0·01 by combining the chord and tangent methods.

1492. Show that the equation $xe^x = 2$ has only one real root, which belongs to the interval (0, 1), and find the root to an accuracy of 0·01.

1493. Prove that the equation $x \ln x = a$ has no real roots for $a < -\frac{1}{e}$, has one real double root for $a = -\frac{1}{e}$, two real simple roots for $-\frac{1}{e} < a < 0$ and one real simple root for $a \geqq 0$. Find the root of the equation $x \ln x = 0{\cdot}8$ to an accuracy of 0·01.

1494. Show that Kepler's equation $x = \varepsilon \sin x + a$, where $0 < \varepsilon < 1$, has one simple real root, and find this root to an accuracy of 0·001 when $\varepsilon = 0{\cdot}538$ and $a = 1$.

1495. Show that the equation $a^x = ax$ always has two (and only two) real positive roots for $a > 1$, one root being equal to unity and the other less, greater than or equal to unity, depending on whether a is greater than, less than or equal to e. Find the second root of the equation to an accuracy of 0·001 when $a = 3$.

1496. Show that the equation $x^2 \arctan x = a$, where $a \neq 0$, has one real root. Find the root to an accuracy of 0·001 when $a = 1$.

1497. For what base a of a system of logarithms do numbers exist which are equal to their logarithms? How many such numbers can there be? What is this number (to an accuracy of 0·01) when $a = \frac{1}{2}$?

5. Taylor's Formula and its Applications

Taylor's Formula for Polynomials

1498. Expand the polynomial $x^4 - 5x^3 + x^2 - 3x + 4$ in powers of $x - 4$.

1499. Expand the polynomial $x^3 + 3x^2 - 2x + 4$ in powers of $x + 1$.

1500. Expand $x^{10} - 3x^5 + 1$ in powers of $x - 1$.

1501. Expand the function $f(x) = (x^2 - 3x + 1)^3$ in powers of x by using Taylor's formula.

1502. $f(x)$ is a fourth-degree polynomial. Knowing that $f(2) = -1$, $f'(2) = 0$, $f''(2) = 2$, $f'''(2) = -12$, $f^{\mathrm{IV}}(2) = 24$, find $f(-1)$, $f'(0)$, $f''(1)$.

Taylor's Formula

1503. Find Taylor's formula of order n at $x_0 = -1$ for $y = \frac{1}{x}$.

1504. Find Taylor's formula of order n at $x_0 = 0$ (Maclaurin's formula) for $y = xe^x$.

1505. Find Taylor's formula of order n at $x_0 = 4$ for $y = \sqrt{x}$.

1506. Find Taylor's formula of order $2n$ at $x_0 = 0$ for $y = \dfrac{e^x + e^{-x}}{2}$.

1507. Find Taylor's formula of order n at $x_0 = 1$ for $y = x^3 \ln x$.

1508. Find Taylor's formula of order $2n$ at $x_0 = 0$ for $y = \sin^2 x$.

1509. Find Taylor's formula of order 3 at $x_0 = 2$ for $y = \dfrac{x}{x-1}$ and draw the graph of the function and of its third-degree Taylor polynomial.

1510. Find Taylor's formula of order 2 at $x_0 = 0$ for $y = \tan x$ and draw the graph of the function and of its second-degree Taylor polynomial.

1511. Find Taylor's formula of order 3 at $x_0 = 0$ for $y = \arcsin x$ and draw the graph of the function and of its third-degree Taylor polynomial.

1512. Find Taylor's formula of order 3 at $x_0 = 1$ for $y = \dfrac{1}{\sqrt{x}}$ and draw the graph of the function and of its third-degree Taylor polynomial.

1513*. Prove that the number θ in the remainder term of Taylor's formula of order 1:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a + \theta h)$$

tends to $\frac{1}{3}$ as $h \to 0$ if $f'''(x)$ is continuous at $x = a$ and $f'''(a) \neq 0$. Generalize this result.

Some Applications of Taylor's Formula

Describe the behaviour of the functions of problems 1514–1519 at the points mentioned:

1514. $y = 2x^6 - x^3 + 3$ at the point $x = 0$.

1515. $y = x^{11} + 3x^6 + 1$ at the point $x = 0$.

1516. $y = 2\cos x + x^2$ at the point $x = 0$.

1517. $y = 6\ln x - 2x^3 + 9x^2 - 18x$ at the point $x = 1$.

1518. $y = 6\sin x + x^2$ at the point $x = 0$.

1519. $y = 24e^x - 24x - 12x^2 - 4x^3 - x^4 - 20$ at the point $x = 0$.

1520. $f(x) = x^{10} - 3x^6 + x^2 + 2$. Find the first three terms of the Taylor expansion at $x_0 = 1$. Evaluate approximately $f(1{\cdot}03)$.

1521. $f(x) = x^8 - 2x^7 + 5x^6 - x + 3$. Find the first three terms of the Taylor expansion at $x_0 = 2$. Evaluate approximately $f(2{\cdot}02)$ and $f(1{\cdot}97)$.

1522. $f(x) = x^{80} - x^{40} + x^{20}$. Find the first three terms of the expansion of $f(x)$ in powers of $x - 1$ and find approximately $f(1{\cdot}005)$.

1523. $f(x) = x^5 - 5x^3 + x$. Find the first three terms of the expansion in powers of $x - 2$. Evaluate approximately $f(2{\cdot}1)$. Evaluate $f(2{\cdot}1)$ accurately and find the absolute and relative errors.

1524. Show that the error is less than $0{\cdot}01$ when evaluating e^x for $0 < x < 1$ from the approximate formula

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}\,.$$

Using this, find $\sqrt{e}$ correct to three figures.

1525. By using the approximation formula

$$e^x \approx 1 + x + \frac{x^2}{2}\,,$$

find $\dfrac{1}{\sqrt[4]{e}}$ and estimate the error.

1526. Show that, if we take $x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$ instead of $\sin x$ for angles less than 28°, the error is less than $0{\cdot}000001$. Use this to evaluate $\sin 20°$ to six correct figures.

1527. Find $\cos 10°$ to an accuracy of 0·001. Show that this accuracy can be achieved by taking the second-order Taylor formula.

1528. Use the approximation

$$\ln (1 + x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4},$$

to find ln 1·5, and estimate the error.

6. Curvature

Find the curvatures of the curves of problems 1529–1526:

1529. The hyperbola $xy = 4$ at the point (2, 2).

1530. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at its vertices.

1531. $y = x^4 - 4x^3 - 18x^2$ at the origin.

1532. $y^2 = 8x$ at the point $\left(\frac{9}{8}, 3\right)$.

1533. $y = \ln x$ at the point (1, 0).

1534. $y = \ln (x + \sqrt{1 + x^2})$ at the origin.

1535. $y = \sin x$ at points corresponding to extremals of the function.

1536. The folium of Descartes $x^3 + y^3 = 3axy$ at the point $\left(\frac{3}{2}a, \frac{3}{2}a\right)$.

Find the curvatures of the curves of problems 1537–1542 at an arbitrary point (x, y):

1537. $y = x^3$.

1538. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

1539. $y = \ln \sec x$.

1540. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

1541. $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$.

1542. $y = a \cosh \frac{x}{a}$.

Find the curvatures of the curves of problems 1543–1549:

1543. $x = 3t^2, \quad y = 3t - t^3$ for $t = 1$.

1544. $x = a\cos^3 t, \quad y = a\sin^3 t$ for $t = t_1$.

1545. $x = a(\cos t + t\sin t), \quad y = a(\sin t - t\cos t)$ for $t = \frac{\pi}{2}$.

1546. $x = 2a\cos t - a\cos 2t$, $y = 2a\sin t - a\sin 2t$ at an arbitrary point.

1547. $\varrho = a^{\varphi}$ at the point $\varrho = 1$, $\varphi = 0$.

1548. $\varrho = a\varphi$ at an arbitrary point.

1549. $\varrho = a\varphi^k$ at an arbitrary point.

1550. Find the radius of curvature of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point where the segment of the tangent lying between the coordinate axes is bisected by the point of contact.

1551. Show that the radius of curvature of a parabola is equal to twice the segment of the normal lying between its points of intersection with the parabola and with the directrix.

1552. Show that the radius of curvature of the cycloid at any point is twice the length of the normal at that point.

1553. Show that the radius of curvature of the lemniscate $\varrho^2 = a^2\cos 2\varphi$ is inversely proportional to the corresponding radius vector.

1554. Find the circle of curvature of the parabola $y = x^2$ at the point $(1, 1)$.

1555. Find the circle of curvature of the hyperbola $xy = 1$ at the point $(1, 1)$.

1556. Find the circle of curvature of $y = e^x$ at the point $(0, 1)$.

1557. Find the circle of curvature of $y = \tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$.

1558. Find the circle of curvature of the cissoid $(x^2 + y^2)x - 2ay^2 = 0$ at the point (a, a).

Find the vertices (the points at which the curvature has an extremal value, see *Course*, sec. 93) of the curves of problems 1559—1562:

1559. $\sqrt{x} + \sqrt{y} = \sqrt{a}$. **1560.** $y = \ln x$.

1561. $y = e^x$.

1562. $x = a(3\cos t + \cos 3t)$, $y = a(3\sin t + \sin 3t)$.

1563. Find the maximum radius of curvature of the curve $\varrho = a \sin^3 \dfrac{\varphi}{3}$.

1564. Show that the curvature at a point P of the curve $y = f(x)$ is equal to $|y'' \cos^3 \alpha|$, where α is the angle formed by the tangent at P with the positive direction of the axis of abscissae.

1565. Show that the curvature of a curve at an arbitrary point can be expressed as $k = \left|\dfrac{d \sin \alpha}{dx}\right|$, where α has the same meaning as in the previous problem.

1566. Function $f(x)$ is defined thus: $f(x) = x^3$ in the interval $-\infty < x \leqq 1$, $f(x) = ax^2 + bx + c$ in the interval $1 < x < \infty$. What must be the values of a, b, c for the curve $y = f(x)$ to have a continuous curvature at all points?

1567. Given (Fig. 35) arc AM of a circle of radius 5 with centre at $(0, 5)$ and segment BC of the straight line joining

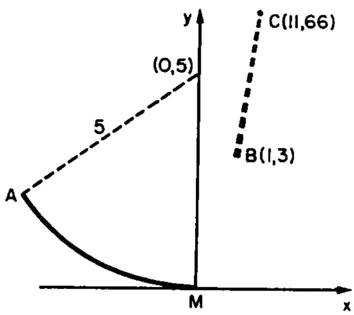

FIG. 35.

B (1, 3) and C (11, 66), we require to join M and B with a parabolic arc such that the curve $AMBC$ has continuous curvature everywhere. Find the equation of the required parabola (take a fifth-order parabola).

Find the coordinates of the centre of curvature and the equation of the evolute for the curves of problems 1568–1574:

1568. Parabola of nth order $y = x^n$.

1569. Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

1570. Astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

1571. Semi-cubical parabola $y^3 = ax^2$.

1572. Parabola $x = 3t,\ y = t^2 - 6$.

1573. Cissoid $y^2 = \frac{x^3}{2a - x}$.

1574. The curve $\begin{cases} x = a\,(1 + \cos^2 t) \sin t, \\ y = a \sin^2 t \cos t. \end{cases}$

1575. Show that the evolute of the tractrix

$$x = -a\left(\ln \tan \frac{t}{2} + \cos t\right), \quad y = a \sin t$$

is a catenary.

1576. Show that the evolute of the logarithmic spiral $\varrho = a^{\varphi}$ is precisely the same spiral except for rotation through a certain angle. Is it possible to choose a so that the evolute coincides with the spiral?

1577. Show that any evolute of a circle can be got by rotation of one of them through a suitable angle.

1578. Show that the distance of a point of the cycloid from the centre of curvature of the corresponding point of the evolute is equal to twice the diameter of the rolling circle.

1579. The evolute of the parabola $y^2 = 4px$ is the semi-cubical parabola

$$py^2 = \frac{4}{27}(x - 2p)^3.$$

Find the length of arc of the semi-cubical parabola from the cusp to the point (x, y).

1580. Find the total length of the evolute of the ellipse with semi-axes a and b.

1581. Show that the evolute of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$ is an astroid of twice the linear dimensions turned through 45°. Use this to find the length of arc of the original astroid.

1582*. Show that the evolute of the cardioid

$$x = 2a\cos t - a\cos 2t;\quad y = 2a\sin t - a\sin 2t$$

is also a cardioid, similar to the first. Use this to find the total arc length of the cardioid.

1583*. Prove the theorem: if the curvature of the arc of a given curve is either only increasing or only decreasing, the circles of curvature corresponding to different points of the arc lie inside each other and do not intersect.

7. Numerical Problems

1584. Find the minimum of the function

$$y = x^4 + x^2 + x + 1$$

to an accuracy of unity.

1585. Find the maximum of the function

$$y = x + \ln x - x^3$$

to an accuracy of 0·001.

1586. Find the greatest and least values of the function

$$y = x^2 + 3\cos x$$

in the interval $\left[0, \frac{\pi}{2}\right]$ to an accuracy of 0·01.

1587. Find the greatest and least values of

$$y = x - e^{x^2}$$

in the interval [0·2, 0·5] to an accuracy of 0·001.

1588. Find the coordinates of the point of inflexion of the curve

$$y = \frac{e^x}{10}(x^3 - 6x^2 + 19x - 30)$$

to an accuracy of 0·01.

1589. Find the coordinates of the point of inflexion of the curve

$$y = 6x^2 \ln x + 2x^3 - 9x^2$$

to an accuracy of 0·01.

1590. Find to an accuracy of 0·0001 the curvature of the curve

$$y = \frac{1}{x^2}$$

at its point of intersection with the straight line $y = x - 1$.

1591. Find to an accuracy of 0·001 the coordinates of the point on the curve $y = \ln x$ at which the radius of curvature of the curve is three times the abscissa of the point.

CHAPTER V

THE DEFINITE INTEGRAL

1. The Definite Integral and its Elementary Properties

1592. Express with the aid of an integral the area bounded by the following curves:

(1) the coordinate axes, the straight line $x = 3$ and the parabola $y = x^2 + 1$;

(2) the axis of abscissae, the straight lines $x = a$, $x = b$ and the curve $y = e^x + 2$ $(b > a)$;

(3) the axis of abscissae and the arc of the sine wave $y = \sin x$ corresponding to the first half-period;

(4) the parabolas $y = x^2$ and $y = 8 - x^2$;

(5) the parabolas $y = x^2$ and $y = \sqrt{x}$;

(6) the curves $y = \ln x$ and $y = \ln^2 x$.

1593. A figure is bounded by the axis of abscissae and the straight lines $y = 2x$, $x = 4$, $x = 6$. By dividing the interval [4, 6] into equal parts, finds the areas of the "inner" and "outer" n-step figures. Show that both the expressions obtained tend on indefinite increase of n to the same limit S, the area of the figure. Find the absolute and relative errors on replacing the given area by the areas of the inner and outer n-step figures.

1594. A curvilinear trapezium with base [2, 3] is bounded by the parabola $y = x^2$. Find the absolute and relative error on replacing the given area by the "inner" 10-step figure.

1595. Find the area of the figure bounded by the parabola $y = \frac{x^2}{2}$, the straight lines $x = 3$, $x = 6$, and the axis of abscissae.

1596. Find the area of the segment cut out of the parabola $y = x^2$ by the straight line $y = 2x + 3$.

1597. Find the area of the parabolic segment with base $a = 10$ cm and "height" $h = 6$ cm (the base is the chord perpendicular to the parabola axis. Fig. 36).

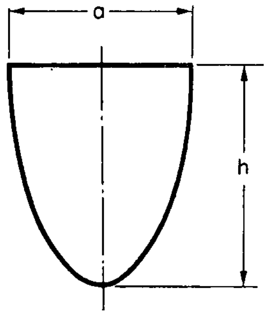

FIG. 36.

1598. Find the area of the figure bounded by the parabola $y = x^2 - 4x + 5$, the axis of abscissae and the straight lines $x = 3$, $x = 5$.

1599. Find the area of the figure bounded by the arcs of parabolas $y = \frac{1}{4}x^2$ and $y = 3 - \frac{x^2}{2}$.

1600. Find the area of the figure bounded by the parabolas $y = x^2 - 6x + 10$ and $y = 6x - x^2$.

1601. Find the area contained between the parabola $y = x^2 - 2x + 2$, the tangent to it at the point (3, 5), the axis of ordinates and the axis of abscissae.

1602. A particle travels with a speed $v = 2t + 4$ cm/sec. Find the path traversed in the first 10 sec.

1603. The velocity of a body falling freely is $v = gt$. Find the distance traversed in the first 5 sec.

1604. If the velocity is proportional to the square of time and is equal to 1 cm/sec at the end of the 4th second, what is the distance travelled in the first 10 sec?

1605. We know from physics that the force opposing the extension of a spring is proportional to its elongation (Hooke's law). The work done on extending a spring 4 cm is 10 kg. How much work is done in extending the spring 10 cm?

1606. The work required to extend a spring 2 cm is 20 kg. How much can the spring be extended on expending work of 80 kg?

1607. The speed v of radioactive decay is a known function of time: $v = v(t)$. Express the amount m of radioactive material disintegrating between time T_0 and time T_1: (a) approximately, by a sum, (b) exactly, by an integral.

1608. The rate of heating of a body is a known function of time $\psi(t)$. How many degrees θ is the body heated from time T_0 to time T_1? Express the solution: (a) approximately, by a sum, (b) exactly, by an integral.

1609. A variable current I is a known function of time: $I = I(t)$. Express (approximately by a sum and exactly by an integral) the quantity of electricity Q that has flowed through the cross-section of the conductor after time T from the start of the experiment.

1610. The voltage E of a variable current is a given function of time $E = \varphi(t)$; the current I is also a given function of time $I = \psi(t)$. Express the work A done by the current between time T_0 and time T_1: (a) approximately, by a sum, (b) exactly, by an integral.

1611. An electrical circuit is supplied from batteries. During 10 min the voltage at the terminals falls uniformly from $E_0 = 60$ V to $E = 40$ V. The circuit resistance $R = 20$ ohm. Find the amount of electricity flowing through the circuit in 10 min.

1612. The voltage drops uniformly in an electrical circuit, at 1·5 V per min. The initial voltage $E_0 = 120$ V. Find the work done by the current in 5 min. The circuit resistance $R = 60$ ohm. Inductance and capacity are neglected.

1613. The input voltage of a circuit rises uniformly, being zero at the start of the experiment. The voltage reaches 120 V during one minute. The circuit resistance is 100 ohm. Inductance and capacity are neglected. Find the work done by the current during one minute.

1614. The water reaches the top of the rectangular wall of an aquarium of base a and height b. Express the pressure of the water over the entire wall: (a) approximately, by a sum, (b) exactly, by an integral.

1615. (a) Evaluate the pressure P exerted by the water in an aquarium on one of its walls. The wall is rectangular. Its length $a = 60$ cm, and height $b = 25$ cm. (b) Divide the wall by a horizontal line so that the pressure on the two parts is the same.

Evaluation of Integrals by Summation

1616. Find $\int\limits_0^1 e^x \, dx$ by direct summation followed by passage to the limit. (Divide the interval of integration into n parts.)

1617. Evaluate $\int\limits_a^b x^k \, dx$, where k is a positive integer, by direct summation followed by passage to the limit (divide the interval of integration so that the abscissae of the points of subdivision form a geometrical progression) (see *Course*, sec. 87).

1618. Use the formula obtained in the previous example to evaluate the integrals:

(1) $\int\limits_0^{10} x \, dx$; (2) $\int\limits_{a-2}^{a+2} dx$; (3) $\int\limits_{\frac{a}{2}}^{a} x^2 \, dx$; (4) $\int\limits_a^{2a} \frac{b^2x^2}{a^2} \, dx$;

(5) $\int\limits_0^a (3x^2 - x + 1) \, dx$; (6) $\int\limits_0^m \frac{x^2 + m^2}{m^2} \, dx$;

(7) $\int\limits_1^{2\cdot5} (2x + 1)^2 \, dx$; (8) $\int\limits_a^b (x - a)(x - b) \, dx$;

(9) $\int_{-a}^{0} \frac{(a+x)^2}{a}\,dx$; (10) $\int_{0}^{1} \left(\frac{ax-b}{a-b}\right)^2 dx$; (11) $\int_{0}^{2} x^3\,dx$;

(12) $\int_{1}^{3} \frac{x^4}{3}\,dx$; (13) $\int_{0}^{1} \left(\frac{x^5}{7}-\frac{x^6}{6}\right) dx$.

1619*. Find $\lim_{n\to\infty} \left(\frac{1^k + 2^k + \ldots + n^k}{n^{k+1}}\right)$ for $k > 0$. Evaluate approximately $1^5 + 2^5 + \ldots + 100^5$.

1620. Find $\int_{1}^{2} \frac{dx}{x}$ by direct summation followed by passage to the limit. (Divide the interval of integration so that the abscissae of the points of subdivision form a geometrical progression.)

1621. Form the integral sum for $\int_{1}^{2} \frac{dx}{x}$ by dividing the interval of integration into n equal parts. By comparing with the previous problem, evaluate:

$$\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right).$$

1622*. Evaluate $\lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{an}\right)$ (a is an integer). Evaluate approximately

$$\left(\frac{1}{100} + \frac{1}{101} + \frac{1}{102} + \ldots + \frac{1}{300}\right).$$

1623*. Use direct summation followed by passage to the limit to evaluate:

(1) $\int_{0}^{a} xe^x\,dx$; (2) $\int_{1}^{a} \ln x\,dx$; (3) $\int_{a}^{b} \frac{\ln x}{x}\,dx$.

(Divide the interval of integration into equal parts in (1), and as in problem 1620 in (2) and (3)).

2. Fundamental Properties of the Definite Integral

Geometrical Interpretation of the Definite Integral

1624. Express with the aid of an integral the area of the figure bounded by the arc of the sine wave corresponding to the interval $0 \leqq x \leqq 2\pi$ and by the axis of abscissae.

1625. Find the area of the figure bounded by the cubical parabola $y = x^3$ and the straight line $y = x$.

1626. Find area of the figure bounded by the parabolas $y = x^2 - 2x - 3$ and $y = -x^2 + 6x - 3$.

1627. Find the area of the figure bounded by the curves $y = x^3 - x$ and $y = x^4 - 1$.

Inequalities for Integrals

1628. Show that the integral $\int\limits_0^{10} \frac{x\,dx}{x^3 + 16}$ is less than $\frac{5}{6}$.

1629. Show that the integral $\int\limits_0^2 e^{x^2-x}\,dx$ lies between $\frac{2}{\sqrt[4]{e}}$ and $2e^2$.

1630. $\int\limits_{1\cdot5}^{3\cdot5} \frac{x^2\,dx}{x-1}$.

1631. $\int\limits_0^2 \frac{x^2+5}{x^2+2}\,dx$.

1632. $\int\limits_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1 + \sin^2 x)\,dx$.

1633. $\int\limits_{\frac{1}{2}}^{\frac{5}{2}} \frac{x}{1+x^2}\,dx$.

1634. $\int\limits_{\frac{\sqrt{3}}{3}}^{\sqrt{3}} x \arctan x\,dx$.

1635. $\int\limits_{\frac{1}{e}}^{e} x^2 e^{-x^2}\,dx$.

1636. Find out, without evaluating them, which of the integrals is the greater:

(1) $\int\limits_0^1 x^2\,dx$ or $\int\limits_0^1 x^3\,dx$? (2) $\int\limits_1^2 x^2\,dx$ or $\int\limits_1^2 x^3\,dx$?

1637. Find out which of the integrals is the greater:

$$(1)\ \int_0^1 2^{x^2}\,dx \text{ or } \int_0^1 2^{x^3}\,dx? \quad (2)\ \int_1^2 2^{x^2}\,dx \text{ or } \int_1^2 2^{x^3}\,dx?$$

$$(3)\ \int_1^2 \ln x\,dx \text{ or } \int_1^2 (\ln x)^2\,dx?$$

$$(4)\ \int_3^4 \ln x\,dx \text{ or } \int_3^4 (\ln x)^2\,dx?$$

1638. Prove that $\int\limits_0^1 \sqrt{1+x^3}\,dx < \frac{\sqrt{5}}{2}$ (use Bunyakovskii's inequality). Show that employment of the general rule yields a cruder estimate.

1639. Prove the following propositions by starting from geometrical considerations:

(a) if function $f(x)$ is increasing in the interval $[a, b]$ and has a concave graph, then

$$(b-a)\,f(a) < \int_a^b f(x)\,dx < (b-a)\,\frac{f(a)+f(b)}{2};$$

(b) if the function $f(x)$ is increasing in interval $[a, b]$ and has a convex graph, then

$$(b-a)\,\frac{f(a)+f(b)}{2} < \int_a^b f(x)\,dx < (b-a)\,f(b).$$

1640*. Estimate the integral $\int\limits_2^3 \frac{x^2\,dx}{1+x^2}$.

1641. Estimate the integral $\int\limits_0^1 \sqrt{1+x^4}\,dx$, by using:

(a) the fundamental theorem on estimation of an integral,
(b) the result of problem 1639,
(c) Bunyakovskii's inequality.

Mean Values of a Function

1642. Find the mean value of the linear function $y = kx + b$ in the interval $[x_1, x_2]$. Find the point at which the function takes this value.

1643. Find the mean value of the quadratic function $y = ax^2$ in the interval $[x_1, x_2]$. At how many points of the interval does the function assume this value?

1644. Find the mean value of $y = 2x^2 + 3x + 3$ in the interval $[1, 4]$.

1645. Starting from geometrical considerations, find the mean value of $y = \sqrt{a^2 - x^2}$ in the interval $[-a, a]$.

1646. Starting from geometrical considerations, obtain the mean value of a continuous odd function in an interval symmetric with respect to the origin.

1647. A gutter section is in the form of a parabolic segment. Its base $a = 1$ m, the depth $h = 1{\cdot}5$ m (see Fig. 36). Find the mean depth of the gutter.

1648. The voltage of an electrical circuit increases uniformly during one minute from $E_0 = 100$ V to $E_1 = 120$ V. Find the average current during this time. The circuit resistance is 10 ohms.

1649. The voltage of an electrical circuit falls uniformly at a rate of 0·4 V per minute. The initial voltage of the circuit is 100 V. The circuit resistance is 5 ohm. Find the average power of the current during the first hour of working.

Integral with Variable Limits

1650. Find the expressions for the following integrals with variable upper limit:

$$(1)\ \int_0^x x^2\,dx; \quad (2)\ \int_a^x x^5\,dx; \quad (3)\ \int_1^x \left(\frac{x^3}{5} - \frac{x^4}{4}\right) dx.$$

1651. The speed of a moving body is proportional to the square of time. Find the relationship between the path traversed s and time t, if it is known that the body moves 18 cm in the first 3 sec and that the motion starts at the instant $t = 0$.

1652. The force acting on a material particle varies uniformly with respect to the path traversed. It is equal to 100 dynes at the start of the path, and increases to 600 dynes when the particle has moved 10 cm. Find the function defining the dependence of the work on the path.

1653. The voltage of an electrical circuit varies uniformly. It is equal to E_1 at $t = t_1$, and equal to E_2 at $t = t_2$. The resistance R is constant, whilst we neglect inductance and capacity. Express the work of the current as a function of time t after the start of the experiment.

1654. The specific heat of a body depends on the temperature as follows: $c = c_0 + \alpha t + \beta t^2$. Find the function that defines the dependence of the quantity of heat, acquired by the body on heating from zero to t, on the temperature t.

1655. A curvilinear trapezium is bounded by the parabola $y = x^2$, the axis of abscissae and a movable ordinate. Find the value of the increment ΔS and differential $\mathrm{d}S$ of the area of the trapezium at $x = 10$ when $\Delta x = 0{\cdot}1$.

1656. A curvilinear trapezium is bounded by the curve $y = \sqrt{x^2 + 16}$, the coordinate axes and a movable ordinate. Find the value of the differential $\mathrm{d}S$ of the area of the trapezium when $x = 3$ and $\Delta x = 0{\cdot}2$.

1657. A curvilinear trapezium is bounded by the curve $y = x^3$, the axis of abscissae and a movable ordinate. Find the values of the increment ΔS of the area, its differential $\mathrm{d}S$, and the absolute (α) and relative $\left(\delta = \frac{\alpha}{\mathrm{d}S}\right)$ errors arising on replacing the increment by the differential, if $x = 4$, and Δx takes the values 1; $0{\cdot}1$ and $0{\cdot}01$.

1658. Find the numerical values of the derivative of

$$y = \int_0^x \frac{1 - t + t^2}{1 + t + t^2}\,\mathrm{d}t \text{ at } x = 1.$$

1659. Find the numerical values of the derivative of

$$y = \int_0^x \sin x \, dx \ \text{ at } \ x = 0, \quad x = \frac{\pi}{4} \ \text{and} \ x = \frac{\pi}{2}.$$

1660. Find the derivative with respect to the lower limit of an integral with variable lower and constant upper limit.

1661. Find the numerical values of the derivative of $y = \int_x^5 \sqrt{1 + x^2}\, dx$ with $x = 0$ and $x = \frac{3}{4}$.

1662. Find the derivative with respect to x of the function

$$y = \int_0^{2x} \frac{\sin x}{x}\, dx.$$

1663. Find the derivative with respect to x of the functions

$$(1) \int_2^{e^x} \frac{\ln z}{z}\, dz; \quad (2) \int_{x^2}^1 \ln x \, dx.$$

1664*. Find the derivative with respect to x of the function $\int_x^{2x} \ln^2 x \, dx$.

1665. Find the derivative y' with respect to x of the function given implicitly:

$$\int_0^y e^t \, dt + \int_0^x \cos t \, dt = 0.$$

1666. Find the derivative of y with respect to x for the functions given parametrically:

$$(1) \ x = \int_0^t \sin t \, dt, \quad y = \int_0^t \cos t \, dt;$$

$$(2) \ x = \int_1^{t^2} t \ln t \, dt, \quad y = \int_{t^2}^1 t^2 \ln t \, dt.$$

1667. Find the value of the second derivative with respect to z of the function $y = \int\limits_0^{z^2} \frac{dx}{1 + x^3}$ for $z = 1$.

1668. For what values of x does the function

$$I(x) = \int\limits_0^x xe^{-x^2}\, dx$$

have extrema?

1669. Find the curvature of the curve given by the equation

$$y = \int\limits_0^x (1 + t) \ln (1 + t)\, dt.$$

1670. Find the extremal points and points of inflexion of the graph of the function $y = \int\limits_0^x (x^2 - 3x + 2)\, dx$. Draw the graph of this function.

1671. Use the graphs of functions illustrated in Fig. 37 and 38 to find the shape of the graphs of their derivatives.

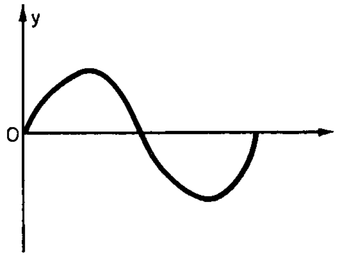

FIG. 37.

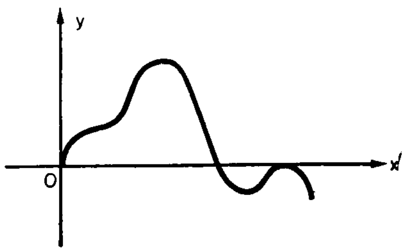

FIG. 38.

Newton–Leibniz Formula

1672. Evaluate the following integrals by a suitable choice of the primitive function for x^k:

$$(1) \int\limits_1^4 \frac{dx}{x^2}; \quad (2) \int\limits_4^1 \frac{dx}{x^3}; \quad (3) \int\limits_1^9 3\sqrt{x}\, dx;$$

(4) $\int\limits_{1}^{2}\left(x+\frac{1}{x}\right)^2 dx$; (5) $\int\limits_{4}^{9}\sqrt{x}\,(1+\sqrt{x})\,dx$;

(6) $\int\limits_{1}^{2}(\sqrt{x}-\sqrt[3]{x})\,dx$; (7) $\int\limits_{a}^{2a}\frac{dx}{\sqrt{2ax}}$; (8) $\int\limits_{1}^{4}\frac{1+t}{\sqrt{t}}\,dt$;

(9) $\int\limits_{a}^{b}\frac{dx}{\sqrt[3]{x^4}}$ $(a>0,\ b>0)$; (10) $\int\limits_{z_0}^{z_1}(\sqrt{z}-1)^2\,dz$.

1673. By using the fundamental tables of derivatives, pick out the primitive and evaluate the integral:

(1) $\int\limits_{0}^{\pi}\sin x\,dx$; (2) $\int\limits_{0}^{\pi}\cos x\,dx$

(interpret geometrically the result obtained),

(3) $\int\limits_{0}^{3}e^x\,dx$; (4) $\int\limits_{0}^{\frac{\pi}{4}}\sec^2 x\,dx$; (5) $\int\limits_{0}^{1}\frac{dx}{1+x^2}$;

(6) $\int\limits_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{dx}{\sqrt{1-x^2}}$.

1674. A function $f(x)$ has equal values at points $x = a$ and $x = b$ and a continuous derivative. What does $\int\limits_{a}^{b} f'(x)\,dx$ equal?

1675. The tangent to the graph of the function $y = f(x)$ at the point with abscissa $x = a$ forms an angle $\frac{\pi}{3}$ with the axis of abscissae, and an angle $\frac{\pi}{4}$ at the point with abscissa $x = b$. Evaluate $\int\limits_{a}^{b} f''(x)\,dx$ and $\int\limits_{a}^{b} f'(x)\,f''(x)\,dx$; $f''(x)$ is assumed continuous.

CHAPTER VI

THE INDEFINITE INTEGRAL. INTEGRAL CALCULUS

1. Elementary Examples of Integration

Find the integrals of problems 1676–1702 by using the basic table of integrals and the theorems on splitting up the integrand and on taking outside a constant factor:

1676. $\int \sqrt{x}\, \mathrm{d}x.$ **1677.** $\int \sqrt[m]{x^n}\, \mathrm{d}x.$

1678. $\int \frac{\mathrm{d}x}{x^2}.$ **1679.** $\int 10^x\, \mathrm{d}x.$

1680. $\int a^x \mathrm{e}^x\, \mathrm{d}x.$ **1681.** $\int \frac{\mathrm{d}x}{2\sqrt{x}}.$

1682. $\int \frac{\mathrm{d}h}{\sqrt{2gh}}.$ **1683.** $\int 3{\cdot}4x^{-0{\cdot}17}\, \mathrm{d}x.$

1684. $\int (1 - 2u)\, \mathrm{d}u.$ **1685.** $\int \sqrt{x} + 1)\,(x - \sqrt{x} + 1)\, \mathrm{d}x.$

1686. $\int \frac{\sqrt{x} - x^3 \mathrm{e}^x + x^2}{x^3}\, \mathrm{d}x.$

1687. $\int (2x^{-1{\cdot}2} + 3x^{-0{\cdot}8} - 5x^{0{\cdot}38})\, \mathrm{d}x.$

1688. $\int \left(\frac{1 - z}{z}\right)^2 \mathrm{d}z.$ **1689.** $\int \frac{(1 - x)^2}{x\sqrt{x}}\, \mathrm{d}x.$

1690. $\int \frac{(1 + \sqrt{x})^3}{\sqrt[3]{x}}\, \mathrm{d}x.$ **1691.** $\int \frac{\sqrt[3]{x^2} - \sqrt[4]{x}}{\sqrt{x}}\, \mathrm{d}x.$

1692. $\int \frac{\mathrm{d}x}{\sqrt{3 - 3x^2}}.$ **1693.** $\int \frac{3 \,.\, 2^x - 2 \,.\, 3^x}{2^x}\, \mathrm{d}x.$

1694. $\int \frac{1 + \cos^2 x}{1 + \cos 2x}\, \mathrm{d}x.$ **1695.** $\int \frac{\cos 2x}{\cos^2 x \sin^2 x}\, \mathrm{d}x.$

1696. $\int \tan^2 x\, dx$.

1697. $\int \cot^2 x\, dx$.

1698. $\int 2\sin^2 \frac{x}{2}\, dx$.

1699. $\int \frac{(1+2x^2)\, dx}{x^2(1+x^2)}$.

1700. $\int \frac{(1+x)^2\, dx}{x(1+x^2)}$.

1701. $\int \frac{dx}{\cos 2x + \sin^2 x}$.

1702. $\int (\arcsin x + \arccos x)\, dx$.

Find the integrals of problems 1703–1780 by using the theorem stating that the form of the integration formula is independent of the nature of the variable of integration:

1703. $\int \sin x\, d(\sin x)$.

1704. $\int \tan^3 x\, d(\tan x)$.

1705. $\int \frac{d(1+x^2)}{\sqrt{1+x^2}}$.

1706. $\int (x+1)^{15}\, dx$.

1707. $\int \frac{dx}{(2x-3)^5}$.

1708. $\int \frac{dx}{(a+bx)^c}$ $(c \neq 1)$.

1709. $\int \sqrt[5]{(8-3x)^6}\, dx$.

1710. $\int \sqrt{8-2x}\, dx$.

1711. $\int \frac{m}{\sqrt[3]{(a+bx)^2}}\, dx$.

1712. $\int 2x\sqrt{x^2+1}\, dx$.

1713. $\int x\sqrt{1-x^2}\, dx$.

1714. $\int x^2 \sqrt[5]{x^3+2}\, dx$.

1715. $\int \frac{x\, dx}{\sqrt{x^2+1}}$.

1716. $\int \frac{x^5\, dx}{\sqrt{4+x^5}}$.

1717. $\int \frac{x^3\, dx}{\sqrt[3]{x^4+1}}$.

1718. $\int \frac{(6x-5)\, dx}{2\sqrt{3x^2-5x+6}}$.

1719. $\int \sin^3 x \cos x\, dx$.

1720. $\int \frac{\sin x\, dx}{\cos^2 x}$.

1721. $\int \frac{\cos x\, dx}{\sqrt[3]{\sin^2 x}}$.

1722. $\int \cos^3 x \sin 2x\, dx$.

1723. $\int \frac{\sqrt{\ln x}}{x}\, dx.$

1724. $\int \frac{(\arctan x)^2\, dx}{1+x^2}.$

1725. $\int \frac{dx}{(\arcsin x)^3 \sqrt{1-x^2}}.$

1726. $\int \frac{dx}{\cos^2 x \sqrt{1+\tan x}}.$

1727. $\int \cos 3x\, d(3x).$

1728. $\int \frac{d\,(1+\ln x)}{\cos^2 (1+\ln x)}.$

1729. $\int \cos 3x\, dx.$

1730. $\int (\cos \alpha - \cos 2x)\, dx.$

1731. $\int \sin (2x-3)\, dx.$

1732. $\int \cos (1-2x)\, dx.$

1733. $\int \left[\cos \left(2x - \frac{\pi}{4}\right)\right]^{-2} dx.$

1734. $\int e^x (\sin e^x)\, dx.$

1735. $\int \frac{d\,(1+x^2)}{1+x^2}.$

1736. $\int \frac{d\,(\arcsin x)}{\arcsin x}$

1737. $\int \frac{(2x-3)\, dx}{x^2-3x+8}.$

1738. $\int \frac{dx}{2x-1}.$

1739. $\int \frac{dx}{cx+m}.$

1740. $\int \frac{x\, dx}{x^2+1}.$

1741. $\int \frac{x^2\, dx}{x^3+1}.$

1742. $\int \frac{e^x\, dx}{e^x+1}.$

1743. $\int \frac{e^{2x}\, dx}{e^{2x}+a^2}.$

1744. $\int \tan x\, dx.$

1745. $\int \cot x\, dx.$

1746. $\int \tan 3x\, dx.$

1747. $\int \cot (2x+1)\, dx;$

1748. $\int \frac{\sin 2x}{1+\cos^2 x}\, dx.$

1749. $\int \frac{dx}{x \ln x}.$

1750. $\int \frac{(\ln x)^m}{x}\, dx.$

1751. $\int e^{\sin x}\, d(\sin x).$

1752. $\int e^{\sin x} \cos x\, dx.$

1753. $\int a^{3x}\, dx.$

1754. $\int a^{-x}\, dx.$

1755. $\int e^{-3x+1}\, dx.$

1756. $\int e^{x^2} x\, dx.$

1757. $\int e^{-x^3} x^2 \, dx$.

1758. $\int \frac{d\left(\frac{x}{3}\right)}{\sqrt{1-\left(\frac{x}{3}\right)^2}}$.

1759. $\int \frac{dx}{\sqrt{1-25x^2}}$.

1760. $\int \frac{dx}{1+9x^2}$.

1761. $\int \frac{dx}{\sqrt{4-x^2}}$.

1762. $\int \frac{dx}{2x^2+9}$.

1763. $\int \frac{dx}{\sqrt{4-9x^2}}$.

1764. $\int \frac{x \, dx}{x^4+1}$.

1765. $\int \frac{x \, dx}{\sqrt{a^2-x^4}}$.

1766. $\int \frac{x^2 \, dx}{x^6+4}$.

1767. $\int \frac{x^3 \, dx}{\sqrt{1-x^8}}$.

1768. $\int \frac{e^x \, dx}{e^{2x}+4}$.

1769. $\int \frac{2^x \, dx}{\sqrt{1-4^x}}$.

1770. $\int \frac{\cos \alpha \, d\alpha}{a^2+\sin^2 \alpha}$.

1771. $\int \frac{e^{2x}-1}{e^x} \, dx$.

1772. $\int (e^x+1)^3 \, dx$.

1773. $\int \frac{1+x}{\sqrt{1-x^2}} \, dx$.

1774. $\int \frac{3x-1}{x^2+9} \, dx$.

1775. $\int \sqrt{\frac{1-x}{1+x}} \, dx$.

1776. $\int \frac{x(1-x^2)}{1+x^4} \, dx$.

1777. $\int \frac{1+x-x^2}{\sqrt{(1-x^2)^3}} \, dx$.

1778. $\int \frac{dx}{(x+\sqrt{x^2-1})^2}$.

1779. $\int \frac{2x-\sqrt{\arcsin x}}{\sqrt{1-x^2}} \, dx$.

1780. $\int \frac{x+(\arccos 3x)^2}{\sqrt{1-9x^2}} \, dx$.

Find the integrals of problems 1781–1790 by dividing out the integrand fraction:

1781. $\int \frac{x}{x+4} \, dx$.

1782. $\int \frac{x}{2x+1} \, dx$.

1783. $\int \frac{Ax}{a+bx} \, dx$.

1784. $\int \frac{3+x}{3-x} \, dx$.

1785. $\int \frac{(2x-1)\,dx}{x-2}$.

1786. $\int \frac{x+2}{2x-1}\,dx$.

1787. $\int \frac{(1+x)^2}{x^2+1}\,dx$.

1788. $\int \frac{x^2-1}{x^2+1}\,dx$.

1789. $\int \frac{x^4}{1-x}\,dx$.

1790. $\int \frac{x^4\,dx}{x^2+1}$.

Find the integrals of problems 1791–1807 by using the method of partial fractions and the method of completing the square:

1791. $\int \frac{dx}{x(x-1)}$.

1792. $\int \frac{dx}{x(x+1)}$.

1793. $\int \frac{dx}{(x+1)(2x-3)}$.

1794. $\int \frac{dx}{(a-x)(b-x)}$.

1795. $\int \frac{x^2+1}{x^2-1}\,dx$.

1796. $\int \frac{dx}{x^2-7x+10}$.

1797. $\int \frac{dx}{x^2+3x-10}$.

1798. $\int \frac{dx}{4x^2-9}$.

1799. $\int \frac{dx}{2-3x^2}$.

1800. $\int \frac{dx}{(x-1)^2+4}$.

1801. $\int \frac{dx}{x^2+2x+3}$.

1802. $\int \frac{dx}{x-x^2-2{\cdot}5}$.

1803. $\int \frac{dx}{4x^2+4x+5}$.

1804. $\int \frac{dx}{\sqrt{1-(2x+3)^2}}$.

1805. $\int \frac{dx}{\sqrt{4x-3-x^2}}$.

1806. $\int \frac{dx}{\sqrt{8+6x-9x^2}}$.

1807. $\int \frac{dx}{\sqrt{2-6x-9x^2}}$.

Find the integrals of problems 1808–1831 by using trigonometric formulae for transforming the integrand:

1808. $\int \cos^2 x\,dx$.

1809. $\int \sin^2 x\,dx$.

1810. $\int \frac{dx}{1 - \cos x}$.

1811. $\int \frac{dx}{1 + \sin x}$.

1812. $\int \frac{1 - \cos x}{1 + \cos x}\, dx$.

1813. $\int \frac{1 + \sin x}{1 - \sin x}\, dx$.

1814. $\int (\tan^2 x + \tan^4 x)\, dx$.

1815. $\int \frac{\cos 2x\, dx}{1 + \sin x \cos x}$.

1816. $\int \cos x \sin 3x\, dx$.

1817. $\int \cos 2x \cos 3x\, dx$.

1818. $\int \sin 2x \sin 5x\, dx$.

1819. $\int \cos x \cos 2x \cos 3x\, dx$.

1820. $\int \frac{dx}{\cos x}$.

1821. $\int \frac{1 - \sin x}{\cos x}\, dx$.

1822. $\int \frac{\sin^3 x}{\cos x}\, dx$.

1823. $\int \frac{\cos^3 x\, dx}{\sin^4 x}$.

1824. $\int \frac{\sin^3 \alpha}{\sqrt{\cos \alpha}}\, d\alpha$.

1825. $\int \frac{dx}{\cos^4 x}$.

1826. $\int \cos^3 x\, dx$.

1827. $\int \tan^4 x\, dx$.

1828. $\int \sin^5 x\, dx$.

1829. $\int \sin^4 x\, dx$.

1830. $\int \tan^3 x\, dx$.

1831. $\int \frac{dx}{\sin^6 x}$.

2. Basic Methods of Integration

Integration by Parts

Find the integrals of problems 1832–1868:

1832. $\int x \sin 2x\, dx$.

1833. $\int x \cos x\, dx$.

1834. $\int x e^{-x}\, dx$.

1835. $\int x 3^x\, dx$.

1836. $\int x^n \ln x\, dx\ (n \neq -1)$.

1837. $\int x \arctan x\, dx$.

1838. $\int \arccos x\, dx$.

1839. $\int \arctan \sqrt{x}\, dx$.

1840. $\int \frac{\arcsin x}{\sqrt{x + 1}}\, dx$.

1841. $\int x \tan^2 x\, dx$.

1842. $\int x \cos^2 x \, dx$.

1843. $\int \frac{\log x}{x^3} \, dx$.

1844. $\frac{x \arctan x}{\sqrt{1 + x^2}} \, dx$.

1845. $\int \arcsin \frac{\sqrt{x}}{\sqrt{1 - x}} \, dx$.

1846. $\int \ln (x^2 + 1) \, dx$.

1847. $\int \frac{x^2 \, dx}{(1 + x^2)^2}$.

1848. $\int \frac{x^3 \, dx}{\sqrt{1 + x^2}}$.

1849. $\int x^2 \ln (1 + x) \, dx$.

1850. $\int x^2 e^{-x} \, dx$.

1851. $\int x^3 e^x \, dx$.

1852. $\int x^2 a^x \, dx$.

1853. $\int x^3 \sin x \, dx$.

1854. $\int x^2 \cos^2 x \, dx$.

1855. $\int \ln^2 x \, dx$.

1856. $\int \frac{\ln^3 x}{x^2} \, dx$.

1857. $\int \frac{\ln^2 x}{\sqrt{x^5}} \, dx$.

1858. $\int (\arcsin x)^2 \, dx$.

1859. $\int (\arctan x)^2 x \, dx$.

1860. $\int e^x \sin x \, dx$.

1861. $\int e^{3x} (\sin 2x - \cos 2x) \, dx$.

1862. $\int e^{ax} \cos nx \, dx$.

1863. $\int \sin \ln x \, dx$.

1864. $\int \cos \ln x \, dx$.

1865*. $\int \frac{x^2 \, dx}{\sqrt{1 - x^2}}$.

1866*. $\int \sqrt{a^2 + x^2} \, dx$.

1867. $\int \frac{x^2 e^x \, dx}{(x + 2)^2}$.

1868. $\int x^2 e^x \sin x \, dx$.

Change of Variable

Find the integrals of problems 1869–1904:

1869. $\int \frac{dx}{1 + \sqrt{x + 1}}$ (substitute $x + 1 = z^2$).

1870. $\int \frac{x^3 \, dx}{\sqrt{x - 1}}$.

1871. $\int \frac{4x + 3}{(x - 2)^3} \, dx$.

1872. $\int \frac{dx}{x\sqrt{x+1}}$.

1873. $\int \frac{x+1}{x\sqrt{x-2}}\,dx$.

1874. $\int \frac{dx}{1+\sqrt{x}}$.

1875. $\int \frac{\sqrt{x}}{x(x+1)}\,dx$.

1876. $\int \frac{\sqrt{x}}{x+1}\,dx$.

1877. $\int \frac{dx}{1+\sqrt[3]{x+1}}$.

1878. $\int \frac{dx}{\sqrt{ax+b}+m}$.

1879. $\int \frac{\sqrt{x}\,dx}{\sqrt{x}-\sqrt[3]{x}}$ (substitute $x = z^6$).

1880. $\int \frac{dx}{\sqrt[3]{x}\left(\sqrt[3]{x}-1\right)}$.

1881. $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}}$.

1882. $\int \frac{\sqrt{x}}{\sqrt[3]{x^2}-\sqrt[4]{x}}\,dx$.

1883. $\int \frac{e^{2x}\,dx}{\sqrt[4]{e^x+1}}$ (substitute $e^x + 1 = z^4$).

1884. $\int \frac{dx}{\sqrt{1+e^x}}$.

1885. $\int \frac{\sqrt{1+\ln x}}{x\ln x}\,dx$.

1886. $\int \sqrt{1+\cos^2 x}\,\sin 2x\cos 2x\,dx$.

1887. $\int \frac{\ln\tan x}{\sin x\cos x}\,dx$.

1888. $\int \frac{x^5\,dx}{\sqrt{a^3-x^3}}$.

1889. $\int \frac{x^5\,dx}{(x^2-4)^2}$.

1890. $\int \frac{dx}{x^2\sqrt{x^2+a^2}}$ (substitute $x = \frac{1}{z}$, or $x = a\tan z$, or $x = a\sinh z$).

1891. $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}}$ (substitute $x = a\sin z$).

1892. $\int \frac{dx}{x\sqrt{x^2 - a^2}}$ (substitute $x = \frac{1}{z}$, or $x = \frac{a}{\cos z}$, or $x = a \cosh z$).

1893. $\int \frac{\sqrt{1 + x^2}}{x^4}\, dx$.

1894. $\int \frac{\sqrt{1 - x^2}}{x^2}\, dx$.

1895. $\int \frac{dx}{\sqrt{(a^2 + x^2)^3}}$.

1896. $\int \frac{\sqrt{(9 - x^2)^3}}{x^6}\, dx$.

1897. $\int \frac{dx}{x^2 \sqrt{x^2 - 9}}$.

1898. $\int \frac{dx}{x\sqrt{1 + x^2}}$.

1899. $\int \frac{dx}{\sqrt{(x^2 - a^2)^3}}$.

1900. $\int x^2 \sqrt{4 - x^2}\, dx$.

1901. $\int \frac{dx}{(x^2 + 4)\sqrt{4x^2 + 1}}$.

1902*. $\int \sqrt{\frac{x - 1}{x + 1}}\, \frac{dx}{x^2}$.

1903*. $\int \frac{dx}{\sqrt{x - x^2}}$.

1904*. $\int \frac{(x + 1)\, dx}{x(1 + xe^x)}$.

Find the integrals of problems 1905–1909 by first changing the variable and then integrating by parts:

1905. $\int e^{\sqrt{x}}\, dx$.

1906. $\sin \sqrt[3]{x}\, dx$.

1907. $\int \frac{\arcsin x}{\sqrt{(1 - x^2)^3}}\, dx$.

1908. $\int \frac{x^2 \arctan x}{1 + x^2}\, dx$.

1909. $\int \frac{\arctan x}{x^2 (1 + x^2)}\, dx$.

Miscellaneous Problems

Find the integrals of problems 1910–2011:

1910. $\int (x + 1)\sqrt{x^2 + 2x}\, dx$.

1911. $\int (1 + e^{3x})^2 e^{3x}\, dx$.

1912. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$.

1913. $\int \frac{\sin x}{e^{\cos x}}\, dx$.

1914. $\int \sqrt{1 - e^x}\, e^x\, dx$.

1915. $\int x \cos x^2\, dx$.

1916. $\int (2 - 3x^{\frac{4}{3}})^{\frac{1}{5}} x^{\frac{1}{3}}\, dx.$

1917. $\int \frac{2x^5 - 3x^2}{1 + 3x^3 - x^6}\, dx.$

1918. $\int \frac{\sqrt{x}\, dx}{1 + x^{\frac{3}{2}}}.$

1919. $\int \frac{dx}{e^x(3 + e^{-x})}.$

1920. $\int \frac{dx}{e^x \sqrt{1 - e^{-2x}}}.$

1921. $\int \frac{2x + 3}{\sqrt{1 + x^2}}\, dx.$

1922. $\int \frac{2x - 1}{\sqrt{9x^2 - 4}}\, dx.$

1923. $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\, dx.$

1924. $\int \frac{dx}{x \sqrt{3 - \ln^2 x}}.$

1925. $\int \frac{\ln x\, dx}{x(1 - \ln^2 x)}.$

1926. $\int \frac{x^2 - x + 1}{\sqrt{(x^2 + 1)^3}}\, dx.$

1927. $\int \frac{(\arctan x)^n}{1 + x^2}\, dx.$

1928. $\int \frac{d\varphi}{\sin^2 \varphi \cos^2 \varphi}.$

1929. $\int \frac{\cos 2x}{\cos^2 x}\, dx.$

1930. $\int \frac{\sin^4 x\, dx}{\cos^6 x}.$

1931. $\int \sqrt{\tan^3 x}\, \sec^4 x\, dx.$

1932. $\int (1 - \tan 3x)^2\, dx.$

1933. $\int \frac{x^3\, dx}{x + 1}.$

1934. $\int \frac{x\, dx}{(x - 1)^3}.$

1935. $\int \frac{x\, dx}{\sqrt{2 + 4x}}.$

1936. $\int \frac{x\, dx}{\sqrt{1 + 2x}}.$

1937. $\int x \sqrt{a + x}\, dx.$

1938. $\int (\sqrt{\sin x} + \cos x)^2\, dx.$

1939. $\int a^{mx} b^{nx}\, dx.$

1940. $\int \frac{dx}{\sqrt{5 - 2x + x^2}}.$

1941. $\int \frac{dx}{\sqrt{9x^2 - 6x + 2}}.$

1942. $\int \frac{dx}{\sqrt{12x - 9x^2 - 2}}.$

1943. $\int \frac{(8x - 11)\, dx}{\sqrt{5 + 2x - x^2}}.$

1944. $\int \frac{(x + 2)\, dx}{x^2 + 2x + 2}.$

1945. $\int \frac{(x - 3)\, dx}{\sqrt{3 - 2x - x^2}}.$

1946. $\int \frac{(3x - 1)\, dx}{4x^2 - 4x + 17}.$

1947. $\int \frac{(3x - 1)\, dx}{\sqrt{x^2 + 2x + 2}}.$

1948. $\int \frac{(x-2)\,dx}{x^2-7x+12}$.

1949. $\int \frac{2x+5}{\sqrt{9x^2+6x+2}}\,dx$.

1950. $\int \frac{3-4x}{2x^2-3x+1}\,dx$.

1951. $\int \frac{(4-3x)\,dx}{5x^2+6x+18}$.

1952. $\int \frac{(2-5x)\,dx}{\sqrt{4x^2+9x+1}}$.

1953. $\int \frac{x\,dx}{\sqrt{3x^2-11x+2}}$.

1954. $\int \frac{\sqrt{x}\,dx}{\sqrt{2x+3}}$.

1955. $\int \sqrt{\frac{a-x}{x-b}}\,dx$.

1956. $\int \arctan x\,dx$.

1957. $\int x \sin x \cos x\,dx$.

1958. $\int x^2 \cos \omega x\,dx$.

1959. $\int e^{2x} x^3\,dx$.

1960. $\int \frac{\ln \cos x\,dx}{\cos^2 x}$.

1961. $\int \frac{\cot x}{\ln \sin x}\,dx$.

1962. $\int \frac{x^7\,dx}{(1+x^4)^2}$.

1963. $\int \frac{\cos^3 3x}{\sin 3x}\,dx$.

1964. $\int \frac{dx}{1-\sin 3x}$.

1965. $\int \frac{\sin 2x\,dx}{4-\cos^2 2x}$.

1966. $\int \frac{dx}{e^x+1}$.

1967. $\int \frac{e^x-1}{e^x+1}\,dx$.

1968. $\int e^{e^x+x}\,dx$.

1969. $\int e^{2x^2+\ln x}\,dx$.

1970. $\int \frac{3+x^3}{\sqrt{2+2x^2}}\,dx$.

1971. $\int \frac{x \arcsin x}{\sqrt{1-x^2}}\,dx$.

1972. $\int \frac{x \cos x}{\sin^3 x}\,dx$.

1973. $\int e^x \sin^2 x\,dx$.

1974. $\int \frac{(1+\tan x)\,dx}{\sin 2x}$.

1975. $\int \frac{1-\tan x}{1+\tan x}\,dx$.

1976. $\int \frac{d\varphi}{\sqrt{3}\cos\varphi+\sin\varphi}$.

1977. $\int \frac{\sin x\,dx}{1+\sin x}$.

1978. $\int \frac{\sin^2 x \cos x}{(1+\sin^2 x)}\,dx$.

1979. $\int \frac{\sqrt{1+\cos x}}{\sin x}\,dx$.

1980. $\int \frac{\ln \ln x}{x}\, dx.$

1981. $\int x^3 e^{x^2}\, dx.$

1982. $\int e^{-x^2} x^5\, dx.$

1983. $\int \frac{x^3\, dx}{\sqrt{1+2x^2}}.$

1984. $\int \frac{x^4\, dx}{\sqrt{(1-x^2)^3}}.$

1985. $\int \frac{\sqrt{(x^2-a^2)^5}}{x}\, dx.$

1986. $\int \frac{dx}{x^4\sqrt{x^2+4}}.$

1987. $\int \frac{\sqrt{x^2-8}}{x^4}\, dx.$

1988. $\int \frac{\sqrt{4+x^2}}{x^6}\, dx.$

1989. $\int \frac{dx}{x^4\sqrt{x^2-3}}.$

1990. $\int \frac{\sqrt{x}\, dx}{\sqrt[4]{x^3}+1}.$

1991. $\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}\, dx.$

1992. $\int \frac{dx}{(2+x)\sqrt{1+x}}.$

1993. $\int \frac{\sqrt[3]{x}\, dx}{x(\sqrt{x}+\sqrt[3]{x})}.$

1994. $\int \frac{\sqrt{x^2+2x}}{x}\, dx.$

1995*. $\int \frac{x^7\, dx}{(1-x^2)^5}.$

1996. $\int \frac{dx}{(ax+b)\sqrt{x}}.$

1997. $\int \frac{\sqrt{1+x^8}}{x^{13}}\, dx.$

1998. $\int \frac{x\, dx}{(1-x^4)^{\frac{3}{2}}}.$

1999. $\int \frac{x^5\, dx}{\sqrt{x^4+4}}.$

2000. $\int \frac{dx}{\sqrt{x}(x-1)}.$

2001. $\int \frac{\sqrt{1-x^3}}{x^2\sqrt{x}}\, dx.$

2002. $\int \frac{x^4\, dx}{(1+x^2)^3}.$

2003. $\int \frac{3x^2-1}{2x\sqrt{x}} \arctan x\, dx.$

2004. $\int \frac{e^x(1+e^x)\, dx}{\sqrt{1-e^{2x}}}.$

2005. $\int \sqrt{e^x-1}\, dx.$

2006*. $\int \frac{\ln(x+1)-\ln x}{x(x+1)}\, dx.$

2007. $\int \frac{dx}{x^6+x^4}.$

2008. $\int \arccos\sqrt{\frac{x}{x+1}}\, dx.$

2009. $\int \ln\left(x + \sqrt{1 + x^2}\right) dx.$ **2010.** $\int \sqrt[3]{\frac{\sin^2 x}{\cos^{14} x}}\, dx.$

2011. $\int \frac{dx}{\cos^3 x \sqrt{\sin 2x}}.$

3. Basic Classes of Integrable Functions

Rational Fractions

Find the integrals of problems 2012–2067:

(1) The denominator has only real distinct roots.

2012. $\int \frac{x\, dx}{(x + 1)(2x + 1)}.$ **2013.** $\int \frac{x\, dx}{2x^2 - 3x - 2}.$

2014. $\int \frac{2x^2 + 41x - 91}{(x - 1)(x + 3)(x - 4)}\, dx.$

2015. $\int \frac{dx}{6x^3 - 7x^2 - 3x}.$ **2016.** $\int \frac{x^5 + x^4 - 8}{x^3 - 4x}\, dx.$

2017. $\int \frac{x^3 - 1}{4x^3 - x}\, dx.$

2018. $\int \frac{32x\, dx}{(2x - 1)(4x^2 - 16x + 15)}.$

2019. $\int \frac{x\, dx}{x^4 - 3x^2 + 2}.$ **2020.** $\int \frac{(2x^2 - 5)\, dx}{x^4 - 5x^2 + 6}.$

2021. $\int \frac{x^6 - 2x^4 + 3x^3 - 9x^2 + 4}{x^5 - 5x^3 + 4x}\, dx.$

(2) The denominator has only real roots; some roots are multiple.

2022. $\int \frac{(x^2 - 3x + 2)\, dx}{x(x^2 + 2x + 1)}.$ **2023.** $\int \left(\frac{x + 2}{x - 1}\right)^2 \frac{dx}{x}.$

2024. $\int \frac{x^2\, dx}{x^3 + 5x^2 + 8x + 4}.$ **2025.** $\int \frac{x^3 + 1}{x^3 - x^2}\, dx.$

2026. $\int \frac{x^3 - 6x^2 + 11x - 5}{(x - 2)^4}\, dx.$

2027. $\int \frac{dx}{x^4 - x^2}$.

2028. $\int \frac{x^2\,dx}{(x+2)^2(x+4)^2}$.

2029. $\int \frac{x^3 - 6x^2 + 9x + 7}{(x-2)^3(x-5)}\,dx$.

2030. $\int \frac{1}{8}\left(\frac{x-1}{x+1}\right)^4 dx$.

2031. $\int \frac{x^5\,dx}{(x-1)^2(x^2-1)}$.

2032. $\int \frac{(x^2 - 2x + 3)\,dx}{(x-1)(x^3 - 4x^2 + 3x)}$.

2033. $\int \frac{(7x^3 - 9)\,dx}{x^4 - 5x^3 + 6x^2}$.

2034. $\int \frac{x^3 - 2x^2 + 4}{x^3(x-2)^2}\,dx$.

2035. $\int \frac{3x^2 + 1}{(x^2 - 1)^3}\,dx$.

(3) The denominator has complex distinct roots.

2036. $\int \frac{dx}{x(x^2+1)}$.

2037. $\int \frac{dx}{1 + x^3}$.

2038. $\int \frac{x\,dx}{x^3 - 1}$.

2039. $\int \frac{(2x^2 - 3x - 3)\,dx}{(x-1)(x^2 - 2x + 5)}$.

2040. $\int \frac{(x^4 + 1)\,dx}{x^3 - x^2 + x - 1}$.

2041. $\int \frac{x^2\,dx}{1 - x^4}$.

2042. $\int \frac{dx}{(x^2+1)(x^2+x)}$.

2043. $\int \frac{dx}{(x+1)^2(x^2+1)}$.

2044. $\int \frac{(3x^2 + x + 3)\,dx}{(x-1)^3(x^2+1)}$.

2045. $\int \frac{x^5 + 2x^3 + 4x + 4}{x^4 + 2x^3 + 2x^2}\,dx$.

2046. $\int \frac{(x^3 - 6)\,dx}{x^4 + 6x^2 + 8}$.

2047*. $\int \frac{dx}{1 + x^4}$.

(4) The denominator has complex multiple roots.

2048. $\int \frac{x^3 + x - 1}{(x^2+2)^2}\,dx$.

2049. $\int \frac{dx}{x(4+x^2)^2(1+x^2)}$.

2050. $\int \frac{(5x^2 - 12)\,dx}{(x^2 - 6x + 13)^2}$.

2051. $\int \frac{(x+1)^4\,dx}{(x^2 + 2x + 2)^3}$.

2052. $\int \frac{dx}{(x^2+9)^3}$.

2053. $\int \frac{2x\,dx}{(1+x)(1+x^2)^2}$.

2054. $\displaystyle\int \frac{dx}{(1+x^2)^4}$. **2055.** $\displaystyle\int \frac{x^9\,dx}{(x^4-1)^2}$.

(5) Ostrogradskii's method.

2056. $\displaystyle\int \frac{x^7+2}{(x^2+x+1)^2}\,dx$. **2057.** $\displaystyle\int \frac{(4x^2-8x)\,dx}{(x-1)^2(x^2+1)^2}$.

2058. $\displaystyle\int \frac{x^2+x+1}{x^5-2x^4+x^3}\,dx$. **2059.** $\displaystyle\int \frac{x^6+x^4-4x^2-2}{x^3(x^2+1)^2}\,dx$.

2060. $\displaystyle\int \frac{(x^2-1)^2\,dx}{(1+x)(1+x^2)^3}$. **2061.** $\displaystyle\int \frac{dx}{x^4(x^3+1)^2}$.

2062. $\displaystyle\int \frac{dx}{(x^2+2x+10)^3}$. **2063.** $\displaystyle\int \frac{(x+2)\,dx}{(x^2+2x+2)^3}$.

2064. $\displaystyle\int \frac{x^5-x^4-26x^2-24x-25}{(x^2+4x+5)^2(x^2+4)^2}\,dx$.

2065. $\displaystyle\int \frac{3x^4+4}{x^2(x^2+1)^3}\,dx$.

2066. $\displaystyle\int \frac{5-3x+6x^2+5x^3-x^4}{x^5-x^4-2x^3+2x^2+x-1}\,dx$.

2067. $\displaystyle\int \frac{9\,dx}{5x^2(3-2x^2)^3}$.

Some Irrational Functions

Find the integrals in problems 2068–2989:

(1) Functions of the form

$$R\left(x, \sqrt[m]{\frac{ax+b}{a_1x+b_1}}, \sqrt[p]{\frac{ax+b}{a_1x+b_1}}, \ldots\right).$$

2068. $\displaystyle\int \frac{dx}{x(\sqrt{x}+\sqrt[5]{x^2})}$. **2069.** $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}+2\sqrt[4]{x}}$.

2070. $\displaystyle\int \frac{x\,dx}{(x+1)^{\frac{1}{2}}+(x+1)^{\frac{1}{3}}}$. **2071.** $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\,\frac{dx}{x}$.

2072. $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx.$ **2073.** $\int \frac{x^2+\sqrt{1+x}}{\sqrt[3]{1+x}}\, dx.$

2074. $\int \sqrt[3]{\frac{1-x}{1+x}}\, \frac{dx}{x}.$ **2075*.** $\int \frac{dx}{\sqrt[4]{(x-1)^3 (x+2)^5}}.$

(2) Binomial differentials $x^m(a+bx^n)^p\, dx$.

2076. $\int \sqrt{x}\,(1+\sqrt[3]{x})^4\, dx.$ **2077.** $\int x^{-1}\,(1+x^{\frac{1}{3}})^{-3}\, dx.$

2078. $\int \frac{dx}{x\sqrt[3]{x^2+1}}.$ **2079.** $\int x^5\sqrt[3]{(1+x^3)^2}\, dx.$

2080. $\int \frac{dx}{\sqrt[3]{1+x^3}}.$ **2081.** $\int \frac{dx}{\sqrt[4]{1+x^4}}.$

2082. $\int \frac{\sqrt{1-x^4}}{x^5}\, dx.$ **2083.** $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\, dx.$

2084. $\int \frac{\sqrt[3]{1+\sqrt{x}}}{x}\, dx.$ **2085.** $\int \frac{dx}{x\sqrt[3]{1+x^5}}.$

2086. $\int \frac{\sqrt[3]{1+x^3}}{x^2}\, dx.$ **2087.** $\int \frac{dx}{x^{11}\sqrt{1+x^4}}.$

2088. $\int \sqrt[3]{x\,(1-x^2)}\, dx.$ **2089.** $\int \sqrt[3]{1+\sqrt[4]{x}}\, dx.$

Trigonometric Functions

Find the integrals in problems 2090–2131:

2090. $\int \sin^3 x \cos^2 x\, dx.$ **2091.** $\int \frac{\sin^3 x}{\cos^4 x}\, dx.$

2092. $\int \frac{dx}{\cos x \sin^3 x}.$ **2093.** $\int \frac{\sin^4 x}{\cos^2 x}\, dx.$

2094. $\int \frac{dx}{\cos^3 x \sin^3 x}.$ **2095.** $\int \frac{dx}{\sin^4 x \cos^4 x}.$

2096. $\int \frac{\sin x\,dx}{(1-\cos x)^2}$.

2097. $\int \frac{\cos x\,dx}{(1-\cos x)^2}$.

2098. $\int \cos^6 x\,dx$.

2099. $\int \cot^4 x\,dx$.

2100. $\int \tan^5 x\,dx$.

2101. $\int \frac{dx}{\tan^8 x}$.

2102. $\int \frac{dx}{\sin^3 x}$.

2103. $\int \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x}\,dx$.

2104. $\int \frac{dx}{(\sin x + \cos x)^2}$.

2105. $\int \frac{dx}{\sin x + \cos x}$.

2106. $\int \frac{dx}{a\cos x + b\sin x}$.

2107. $\int \frac{dx}{\tan x \cos 2x}$

2108. $\int \frac{\cos^2 x\,dx}{\sin x \cos 3x}$.

2109. $\int \frac{dx}{1+\tan x}$.

2110. $\int \frac{dx}{5 - 3\cos x}$.

2111. $\int \frac{dx}{5 + 4\sin x}$.

2112. $\int \frac{2-\sin x}{2+\cos x}\,dx$.

2113. $\int \frac{\sin^2 x\,dx}{1-\tan x}$.

2114. $\int \frac{dx}{4 + \tan x + 4\cot x}$.

2115. $\int \frac{dx}{(\sin x + 2\sec x)^2}$.

2116. $\int \frac{dx}{5 - 4\sin x + 3\cos x}$.

2117. $\int \frac{dx}{4 - 3\cos^2 x + 5\sin^2 x}$.

2118. $\int \frac{dx}{1+\sin^2 x}$.

2119. $\int \frac{dx}{1-\sin^4 x}$.

2120. $\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$.

2121. $\int \frac{dx}{\sin^2 x + \tan^2 x}$.

2122. $\int \frac{\cos x\,dx}{\sin^3 x - \cos^3 x}$.

2123. $\int \sqrt{1+\sin x}\,dx$.

2124. $\int \frac{\sqrt{\tan x}}{\sin x \cos x}\,dx$.

2125*. $\int \frac{\sqrt{\sin^3 2x}}{\sin^5 x}\,dx.$

2126. $\int \frac{dx}{\sqrt[4]{\sin^3 x \cos^5 x}}.$

2127. $\int \frac{dx}{\sqrt{1 - \sin^4 x}}.$

2128. $\int \sqrt{1 + \operatorname{cosec} x}\,dx.$

2129. $\int \frac{(\cos 2x - 3)\,dx}{\cos^4 x \sqrt{4 - \cot^2 x}}.$

2130. $\int \frac{dx}{\sin \frac{x}{2} \sqrt{\cos^3 \frac{x}{2}}}.$

2131. $\int \sqrt{\tan x}\,dx.$

Hyperbolic Functions

Find the integrals of problems 2132–2150:

2132. $\int \cosh x\,dx.$

2133. $\int \sinh x\,dx.$

2134. $\int \frac{dx}{\cosh^2 x}.$

2135. $\int \frac{e^x\,dx}{\cosh x + \sinh x}.$

2136. $\int (\cosh^2 ax + \sinh^2 ax)\,dx.$

2137. $\int \sinh^2 x\,dx.$

2138. $\int \tanh^2 x\,dx.$

2139. $\int \coth^2 x\,dx.$

2140. $\int \sinh^3 x\,dx.$

2141. $\int \cosh^3 x\,dx.$

2142. $\int \tanh^4 x\,dx.$

2143. $\int \sinh^2 x \cosh^3 x\,dx.$

2144. $\int \coth^5 x\,dx.$

2145. $\int \frac{dx}{\sinh x \cosh x}.$

2146. $\int \frac{dx}{\sinh x}.$

2147. $\int \frac{dx}{(1 + \cosh x)^2}.$

2148. $\int \sqrt{\tanh x}\,dx.$

2149. $\int \frac{x\,dx}{\cosh^2 x}.$

2150. $\int \frac{e^{2x}\,dx}{\sinh^4 x}.$

Rational Functions of x and $\sqrt{ax^2 + bx + c}$

Find the integrals of problems 2151–2174:

2151*. $\int \frac{dx}{x\sqrt{x^2 + x + 1}}.$

2152. $\int \frac{dx}{x\sqrt{x^2 + 4x - 4}}.$

2153. $\int \frac{dx}{x\sqrt{x^2+2x-1}}$.

2154. $\int \frac{dx}{x\sqrt{2+x-x^2}}$.

2155. $\int \frac{\sqrt{2x+x^2}}{x^2}\,dx$.

2156. $\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$.

2157. $\int \frac{dx}{(2x-3)\sqrt{4x-x^2}}$.

2158. $\int \sqrt{x^2-2x-1}\,dx$.

2159. $\int \sqrt{3x^2-3x+1}\,dx$.

2160. $\int \sqrt{1-4x-x^2}\,dx$.

2161. $\int \frac{dx}{x-\sqrt{x^2-x+1}}$.

2162. $\int \frac{dx}{x^2(x+\sqrt{1+x^2})}$.

2163. $\int \frac{dx}{1+\sqrt{x^2+2x+2}}$.

2164. $\int \frac{x^2\,dx}{\sqrt{1-2x-x^2}}$.

2165. $\int \frac{(2x^2-3x)\,dx}{\sqrt{x^2-2x+5}}$.

2166. $\int \frac{3x^2-5x}{\sqrt{3-2x-x^2}}\,dx$.

2167. $\int \frac{3x^3\,dx}{\sqrt{x^2+4x+5}}$.

2168. $\int \frac{x^3-x+1}{\sqrt{x^2+2x+2}}\,dx$.

2169. $\int \frac{3x^3-8x+5}{\sqrt{x^2-4x-7}}\,dx$.

2170. $\int \frac{x^4\,dx}{\sqrt{x^2+4x+5}}$.

2171. $\int \frac{dx}{(x^3+3x^2+3x+1)\sqrt{x^2+2x-3}}$.

2172. $\int \frac{\sqrt{1+x^2}}{2+x^2}\,dx$.

2173. $\int \frac{(x-1)\,dx}{x^2\sqrt{2x^2-2x+1}}$.

2174. $\int \frac{(2x+3)\,dx}{(x^2+2x+3)\sqrt{x^2+2x+4}}$.

Various Functions

Find the integrals of problems 2175–2230:

2175. $\int \frac{x^3\,dx}{(x-1)^{12}}$.

2176. $\int \frac{x\,dx}{x-\sqrt{x^2-1}}$.

2177. $\int x\sqrt[3]{a+x}\,dx$.

2178. $\int \frac{dx}{ae^{mx}+be^{-mx}}$.

2179. $\int \frac{x\sqrt{1+x}}{\sqrt{1-x}}\,dx$.

2180. $\int \frac{x^4\,dx}{(x^2-1)(x+2)}$.

2181. $\int \frac{dx}{1 - x^4}$.

2182. $\int \frac{dx}{(x^4 - 1)^2}$.

2183. $\int \frac{\ln (x + 1)\, dx}{\sqrt{x + 1}}$.

2184. $\int (x^2 + 3x + 5) \cos 2x\, dx$.

2185. $\int x^2 \sinh x\, dx$

2186. $\int \arctan (1 + \sqrt{x})\, dx$.

2187. $\int \frac{\arcsin x\, dx}{x^2}$.

2188. $\int e^{\sqrt[3]{x}}\, dx$.

2189. $\int x e^{\sqrt[3]{x}}\, dx$.

2190. $\int (x^3 - 2x^2 + 5)\, e^{3x}\, dx$.

2191. $\int \sin \sqrt{x}\, dx$.

2192. $\int \frac{dx}{x^3 (x - 1)^{\frac{1}{2}}}$.

2193. $\int \frac{dx}{x - \sqrt{x^2 - 1}}$.

2194. $\int \frac{\sqrt{(1 + x^2)^5}}{x^6}\, dx$.

2195. $\int \frac{x^4\, dx}{\sqrt{x^2 + 1}}$.

2196. $\int \sqrt{\frac{1 - \sqrt[3]{x}}{1 + \sqrt[3]{x}}}\, \frac{dx}{x}$.

2197. $\int \frac{dx}{x^3 \sqrt{(1 + x)^3}}$.

2198. $\int \frac{\sqrt{2x + 1}}{x^2}\, dx$.

2199. $\int \frac{x^4\, dx}{x^{15} - 1}$.

2200. $\int \frac{dx}{\sin 2x - 2 \sin x}$.

2201. $\int \frac{dx}{1 + \cos^2 x}$.

2202. $\int \frac{dx}{a^2 - b^2 \cos^2 x}$.

2203. $\int x \ln (1 + x^3)\, dx$.

2204. $\int \frac{(\ln x - 1)\, dx}{\ln^2 x}$.

2205. $\int \frac{x \ln x}{\sqrt{(x^2 - 1)^3}}\, dx$.

2206. $\int x^2 e^x \cos x\, dx$.

2207. $\int x e^{x^2} (x^2 + 1)\, dx$.

2208. $\frac{dx}{\sqrt{\sin^3 x \cos^5 x}}$.

2209. $\int \frac{dx}{\sin^5 x \cos^5 x}$.

2210. $\int \frac{\sin 2x\, dx}{\cos^4 x + \sin^4 x}$.

2211. $\int \frac{dx}{1+\sin x+\cos x}$.

2212. $\int \sqrt{\tan^2 x + 2}\, dx$.

2213. $\int \frac{(x^2-1)\,dx}{x\sqrt{x^4+3x^2+1}}$.

2214. $\int \frac{dx}{(2x-3)\sqrt{4x-x^2}}$.

2215. $\int \frac{xe^x\,dx}{(1+x)^2}$.

2216. $\int \frac{xe^x\,dx}{\sqrt{1+e^x}}$.

2217. $\int \frac{\arctan x\,dx}{x^4}$.

2218. $\int \frac{x\arctan x}{(1+x^2)^2}\,dx$.

2219. $\int \frac{\arctan x}{(1+x)^3}\,dx$.

2220. $\int \frac{dx}{(1-2^x)^4}$.

2221. $\int \frac{(e^{3x}+e^x)\,dx}{e^{4x}-e^{2x}+1}$.

2222. $\int \frac{dx}{\sqrt{1+e^x+e^{2x}}}$.

2223. $\int \frac{\tan x\,dx}{1+\tan x+\tan^2 x}$.

2224. $\int \sin^8 x\,dx$.

2225. $\int \frac{(3+x^2)^2\,x^3\,dx}{(1+x^2)^3}$.

2226. $\int \frac{x^2-8x+7}{(x^2-3x-10)^2}\,dx$.

2227. $\int \frac{dx}{\sin^4 x+\cos^4 x}$.

2228. $\int \frac{(x+\sin x)\,dx}{1+\cos x}$.

2229*. $\int \frac{x^2-1}{x^2+1}\cdot\frac{dx}{\sqrt{1+x^4}}$.

2230. $\int e^{\sin x}\,\frac{x\cos^3 x-\sin x}{\cos^2 x}\,dx$.

CHAPTER VII

METHODS OF EVALUATING DEFINITE INTEGRALS. IMPROPER INTEGRALS

1. Methods of Exact Evaluation of Integrals

Direct Application of the Newton–Leibniz Formula

Evaluate the integrals in problems 2231–2258:

2231. $\int\limits_0^1 \sqrt{1+x}\,dx.$

2232. $\int\limits_{-2}^{-1} \frac{dx}{(11+5x)^3}.$

2233. $\int\limits_2^{-13} \frac{dx}{\sqrt[5]{(3-x)^4}}.$

2234. $\int\limits_4^9 \frac{y-1}{\sqrt{y}+1}\,dy.$

2235. $\int\limits_0^{\frac{T}{2}} \sin\left(\frac{2\pi t}{T}-\varphi_0\right)dt.$

2236. $\int\limits_0^{16} \frac{dx}{\sqrt{x+9}-\sqrt{x}}.$

2237. $\int\limits_0^1 (e^x-1)^4 e^x\,dx.$

2238. $\int\limits_0^{2a} \frac{3\,dx}{2b-x}$ $(b>a>0).$

2239. $\int\limits_0^1 \frac{x\,dx}{(x^2+1)^2}.$

2240. $\int\limits_1^e \frac{dx}{x\sqrt{1-(\ln x)^2}}.$

2241. $\int\limits_1^e \frac{1+\log x}{x}\,dx.$

2242. $\int\limits_1^2 \frac{e^{\frac{1}{x}}\,dx}{x^2}.$

2243. $\int\limits_0^{\sqrt[n]{\frac{a}{2}}} \frac{x^{n-1}\,dx}{\sqrt{a^2-x^{2n}}}.$

2244. $\int\limits_1^{e^3} \frac{dx}{x\sqrt{1+\ln x}}.$

2245. $\displaystyle\int\limits_{0\cdot5}^{\frac{\sqrt{3}}{2}} \frac{x^3\,dx}{\left(\frac{5}{8}-x^4\right)\sqrt{\frac{5}{8}-x^4}}$.

2246. $\displaystyle\int\limits_{0}^{\frac{a}{2}} \frac{a\,dx}{(x-a)(x-2a)}$.

2247. $\displaystyle\int\limits_{2}^{3} \frac{dx}{2x^2+3x-2}$.

2248. $\displaystyle\int\limits_{0}^{1} \frac{dx}{x^2+4x+5}$.

2249. $\displaystyle\int\limits_{1}^{2} \frac{dx}{x+x^3}$.

2250. $\displaystyle\int\limits_{-0\cdot5}^{1} \frac{dx}{\sqrt{8+2x-x^2}}$.

2251. $\displaystyle\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$.

2252. $\displaystyle\int\limits_{0}^{\frac{\pi}{2}} \cos^5 x \sin 2x\,dx$.

2253. $\displaystyle\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x-\cos^3 x}\,dx$.

2254. $\displaystyle\int\limits_{0}^{\frac{\pi}{\omega}} \sin^2(\omega x+\varphi_0)\,dx$.

2255. $\displaystyle\int\limits_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \frac{\cos^3 x\,dx}{\sqrt[3]{\sin x}}$.

2256. $\displaystyle\int\limits_{\alpha}^{\frac{\pi}{4}} \cot^4\varphi\,d\varphi$.

2257. $\displaystyle\int\limits_{\frac{1}{\pi}}^{\frac{2}{\pi}} \frac{\sin\frac{1}{x}}{x^2}\,dx$.

2258. $\displaystyle\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \sin\left(2t-\frac{\pi}{4}\right)dt$.

Definite Integration by Parts

Find the integrals in problems 2259–2268:

2259. $\displaystyle\int\limits_{0}^{1} xe^{-x}\,dx$.

2260. $\displaystyle\int\limits_{0}^{\frac{\pi}{2}} x\cos x\,dx$.

2261. $\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x\,dx}{\sin^2 x}$.

2262. $\int\limits_{0}^{\pi} x^3 \sin x\,dx$.

2263. $\int\limits_{1}^{2} x \log_2 x\,dx$.

2264. $\int\limits_{0}^{e-1} \ln(x+1)\,dx$.

2265. $\int\limits_{0}^{a\sqrt{7}} \frac{x^3\,dx}{\sqrt[3]{a^2+x^2}}\,dx$.

2266. $\int\limits_{0}^{a} \sqrt{a^2-x^2}\,dx$.

2267. $\int\limits_{0}^{\frac{\pi}{2}} e^{2x} \cos x\,dx$.

2268. $\int\limits_{1}^{e} \ln^3 x\,dx$.

2269. Form recurrence formulae for $\int\limits_{0}^{\frac{\pi}{2}} \cos^n x\,dx$ and $\int\limits_{0}^{\frac{\pi}{2}} \sin^n x\,dx$ (n is a positive integer or zero; see *Course*, sec. 106) and evaluate the integrals:

(a) $\int\limits_{0}^{\frac{\pi}{2}} \sin^5 x\,dx$; (b) $\int\limits_{0}^{\frac{\pi}{2}} \cos^8 x\,dx$; (c) $\int\limits_{0}^{\frac{\pi}{2}} \sin^{11} x\,dx$.

2270. Form a recurrence formula for the integral $\int\limits_{0}^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx$ (m and n are positive integers or zero; investigate the particular cases of even and odd values of m and n).

2271. Form a recurrence formula for $\int\limits_{-1}^{0} x^n e^x\,dx$ (n is a positive integer or zero).

2272. Obtain the recurrence formula

$$\int \frac{dx}{(1+x^2)^n} = \frac{x}{2(n-1)(1+x^2)^{n-1}} + \frac{2n-3}{2(n-1)} \int \frac{dx}{(1+x^2)^{n-1}}$$

(n is a positive integer or zero) and evaluate with its aid the integral

$$\int_0^1 \frac{dx}{(1+x^2)^4}.$$

2273. Prove that, if $J_m = \int_1^e \ln^m x\, dx$ we have $J_m = e - mJ_{m-1}$ (m is a positive integer or zero).

2274*. Find $\int_0^1 x^p(1-x)^q\, dx$ (p and q are positive integers).

Change of Variable in a Definite Integral

Evaluate the integrals in problems 2275–2295:

2275. $\int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1}\, dx.$

2276. $\int_0^1 \frac{\sqrt{x}\, dx}{1+x}.$

2277. $\int_3^8 \frac{x\, dx}{\sqrt{1+x}}.$

2278. $\int_0^1 \frac{x\, dx}{1+\sqrt{x}}.$

2279. $\int_0^1 \frac{\sqrt{e^x}\, dx}{\sqrt{e^x+e^{-x}}}.$

2280. $\int_3^{29} \frac{\sqrt[3]{(x-2)^2}\, dx}{3+\sqrt[3]{(x-2)^2}}.$

2281*. $\int_0^\pi \sin^6 \frac{x}{2}\, dx.$

2282*. $\int_0^{\frac{\pi}{4}} \cos^7 2x\, dx.$

2283. $\int_0^1 \frac{x^2\, dx}{(1+x^2)^3}.$

2284. $\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}\, dx.$

2285. $\int_{\frac{\sqrt{2}}{2}}^1 \frac{\sqrt{1-x^2}}{x^6}\, dx.$

2286. $\int_1^2 \frac{\sqrt{x^2-1}}{x}\, dx.$

2287. $\int\limits_1^2 \frac{dx}{x^5\sqrt{x^2-1}}$.

2288. $\int\limits_0^1 \sqrt{(1-x^2)^3}\,dx$.

2289. $\int\limits_0^1 x^2\sqrt{1-x^2}\,dx$.

2290. $\int\limits_0^{-\ln 2} \sqrt{1-e^{2x}}\,dx$.

2291. $\int\limits_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$.

2292. $\int\limits_0^3 \frac{dx}{(x^2+3)^{\frac{5}{2}}}$.

2293. $\int\limits_{2\cdot 5}^5 \frac{(\sqrt{25-x^2})^3}{x^4}\,dx$.

2294. $\int\limits_0^{\frac{1}{\sqrt{3}}} \frac{dx}{(2x^2+1)\sqrt{x^2+1}}$.

2295. $\int\limits_{\sqrt{\frac{8}{3}}}^{2\sqrt{2}} \frac{dx}{x\sqrt{(x^2-2)^5}}$.

Miscellaneous Problems

2296. Find the mean value of the function $y=\sqrt{x}+\frac{1}{\sqrt{x}}$ in the interval [1, 4].

2297. Find the mean value of the function

$$f(x)=\frac{1}{x^2+x}$$

in the interval [1, 1·5].

2298. Find the mean value of the functions $f(x)=\sin x$ and $f(x)=\sin^2 x$ in the interval $[0, \pi]$.

2299. Find the mean value of the function

$$f(x)=\frac{1}{e^x+1}$$

in the interval [0, 2].

2300. For what a is the mean value of the function $y = \ln x$ in the interval $[1, a]$ equal to the mean rate of change of the function in this interval?

Evaluate the integrals of problems 2301–2317:

2301. $\displaystyle\int_1^2 \frac{dx}{x + x^3}$.

2302. $\displaystyle\int_0^{\sqrt[5]{2}} \frac{x^9\,dx}{(1 + x^5)^3}$.

2303. $\displaystyle\int_0^{\frac{1}{2}} \frac{x^3\,dx}{x^2 - 3x + 2}$.

2304. $\displaystyle\int_0^{\sqrt[4]{2}} \frac{x^{15}\,dx}{(1 + x^8)^{\frac{2}{5}}}$.

2305. $\displaystyle\int_0^2 \frac{dx}{\sqrt{x + 1} + \sqrt{(x + 1)^3}}$.

2306. $\displaystyle\int_{-a}^{+a} \frac{x^2\,dx}{\sqrt{a^2 + x^2}}$.

2307. $\displaystyle\int_0^1 \sqrt{2x + x^2}\,dx$.

2308. $\displaystyle\int_0^{\sqrt{3}} x^5 \sqrt{1 + x^2}\,dx$.

2309. $\displaystyle\int_0^{\ln 5} \frac{e^x \sqrt{e^x - 1}}{e^x + 3}\,dx$.

2310. $\displaystyle\int_1^3 \frac{dx}{x\sqrt{x^2 + 5x + 1}}$.

2311. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{x \sin x}{\cos^3 x}\,dx$.

2312. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x + 3}$.

2313. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \frac{1}{6}\sin^2 x}$.

2314. $\displaystyle\int_0^1 (\arcsin x)^4\,dx$.

2315. $\displaystyle\int_1^{16} \arctan\sqrt{\sqrt{x} - 1}\,dx$.

2316. $\displaystyle\int_0^1 \frac{(3x + 2)\,dx}{(x^2 + 4x + 1)^{\frac{5}{2}}}$.

2317. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$.

2318. Prove that $\int\limits_0^{\frac{\pi}{2}} \frac{|ab|\,dx}{a^2\cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2}$, where a and b are any real non-zero numbers.

2319. Solve the equation $\int\limits_{\sqrt{2}}^{x} \frac{dx}{x\sqrt{x^2-1}} = \frac{\pi}{12}$.

2320. Solve the equation $\int\limits_{\ln 2}^{x} \frac{dx}{\sqrt{e^x - 1}} = \frac{\pi}{6}$.

2321. Having verified the inequalities $\frac{x}{e} > \ln x > 1$ for $x > e$, prove that the integral $\int\limits_3^4 \frac{dx}{\sqrt[3]{\ln x}}$ is less than unity but greater than 0·92.

2322*. Prove that

$$\frac{\pi}{6} \approx 0{\cdot}523 < \int\limits_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi}{4\sqrt{2}} \approx 0{\cdot}555.$$

2323*. Prove that

$$0{\cdot}5 < \int\limits_0^{0\cdot 5} \frac{dx}{\sqrt{1 - x^{2n}}} \leqq \frac{\pi}{6} \approx 0{\cdot}523 \quad (n \geqq 1).$$

2324. By using the inequality $\sin x > x - \frac{x^3}{6}$, valid for $x > 0$, and Bunyakovskii's inequality, estimate the integral $\int\limits_0^{\frac{\pi}{2}} \sqrt{x \sin x}\, dx$.

2325*. Prove that

$$0{\cdot}78 < \int\limits_0^1 \frac{dx}{\sqrt{1 + x^4}} < 0{\cdot}93.$$

2326. Find the maximum and minimum of the function $J(x) = \int\limits_0^x \frac{2x+1}{x^2-2x+2}\,dx$ in the interval $[-1, 1]$.

2327. Find the extremal points and points of inflexion of the graph of the function $y = \int\limits_0^x (x-1)(x-2)^2\,dx$.

In problems 2328–2331, prove without evaluating the integrals that the equalities hold (see *Course*, sec. 107):

2328. $\int\limits_{-\frac{\pi}{8}}^{\frac{\pi}{8}} x^{10}\sin^9 x\,dx = 0.$

2329. $\int\limits_{-1}^{1} \frac{x^7-3x^5+7x^3-x}{\cos^2 x}\,dx = 0.$

2330. $\int\limits_{-1}^{1} e^{\cos x}\,dx = 2\int\limits_0^1 e^{\cos x}\,dx.$

2331. $\int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \ln\frac{1+x}{1-x}\,dx = 0.$

2332*. (a) Prove that, if $f(t)$ is an odd function, $\int\limits_a^x f(t)\,dt$ is an even function, i.e. that $\int\limits_a^x f(t)\,dt = \int\limits_a^{-x} f(t)\,dt$.

(b) Will $\int\limits_a^x f(t)\,dt$ be an odd function, if $f(t)$ is an even function?

2333*. Prove the equality

$$\int\limits_x^1 \frac{dt}{1+t^2} = \int\limits_1^{\frac{1}{x}} \frac{dt}{1+t^2} \quad (x > 0).$$

2334. Prove the identity

$$\int_{\frac{1}{e}}^{\tan x} \frac{t\,dt}{1+t^2} + \int_{\frac{1}{e}}^{\cot x} \frac{dt}{t(1+t^2)} = 1.$$

2335. Prove the identity

$$\int_0^{\sin^2 x} \arcsin\sqrt{t}\,dt + \int_0^{\cos^2 x} \arccos\sqrt{t}\,dt = \frac{\pi}{4}.$$

2336. Prove the equality

$$\int_0^1 x^m(1-x)^n\,dx = \int_0^1 x^n(1-x)^m\,dx.$$

2337. Prove the equality

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$$

2338. Show that

$$\int_0^{\frac{\pi}{2}} f(\cos x)\,dx = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx$$

(see *Course,* sec. 107).

Apply the result obtained to evaluation of the integrals

$\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ and $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$.

2339*. Prove that

$$\int_0^{\pi} xf(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx =$$

$$= \frac{\pi}{2} \times 2\int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \pi\int_0^{\frac{\pi}{2}} f(\sin x)\,dx.$$

Apply the result obtained to evaluation of the integral

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx.$$

2340*. Show that, if $f(x)$ is a periodic function of period T, $\int_a^{a+T} f(x)\, dx$ is independent of a.

2341*. We know in regard to the function $f(x)$ that it is odd in the interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$ and has the period T. Show that $\int_a^x f(x)\, dx$ is also a periodic function with the same period.

2342. Evaluate $\int_0^1 (1 - x^2)^n\, dx$, where n is a positive integer, by two methods: by using Newton's formula to expand the integrand as a series, and by substituting $x = \sin \varphi$. By comparing the results, obtain the following summation formula (C_n^k are binomial coefficients):

$$C_n^0 - \frac{C_n^1}{3} + \frac{C_n^2}{5} - \frac{C_n^3}{7} + \ldots + \frac{(-1)^n C_n^n}{2n+1} = \frac{2 \,.\, 4 \,.\, 6 \ldots 2n}{1 \,.\, 3 \,.\, 5 \ldots (2n+1)}.$$

2343. The integral $\int_0^{2\pi} \frac{dx}{5 - 3\cos x}$ is easily obtained with the aid of the substitution $\tan \frac{x}{2} = z$. We have:

$$\int_0^{2\pi} \frac{dx}{5 - 3\cos x} = \int_0^0 \frac{2dz}{(1 + z^2)\left(5 - 3\,\frac{1 - z^2}{1 + z^2}\right)} = 0.$$

But on the other hand, $-3 < -3\cos x < +3$, so that $2 < 5 - 3\cos x < 8$ and $\frac{1}{2} > \frac{1}{5 - 3\cos x} > \frac{1}{8}$. Hence

$$\int_0^{2\pi} \frac{1}{2}\, dx > \int_0^{2\pi} \frac{dx}{5 - 3\cos x} > \int_0^{2\pi} \frac{1}{8}\, dx,$$

i.e. $\displaystyle\int_0^{2\pi} \frac{dx}{5 - 3\cos x} > \frac{\pi}{4}$. Find the mistake in the argument.

2344*. Let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x\, dx$ ($n > 1$ and an integer). Prove that $I_n + I_{n-2} = \dfrac{1}{n-1}$. Show that $\dfrac{1}{2n+2} < I_n < \dfrac{1}{2n-2}$.

2345*. Prove the equality

$$\int_0^x e^{zx} e^{-z^2}\, dz = e^{\frac{x^2}{4}} \int_0^x e^{-\frac{z^2}{4}}\, dz.$$

2346*. Prove that

$$\lim_{\omega\to\infty} \frac{e^{k\omega^2 x^2}}{\int_a^b e^{k\omega^2 x^2}\, dx} = \begin{cases} 0, \text{ if } x < b^a, \\ \infty, \text{ if } x = b, \end{cases} \quad (\omega > 0,\ k > 0,\ b > 0).$$

2. Approximation Methods

Carry out the working to an accuracy of 0·001 in problems 2347–2349.

2347. The area of the quadrant of a circle of unit radius is equal to $\dfrac{\pi}{4}$. On the other hand, on taking the unit circle with centre at the origin, the equation of which is $x^2 + y^2 = 1$, and using integration to evaluate the area of a quadrant of this circle, we get:

$$\frac{\pi}{4} = \int_0^1 \sqrt{1 - x^2}\, dx, \quad \text{i.e.} \quad \pi = 4\int_0^1 \sqrt{1 - x^2}\, dx.$$

By using the rectangle, trapezium and Simpson rules, evaluate approximately the number π, the interval of integration $[0, 1]$ being divided into 10 parts. Compare the results with each other and with the tabulated value of the number π.

2348. Knowing that $\int\limits_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$, evaluate approximately the number π. Split the interval of integration into 10 parts and compare the results obtained by the various methods with each other and with the results of the previous problem.

2349. Evaluate $\ln 10 = \int\limits_1^{10} \frac{dx}{x}$, by using Simpson's rule with $n = 10$. Find the modulus of transition from natural to common logarithms. Compare with the tabulated value.

Use Simpson's rule to evaluate approximately the integrals given in problems 2350–2355; these integrals cannot be found in a finite form with the aid of elementary functions. The number (n) of sub-intervals is quoted in brackets.

2350. $\int\limits_0^1 \sqrt{1-x^3}\, dx \quad (n = 10)$. **2351.** $\int\limits_0^1 \sqrt{1+x^4}\, dx \quad (n = 10)$.

2352. $\int\limits_2^5 \frac{dx}{\ln x} \quad (n = 6)$. **2353.** $\int\limits_0^{\frac{\pi}{3}} \sqrt{\cos\varphi}\, d\varphi \quad (n = 10)$.

2354. $\int\limits_0^{\frac{\pi}{2}} \sqrt{1 - 0{\cdot}1 \sin^2\varphi}\, d\varphi \quad (n = 6)$.

2355. $\int\limits_0^{\frac{\pi}{3}} \frac{\sin x}{x}\, dx \quad (n = 10)$.

2356. Evaluate from Simpson's formula the integral $\int\limits_{1{\cdot}05}^{1{\cdot}35} f(x)\, dx$, by using the following table of values of $f(x)$:

x	1·05	1·10	1·15	1·20	1·25	1·30	1·35
$f(x)$	2·36	2·50	2·74	3·04	3·46	3·98	4·60

2357. A straight line touches a river bank at points A and B. To measure the area lying between the river and AB, 11 perpendiculars 5 m apart are drawn to AB from points along the river (hence AB has a length of 60 m). The lengths of these perpendiculars turn out to be 3·28; 4·02; 4·64; 5·26; 4·98; 3·62; 3·82; 4·68; 5·26; 3·82; 3·24 m. Work out approximately the area in question.

2358. Work out the cross-section at the widest part of a ship (middle rib section) from the following data (Fig. 39):

$$AA_1 = A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5 = A_5A_6 = A_6A_7 = \\ = 0{\cdot}4 \text{ m},$$

$$AB = 3 \text{ m}, A_1B_1 = 2{\cdot}92 \text{ m}, A_2B_2 = 2{\cdot}75 \text{ m}, A_3B_3 = 2{\cdot}52 \text{ m} \\ A_4B_4 = 2{\cdot}30 \text{ m}, A_5B_5 = 1{\cdot}84 \text{ m}, A_6B_6 = 0{\cdot}92 \text{ m}.$$

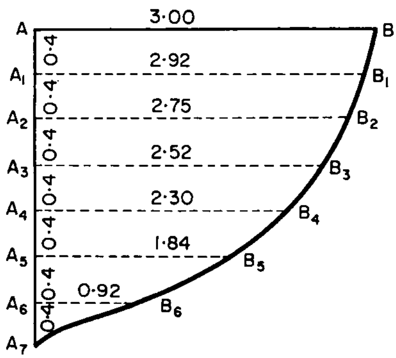

FIG. 39.

2359. The work done by the steam in the cylinder of a steam engine is worked out by finding the area of the indicator diagram, representing graphically the relationship between the steam pressure in the cylinder and the movement of the piston. The indicator diagram of a steam engine is

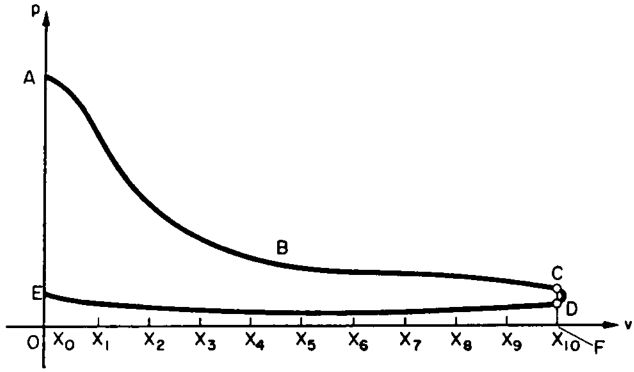

FIG. 40.

illustrated in Fig. 40. The ordinates of the points of curves ABC and ED, corresponding to abscissae $x_0, x_1\, x_2, \ldots, x_{10}$, are given by the following tables:

Abscissa	x_0	x_1	x_2	x_3	x_4	x_5
Ordinate of curve ABC	60·6	53·0	32·2	24·4	19·9	17·0
Ordinate of curve ED	5·8	1·2	0·6	0·6	0·7	0·8

Abscissa	x_6	x_7	x_8	x_9	x_{10}
Ordinate of curve ABC	15·0	13·3	12·0	11·0	6·2
Ordinate of curve ED	0·9	1·0	1·3	1·8	5·7

Evaluate the area $ABCDE$ with the aid of Simpson's formula. The ordinates are given in millimetres. Length $OF = 88{\cdot}7$ mm.

In problems 2360–2363 it is necessary to use methods of approximate solution of equations for finding the limits of integration.

2360. Find the area of the figure bounded by the arcs of parabolas $y = x^3 - 7$ and $y = -2x^2 + 3x$ and the axis of ordinates.

2361. Find the area of the figure bounded by the parabola $y = x^3$ and the straight line $y = 7(x + 1)$.

2362. Find the area of the figure bounded by the parabola $y = 16 - x^2$ and the semi-cubical parabola $y = -\sqrt[3]{x^2}$.

2363. Find the area of the figure bounded by the curves $y = 4 - x^4$ and $y = \sqrt[3]{x}$.

2364. A steam engine indicator diagram (simplified) is shown in Fig. 41. Starting from the dimensions quoted in the figure (in mm), evaluate the area $ABCDO$, if it is known that the equation of curve BC is: $pv^{\gamma} = \text{const}$ (curve BC is called an adiabat), $\gamma = 1{\cdot}3$, AB being a straight line parallel to the Ov axis.

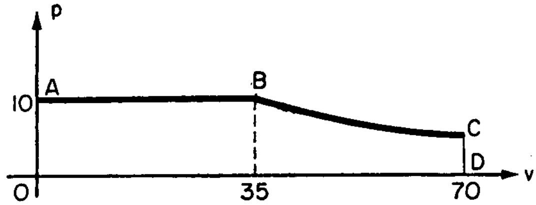

FIG. 41.

2365. The indicator diagram of a Diesel engine is shown in Fig. 42. Segment AB corresponds to the mixture combustion process, adiabat BC to expansion, segment CD to exhaust and adiabat DA to compression. The equation of adiabat BC is $pv^{1{\cdot}3} = \text{const}$, and of adiabat $AD : pv^{1{\cdot}35} = \text{const}$. Starting from the dimensions given in the figure (in mm), find the area $ABCD$.

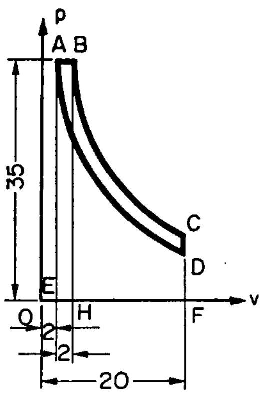

FIG. 42.

3. Improper Integrals

Integrals with Infinite Limits

Evaluate the improper integrals in problems 2366–2385 (or establish their divergence):

2366. $\int\limits_1^\infty \frac{dx}{x^4}$.

2367. $\int\limits_1^\infty \frac{dx}{\sqrt{x}}$.

2368. $\int\limits_0^\infty e^{-ax}\,dx \quad (a > 0)$.

2369. $\int\limits_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$.

2370. $\int\limits_{-\infty}^\infty \frac{dx}{x^2+2x+2}$.

2371. $\int\limits_2^\infty \frac{\ln x}{x}\,dx$.

2372. $\int\limits_1^\infty \frac{dx}{x^2(x+1)}$.

2373. $\int\limits_0^\infty \frac{x}{(1+x)^3}\,dx$.

2374. $\int\limits_1^\infty \frac{dx}{x\sqrt{x^2-1}}$.

2375. $\int\limits_{a^2}^\infty \frac{dx}{x\sqrt{1+x^2}}$.

2376. $\int\limits_0^\infty x\,e^{-x^2}\,dx$.

2377. $\int\limits_0^\infty x^3\,e^{-x^2}\,dx$.

2378. $\int\limits_0^\infty x \sin x\,dx$.

2379. $\int\limits_0^\infty e^{-\sqrt{x}}\,dx$.

2380. $\int\limits_0^\infty e^{-x}\sin x\,dx$.

2381. $\int\limits_0^\infty e^{-ax}\cos bx\,dx$.

2382. $\int\limits_1^\infty \frac{\arctan x}{x^2}\,dx$.

2383. $\int\limits_0^\infty \frac{dx}{1+x^3}$.

2384. $\int\limits_{-\infty}^\infty \frac{dx}{(x^2+1)^2}$.

2385. $\int\limits_1^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx$.

Investigate the convergence of the integrals of problems 2386–2393:

2386. $\int\limits_0^{\infty} \frac{x}{x^3+1}\,dx.$

2387. $\int\limits_1^{\infty} \frac{x^3+1}{x^4}\,dx.$

2388. $\int\limits_0^{\infty} \frac{x^{13}}{(x^5+x^3+1)^3}\,dx.$

2389. $\int\limits_1^{\infty} \frac{\ln(x^2+1)}{x}\,dx.$

2390. $\int\limits_0^{\infty} \sqrt{x}\,e^{-x}\,dx.$

2391. $\int\limits_0^{\infty} \frac{x \arctan x}{\sqrt[3]{1+x^4}}\,dx.$

2392. $\int\limits_{e^2}^{\infty} \frac{dx}{x \ln \ln x}.$

2393. $\int\limits_{e}^{\infty} \frac{dx}{x(\ln x)^{\frac{3}{2}}}.$

Integrals of Functions with Infinite Discontinuities

Evaluate the improper integrals of problems 2394–2411 (or establish their divergence):

2394. $\int\limits_0^{1} \frac{dx}{\sqrt{1-x^2}}.$

2395. $\int\limits_0^{2} \frac{dx}{x^2-4x+3}.$

2396. $\int\limits_1^{2} \frac{x\,dx}{\sqrt{x-1}}.$

2397. $\int\limits_0^{1} x \ln x\,dx.$

2398. $\int\limits_0^{\frac{1}{e}} \frac{dx}{x \ln^2 x}.$

2399. $\int\limits_1^{2} \frac{dx}{x \ln x}.$

2400. $\int\limits_1^{e} \frac{dx}{x\sqrt{\ln x}}.$

2401. $\int\limits_a^{b} \frac{dx}{\sqrt{(x-a)(b-x)}}$ $(a<b).$

2402. $\int\limits_a^{b} \frac{x\,dx}{\sqrt{(x-a)(b-x)}}$ $(a<b).$

2403. $\int\limits_3^5 \frac{x^2\,dx}{\sqrt{(x-3)(5-x)}}$.

2404. $\int\limits_0^1 \frac{dx}{1-x^2+2\sqrt{1-x^2}}$.

2405. $\int\limits_{-1}^1 \frac{dx}{(2-x)\sqrt{1-x^2}}$.

2406. $\int\limits_{-1}^1 \frac{3x^2+2}{\sqrt[3]{x^2}}\,dx$.

2407. $\int\limits_{-1}^1 \frac{x+1}{\sqrt[5]{x^3}}\,dx$.

2408. $\int\limits_{-1}^1 \frac{x-1}{\sqrt[3]{x^5}}\,dx$.

2409. $\int\limits_{-1}^1 \frac{\ln(2+\sqrt[3]{x})}{\sqrt[3]{x}}\,dx$.

2410. $\int\limits_{-1}^0 \frac{e^{\frac{1}{x}}}{x^3}\,dx$.

2411. $\int\limits_0^1 \frac{e^{\frac{1}{x}}}{x^3}\,dx$.

Investigate the convergence of the integrals of problems 2412–2417:

2412. $\int\limits_0^1 \frac{\sqrt{x}}{\sqrt{1-x^4}}\,dx$.

2413. $\int\limits_0^1 \frac{x^2\,dx}{\sqrt[3]{(1-x^2)^5}}$.

2414. $\int\limits_0^1 \frac{dx}{e^{\sqrt{x}}-1}$.

2415. $\int\limits_0^1 \frac{\sqrt{x}\,dx}{e^{\sin x}-1}$.

2416. $\int\limits_0^1 \frac{dx}{e^x-\cos x}$.

2417. $\int\limits_0^{\frac{\pi}{2}} \frac{\ln\sin x}{\sqrt{x}}\,dx$.

Various Problems

2418. Function $f(x)$ is continuous in the interval $[a, \infty)$ and $f(x) \to A \neq 0$ as $x \to \infty$. Can $\int\limits_a^\infty f(x)\,dx$ be convergent?

2419. For what values of k will $\int\limits_1^\infty x^k \frac{x+\sin x}{x-\sin x}\,dx$ be convergent?

2420. For what values of k will the following integrals be convergent:

$$\int_2^{\infty} \frac{dx}{x^k \ln x} \quad \text{and} \quad \int_2^{\infty} \frac{dx}{x (\ln x)^k} \ ?$$

2421. For what values of k is $\int_a^b \frac{dx}{(b-x)^k}$ $(b > a)$ convergent?

2422. Can a k be found such that $\int_0^{\infty} x^k \, dx$ is convergent?

2423. For what values of k and t is $\int_0^{\infty} \frac{x^k}{1+x^t} \, dx$ convergent?

2424. For what values of m is $\int_0^{\frac{\pi}{2}} \frac{1-\cos x}{x^m} \, dx$ convergent?

2425. For what values of k is $\int_0^{\pi} \frac{dx}{\sin^k x}$ convergent?

Evaluate the improper integrals in problems 2426–2435:

2426. $\int_1^{\infty} \frac{dx}{x\sqrt{x-1}}$.

2427*. $\int_{-1}^{1} \ln \frac{1+x}{1-x} \frac{x^3 \, dx}{\sqrt{1-x^2}}$.

2428. $\int_0^{\infty} \frac{\operatorname{arc}\tan (x-1) \, dx}{\sqrt[3]{(x-1)^4}}$.

2429. $\int_0^{\infty} \frac{dx}{(a^2+x^2)^n}$ (n is a positive integer).

2430. $\int_0^{\infty} x^n e^{-x} \, dx$ (n is a positive integer).

2431. $\int_0^{\infty} x^{2n+1} e^{-x^2} \, dx$ (n is a positive integer).

2432. $\int\limits_0^1 (\ln x)^n \, dx$ (n is a positive integer).

2433*. $\int\limits_0^1 \frac{x^m \, dx}{\sqrt{1 - x^2}}$ for m: (a) even, (b) odd ($m > 0$).

2434*. $\int\limits_0^1 \frac{(1 - x)^n}{\sqrt{x}} \, dx$ (n is a positive integer).

2435. $\int\limits_1^\infty \frac{dx}{(x - \cos\alpha)\sqrt{x^2 - 1}}$ $(0 < \alpha < 2\pi)$.

2436*. Prove that $\int\limits_0^\infty \frac{dx}{1 + x^4} = \int\limits_0^\infty \frac{x^2 \, dx}{1 + x^4} = \frac{\pi}{2\sqrt{2}}$.

2437*. Prove that $\int\limits_0^\infty \frac{x \ln x}{(1 + x^2)^2} \, dx = 0$.

2438. Evaluate the integral $\int\limits_1^\infty \frac{x^2 - 2}{x^3 \sqrt{x^2 - 1}} \, dx$.

Evaluate the integrals of problems 2439–2448 by using the formulae (see *Course*, secs. 111, 180, 181)

$$\int\limits_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} \text{ — Poisson's integral,}$$

$$\int\limits_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2} \text{ — Dirichlet's integral.}$$

2439. $\int\limits_0^\infty e^{-ax^2} \, dx$ $(a > 0)$. **2440.** $\int\limits_0^\infty \frac{e^{-x}}{\sqrt{x}} \, dx$.

2441*. $\int\limits_0^\infty x^2 e^{-x^2} \, dx$.

2442. $\int_0^\infty x^{2n}e^{-x^2}\,dx$ (n is a positive integer).

2443. $\int_0^\infty \frac{\sin 2x}{x}\,dx$.

2444. $\int_0^\infty \frac{\sin ax}{x}\,dx$.

2445. $\int_0^\infty \frac{\sin ax \cos bx}{x}\,dx \quad (a > 0,\ b > 0)$.

2446*. $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$.

2447*. $\int_0^\infty \frac{\sin^3 x}{x}\,dx$.

2448*. $\int_0^\infty \frac{\sin^4 x}{x^2}\,dx$.

2449*. We put $\varphi(x) = -\int_0^x \ln\cos y\,dy$. (This is known as Lobachevskii's integral.) Prove the relationship

$$\varphi(x) = 2\varphi\left(\frac{\pi}{4} + \frac{x}{2}\right) - 2\varphi\left(\frac{\pi}{4} - \frac{x}{2}\right) - x\ln 2.$$

Evaluate with the aid of this relationship:

$$\varphi\left(\frac{\pi}{2}\right) = -\int_0^{\frac{\pi}{2}} \ln\cos y\,dy$$

(The quantity $\varphi\left(\frac{1}{2}\pi\right)$ was first evaluated by Euler.)

Evaluate the integrals of problems 2450–2454:

2450. $\int_0^{\frac{\pi}{2}} \ln\sin x\,dx$.

2451. $\int_0^{\pi} x\ln\sin x\,dx$.

2452*. $\int_0^{\frac{\pi}{2}} x\cot x\,dx$.

2453*. $\int_0^1 \frac{\arcsin x}{x}\,dx$.

2454. $\int_0^1 \frac{\ln x\,dx}{\sqrt{1-x^2}}$.

CHAPTER VIII

APPLICATIONS OF THE INTEGRAL

1. Some Problems of Geometry and Statics

Areas of Figures

2455. Find the area of the figure bounded by the curves whose equations are $y^2 = 2x + 1$ and $x - y - 1 = 0$.

2456. Find the area of the figure lying between the parabola $y = -x^2 + 4x - 3$ and the tangents to it at the points $(0, -3)$ and $(3, 0)$.

2457. Find the area of the figure bounded by the parabola $y^2 = 2px$ and the normal to it inclined at 135° to the axis of abscissae.

2458. Find the area of the figure bounded by the parabolas $y = x^2$ and $y = \sqrt{x}$.

2459. Find the area of the figure bounded by the parabolas $y^2 + 8x = 16$ and $y^2 - 24x = 48$.

2460. Find the area of the figure bounded by the parabolas

$$y = x^2 \text{ and } y = \frac{x^3}{3}.$$

2461. The circle $x^2 + y^2 = 8$ is divided into two parts by the parabola $y = \frac{x^2}{2}$. Find the area of each part.

2462. Find the areas of the figures into which the parabola $y^2 = 6x$ divides the circle $x^2 + y^2 = 16$.

2463. An ellipse is cut out from a circular disc of radius a such that its major axis coincides with a diameter of the circle and its minor axis is equal to $2b$.

Prove that the area of the remaining part is equal to the area of the ellipse with semi-axes a and $a - b$.

2464. Find the area of the figure bounded by the arc of a hyperbola and a chord passing through a focus perpendicularly to the transverse axis.

2465. The circle $x^2 + y^2 = a^2$ is cut into three parts by the hyperbola $x^2 - 2y^2 = \frac{a^2}{4}$. Find the areas of these parts.

2466. Find the areas of the curvilinear figures formed by the intersection of the ellipse $\frac{x^2}{4} + y^2 = 1$ and the hyperbola $\frac{x^2}{2} - y^2 = 1$.

2467. Find the area of the figure lying between the curve $y = \frac{1}{(1+x^2)}$ and the parabola $y = \frac{x^2}{2}$.

2468. Find the area of the figure bounded by the curve $y = x(x-1)^2$ and the axis of abscissae.

2469. Find the area of the figure bounded by the axis of ordinates and the curve $x = y^2(y-1)$.

2470. Find the area of the piece of the figure bounded by the curves $y^m = x^n$ and $y^n = x^m$, where m and n are positive integers, lying in the first quadrant. Consider the area of the whole figure from the point of view of the property of numbers m and n of being even or odd.

2471. (a) Find the area of the curvilinear trapezium bounded by the axis of abscissae and the curve $y = x - x^2\sqrt{x}$.

(b) Work out the area of the figure bounded by the two branches of the curve $(y-x)^2 = x^5$ and the straight line $x = 4$.

2472. Find the area of the figure bounded by the curve $(y - x - 2)^2 = 9x$ and the coordinate axes.

2473. Find the area of the loop of the curve $y^2 = x(x-1)^2$.

2474. Find the area of the figure bounded by the closed curve $y^2 = (1-x^2)^3$.

2475. Find the area of the figure bounded by the closed curve $y^2 = x^2 - x^4$.

2476. Find the area of the figure bounded by the closed curve $x^4 - ax^3 + a^2y^2 = 0$.

2477. Find the area of the finite part of the figure bounded by the curve $x^2y^2 = 4(x - 1)$ and the straight line passing through its points of inflexion.

2478. Find the area of the figure bounded by the curves $y = e^x$, $y = e^{-x}$ and the straight line $x = 1$.

2479. Find the area of the curvilinear trapezium bounded by the curve $y = (x^2 + 2x) e^{-x}$ and the axis of abscissae.

2480. Find the area of the curvilinear trapezium bounded by the curve $y = e^{-x} (x^2 + 3x + 1) + e^2$, axis Ox and the two straight lines parallel to Oy passing through the extremal points of function y.

2481. Find the area of the finite part of the figure bounded by the curves $y = 2x^2e^x$ and $y = -x^3 e^x$.

2482. (a) Work out the area of the curvilinear trapezium with base $[a, b]$ bounded by curve $y = \ln x$.

(b) Work out the area of the figure bounded by the curve $y = \ln x$, the axis of ordinates and the straight lines $y = \ln a$ and $y = \ln b$.

2483. Work out the area of the figure bounded by the curves $y = \ln x$ and $y = \ln^2 x$.

2484. Find the area of the figure bounded by the curves

$$y = \frac{\ln x}{4x} \text{ and } y = x \ln x.$$

2485. Find the area of one of the curvilinear triangles bounded by the axis of abscissae and the curves

$$y = \sin x \text{ and } y = \cos x.$$

2486. Find the area of the curvilinear triangle bounded by the axis of ordinates and the curves

$$y = \tan x \text{ and } y = \frac{2}{3} \cos x.$$

2487. Find the area of the figure bounded by the curve $y = \sin^3 x + \cos^3 x$ and the segment of the axis of abscissae joining two successive points of intersection of the curve with the axis of abscissae.

2488. Work out the area of the figure bounded by the axis of abscissae and the curves

$$y = \arcsin x \quad \text{and} \quad y = \arccos x.$$

2489. Find the area of the figure bounded by the closed curve $(y - \arcsin x)^2 = x - x^2$.

2490. Find the area of the figure bounded by one arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ and the axis of abscissae.

2491. Work out the area of the figure bounded by the astroid $x = a \cos^3 t$, $y = a \sin^3 t$.

2492. Find the area of the figure bounded by the cardioid $x = 2a \cos t - a \cos 2t$, $y = 2a \sin t - a \sin 2t$.

2493. Find the area of the figure bounded: (1) by the epicycloid

$$x = (R + r) \cos t - r \cos \frac{R + r}{r} t,$$

$$y = (R + r) \sin t - r \sin \frac{R + r}{r} t,$$

(2) by the hypocycloid

$$x = (R - r) \cos t + r \cos \frac{R - r}{r} t,$$

$$y = (R - r) \sin t - r \sin \frac{R - r}{r} t,$$

where $R = nr$ (n is an integer). Here R is the radius of the fixed, and r the radius of the moving circle; the centre of the fixed circle coincides with the origin, whilst t denotes the angle of rotation of the radius from the centre of the fixed circle to the point of contact (see *Course*, sec. 83).

2494. Find the area of the loop of the curve:

(1) $x = 3t^2$, $y = 3t - t^3$; (2) $x = t^2 - 1$, $y = t^3 - t$.

2495. (a) Find the area swept out by the radius vector of the spiral of Archimedes $\varrho = a\varphi$ during one rotation, if the start of the motion corresponds to $\varphi = 0$.

(b) Find the area of the figure bounded by the second and third turns of the spiral and the segment of the polar axis.

2496. Find the area of the figure bounded by the curve $\varrho = a \sin 2\varphi$.

2497. Find the area of the figure bounded by the curve $\varrho = a \cos 5\varphi$.

2498. Find the area of the figure bounded by the limaçon of Pascal $\varrho = 2a(2 + \cos \varphi)$.

2499. Find the area of the figure bounded by the curve $\varrho = a \tan \varphi (a > 0)$ and the straight line $\varphi = \frac{\pi}{4}$.

2500. Find the area of the common part of the figures bounded by the curves $\varrho = 3 + \cos 4\varphi$ and $\varrho = 2 - \cos 4\varphi$.

2501. Find the area of the piece of the figure bounded by the curve $\varrho = 2 + \cos 2\varphi$ lying outside the curve $\varrho = 2 + \sin \varphi$.

2502. Find the area of the figure bounded by the curve $\varrho^2 = a^2 \cos n\varphi$ (n is a positive integer).

2503. Prove that the area of the figure bounded by any two radius vectors of the hyperbolic spiral $\varrho\varphi = a$ and its arc is proportional to the difference between these radius vectors.

2504. Prove that the area of the figure bounded by any two radius vectors of the logarithmic spiral $\varrho = ae^{m\varphi}$ and its arc is proportional to the difference between the squares of these radius vectors.

2505*. Find the area of the figure lying between the exterior and interior parts of the curve $\varrho = a \sin^3 \frac{\varphi}{3}$.

2506. Find the area of the figure bounded by the curve

$$\varrho = \sqrt{1 - t^2}, \quad \varphi = \arcsin t + \sqrt{1 - t^2}.$$

It is convenient to pass first to polar coordinates in problems 2507–2511.

2507. Find the area of the figure bounded by the lemniscate of Bernoulli $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

2508. Find the area of the part of the figure bounded by the lemniscate of Bernoulli (see problem 2507) lying inside the circle $x^2 + y^2 = \frac{a^2}{2}$.

2509. Find the area of figure bounded by the curve $(x^2 + y^2)^2 - a^2x^2 - b^2y^2 = 0$ ("pedal of ellipse").

2510. Find the area of the figure bounded by the curve

$$(x^2 + y^2)^3 = 4a^2xy(x^2 - y^2).$$

2511. Find the area of the figure bounded by the curve $x^4 + y^4 = x^2 + y^2$.

2512. Find the area of the figure lying between the curve $y = \frac{1}{(x^2 + 1)}$ and its asymptote.

2513. Find the area of the figure lying between the curve $y = xe^{-\frac{x^2}{2}}$ and its asymptote.

2514. Find the area of the figure contained between the cissoid $y^2 = \frac{x^3}{(2a - x)}$ and its asymptote.

2515. Find the area of the figure lying between the curve $xy^2 = 8 - 4x$ and its asymptote.

2516*. (1) Find the area of the figure bounded by the curve $y = x^2e^{-x^2}$ and its asymptote.

(2) Find the area of the figure bounded by the curve $y^2 = x\,e^{-2x}$.

2517. Find the area of the figure lying between the tractrix $x = a\left(\cos t + \ln\tan\frac{t}{2}\right)$, $y = a\sin t$ and the axis of abscissae.

2518. For the curve $\varrho = \frac{\cos 2\varphi}{\cos\varphi}$, find the area of the loop and the area of the figure lying between the curve and its asymptote.

Length of Arc†

2519. Find the length of arc of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) \text{ (from } x_1 = 0 \text{ to } x_2 = b).$$

2520. Find the length of arc of the parabola $y^2 = 2px$ from the vertex to its point $M(x, y)$. (Take y as the independent variable.)

2521. Find the length of arc of the curve

$$y = \ln x \text{ (from } x_1 = \sqrt{3} \text{ to } x_2 = \sqrt{8}).$$

2522. Find the length of arc of the curve

$$y = \ln(1 - x^2) \left(\text{from } x_1 = 0 \text{ to } x_2 = \frac{1}{2}\right).$$

2523. Find the length of arc of the curve

$$y = \ln \frac{e^x + 1}{e^x - 1} \text{ (from } x_1 = a \text{ to } x_2 = b).$$

2524. Find the length of arc of the semi-cubical parabola $y^2 = \frac{2}{3}(x - 1)^3$ lying inside the parabola $y^2 = \frac{x}{3}$.

2525. Find the length of arc of the semi-cubical parabola $5y^3 = x^2$ lying inside the circle $x^2 + y^2 = 6$.

2526. Find the length of the loop of the curve

$$9ay^2 = x(x - 3a)^2.$$

2527. Find the perimeter of one of the curvilinear trapezia bounded by the axis of abscissae and the curves $y = \ln \cos x$ and $y = \ln \sin x$.

2528. Find the length of arc of the curve $y = \frac{x^2}{4} - \frac{\ln x}{2}$ lying between its lowest point and the vertex (the point of the curve of extremal curvature).

† In the problems on evaluating the length of arc, the interval of variation of the independent variable corresponding to the rectified arc is indicated where necessary.

2529. Find the length of the curve $y = \sqrt{x - x^2} + \arcsin \sqrt{x}$.

2530. Find the length of the curve $(y - \arcsin x)^2 = 1 - x^2$.

2531. Find the point that divides the length of the first arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ in the ratio $1 : 3$.

2532. Given the astroid $x = R\cos^3 t$, $y = R\sin^3 t$ and the points $A(R, 0)$, $B(0, R)$ on it, find the point M on arc $\breve{AB}$ such that the length of arc $\breve{AM}$ amounts to a quarter of the length of arc $\breve{AB}$.

2533*. Find the length of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$$

2534. Find the length of the curve

$$x = a\cos^5 t, \ y = a\sin^5 t.$$

2535. Find the length of arc of the tractrix

$$x = a\left(\cos t + \ln \tan \frac{t}{2}\right), \quad y = a \sin t$$

from the point $(0, a)$ to the point (x, y).

2536. Find the length of arc of the involute of the circle

$$x = R(\cos t + t\sin t), \ y = R(\sin t - t\cos t)$$

(from $t_1 = 0$ to $t_2 = \pi$).

2537. Find the length of arc of the curve

$$x = (t^2 - 2)\sin t + 2t\cos t, \ y = (2 - t^2)\cos t + 2t\sin t$$

(from $t_1 = 0$ to $t_2 = \pi$).

2538. Find the length of the loop of the curve $x = t^2$, $y = t - \frac{t^2}{3}$.

2539. Two circles of radii equal to b roll without slipping with the same angular velocity on the inside and outside of a circle of radius a. At the instant $t = 0$ they touch the point M of the fixed circle with their points M_1 and M_2. Show that the ratio of the paths traversed by points M_1 and M_2 after an arbitrary interval of time t is a constant equal to $\frac{a+b}{a-b}$ (see problem 2493).

2540. Show that the length of arc of the curve

$$x = f''(t)\cos t + f'(t)\sin t,\ y = -f''(t)\sin t + f'(t)\cos t,$$

corresponding to the interval (t_1, t_2) is equal to

$$|[f(t) + f''(t)]|_{t_1}^{t_2}.$$

2541. Apply the result of the previous problem to evaluating the length of arc of the curve $x = e^t(\cos t + \sin t)$, $y = e^t(\cos t - \sin t)$ (from $t_1 = 0$ to $t_2 = t$).

2542. Show that the arcs of the curves

$$x = f(t) - \varphi'(t),\quad y = \varphi(t) + f'(t)$$

and

$$x = f'(t)\sin t - \varphi'(t)\cos t,\quad y = f'(t)\cos t + \varphi'(t)\sin t,$$

corresponding to the same interval of variation of parameter t, have equal lengths.

2543. Find the length of arc of the spiral of Archimedes $\varrho = a\varphi$ from the origin to the end of the first turn.

2544. Show that the arc of the parabola $y = \frac{x^2}{2p}$ corresponding to the interval $0 \leqq x \leqq a$ has the same length as the arc of the spiral $\varrho = p\varphi$ corresponding to the interval $0 \leqq \varrho \leqq a$.

2545. Find the length of arc of the hyperbolic spiral $\varrho\varphi = 1$, $\left(\text{from } \varphi_1 = \frac{3}{4} \text{ to } \varphi_2 = \frac{4}{3}\right)$.

2546. Find the length of the cardioid $\varrho = a(1 + \cos\varphi)$.

2547. Find the length of the curve $\varrho = a\sin^3\frac{\varphi}{3}$ (see problem 2505).

2548. Show that the length of the curve $\varrho = a \sin^m \dfrac{\varphi}{m}$ (m is an integer) is commensurate with a when m is even and commensurate with the circumference of a circle of radius a when m is odd.

2549. For what values of the exponent k ($k \neq 0$) is the length of arc of the curve $y = ax^k$ expressed in elementary functions? (Take as basis Chebyshev's theorem on the conditions for integrability in a finite form of the differential binomial; see *Course,* sec. 102.)

2550. Find the length of the curve given by the equation

$$y = \int_{-\frac{\pi}{2}}^{x} \sqrt{\cos x}\, dx.$$

2551. Find the length of arc of the curve

$$x = \int_1^t \frac{\cos z}{z}\, dz, \quad y = \int_1^t \frac{\sin z}{z}\, dz$$

from the origin to the nearest point with vertical tangent.

2552. Show that the length of arc of the sine wave $y = \sin x$ corresponding to one period is equal to the length of the ellipse whose semi-axes are equal to $\sqrt{2}$ and 1.

2553. Show that the length of arc of the curtate or prolate cycloid $x = mt - n \sin t$, $y = m - n \cos t$ (m and n are positive numbers) in the interval from $t_1 = 0$ to $t_2 = 2\pi$ is equal to the length of the ellipse with semi-axes $a = m + n$, $b = |m - n|$.

2554*. Show that the length of the ellipse with semi-axes a and b satisfies the inequality $\pi(a + b) < L < \pi\sqrt{2} \times \times \sqrt{a^2 + b^2}$ (Bernoulli's problem).

Volume of a Solid

2555. Find the volume of the solid, bounded by the surface which is formed by revolution of the parabola $y^2 = 4x$ about its axis (paraboloid of revolution), and by the plane perpendicular to the axis and at a distance equal to unity from the vertex.

2556. An ellipse with major axis equal to $2a$ and minor axis $2b$ revolves (1) about the major axis, (2) about the minor axis. Find the volumes of the ellipsoids of revolution thus obtained. Obtain the volume of a sphere as a particular case.

2557. A symmetric parabolic segment of base a and height h revolves about the base. Find the volume of the solid of revolution thus obtained. (Cavalieri's "lemon".)

2558. The figure bounded by the hyperbola $x^2 - y^2 = a^2$ and the straight line $x = a + h (h > 0)$ revolves about the axis of abscissae. Find the volume of the solid of revolution.

2559. The curvilinear trapezium bounded by the curve $y = xe^x$ and the straight lines $x = 1$ and $y = 0$ revolves about the axis of abscissae. Find the volume of the solid thus obtained.

2560. The catenary $y = \frac{e^x + e^{-x}}{2}$ revolves about the axis of abscissae. The surface thus obtained is called a catenoid. Find the volume of the solid bounded by the catenoid and two planes at distances of a and b units from the origin and perpendicular to the axis of abscissae.

2561. The figure bounded by the arcs of parabolas $y = x^2$ and $y^2 = x$ revolves about the axis of abscissae. Find the volume of the solid thus obtained.

2562. Find the volume of the solid obtained by revolution about the axis of abscissae of the trapezium lying above Ox and bounded by the curve $(x - 4) y^2 = x (x - 3)$.

2563. Find the volume of the solid obtained by revolution of the curvilinear trapezium bounded by the curve $y =$ $= \arcsin x$, and with base $[0, 1]$, about the Ox axis.

2564. Find the volume of the solid obtained by revolution of the figure, bounded by the parabola $y = 2x - x^2$ and the axis of abscissae, about the axis of ordinates.

2565. Find the volume of the body which is obtained on revolution about the axis of ordinates of the curvilinear trapezium bounded by the arc of the sine wave $y = \sin x$ corresponding to a half period.

2566. The lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ revolves about the axis of abscissae. Find the volume of the solid of revolution thus formed.

2567. Find the volume of the solid formed by revolution about the axis of abscissae of the figure bounded by the curve: (1) $x^4 + y^4 = a^2x^2$; (2) $x^4 + y^4 = x^3$.

2568. One arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ revolves about its base. Find the volume of the solid bounded by the surface obtained.

2569. The figure, bounded by an arc of the cycloid (see previous problem) and its base, revolves about the perpendicular bisector of the base (the axis of symmetry). Find the volume of the solid thus obtained.

2570. Find the volume of the solid obtained on revolution of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about its axis of symmetry.

2571. The figure bounded by the arc of the curve $x = \frac{c^2}{a}\cos^3 t$, $y = \frac{c^2}{b}\sin^3 t$ (evolute of the ellipse), lying in the first quadrant, and by the coordinate axes, revolves about the axis of abscissae. Find the volume of the solid thus obtained.

2572. Find the volume of the solid bounded by the surface of the infinite "spindle", formed by revolution of the curve $y = \frac{1}{(1 + x^2)}$ about its asymptote.

2573. The curve $y^2 = 2xe^{-2x}$ revolves about its asymptote. Find the volume of the solid bounded by the resulting surface.

2574*. (1) The figure bounded by the curve $y = e^{-x^2}$ and its asymptote revolves about the axis of ordinates. Find the volume of the resulting solid.

(2) The same figure revolves about the axis of abscissae. Find the volume of the resulting solid.

2575*. Find the volume of the solid formed by revolution of the curve $y = x^2 e^{-x^2}$ about its asymptote.

2576*. The figure bounded by the curve $y = \dfrac{\sin x}{x}$ and the axis of abscissae revolves about the axis of abscissae. Find the volume of the resulting solid.

2577*. Find the volume of the solid bounded by the surface produced by revolution of the cissoid $y^2 = \dfrac{x^3}{(2a - x)}$ $(a > a)$ about its asymptote.

2578. Find the volume of the solid whose boundary surface is obtained by revolution of the tractrix $x = a\left(\cos t + \ln \tan \dfrac{t}{2}\right)$, $y = a \sin t$ about its asymptote.

2579*. Find the volume of the solid bounded by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

2580. (1) Find the volume of the solid bounded by the elliptic paraboloid $z = \dfrac{x^2}{4} + \dfrac{y^2}{2}$ and the plane $z = 1$.

(2) Find the volume of the solid bounded by the hyperboloid of one sheet $\dfrac{x^2}{4} + \dfrac{y^2}{9} - z^2 = 1$ and the planes $z = -1$ and $z = 2$.

2581. Find the volumes of the solids bounded by the paraboloid $z = x^2 + 2y^2$ and by the ellipsoid $x^2 + 2y^2 + z^2 = 6$.

2582. Find the volumes of the solids formed by intersection of the hyperboloid of two sheets $\dfrac{x^2}{3} - \dfrac{y^2}{4} - \dfrac{z^2}{9} = 1$ and the ellipsoid $\dfrac{x^2}{6} + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$.

2583. Find the volume of the solid bounded by the conical surface $(z - 2)^2 = \dfrac{x^2}{3} + \dfrac{y^2}{2}$ and the plane $z = 0$.

2584. Find the volume of the solid bounded by the paraboloid $2z = \frac{x^2}{4} + \frac{y^2}{9}$ and the cone $\frac{x^2}{4} + \frac{y^2}{9} = z^2$.

2585*. Find the volume of the solid cut out from a circular cylinder by a plane through a base diameter ("special ungula of cylinder", Fig. 43). In particular, put $R = 10$ cm, $H = 6$ cm.

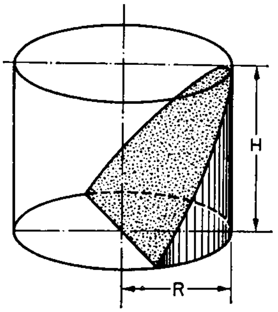

FIG. 43.

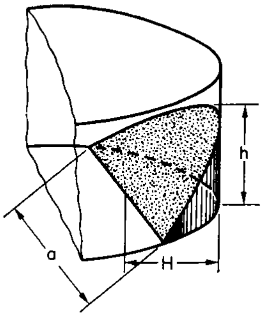

FIG. 44.

2586. A parabolic cylinder is cut by two planes, one of which is perpendicular to the generators. The resulting solid is illustrated in Fig. 44. The common base of the parabolic segments is $a = 10$ cm, the height of the parabolic segment lying in the base is $H = 8$ cm, and the height of the solid is $h = 6$ cm. Find the volume of the solid.

2587. A cylinder, whose base is an ellipse, is cut by an inclined plane through the major axis of the ellipse. Find the

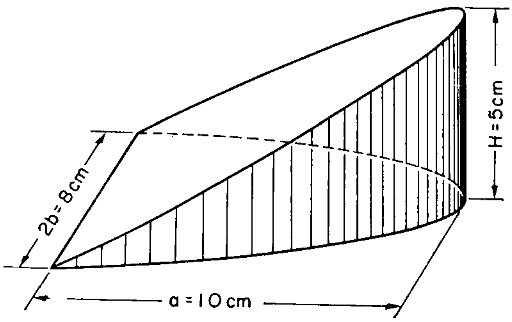

FIG. 45.

volume of the resulting solid. The linear dimensions are given in Fig. 45.

2588*. Symmetrical parabolic segments, of constant height H, are constructed on all the chords of a circle of radius R parallel to a single direction. The planes of the segments are perpendicular to the plane of the circle. Find the volume of the solid thus obtained.

2589*. A right circular cone of radius R and height H is cut into two pieces by a plane through the centre of the base parallel to a generator (Fig. 46). Find the volumes of the two pieces. (The sections of a cone by planes parallel to a generator are parabolic segments.)

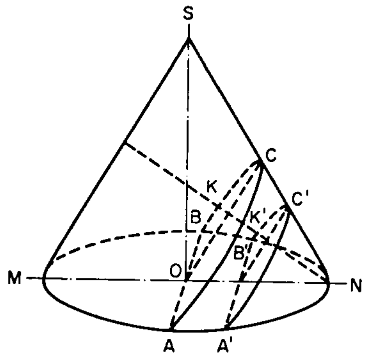

FIG. 46.

2590. The centre of a square moves along a diameter of a circle of radius a, the plane of the square remains perpendicular to the plane of the circle, whilst two opposite vertices of the square move round the circumference. Find the volume of the solid formed by this moving square.

2591. A circle of variable radius moves in such a way that a point of its circumference remains on the axis of abscissae, whilst the centre moves along the circle $x^2 + y^2 = r^2$, and the plane of the circle is perpendicular to the axis of abscissae. Find the volume of the solid thus obtained.

2592. The axes of two equal cylinders intersect at right angles. Find the volume of the solid consisting of the common part of the cylinders ($^1/_8$ of the solid is illustrated in Fig. 47). (Consider the sections formed by planes parallel to the axes of the two cylinders.)

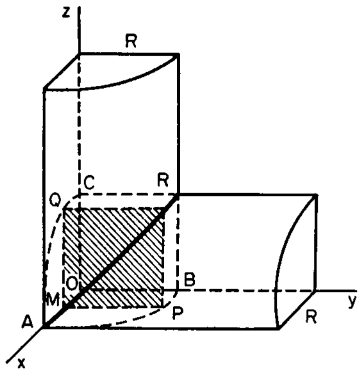

FIG. 47.

2593. Two circular cylinders have the same height H and a common upper base of radius R, whilst the lower bases touch (Fig. 48). Find the volume of the common part of the cylinders.

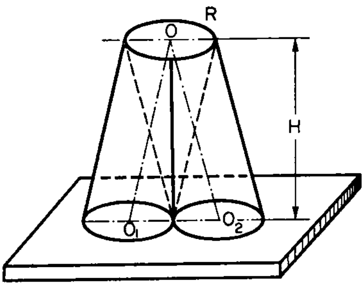

FIG. 48.

Area of a Surface of Revolution

2594. Find the area of the surface formed by revolution of the parabola $y^2 = 4ax$ about the axis of abscissae from the vertex to the point with abscissa $x = 3a$.

2595. Find the area of the surface formed by revolution of the cubical parabola $3y - x^3 = 0$ about the axis of abscissae (from $x_1 = 0$ to $x_2 = a$).

2596. Find the area of the catenoid — the surface formed by revolution of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

about the axis of abscissae (from $x_1 = 0$ to $x_2 = 2$).

2597. When the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

revolves about its major axis a surface is obtained which is called a prolate ellipsoid of revolution, whilst when it revolves about the minor axis the surface is an oblate ellipsoid of revolution. Find the surface areas of the prolate and oblate ellipsoids of revolution.

2598. Find the area of the spindle-shaped surface formed by revolution of one arc of the sine wave $y = \sin x$ about the axis of abscissae.

2599. The arc of the tangent curve $y = \tan x$ from the point (0, 0) to the point $\left(\frac{\pi}{4}, 1\right)$ revolves about the axis of abscissae. Find the area of the surface thus obtained.

2600. Find the area of the surface formed by revolution about the axis of abscissae of the loop of the curve $9ay^2 = x(3a - x)^2$.

2601. The arc of the circle $x^2 + y^2 = a^2$ lying in the first quadrant revolves about the chord subtending it. Find the area of the resulting surface.

2602. Find the area of the surface formed by revolution about the axis of abscissae of the arc of the curve

$$x = e^t \sin t,\ y = e^t \cos t \left(\text{from } t_1 = 0 \text{ to } t_2 = \frac{\pi}{2}\right).$$

2603. Find the area of the surface formed by revolution of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$ about the axis of abscissae.

2604. An arc of the cycloid revolves about its axis of symmetry. Find the area of the surface thus obtained. (See problem 2568.)

2605. Find the area of the surface formed by revolution about the polar axis of the cardioid $\varrho = a(1 + \cos\varphi)$.

2606. The circle $\varrho = 2r\sin\varphi$ revolves about the polar axis. Find the area of the surface thus formed.

2607. The lemniscate $\varrho^2 = a^2\cos 2\varphi$ revolves about the polar axis. Find the area of the resulting surface.

2608. The infinite arc of the curve $y = e^{-x}$, corresponding to positive values of x, revolves about the axis of abscissae. Find the area of the surface thus obtained.

2609. The tractrix $x = a\left(\cos t + \ln\tan\frac{t}{2}\right)$, $y = a\sin t$ revolves about the axis of abscissae. Find the area of the resulting infinite surface.

Moments and Centres of Gravity†

2610. Find the statical moment of a rectangle of base a and height h about its base.

2611. Find the statical moment of a right-angled isosceles triangle, whose adjacent sides are equal to a, with respect to each of its sides.

2612. Prove that the following formula holds:

$$\int_a^b (ax + b)\,f(x)\,dx = (a\xi + b)\int_a^b f(x)\,dx,$$

where ξ is the abscissa of the centre of gravity of the curvilinear trapezium with base $[a, b]$, bounded by the curve $y = f(x)$. (Vereshchagin's rule.)

2613. Find the centre of gravity of the symmetrical parabolic segment with base equal to a and height h.

† The density is taken as equal to unity in all the problems of this section (2610–2662).

2614. A rectangle of sides a and b is divided into two parts by the arc of the parabola whose vertex coincides with one corner of the rectangle and which passes through the opposite corner (Fig. 49). Find the centres of gravity of the two parts S_1 and S_2 of the rectangle.

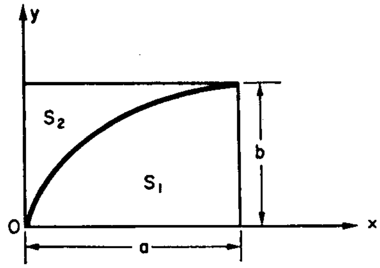

FIG. 49.

2615. Find the coordinates of the centre of gravity of the semi-circular arc

$$y = \sqrt{r^2 - x^2}.$$

2616. Find the coordinates of the centre of gravity of the semi-circular area bounded by the axis of abscissae and

$$y = \sqrt{r^2 - x^2}.$$

2617. Find the centre of gravity of the circular arc of radius R subtending an angle α at the centre.

2618. Find the coordinates of the centre of gravity of the figure bounded by the coordinate axes and the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

2619. Find the coordinates of the centre of gravity of the figure bounded by the coordinate axes and the arc of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, lying in the first quadrant, with respect to the axis of abscissae.

2620. Find the statical moment of the arc of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, lying in the first quadrant, with respect to the axis of abscissae.

2621. Find the coordinates of the centre of gravity of the figure bounded by the arc of the sine wave $y = \sin x$ and the segment of the axis of abscissae (from $x_1 = 0$ to $x_2 = \pi$).

2622. Find the statical moment of the figure bounded by the curves $y = \frac{2}{(1 + x^2)}$ and $y = x^2$ with respect to the axis of abscissae.

2623. The same for the curves $y = \sin x$ and $y = \frac{1}{2}$ (one segment) with respect to the axis of abscissae.

2624. The same for the curves $y = x^2$ and $y = \sqrt{x}$ with respect to the axis of abscissae.

2625. Find the coordinates of the centre of gravity of the figure bounded by the closed curve $y^2 = ax^3 - x^4$.

2626. Find the coordinates of the centre of gravity of the arc of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, lying between the points with abscissae $x_1 = -a$ and $x_2 = a$.

2627. Prove the theorem: the statical moment of an arbitrary arc of a parabola with respect to the parabola axis is proportional to the difference between the radii of curvature at the ends of the arc. The coefficient of proportionality is equal to $\frac{p}{3}$, where p is the parameter of the parabola.

2628. Find the coordinates of the centre of gravity of the first arc of the cycloid

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

2629. Find the coordinates of the centre of gravity of the figure bounded by the first arc of the cycloid and the axis of abscissae.

2630. Find the coordinates of the centre of gravity of the arc of the astroid $x = a \cos^3 t$, $y = a \sin^3 t$, lying in the first quadrant.

2631. Find the coordinates of the centre of gravity of the figure bounded by the coordinate axes and the arc of the astroid (in the first quadrant).

2632. Prove that the abscissa and ordinate of the centre of gravity of the sector bounded by two radius vectors and the curve whose equation is given in polar coordinates $\varrho = \varrho(\varphi)$, is given by

$$x = \frac{2}{3}\,\frac{\int\limits_{\varphi_1}^{\varphi_2} \varrho^3 \cos\varphi \, d\varphi}{\int\limits_{\varphi_1}^{\varphi_2} \varrho^2 \, d\varphi}, \qquad y = \frac{2}{3}\,\frac{\int\limits_{\varphi_1}^{\varphi_2} \varrho^3 \sin\varphi \, d\varphi}{\int\limits_{\varphi_1}^{\varphi_2} \varrho^2 \, d\varphi}.$$

2633. Find the Cartesian coordinates of the centre of gravity of the sector bounded by one half turn of the spiral of Archimedes $\varrho = a\varphi$ (from $\varphi_1 = 0$ to $\varphi_2 = \pi$).

2634. Find the centre of gravity of the circular sector of radius R subtending an angle 2α at the centre.

2635. Find the Cartesian coordinates of the centre of gravity of the figure bounded by the cardioid $\varrho = a(1 + \cos\varphi)$.

2636. Find the Cartesian coordinates of the centre of gravity of the figure bounded by the right-hand loop of the lemniscate of Bernoulli

2637. Prove that the Cartesian coordinates of the centre of gravity of the arc of the curve whose equation is given in polar coordinates as $\varrho = \varrho(\varphi)$ is given by

$$x = \frac{\int\limits_{\varphi_1}^{\varphi_2} \varrho \cos\varphi \sqrt{\varrho^2 + \varrho'^2}\, d\varphi}{\int\limits_{\varphi_1}^{\varphi_2} \sqrt{\varrho^2 + \varrho'^2}\, d\varphi}, \quad y = \frac{\int\limits_{\varphi_1}^{\varphi_2} \varrho \sin\varphi \sqrt{\varrho^2 + \varrho'^2}\, d\varphi}{\int\limits_{\varphi_1}^{\varphi_2} \sqrt{\varrho^2 + \varrho'^2}\, d\varphi}.$$

2638. Find the Cartesian coordinates of the centre of gravity of the arc of the logarithmic spiral $\varrho = ae^{\varphi}$ (from $\varphi_1 = \frac{\pi}{2}$ to $\varphi_2 = \pi\Big)$.

2639. Find the Cartesian coordinates of the centre of gravity of the arc of the cardioid $\varrho = a(1 + \cos\varphi)$ (from $\varphi_1 = 0$ to $\varphi_2 = \pi$).

2640. At what distance from the geometrical centre is the centre of gravity of a solid hemisphere of radius R?

2641. Find the centre of gravity of the surface of a hemisphere.

2642. The base radius of a right circular cone is R, its height is H. Find the distance from the base of the centre of gravity of its lateral surface, of its total surface and of its volume.

2643. How far from the base is the centre of gravity of the solid bounded by a paraboloid of revolution and a plane perpendicular to its axis? The height of the solid is h.

2644. Find the moment of inertia of the segment $AB = 1$ with respect to an axis lying in the same plane, given that the distance of end A of the segment from the axis is a units and the distance of end B from the axis is b units.

2645. Find the moment of inertia of the semi-circular arc of radius R with respect to the diameter.

2646. Find the moment of inertia of the arc of the curve $y = e^x \left(0 \leqq x \leqq \frac{1}{2}\right)$ with respect to the axis of abscissae.

2647. Find the moment of inertia with respect to both coordinate axes of an arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

2648. Find the moment of inertia of a rectangle with sides a and b with respect to side a.

2649. Find the moment of inertia of a triangle of base a and height h with respect to:

(1) the base;

(2) a straight line parallel to the base through the vertex;

(3) a straight line parallel to the base through the centre of gravity of the triangle.

2650. Find the moment of inertia of a semi-circular area of radius R with respect to its diameter.

2651. Find the moment of inertia of a circular disc of radius R with respect to the centre.

2652. Find the moment of inertia of the ellipse with semi-axes a and b with respect to both its axes.

2653. Find the moment of inertia of a cylinder of base radius R and height H with respect to its axis.

2654. Find the moment of inertia of the cone of base radius R and height H with respect to its axis.

2655. Find the moment of inertia of a sphere of radius R with respect to a diameter.

2656. An ellipse revolves about one of its axes. Find the moment of inertia of the resulting solid (ellipsoid of revolution) with respect to the axis of revolution.

2657. Find the moment of inertia with respect to the axis of revolution of the paraboloid of revolution, the base radius of which is R, and the height H.

2658. Find the moment of inertia with respect to Oz of the solid bounded by the hyperboloid of one sheet

$$\frac{x^2}{2} + \frac{y^2}{2} - z^2 = 1$$

and the planes

$$z = 0 \text{ and } z = 1.$$

2659. The curvilinear trapezium bounded by the curves

$$y = e^x, \ y = 0, \ x = 0 \text{ and } x = 1,$$

revolves (1) about Ox, (2) about Oy.

Find the moment of inertia of the resulting solid with respect to the axis of revolution.

2660. Find the moment of inertia of the lateral surface of a cylinder (base radius R, height H) with respect to its axis.

2661. Find the moment of inertia of the lateral surface of a cone (base radius R, height H) with respect to its axis.

2662. Find the moment of inertia of the spherical surface of radius R with respect to a diameter.

Guldin's Theorem

2663. A regular hexagon of side a revolves about one of the sides. Find the volume of the solid thus obtained.

2664. An ellipse with axes $AA_1 = 2a$, $BB_1 = 2b$, revolves about a straight line parallel to axis AA_1 and at a distance $3b$ from it. Find the volume of the solid thus obtained.

2665. An astroid revolves about an axis through two neighbouring cusps. Find the volume and surface of the body thus formed (see problem 2630).

2666. The figure formed by the first arcs of the cycloids

$$x = a(t - \sin t), \quad y = a(1 - \cos t)$$

and

$$x = a(t - \sin t), \quad y = -a(1 - \cos t),$$

revolves about the axis of ordinates. Find the volume and surface of the solid thus obtained.

2667. A square revolves about an axis, lying in its plane and passing through one of its corners. For what position of the axis with respect to the square is the volume of the solid thus obtained a maximum? The same problem for a triangle.

2. Some Problems of Physics

2668. The speed of a body is given by $v = \sqrt{1 + t}$ m/sec. Find the path traversed by the body during the first 10 sec. from the start of the motion.

2669. The speed $\frac{dx}{dt}$ for a harmonic vibration along the axis of abscissae about the origin is given by

$$\frac{dx}{dt} = \frac{2\pi}{T} \cos\left(\frac{2\pi t}{T} + \varphi_0\right)$$

(t is time, T the period of vibration, φ_0 the initial phase). Find the position of the point at the instant t_2, if it is known that it was at the point $x = x_1$ at the instant t_1.

The force f of interaction of two material particles is given by the formula $f = k\frac{mM}{r^2}$, where m and M are the masses of the particles, r is the distance between them, and k is a coefficient of proportionality, equal in the CGS system to $6{\cdot}66\times10^{-8}$ (Newton's law). Use this in solving problems 2670–2678. (The density is assumed constant.)

2670. A rod AB of length l and mass M attracts a particle C of mass m which lies on the continuation of the rod at a distance a from its nearest end B. Find the force of interaction of the rod and particle. What material particle must be located at A in order for it to act on C with the same force as rod AB? How much work is done by the force of attraction when the particle, situated at a distance r_1 from the rod, approaches along the straight line forming the prolongation of the rod until its distance from the rod is r_2?

2671. With what force does a half-ring of radius r and mass M act on a material particle of mass m situated at its centre?

2672. With what force does a wire ring of mass M and radius R act on a material particle C of mass m, located on the straight line through the centre of the ring perpendicular to its plane? The distance from the particle to the ring centre is equal to a. What work is done by the attraction force when the particle moves from infinity to the ring centre?

2673. Using the result of the previous problem, find the force that a plane disc of radius R and mass M exerts on a material particle of mass m, which lies on its axis at a distance a from the centre.

2674. Using the result of the previous problem, find the force exerted on a material particle of mass m by an infinite plane on which mass is uniformly distributed with surface density σ. The distance from the particle to the plane is equal to a.

2675. The base radii of the frustum of a right circular cone are equal to R and r, its height is h and density γ. What force does it exert on a material particle of mass m located at its vertex?

2676. With what force does the material step-line $y = |x| + 1$ attract a material particle of mass m, located at the origin? (The linear density is equal to γ.)

2677. Prove that the material step-line $y = a|x| + 1$ ($a \geqq 0$) attracts a material particle, situated at the origin, with a force independent of a, i.e. independent of the angle between the sides of the step-line.

2678*. Two equal rods (each of length l and mass M) lie on the same straight line at a distance l from each other. Work out the force of mutual attraction.

2679. A drop with initial mass M falls under the action of gravity and evaporates uniformly, losing mass m per second. What is the work done by gravity from the start of the motion to the complete evaporation of the drop? (The air resistance is neglected.)

2680. How much work must be done in producing a conical heap of sand of base radius 1·2 m and height 1 m? The specific weight of sand is 2 g/cm^3 (the sand is taken from the surface of the earth).

2681. The dimensions of the pyramid of Cheops are roughly as follows: height 140 m, side of the (square) base 200 m. The specific weight of the stone of which it is made is approximately 2·5 g/cm^3. Find the work done during its construction in overcoming the force of gravity.

2682. Find the work required when pumping out the water from a cylindrical reservoir of height $H = 5$ m, having a circular base of radius $R = 3$ m.

2683. Find the work that must be expended in pumping out liquid of specific weight d from a reservoir, having the shape of an inverted cone with vertex downwards, the height of which is H and base radius R. How is the result affected if the one has ite vsertx upwards?

2684. Find the work that must be expended to pump out the water filling a hemispherical vessel of radius $R = 0{\cdot}6$ m.

2685. A boiler has the shape of a paraboloid of revolution (Fig. 50). The base radius $R = 2$ m, the depth of the boiler $H = 4$ m. It is filled with liquid of specific weight $d = 0{\cdot}8$ g/cm³. Find the work which must be done to pump the liquid out of the boiler.

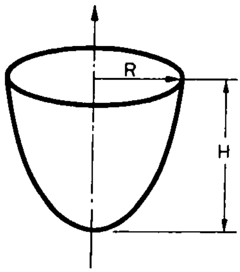

FIG. 50.

2686. Find the work which must be expended to pump out the water from a trough which has the following dimensions (Fig. 51): $a = 0{\cdot}75$ m, $b = 1{\cdot}2$ m, $H = 1$ m. The surface bounding the trough is a parabolic cylinder.

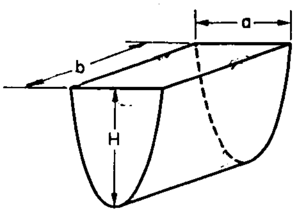

FIG. 51.

The kinetic energy of a body rotating about a fixed axis is equal to $\frac{1}{2}J\omega^2$, where ω is the angular velocity and J is the moment of inertia with respect to the axis of rotation. Knowing this, solve problems 2687–2692.

2687. A rod AB (Fig. 52) rotates in a horizontal plane about axis OO' with angular velocity $\omega = 10\pi$ rad/sec. The cross-section of the rod $S = 4$ cm^2, its length $l = 20$ cm, the density of the material of which it is made is $\gamma = 7{\cdot}8$ g/cm^3. Find the kinetic energy of the rod.

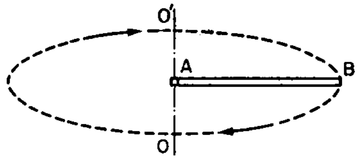

FIG. 52.

2688. A rectangular plate with sides $a = 50$ cm and $b = 40$ cm rotates with constant angular velocity ω, equal to 3π sec^{-1}, about the side a. Find the kinetic energy of the plate. The plate thickness d is equal to $0{\cdot}3$ cm, the density of its material γ is equal to 8 g/cm^3.

2689. A triangular plate, whose base $a = 40$ cm and height $h = 30$ cm, rotates about its base with constant angular velocity $\omega = 5\pi$ sec^{-1}. Find the kinetic energy of the plate, if its thickness $d = 0{\cdot}2$ cm, and the density of its material $\gamma = 2{\cdot}2$ g/cm^3.

2690. A plate in the shape of a parabolic segment (Fig. 53) rotates about the parabola axis with constant angular velocity $\omega = 4\pi$ sec^{-1}. The base of the segment $a = 20$ cm, the height $h = 30$ cm, the thickness of the plate $d = 0{\cdot}3$ cm, the density of the material $\gamma = 7{\cdot}8$ g/cm^3. Find the kinetic energy of the plate.

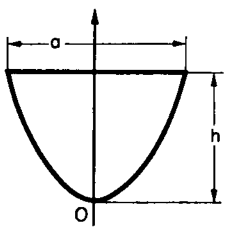

FIG. 53.

2691. A circular cylinder of base radius R and height H rotates about its axis with constant angular velocity ω. The density of the material of which the cylinder is made is equal to γ. Find the kinetic energy of the cylinder.

2692. A thin wire of mass M is bent to form a semi-circle of radius R and rotates about an axis passing through the ends of the semi-circle, performing n revolutions per minute. Find its kinetic energy.

Work out the kinetic energy if the axis of rotation is the tangent at the mid-point of the semi-circle.

2693. A plate of triangular shape is submerged vertically in water so that its base lies at the surface of the water. The plate base is a, its height h.

(a) Find the force of the water pressure on each side of the plate.

(b) How many times is the force increased if the plate is turned over so that the vertex is at the water surface and the base is parallel to the water surface?

2694. A square plate is submerged vertically in water so that one corner lies at the water surface and a diagonal is parallel to the surface. The side of the square is a. What is the water pressure on each side of the plate?

2695. Calculate the water pressure on a dam having the shape of an isosceles trapezoid, whose upper base $a = 6{\cdot}4$ m, lower base $b = 4{\cdot}2$ m, and height $H = 3$ m.

2696. A plate in the form of an ellipse is half submerged in liquid (vertically), so that one of its axes (of length $2b$) lies at the surface of the liquid. How great is the fluid pressure on each of the sides of the plate if the length of the submerged semi-axis of the ellipse is equal to a, whilst the specific weight of the fluid is d?

2697. A rectangular plate with sides a and b $(a > b)$ is submerged in fluid at an angle α to the fluid surface. The longer side is parallel to the surface and lies at a depth h. Calculate the fluid pressure on each of the plate sides, if the specific weight of the fluid is d.

2698. A rectangular vessel is filled with equal parts by volume of water and oil, the oil being twice as light as water. Show that the pressure on each wall of the vessel is diminished by a fifth if oil only is taken instead of the mixture. (Take into account the fact that all the oil is on top.)

The solutions of problems 2699–2700 must be based on Archimedes' law: the buoyancy force acting on a solid body immersed in a fluid is equal to the weight of displaced fluid.

2699. A wooden float of cylindrical shape, the base area of which $S = 4000$ cm^2, and height $H = 50$ cm, floats on water. The specific gravity of wood $d = 0{\cdot}8$ g/cm^3. (a) What work must be done in order to pull the float out of the water? (b) Find how much work must be expended to submerge the float completely.

2700. A sphere of radius R with specific weight 1 is submerged in water so that it touches the surface. How much work must be done in order to pull the sphere from the water?

Problems 2701–2706 are connected with the flow of a fluid from a small orifice. The velocity of flow of the fluid is defined by Torricelli's law: $v = \sqrt{2gh}$, where h is the height of the column of fluid above the orifice, g is the acceleration due to gravity† (see *Course*, sec. 116).

2701. There is an orifice at the bottom of a cylindrical vessel, the base area of which is 100 cm, and the height 30 cm. Find the area of the orifice if it is known that water filling the vessel flows out in the course of 2 min.

2702. Water fills a conical funnel of height $H = 20$ cm. The radius of the upper orifice $R = 12$ cm. The lower orifice, through which the water flows from the funnel, has radius $r = 0{\cdot}3$ cm. (a) How long does it take the level of water in the funnel to fall by 5 cm? (b) When will the funnel be empty?

† Torricelli's law in the form given here is only applicable to an ideal fluid. The answers to the problems are given for this ideal fluid. (In practice, the formula $v = \mu\sqrt{2gh}$ is used, where μ is a coefficient depending on the fluid viscosity and the nature of the orifice. For water in the simplest case, $\mu = 0{\cdot}6$.

2703. A hole of area $S = 0{\cdot}2$ cm^2 has formed in the bottom of a boiler, of hemispherical shape with radius $R = 43$ cm. If the boiler is filled with water, how long will it take the water to flow out?

2704. A boiler has the form of an elliptic cylinder with horizontal axis. The semi-axes of the elliptic section (perpendicular to the cylinder axis) are b (horizontal) and a (vertical); the cylinder generator is of length l (Fig. 54). The boiler

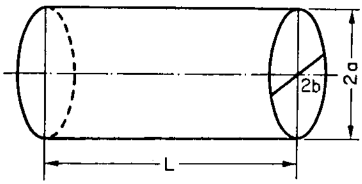

FIG. 54.

is half filled with water. How long does it take the water to flow from the boiler through an orifice of area S at the bottom?

2705. A rectangular vertical slit, of height h and width b, is made in the vertical wall of a prismatic vessel filled with water. The upper edge of the slit, parallel to the water surface, is at a distance H from the surface. What amount of water flows from the vessel in 1 sec., if the water level is assumed always to be maintained at the same height? Take the case $H = 0$ (problem of a spill-way).

2706. A vessel filled to the brim with water has the shape of a parallelepiped with base area 100 cm^2. There is a narrow

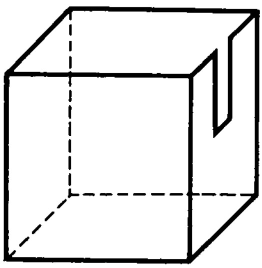

FIG. 55.

slit in the side wall, of height 20 cm and width 0·1 cm (Fig. 55). How long does it take the water level in the vessel to fall by (a) 5 cm? (b) 10 cm? (c) 19 cm? (d) 20 cm? (The result of the previous problem should be used.)

The equation of state of an ideal gas has the form $pv = RT$, where p is the pressure, v the volume, T the absolute temperature, and R a constant for a given mass of gas. Solve problems 2707–2709 on the assumption that the gases are ideal.

2707. Atmospheric air is contained in a cylindrical vessel of base area 10 cm² and height 30 cm. What work must be expended in order to drive in a piston 20 cm, i.e. so that the piston is 10 cm from the cylinder bottom (Fig. 56)? Atmospheric pressure is 1·033 kg/cm². The process is carried out isothermally, i.e. at constant temperature. (To find the work in kgm, the pressure must be taken in kg/m² and the volume in m³.)

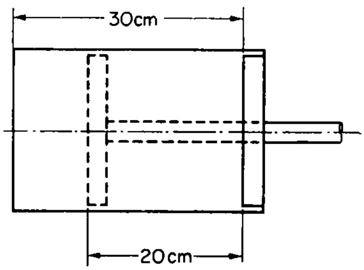

FIG. 56.

2708. Air at atmospheric pressure is contained in a cylindrical vessel of cross-section 100 cm². There is a piston in the vessel. Its initial distance from the vessel bottom is 0·1 m. The vessel is placed in a vacuum, as a result of which the air in it expands and pushes out the piston. (1) Find the work done by the air in the cylinder when it raises the piston a height (a) 0·2 m, (b) 0·5 m, (c) 1 m. (2) Can this work increase indefinitely on indefinite expansion of the gas? (The process is isothermal, as in the previous example.)

2709. Atmospheric air is contained in a cylindrical vessel of volume $v_0 = 0{\cdot}1$ m^3 and is subjected to compression by rapidly driving in a piston (it is assumed here that the process is carried out without the influx or transmission of heat, i.e. adiabatically). What work must be expended to compress the air in the vessel to a volume $v = 0{\cdot}03$ m^3? (Atmospheric pressure is $1{\cdot}033$ kg/cm^2.) In the case of an adiabatic process the pressure and volume of the gas are connected by the relationship $pv^\gamma = p_0 v_0^\gamma$ (Poisson's equation). For diatomic gases (as also for air) $\gamma \approx 1{\cdot}40$.

By Newton's law of cooling, the rate of cooling of a body is proportional to the difference in temperature between the body and the surrounding medium. Solve problems 2710–2711 on the basis of this law.

2710. A body whose temperature is 25° is placed in a thermostat (the temperature of which is maintained at 0°). How long does it take the body to cool to 10°, if it has cooled to 20° after 20 min.?

2711. A body whose temperature is 30° reaches a temperature of $22{\cdot}5°$ after being placed for 30 min in a thermostat whose temperature is 0°. What will the temperature of the body be 3 hours after the start of the experiment?

The force of interaction of two electric charges is $\dfrac{e_1 e_2}{\varepsilon r^2}$ dynes, where e_1 and e_2 are the charges in electrostatic units, r is the separation in cm, and ε is the dielectric constant (Coulomb's law). Solve problems 2712–2714 on the basis of this law.

2712. An infinite straight line is uniformly charged with positive electricity (the linear density of electricity is σ). What force does this straight line exert on a unit charge located at a point A distant a from it? The dielectric constant of the medium is equal to unity (see *Course*, sec. 116).

2713. Two electric charges: $e_1 = 20$ electrostatic units and $e_2 = 30$ electrostatic units, are separated by a distance of 10 cm. The medium between them is air. Both charges are first held fixed, then charge e_2 is freed. Under the action of the force of repulsion, charge e_2 starts to move away from charge e_1. How much work is done by the repulsion force when the charge (a) moves away to a distance of 30 cm? (b) moves away to infinity?

2714. Two electric charges: $e_1 = 100$ electrostatic units and $e_2 = 120$ electrostatic units, are at a distance of 20 cm from each other. What will the distance be between the charges if we bring the second closer to the first whilst expending 1800 ergs of work? (Air is the separating medium.)

2715. The voltage is $v = 120$ V at the terminals of an electrical circuit. Resistance is introduced into the circuit at a uniform rate of 0·1 ohm per sec. Furthermore, a constant resistance of $r = 10$ ohm is included in the circuit. How many Coulombs of electricity pass through the circuit during two minutes?

2716. The voltage at the terminals of an electrical circuit, initially equal to 120 V, falls uniformly, decreasing by 0·01 V in a second. Simultaneously with this, resistance is introduced into the circuit, also at a uniform rate, viz. 0·1 ohm per sec. Moreover, constant resistance equal to 12 ohm is present in the circuit. How many Coulombs of electricity flow through the circuit during 3 min?

2717. When the temperature changes, the resistance of a metallic conductor varies (at normal temperatures) in accordance with the law $R = R_0\,(1 + 0{\cdot}004\ \theta)$, where R_0 is the resistance at 0° C and θ is the temperature in centigrade. (This law holds for the majority of pure metals.) A conductor whose resistance at 0° C is equal to 10 ohm is uniformly heated from $\theta_1 = 20°$ to $\theta_2 = 200°$ in the course of 10 min. A current flows along it in this time at a voltage of 120 V. How many Coulombs of electricity flow through the conductor during this time?

2718. The law of variation of the voltage of ordinary alternating (urban) current, of 50 cycles per sec, is given by the formula: $E = E_0 \sin 100\,\pi t$, where E_0 is the maximum voltage and t is time. Find the mean value of the square of the voltage during 1 period (0·02 sec). Show that, when the resistance is constant, alternating current produces as much heat during 1 cycle as a constant current having a voltage equal to $\sqrt{(E^2)_{aV}}$. (In view of this, expression $\sqrt{(E^2)_{aV}}$ is termed the effective voltage of the alternating current.)

2719. The voltage of a sinusoidal electric current is given by

$$E = E_0 \sin\left(\frac{2\pi}{T}t\right),$$

whilst the current is given by

$$I = I_0 \sin\left(\frac{2\pi t}{T} - \varphi_0\right),$$

where E_0 and I_0 are constant quantities (the peak values of the voltage and current), T is the period, and φ_0 the phase difference. Find the work done by the current during the time from $t_1 = 0$ to $t_2 = T$ and show that the peak value of this work will be obtained when the phase difference φ_0 is zero.

2720. Find the time required to heat 1 kg of water electrically from 20 to 100° C, if the voltage is 120 V, the spiral resistance is 14·4 ohm, the air temperature in the room is 20° C and it is known that 1 kg of water cools from 40° C to 30° C in 10 min. (By the Joule–Lenz law, $Q = 0{\cdot}24 I^2 Rt$, where Q is the amount of heat in small calories, I is the current in amperes, R is the resistance in ohms, and t is the time in seconds. In addition, use is made of Newton's law of cooling (see problem 2710).

2721. Air filling a vessel of capacity 3 l., contains 20 per cent oxygen. The vessel has two pipes. Pure oxygen is now pumped into the vessel through one of them, whilst air passes out through the other, the amount of air leaving being the same as the amount of oxygen flowing in. How much oxygen will the vessel contain after 10 l. of gas have flowed through it? The concentration of oxygen is kept the same in the vessel at each instant with the aid of a mixer.

2722. Air contains a per cent ($=8$ per cent) CO_2; it is filtered through a cylindrical vessel with an absorbent medium. A thin layer of the medium absorbs an amount of gas proportional to its concentration and the layer thickness. (a) If air that has passed through a layer H cm ($=10$ cm) thick contains b per cent ($=2$ per cent) CO_2, what thickness H_1 must the layer have for the air leaving the filter to contain only c per cent ($=1$ per cent) carbon dioxide? (b) How much carbon dioxide (in per cent) remains in the air after passing through the filter if the thickness of the absorbent layer is 30 cm?

2723. If half the initial quantity of light is absorbed on passing through a layer of water 3 m thick, what part of this quantity remains at a depth of 30 m? The quantity of light absorbed on passing through a thin layer of water is proportional to the layer thickness and the quantity of light incident on its surface.

2724. If an initial quantity of ferment equal to 1 g becomes 1·2 g after an hour, what will it be 5 hours after the start of the fermentation, if the rate of growth of the ferment is assumed proportional to the initial quantity?

2725. If the quantity of ferment present is 2 g two hours after the start of the fermentation, and is 3 g after 3 hours, what was the initial quantity of ferment? (See previous problem.)

2726. Two kilogrammes of salt are dissolved in 30 l. water. One kilogramme of salt passes into solution after 5 min. How long will it take 99 per cent of the initial quantity of salt to pass into solution? (The rate of solution is proportional to the amount of undissolved salt and the difference between the concentration of a saturated solution, which is equal to 1 kg per 3 l., and the concentration at the given instant.)

CHAPTER IX

SERIES

1. Numerical Series

Convergence of Numerical Series

For each series of problems 2727–2736: (1) find the sum (S_n) of the first n terms of the series, (2) show directly from the definition that the series is convergent and (3) find the sum (S) of the series.

2727*. $\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\ldots+\frac{1}{n(n+1)}+\ldots$

2728. $\frac{1}{1\,.\,3}+\frac{1}{3\,.\,5}+\ldots+\frac{1}{(2n-1)(2n+1)}+\ldots$

2729. $\frac{1}{1\,.\,4}+\frac{1}{4\,.\,7}+\ldots+\frac{1}{(3n-2)(3n+1)}+\ldots$

2730. $\frac{1}{1\,.\,4}+\frac{1}{2\,.\,5}+\ldots+\frac{1}{n(n+3)}+\ldots$

2731. $\frac{1}{1\,.\,7}+\frac{1}{3\,.\,9}+\ldots+\frac{1}{(2n-1)(2n+5)}+\ldots$

2732. $\frac{1}{1\,.\,2\,.\,3}+\frac{1}{2\,.\,3\,.\,4}+\ldots+\frac{1}{n(n+1)(n+2)}+\ldots$

2733. $\frac{5}{6}+\frac{13}{36}+\ldots+\frac{3^n+2^n}{6^n}+\ldots$

2734. $\frac{3}{4}+\frac{5}{36}+\ldots+\frac{2n+1}{n^2(n+1)^2}+\ldots$

2735. $\frac{1}{9}+\frac{2}{225}+\ldots+\frac{1}{(2n-1)^2\,(2n+1)^2}+\ldots$

2736. $\operatorname{arc\,tan}\frac{1}{2}+\operatorname{arc\,tan}\frac{1}{8}+\ldots+\operatorname{arc\,tan}\frac{1}{2\,.\,n^2}+\ldots$

Series with Positive Terms

Solve the question of the convergence of the series of problems 2737–2753 with the aid of the theorems on comparison of series:

2737. $\frac{1}{1\,.\,2}+\frac{1}{3\,.\,2^3}+\ldots+\frac{1}{(2n-1)\,.\,2^{2n-1}}+\ldots$

2738. $\sin\frac{\pi}{2}+\sin\frac{\pi}{4}+\ldots+\sin\frac{\pi}{2^n}+\ldots$

2739. $1+\frac{1+2}{1+2^2}+\ldots+\frac{1+n}{1+n^2}+\ldots$

2740. $\frac{1}{2\,.\,5}+\frac{1}{3\,.\,6}+\ldots+\frac{1}{(n+1)(n+4)}+\ldots$

2741. $\frac{2}{3}+\frac{3}{8}+\ldots+\frac{n+1}{(n+2)n}+\ldots$

2742. $\tan\frac{\pi}{4}+\tan\frac{\pi}{8}+\ldots+\tan\frac{\pi}{4n}+\ldots$

2743. $\frac{1}{2}+\frac{1}{5}+\ldots+\frac{1}{n^2+1}+\ldots$

2744. $\frac{1}{2}+\frac{1}{5}+\ldots+\frac{1}{3n-1}+\ldots$

2745. $\frac{1}{\ln 2}+\frac{1}{\ln 3}+\ldots+\frac{1}{\ln(n+1)}+\ldots$

2746. $\sum\limits_{n=1}^{n=\infty}\frac{1}{n^2-4n+5}$.

2747. $\sum\limits_{n=1}^{n=\infty}\left(\frac{1+n^2}{1+n^3}\right)^2$.

2748. $\sum\limits_{n=1}^{n=\infty}\frac{1}{\sqrt{n^2+2n}}$.

2749. $\sum\limits_{n=1}^{n=\infty}\frac{\ln n}{\sqrt[4]{n^5}}$.

2750. $\sum\limits_{n=1}^{n=\infty}(\sqrt{n}-\sqrt{n-1})$.

2751. $\sum\limits_{n=1}^{n=\infty}\sqrt{\frac{n}{n^4+1}}$.

2752. $\sum\limits_{n=1}^{n=\infty}\frac{1}{n}(\sqrt{n+1})-(\sqrt{n-1})$.

2753. $\sum\limits_{n=1}^{n=\infty}\frac{1}{n}(\sqrt{n^2+n+1}-\sqrt{n^2-n-1})$.

Prove the convergence of the series of problems 2754–2762 with the aid of d'Alembert's test:

2754. $\frac{1}{3!}+\frac{1}{5!}+\ldots+\frac{1}{(2n+1)!}+\ldots$

2755. $\frac{1}{2}+\frac{2}{2^2}+\ldots+\frac{n}{2^n}+\ldots$

2756. $\tan\frac{\pi}{4}+2\tan\frac{\pi}{8}+\ldots+n\tan\frac{\pi}{2^{n+1}}+\ldots$

2757. $\frac{2}{1}+\frac{2\,.\,5}{1\,.\,5}+\ldots+\frac{2\,.\,5\ldots(3n-1)}{1\,.\,5\ldots(4n-3)}+\ldots$

2758. $\frac{1}{3}+\frac{4}{9}+\ldots+\frac{n^2}{3^n}+\ldots$

2759. $\frac{1}{3}+\frac{1\,.\,3}{3\,.\,6}+\ldots+\frac{1\,.\,3\ldots(2n-1)}{3^n\,.\,n!}+\ldots$

2760. $\sin\frac{\pi}{2}+4\sin\frac{\pi}{4}+\ldots+n^2\sin\frac{\pi}{2^n}+\ldots$

2761. $\frac{1}{2!}+\frac{2}{3!}+\ldots+\frac{n}{(n+1)!}+\ldots$

2762. $\frac{2}{2}+\frac{2\,.\,3}{2\,.\,4}+\ldots+\frac{(n+1)!}{2^n\,.\,n!}+\ldots$

Prove the convergence of the series of problems 2763–2766 with the aid of Cauchy's test:

2763. $\frac{1}{\ln 2}+\frac{1}{\ln^2 3}+\ldots+\frac{1}{\ln^n(n+1)}+\ldots$

2764. $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\ldots+\left(\frac{n}{2n+1}\right)^n+\ldots$

2765. $\operatorname{arc\,sin} 1+\operatorname{arc\,sin}^2\frac{1}{2}+\ldots+\operatorname{arc\,sin}^n\frac{1}{n}+\ldots$

2766. $\frac{2}{3}+\frac{\left(\frac{3}{2}\right)^4}{9}+\ldots+\frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}+\ldots$

Solve the question of the convergence of the series of problems 2767–2770 with the aid of the integral test for convergence:

2767. $\frac{1}{2\ln^2 2}+\frac{2}{3\ln^2 3}+\ldots+\frac{1}{(n+1)\ln^2(n+1)}+\ldots$

2768. $\frac{1}{2\ln 2}+\frac{1}{3\ln 3}+\ldots+\frac{1}{n\ln n}+\ldots$

2769. $\left(\frac{1+1}{1+1^2}\right)^2+\left(\frac{1+2}{1+2^2}\right)^2+\ldots+\left(\frac{1+n}{1+n^2}\right)^2+\ldots$

2770. $\sum\limits_{n=2}^{n=\infty}\frac{1}{\sqrt{n}}\ln\frac{n+1}{n-1}$.

Examine the series of problems 2771–2784 for convergence or divergence:

2771. $\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\ldots+\frac{1}{(n+1)\sqrt{n+1}}+\ldots$

2772. $1+\frac{2}{3}+\ldots+\frac{n}{2n-1}+\ldots$

2773. $\sqrt{2}+\sqrt{\frac{3}{2}}+\ldots+\sqrt{\frac{n+1}{n}}+\ldots$

2774. $1+\frac{4}{1\,.\,2}+\ldots+\frac{n^2}{n!}+\ldots$

2775. $2+\frac{5}{8}+\ldots+\frac{n^2+1}{n^3}+\ldots$

2776. $\frac{1}{1001}+\frac{2}{2001}+\ldots+\frac{n}{1000n+1}+\ldots$

2777. $\frac{1}{1+1^2}+\frac{2}{1+2^2}+\ldots+\frac{n}{1+n^2}+\ldots$

2778. $\frac{1}{3}+\frac{3}{3^2}+\ldots+\frac{2n-1}{3^n}+\ldots$

2779. $\arctan 1+\arctan^2\frac{1}{2}+\ldots+\arctan^n\frac{1}{n}+\ldots$

2780. $2+\frac{4}{16}+\ldots+\frac{2^n}{n^4}+\ldots$

2781. $\frac{1}{1\,.\,3} + \frac{1}{6\,.\,7} + \ldots + \frac{1}{(5n-4)(4n-1)} + \ldots$

2782. $\frac{3}{2} + \frac{9}{8} + \ldots + \frac{3^n}{n\,.\,2^n} + \ldots$

2783. $1 + \frac{1\,.\,2}{2^2} + \ldots + \frac{n!}{n^n} + \ldots$

2784*. $\sin\frac{\pi}{2} + \sin\frac{\pi}{4} + \ldots + \sin\frac{\pi}{2n} + \ldots$

Prove each of the relationships of problems 2785–2789 with the aid of the series whose general term is the given function:

2785. $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0.$

2786. $\lim\limits_{n\to\infty} \frac{(2n)!}{a^{n!}} = 0 \ (a > 1).$

2787. $\lim\limits_{n\to\infty} \frac{n^n}{(2n)!} = 0.$

2788. $\lim\limits_{n\to\infty} \frac{n^n}{(n!)^2} = 0.$

2789. $\lim\limits_{n\to\infty} \frac{(n!)^n}{n^{n^2}} = 0.$

Series with Arbitrary Terms. Absolute Convergence

Examine the series of problems 2790–2799 for absolute convergence, non-absolute convergence, or divergence:

2790. $1 - \frac{1}{3} + \ldots + (-1)^{n+1}\frac{1}{2n-1} + \ldots$

2791. $1 - \frac{1}{3^3} + \ldots + (-1)^{n+1}\frac{1}{(2n-1)^3} + \ldots$

2792. $\frac{1}{\ln 2} - \frac{1}{\ln 3} + \ldots + (-1)^{n+1}\frac{1}{\ln(n+1)} + \ldots$

2793. $\frac{\sin\alpha}{1} + \frac{\sin 2\alpha}{4} + \ldots + \frac{\sin n\alpha}{n^2} + \ldots$

2794. $\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2^2} + \ldots + (-1)^{n+1}\frac{1}{n}\cdot\frac{1}{2^n} + \ldots$

2795. $2 - \frac{3}{2} + \ldots + (-1)^{n+1}\frac{n+1}{n} + \ldots$

2796. $-1 + \frac{1}{\sqrt{2}} - \ldots + (-1)^n \frac{1}{\sqrt{n}} + \ldots$

2797. $\frac{1}{2} - \frac{8}{4} + \ldots + (-1)^{n+1} \frac{n^3}{2^n} + \ldots$

2798. $\sum_{n=1}^{n=\infty} \frac{(-1)^n}{n - \ln n}$.

2799. $\sum_{n=1}^{n=\infty} \frac{2^{n^2}}{n!} (-1)^{n+1}$.

2800. Prove that, if the series $\sum_{n=1}^{n=\infty} a_n^2$ and $\sum_{n=1}^{n=\infty} b_n^2$ are convergent, the series $\sum_{n=1}^{n=\infty} a_n b_n$ is absolutely convergent.

2801. Prove that, if the series $\sum_{n=1}^{n=\infty} a_n$ is absolutely convergent, the series $\sum_{n=1}^{n=\infty} \frac{n+1}{n} a_n$ is also absolutely convergent.

2. Functional Series

Convergence of Functional Series

Find the domains of convergence of the series of problems 2802–2816:

2802. $1 + x + \ldots + x^n + \ldots$

2803. $\ln x + \ln^2 x + \ldots + \ln^n x + \ldots$

2804. $x + x^4 + \ldots + x^{n^2} + \ldots$

2805. $x + \frac{x^2}{2^2} + \ldots + \frac{x^n}{n^2} + \ldots$

2806. $x + \frac{x^2}{\sqrt{2}} + \ldots + \frac{x^n}{\sqrt{n}} + \ldots$

2807. $\frac{1}{1+x} + \frac{1}{1+x^2} + \ldots + \frac{1}{1+x^n} + \ldots$

2808. $2x + 6x^2 + \ldots + n(n+1)\, x^n + \ldots$

2809. $\frac{x}{2} + \frac{x^2}{2+\sqrt{2}} + \ldots + \frac{x^n}{n+\sqrt{n}} + \ldots$

2810. $\frac{x}{1+x^2} + \frac{x^2}{1+x^4} + \ldots + \frac{x^n}{1+x^{2n}} + \ldots$

2811. $\sin\frac{x}{2}+\sin\frac{x}{4}+\ldots+\sin\frac{x}{2^n}+\ldots$

2812. $x\tan\frac{x}{2}+x^2\tan\frac{x}{4}+\ldots+x^n\tan\frac{x}{2^n}+\ldots$

2813. $\sin x+\frac{\sin 2x}{2^2}+\ldots+\frac{\sin nx}{n^2}+\ldots$

2814. $\frac{\cos x}{e^x}+\frac{\cos 2x}{e^{2x}}+\ldots+\frac{\cos nx}{e^{nx}}+\ldots$

2815. $e^{-x}+e^{-4x}+\ldots+e^{-n^2x}+\ldots$

2816. $\frac{x}{e^x}+\frac{2x}{e^{2x}}+\ldots+\frac{nx}{e^{nx}}+\ldots$

Uniform Convergence

Verify that the series of problems 2817–2820 are uniformly convergent throughout the Ox axis:

2817. $1+\frac{\sin x}{1!}+\ldots+\frac{\sin nx}{n!}+\ldots$

2818. $\sum_{n=1}^{n=\infty}\frac{1}{n^2[1+(nx)^2]}$. **2819.** $\sum_{n=1}^{n=\infty}\frac{\sin nx}{2^n}$.

2820. $\sum_{n=1}^{n=\infty}\frac{e^{-n^2x^2}}{n^2}$.

2821. Show that the series $\frac{1}{1+[\varphi(x)]^2}+\frac{1}{4+[\varphi(x)]^2}+\ldots$ $\ldots+\frac{1}{n^2+[\varphi(x)]^2}+\ldots$ is uniformly convergent in any interval in which the function $\varphi(x)$ is defined.

2822. Show that the series $\frac{1}{\sqrt{1+x}}+\frac{1}{2\sqrt{1+2x}}+\ldots$ $+\ldots\frac{1}{2^{n-1}\sqrt{1+nx}}+\ldots$ is uniformly convergent throughout the positive semi-axis ($0\leqq x<\infty$). Given any non-negative x, how many terms of the series must be taken for the sum to be calculable to an accuracy of 0·001?

2823*. Show that the series $\frac{\ln(1+x)}{x} + \frac{\ln(1+2x)}{2x^2} + \ldots + \frac{\ln(1+nx)}{nx^n} + \ldots$ is uniformly convergent in any interval $1+\omega \leqq x < \infty$, where ω is any positive number. Verify that, for any x of the interval $(2 \leqq x \leqq 100)$, it is sufficient to take eight terms in order to obtain the sum of the series to an accuracy of 0·01.

2824. Prove that the series $\sum_{n=1}^{n=\infty} x^n(1-x^n)$ is non-uniformly convergent in the interval $[0, 1]$. (See *Course*, sec. 127).

2825. The function $f(x)$ is given by the series

$$f(x) = \sum_{n=1}^{n=\infty} \frac{\cos nx}{10^n}.$$

Show that $f(x)$ is defined and continuous for any x. Find $f(0)$, $f\left(\frac{\pi}{2}\right)$, and $f\left(\frac{\pi}{3}\right)$. Verify that it is necessary to take three terms of the series in order to compute the approximate values of $f(x)$ for any x to an accuracy of 0·001. Find to this accuracy $f(1)$ and $f(-0{\cdot}2)$.

2826. The function $f(x)$ is defined by the series

$$f(x) = \frac{1}{1+x^2} + \sum_{n=1}^{n=\infty} \frac{1}{1+(x+n\omega)^2} + \sum_{n=1}^{n=\infty} \frac{1}{1+(x-n\omega)^2} \quad (\omega > 0).$$

Show that $f(x)$ is defined and continuous for any x. Establish that $f(x)$ is a periodic function with period ω.

Integration and Differentiation of Series

2827. Show that the series $x^2 + x^6 + \ldots + x^{4n-2} + \ldots$ is uniformly convergent in any interval $-1+\omega \leqq x \leqq 1-\omega$, where ω is any positive number less than unity. By integrating this series, find the sum in the interval $(-1, 1)$ of the series

$$\frac{x^3}{3} + \frac{x^7}{7} + \ldots + \frac{x^{4n-1}}{4n-1} + \ldots$$

2828. Find the sum of the series

$$x + \frac{x^5}{5} + \ldots + \frac{x^{4n-3}}{4n-3} + \ldots$$

2829. Find the sum of the series

$$\frac{x^2}{1 \cdot 2} - \frac{x^3}{2 \cdot 3} + \ldots + (-1)^{n+1} \frac{x^{n+1}}{n(n+1)} + \ldots$$

2830. The function $f(x)$ is defined by the series

$$f(x) = e^{-x} + 2e^{-2x} + \ldots + ne^{-nx} + \ldots$$

Show that $f(x)$ is continuous throughout the positive half of the Ox axis. Evaluate $\int\limits_{\ln 2}^{\ln 3} f(x)\,dx$.

2831. Function $f(x)$ is defined by the series

$$f(x) = 1 + 2 \cdot 3x + \ldots + n3^{n-1}x^{n-1} + \ldots$$

Show that $f(x)$ is continuous in the interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$. Evaluate $\int\limits_{0}^{0.125} f(x)\,dx$.

2832*. Function $f(x)$ is defined by the series

$$f(x) = \frac{1}{2}\tan\frac{x}{2} + \frac{1}{4}\tan\frac{x}{4} + \ldots + \frac{1}{2^n}\tan\frac{x}{2^n} + \ldots$$

Evaluate $\int\limits_{\frac{\pi}{6}}^{\frac{\pi}{2}} f(x)\,dx$, having first established that $f(x)$ is continuous in the interval of integration.

2833*. Function $f(x)$ is defined by the series $f(x) = \sum_{n=1}^{n=\infty} \frac{1}{(n^4+x^2)}$. Show that $f(x)$ is continuous throughout the real axis. Evaluate $\int_1^\infty f(x)\,dx$.

2834. Starting from the relationship $\int_0^1 x^n\,dx = \frac{1}{n+1}$, find the sum of the series:

$$(1)\ 1 - \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{3n-2} + \ldots,$$

$$(2)\ 1 - \frac{1}{5} + \ldots + \frac{(-1)^{n+1}}{4n-3} + \ldots$$

2835. Starting from the relationship $\int_2^\infty \frac{dx}{x^{n+1}} = \frac{1}{n2^n}$, find the sum of the series $\frac{1}{1\,.\,2} + \frac{1}{2\,.\,2^2} + \ldots + \frac{1}{n2^n} + \ldots$

2836. Starting from the relationship

$$\int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx = \frac{\pi}{2} \cdot \frac{(2n-1)(2n-3)\ldots 3\,.\,1}{2n(2n-2)\ldots 4\,.\,2,}$$

find the sum of the series

$$\frac{1}{2} - \frac{1\,.\,3}{2\,.\,4} + \ldots + (-1)^{n+1}\frac{1\,.\,3\ldots(2n-1)}{2\,.\,4\ldots 2n} + \ldots$$

2837. Show that the series

$$\frac{\sin 2\pi x}{2} + \frac{\sin 4\pi x}{4} + \ldots + \frac{\sin 2^n \pi x}{2^n} + \ldots$$

is uniformly convergent throughout the real axis. Prove that this series cannot be differentiated term by term in any interval.

2838. Starting from the progression $1 + x + x^2 + \ldots = \frac{1}{1-x}$ ($|x| < 1$), sum the series $1 + 2x + 3x^2 + \ldots + nx^{n-1} + \ldots$ and the series $1 + 3x + \ldots + \frac{n(n+1)}{2}x^{n-1} + \ldots$ and show that the series $1 + 2x + \ldots + nx^{n-1} + \ldots$ is uniformly convergent in the interval $[-\varrho, \varrho]$, where $|\varrho| < 1$.

2839. Show that the equality

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \ldots + \frac{2^{n-1}x^{2^{n-1}-1}}{1+x^{2^{n-1}}} + \ldots = \frac{1}{1-x}$$

holds for $-1 < x < 1$.

2840. Verify that the function $y = f(x)$ defined by the series $x + x^2 + \frac{x^3}{2!} + \ldots + \frac{x^n}{(n-1)!} + \ldots$ satisfies the relationship $xy' = y(x+1)$.

3. Power Series

Expansion of Functions in Power Series

2841. Expand the function $y = \ln x$ in a Taylor series in the neighbourhood of the point $x = 1$ (with $x_0 = 1$) (see *Course*, sec. 130).

2842. Expand the function $y = \sqrt{x^3}$ in a Taylor series in the neighbourhood of the point $x = 1$.

2843. Expand the function $y = \frac{1}{x}$ in a Taylor series in the neighbourhood of the point $x = 3$.

2844. Expand the function $y = \sin\frac{\pi x}{4}$ in a Taylor series in the neighbourhood of the point $x = 2$.

Expand the functions of problems 2845–2849 in Taylor series in the neighbourhood of the point $x = 0$ (Maclaurin series):

2845. $y = \dfrac{e^x + e^{-x}}{2}$.

2846. $y = x^2 e^x$.

2847. $y = \cos(x + \alpha)$.

2848. $y = e^x \sin x$.

2849. $y = \cos x \cosh x$.

Find the first five terms of the Taylor series of the functions of problems 2850–2854 in the neighbourhood of the point $x = 0$.

2850. $y = \ln(1 + e^x)$.

2851. $y = e^{\cos x}$.

2852. $y = \cos^n x$.

2853. $y = -\ln \cos x$.

2854. $y = (1 + x)^x$.

By using the formulae for the Taylor expansions of functions e^x, $\sin x$, $\cos x$, $\ln(1 + x)$ and $(1 + x)^m$, expand the functions of problems 2855–2868 as Taylor series in the neighbourhood of the point $x = 0$:

2855. $y = e^{2x}$.

2856. $y = e^{-x^2}$

2857. $y = \begin{cases} \dfrac{e^x - 1}{x} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$

2858. $y = \begin{cases} \dfrac{e^{x^3} - e^{-x^3}}{2x^3} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$

2859. $y = \sin \dfrac{x}{2}$.

2860. $y = \cos^2 x$.

2861. $y = \begin{cases} \dfrac{\sin x}{x} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$

2862. $y = (x - \tan x) \cos x$.

2863. $y = \ln(10 + x)$.

2864. $y = x \ln(1 + x)$.

2865. $y = \sqrt{1 + x^2}$.

2866. $y = \sqrt[3]{8 - x^3}$.

2867. $y = \dfrac{1}{\sqrt[3]{1 + x^3}}$.

2868. $y = \dfrac{x^2}{\sqrt{1-x^2}}$.

2869. Expand the function $y = \dfrac{1+x}{(1-x)^3}$ in a Taylor series in the neighbourhood of the point $x = 0$. Use this expansion to find the sum of the series $1 + \frac{4}{2} + \ldots$ $\ldots + \frac{n^2}{2^{n-1}} + \ldots$

2870. By using the Taylor expansion of the function, find the values of:

(1) the seventh derivative of the function $y = \dfrac{x}{1+x^2}$ at $x = 0$,

(2) the fifth derivative of the function $y = x^2 \sqrt[4]{1+x}$ at $x = 0$,

(3) the tenth derivative of the function $y = x^6 e^x$ at $x = 0$,

(4) the curvature of the curve $y = x[\sqrt[3]{(1+x)^4} - 1]$ at the origin.

In problems 2871–2877, use the Taylor expansions of the functions to evaluate the limits:

2871. $\lim\limits_{x\to 0} \dfrac{x + \ln(\sqrt{1+x^2} - x)}{x^3}$.

2872. $\lim\limits_{x\to 0} \dfrac{2(\tan x - \sin x) - x^3}{x^5}$.

2873. $\lim\limits_{x\to 0} \dfrac{\ln(1+x+x^2) + \ln(1-x+x^2)}{x(e^x - 1)}$.

2874. $\lim\limits_{x\to \infty} \left[x - x^2 \ln\left(1 + \dfrac{1}{x}\right)\right]$.

2875. $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2} - \cot^2 x\right)$.

2876. $\lim\limits_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{\cot x}{x}\right)$.

2877. $\lim\limits_{x\to 0} \left(\dfrac{2 + \cos x}{x^3 \sin x} - \dfrac{3}{x^4}\right)$.

Interval of Convergence

Find the intervals of convergence of the power series of problems 2878–2889:

2878. $10x + 100x^2 + \ldots + 10^n x^n + \ldots$

2879. $x - \frac{x^2}{2} + \ldots + (-1)^{n+1}\frac{x^n}{n} + \ldots$

2880. $x + \frac{x^2}{20} + \ldots + \frac{x^n}{n \,.\, 10^{n-1}} + \ldots$

2881. $1 + x + \ldots + n!\, x^n + \ldots$

2882. $1 + 2x^2 + \ldots + 2^{n-1}x^{2(n-1)} + \ldots$

2883. $x - \frac{x^3}{3 \,.\, 3!} + \ldots + (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)\,.\,(2n-1)!} + \ldots$

2884. $1 + 3x + \ldots + (n-1)\, 3^{n-1}x^{n-1} + \ldots$

2885. $\frac{x}{1\,.\,2} + \frac{x^2}{2\,.\,3} + \ldots + \frac{x^n}{n(n+1)} + \ldots$

2886. $x + \frac{(2x)^2}{2!} + \ldots + \frac{(nx)^n}{n!} + \ldots$ (In studying the convergence at the right-hand end of the interval, use the fact that the factorials of large numbers can be approximately expressed in accordance with Stirling's formula:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

2887. $x + 4x^2 + \ldots + (nx)^n + \ldots$

2888. $\frac{\ln 2}{2}x^2 + \frac{\ln 3}{3}x^3 + \ldots + \frac{\ln(n+1)}{n+1}x^{n+1} + \ldots$

2889. $2x + \left(\frac{9}{4}x\right)^2 + \ldots + \left[\left(\frac{n+1}{n}\right)^n x\right]^n + \ldots$

2890. Expand the function $y = \ln(x + \sqrt{1 + x^2})$ in a Taylor series in the neighbourhood of the point $x = 0$, starting from the relationship

$$\ln\left(x + \sqrt{1 + x^2}\right) = \int\limits_0 \frac{dx}{\sqrt{1 + x^2}},$$

and find the interval of convergence of the series obtained.

2891. Expand the function $y = \ln \sqrt{\frac{1 + x}{1 - x}}$ in a Taylor series in the neighbourhood of the point $x = 0$, by starting from the relationship

$$\ln \sqrt{\frac{1 + x}{1 - x}} = \int\limits_0^x \frac{dx}{1 - x^2},$$

and find the interval of convergence of the series obtained.

2892. Expand the function $y = \ln\left[(1 + x)^{1+x}\right] + \ln\left[(1 - x)^{1-x}\right]$ in a Taylor series in the neighbourhood of the point $x = 0$ and find the interval of convergence of the series obtained.

2893. Expand the function $y = (1 + x)\, e^{-x} - (1 - x)\, e^{x}$ in a Taylor series in the neighbourhood of the point $x = 0$ and find the interval of convergence of the series obtained. Use the expansion to find the sum of the series.

$$\frac{1}{3!} + \frac{2}{5!} + \ldots + \frac{n}{(2n + 1)!} + \ldots$$

4. Some Applications of Taylor's Series

Finding Approximate Values of Functions

2894. Find the approximate value of $\sqrt[3]{e}$ by taking three terms of the Taylor expansion of $f(x) = e^x$, and estimate the error.

2895. Find the approximate value of sin 18° by taking three terms of the Taylor expansion of $f(x) = \sin x$, and estimate the error.

2896. Find the approximate value of $\sqrt[3]{10}$ by taking four terms of the Taylor expansion of the function $f(x) = (1 + x)^m$, and estimate the error.

In problems 2897–2904, use the formulae for the Taylor expansions of the functions e^x, $\sin x$ and $\cos x$ to find:

2897. e^2 to an accuracy of 0·001.

2898. $\sqrt{e}$ to an accuracy of 0·001.

2899. $\frac{1}{e}$ to an accuracy of 0·0001.

2900. $\frac{1}{\sqrt[4]{e}}$ to an accuracy of 0·0001.

2901. $\sin 1°$ to an accuracy of 0·0001.

2902. $\cos 1°$ to an accuracy of 0·001.

2903. $\sin 10°$ to an accuracy of 0·00001.

2904. $\cos 10°$ to an accuracy of 0·0001.

In problems 2905–2911, use the formula for the Taylor expansion of the function $(1 + x)^m$ to find to an accuracy of 0·001:

2905. $\sqrt[3]{30}$.

2906. $\sqrt[3]{70}$.

2907. $\sqrt[3]{500}$.

2908. $\sqrt[3]{1{\cdot}015}$.

2909. $\sqrt[5]{250}$.

2910. $\sqrt[3]{129}$.

2911. $\sqrt[10]{1027}$.

In problems 2912–2914, use the formula for the Taylor expansion of the function $\ln \frac{(1+x)}{(1-x)}$ to find:

2912. $\ln 3$ to an accuracy of 0·0001.

2913. $\log e = \frac{1}{\ln 10}$ to an accuracy of 0·000001.

2914. $\log 5$ to an accuracy of 0·0001.

The Solution of Equations

2915. Given the equation $xy + e^x = y$, use the method of undetermined coefficients to find the expansion of function y in a Taylor series in powers of x. Solve the problem by finding the coefficients of the Taylor series by successive differentiation.

2916. Given the equation $y = \ln(1 + x) - xy$, use the method of undetermined coefficients to find the expansion of function y in a Taylor series in powers of x. Solve the problem by finding the coefficients of the Taylor series by successive differentiation.

Solve the equations of problems 2917–2919 with respect to y (find an explicit expression for y) with the aid of Taylor series by two methods: the method of undetermined coefficients and successive differentiation:

2917. $y^3 + xy = 1$ (find three terms of the expansion).

2918. $2 \sin x + \sin y = x - y$ (find two terms of the expansion).

2919. $e^x - e^y = xy$ (find three terms of the expansion).

Integration of Functions

Express the integrals of problems 2920–2929 in the form of a series by using the expansions of the integrands into series; indicate the domains of convergence of the series obtained.

2920. $\int \frac{\sin x}{x}\, dx$.

2921. $\int \frac{\cos x}{x}\, dx$.

2922. $\int \frac{e^x}{x}\, dx$.

2923. $\int \frac{e^x}{x^2}\, dx$.

2924. $\int_0^x e^{-x^2}\, dx$.

2925. $\int_0^x \frac{\arctan x}{x}\, dx$.

2926. $\int_0^x \frac{dx}{\sqrt{1 - x^4}}$.

2927. $\int_0^x \sqrt{1 + x^3}\, dx$.

2928. $\int\limits_0^x \frac{dx}{1-x^9}$.

2929. $\int\limits_0^x \frac{\sqrt[4]{1+x^4}-1}{x^2}\,dx$.

Obtain approximate values for the definite integrals of problems 2930–2934 by taking the indicated number of terms of the expansion of the integrand; indicate the error:

2930. $\int\limits_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos x}{x}\,dx$ (3 terms).

2931. $\int\limits_0^{\frac{1}{4}} e^{-x^2}\,dx$ (3 terms).

2932. $\int\limits_0^{\frac{1}{2}} \frac{dx}{\sqrt{1+x^4}}$ (2 terms).

2933. $\int\limits_{0.1}^{1} \frac{e^x}{x}\,dx$ (6 terms).

2934. $\int\limits_0^{\frac{\sqrt{3}}{3}} x^3 \arctan x\,dx$ (2 terms).

Evaluate the integrals of problems 2935–2938 to an accuracy of 0·001:

2935. $\int\limits_{0.1}^{0.2} \frac{e^{-x}}{x^3}\,dx$.

2936. $\int\limits_0^{0.5} \frac{\arctan x}{x}\,dx$.

2937. $\int\limits_0^{0.8} x^{10} \sin x\,dx$.

2938. $\int\limits_0^{0.5} \frac{dx}{1+x^4}$.

2939. Show that the functions $\int\limits_0^x e^{-x^2}\,dx$ and $\arctan x - \frac{x^5}{10}$ differ by not more than 0·0000001 in the interval (—0·1, 0 1).

2940. By taking into account the identity

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239},$$

evaluate π correct to 10 figures.

2941. Expand in a Taylor series the function $y = e^{x^2}\int\limits_0^x e^{-x^2}\,dx$ by using two methods: direct evaluation of the successive derivatives at $x = 0$ and cross-multiplication of the series.

2942*. Evaluate the integral $\int\limits_0^1 x^x\,dx$.

5. Numerical Problems

2943. Evaluate $\int\limits_0^{0\cdot 5} e^{\sin x}\,dx$ to an accuracy of 0·0001.

2944. Evaluate $\int\limits_0^{\frac{\pi}{6}} \sqrt{\cos x}\,dx$ to an accuracy of 0·001.

2945. Evaluate the area bounded by the curve $y^2 = x^3 + 1$, the axis of ordinates and the straight line $x = \frac{1}{2}$ to an accuracy of 0·001.

2946*. Evaluate the area of the oval $x^4 + y^4 = 1$ to an accuracy of 0·01.

2947. Evaluate the length of arc of the curve $25y^2 = 4x^5$ from the cusp to the point of intersection with the parabola $5y = x^2$ to an accuracy of 0·0001.

2948. Evaluate the length of one half wave of the sine wave $y = \sin x$ to an accuracy of 0·001.

2949. The figure bounded by the curve $y = \operatorname{arc\,tan} x$, the axis of abscissae and the straight line $x = \frac{1}{2}$, revolves about the axis of abscissae. Find the volume of the solid of revolution to an accuracy of 0·001.

2950. The figure bounded by the curves $y^3 - x^3 = 1$, $4y + x^3 = 0$, the straight line $y = \frac{1}{2}$ and the axis of ordinates, revolves about the axis of ordinates. Find the volume of the solid of revolution to an accuracy of 0·001.

2951. Find the coordinates of the centre of gravity of the arc of the hyperbola $y = \frac{1}{x}$, bounded by the points with abscissae $x_1 = \frac{1}{4}$ and $x_2 = \frac{1}{2}$, to an accuracy of 0.001.

2952. Find the coordinates of the centre of gravity of the curvilinear trapezium, bounded by the curve $y = \frac{1}{\ln x}$, the straight lines $x = 1{\cdot}5$ and $x = 2$ and the axis of abscissae, to an accuracy of 0·01.

CHAPTER X

FUNCTIONS OF SEVERAL VARIABLES. DIFFERENTIAL CALCULUS

1. Functions of Several Variables

2953. Express the volume z of a cone as a function of its generator x and height y.

2954. Express the area S of a triangle as a function of its three sides x, y, z.

2955. Form a table of the values of function $z = 2x - 3y + 1$ by giving the independent variables integral values from 0 to 5.

2956. Form a table of values of the function

$$z = \sqrt{x^2 + y^2},$$

by giving the independent variables values spaced 0·1 apart from 0 to 1. Calculate the values of the function to an accuracy of 0·01.

2957. Find the particular value of the function:

(1) $z = \left(\dfrac{\arctan(x+y)}{\arctan(x-y)}\right)^2$ for $x = \dfrac{1+\sqrt{3}}{2}$, $y = \dfrac{1-\sqrt{3}}{2}$;

(2) $z = e^{\sin(x+y)}$ for $x = y = \dfrac{\pi}{2}$;

(3) $z = y^{x^2-1} + x^{y^2-1}$ for $x = 2$, $y = 2$; $x = 1$, $y = 2$; $x = 2$, $y = 1$.

2958. Given the function

$$F(x, y) = \frac{\varphi(x)\,\psi(y) - \psi(x)\,\varphi(y)}{\varphi(xy)\,\psi(xy)},$$

find $F\left(a, \frac{1}{a}\right)$. In particular, put $\varphi(u) = u^3$, $\psi(u) = u^2$ and work out $F\left(a, \frac{1}{a}\right)$.

2959. Given the function $F(x, y) = y^x - \frac{1}{2} x^y$; if x and y have the same rate of change, which function increases the more rapidly for $x = 3$, $y = 2$: the function obtained from F with fixed y (x only varies), or that obtained with fixed x (y only varies)?

2960. Given the function

$$\varphi(x, y, z) = y^2 - (y \cos z + z \cos y)\, x + x^{\frac{y+z}{y-z}},$$

let variables y and z preserve fixed values y_0 and z_0, where $y_0 = 3z_0$. What is the graph of the function $v = \varphi(x, y_0, z_0)$? Is $\varphi(x, y, z)$: (1) a rational function of y? of z? (2) an integral function of x?

2961*. A function $z = f(x, y)$, satisfying identically the relationship

$$f(mx, my) = m^k f(x, y) \text{ for any } m,$$

is called a homogeneous function of the kth degree. Show that the homogeneous function of the kth degree $z = f(x, y)$ can always be written as

$$z = x^k F\left(\frac{y}{x}\right).$$

2962. The homogeneity of a function of any number of independent variables is defined in the same way as for a function of two variables: for instance, $f(x, y, z)$ is a homogeneous function of the kth degree if

$$f(mx, my, mz) = m^k f(x, y, z) \text{ for any } m.$$

The property

$$f(x, y, z) = x^k F\left(\frac{y}{x}, \frac{z}{x}\right)$$

also holds. Prove this.

2963. Prove that the function $z = F(x, y) = xy$ satisfies the functional equation

$$F(ax + bu, cy + dv) = acF(x, y) + bcF(u, y) + \\ + adF(x, v) + bdF(u, v).$$

2964. Prove that the function $z = F(x, y) = \ln x \ln y$ satisfies the functional equation

$$F(xy, uv) = F(x, u) + F(x, v) + F(y, u) + F(y, v)$$
$$(x, y, u, v \text{ are positive}).$$

2965. Find z as an explicit function of x and y from the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Is this function single-valued?

2966. Given the function of a function $z = u^v$, where $u = x + y$, $v = x - y$, find the particular value of the function: (1) when $x = 0$, $y = 1$; (2) when $x = 1$, $y = 1$; (3) when $x = 2$, $y = 3$; (4) when $x = 0$, $y = 0$; (5) when $x = -1$, $y = -1$.

2967. $z = \frac{u + v}{uv}$; $u = w^t$; $v = w^{-t}$; $w = \sqrt{x + y}$; $t = 2(x - y)$. Express z directly as a function of x and y. Is z a rational function of u and v? of w and t? of x and y?

2968. Given the function of a function $z = u^w + w^{u+v}$, where $u = x + y$, $v = x - y$, $w = xy$, express z directly as a function of x and y.

2969. $u = (\xi + \eta)^2 - \xi^3 - \eta^3$; $\xi = \frac{e^\omega + e^\varphi}{2}$; $\eta = \frac{e^\omega + e^\varphi}{2}$; $\omega = \ln(x^2 + y^2 + z^2)$; $\varphi = 2\ln(x + y + z)$. Express u directly as a function of x, y and z. Is u an integral rational function of ξ and η? of ω and φ? of x, y, z?

2970. Write the function of a function

$$z = \left(\frac{x^2 + xy + y^2}{x^2 - xy + y^2}\right)^{xy} + x^2 + y^2$$

by means of a two-link chain of relationships.

2971. Investigate by the method of sections the "graph" of the function $z = \frac{1}{2}(x^2 - y^2)$. What are the sections by the planes $x = \text{const}$? $y = \text{const}$? $z = \text{const}$?

2972. Investigate by the method of sections the "graph" of the function $z = xy$. What are the sections by the planes $x = \text{const}$? $y = \text{const}$? $z = \text{const}$?

2973. Investigate by the method of sections the "graph" of the function $z = y^2 - x^3$.

2974. Investigate by the method of sections the "graph" of the function

$$z^3 = ax^2 + by^2 \quad (a > 0, b > 0).$$

2. Elementary Investigation of a Function

Domain of Definition

2975. A domain is bounded by the parallelogram with sides $y = 0$, $y = 2$, $y = \frac{1}{2}x$, $y = \frac{1}{2}x - 1$; the boundary itself is excluded. Give this domain by means of inequalities.

2976. A domain consists of the figure bounded by the parabolas $y = x^2$ and $x = y^2$ (including the boundary). Specify this domain by inequalities.

2977. Write with the aid of inequalities the open domain consisting of the equilateral triangle with a vertex at the origin and side a, one of the sides being in the direction of positive x.

2978. A domain is bounded by an infinite circular cylinder of radius R (excluding the boundary) with axis parallel to Oz and passing through the point (a, b, c). Specify this domain with the aid of inequalities.

2979. Write with the aid of inequalities the domain bounded by the sphere of radius R with centre at the point (a, b, c) (the boundary included).

2980. The vertices of a right-angled triangle lie inside a circle of radius R. The area S of the triangle is a function of its adjacent sides x and y: $S = \varphi(x, y)$. (a) What is the domain of definition of function φ? (b) What is the domain of definiteness of the corresponding analytic expression?

2981. A pyramid with a rectangular base, the vertex of which projects orthogonally into the point of intersection of the base diagonals, is inscribed in a sphere of radius R. The volume V of the pyramid is a function of sides x and y of its base. Is this function single-valued? Form the analytic expression for it. Find the domain of definition of the function and the domain of definiteness of the corresponding analytic expression.

2982. A square board consists of four square chequers, two black and two white, as shown in Fig. 57; the side of each of

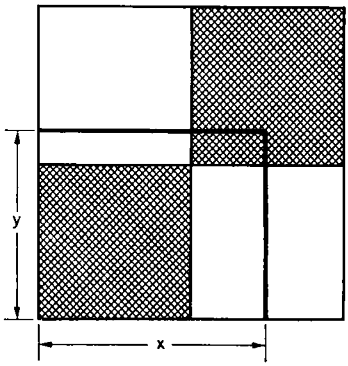

FIG. 57.

them is equal to unit length. We take the rectangle whose sides x and y are parallel to the sides of the board and one of the corners of which coincides with a black corner. The area of the black part of this rectangle will be a function of x and y. What is the domain of definition of this function? Express this function analytically.

Find the domains of definition of the functions of problems 2983–3002:

2983. $z = \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$. **2984.** $z = \ln(y^2 - 4x + 8)$.

2985. $z = \frac{1}{R^2 - x^2 - y^2}$. **2986.** $z = \sqrt{x + y} + \sqrt{x - y}$.

2987. $z = \frac{1}{\sqrt{x + y}} + \frac{1}{\sqrt{x - y}}$.

2988. $z = \arcsin \frac{y - 1}{x}$. **2989.** $z = \ln xy$.

2990. $z = \sqrt{x - \sqrt{y}}$.

2991. $z = \arcsin \frac{x^2 + y^2}{4} + \operatorname{arc\,sec}(x^2 + y^2)$.

2992. $z = \frac{\sqrt{4x - y^2}}{\ln(1 - x^2 - y^2)}$. **2993.** $z = \sqrt{\frac{x^2 + 2x + y^2}{x^2 - 2x + y^2}}$.

2994. $z = xy + \sqrt{\ln \frac{R^2}{x^2 + y^2}} + \sqrt{x^2 + y^2 - R^2}$.

2995. $z = \cot \pi(x + y)$. **2996.** $z = \sqrt{\sin \pi(x^2 + y^2)}$

2997. $z = \sqrt{x \sin y}$. **2998.** $z = \ln x - \ln \sin y$.

2999. $z = \ln[x \ln(y - x)]$.

3000. $z = \arcsin[2y(1 + x^2) - 1]$.

3001. $u = \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}}$.

3002. $u = \sqrt{R^2 - x^2 - y^2 - z^2} + \frac{1}{\sqrt{x^2 + y^2 + z^2 - r^2}} (R > r)$.

Limits. Continuity of a Function

Work out the limits of the functions of problems 3003–3008 on the assumption that the independent variables approach their limiting values in an arbitrary manner:

3003. $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$.

3004. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{\sqrt{x^2y^2+1}-1}{x^2+y^2}$.

3005. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{\sin(x^3+y^3)}{x^2+y^2}$.

3006. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{1-\cos(x^2+y^2)}{(x^2+y^2)\,x^2y^2}$.

3007. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4}$.

3008. $\lim\limits_{\substack{x\to 0\\ y\to 0}} (1+x^2y^2)^{\frac{1}{x^2+y^2}}$.

3009. Verify that the function $u=\dfrac{x+y}{x-y}$ can tend to any limit as $x\to 0$, $y\to 0$ (depending on how x and y tend to zero). Give examples of variations of x and y such that: (a) $\lim u=1$, (b) $\lim u=2$.

3010. Find the point of discontinuity of the function $z=\dfrac{2}{x^2+y^2}$. How does the function behave in the neighbourhood of the point of discontinuity?

3011. Find the point of discontinuity of the function $z=\dfrac{1}{\sin^2\pi x+\sin^2\pi y}$.

3012. Where is the function $z=\dfrac{1}{(x-y)}$ discontinuous?

3013. Where is the function $z=\dfrac{1}{\sin\pi x}+\dfrac{1}{\sin\pi y}$ discontinuous?

3014. Where is the function $z=\dfrac{(y^2+2x)}{(y^2-2x)}$ discontinuous?

3015*. Investigate the continuity of the functions at $x=0$, $y=0$:

(1) $f(x, y)=\dfrac{x^2y^2}{x^2+y^2}$; $f(0, 0)=0$.

(2) $f(x, y)=\dfrac{xy}{x^2+y^2}$; $f(0, 0)=0$.

(3) $f(x, y)=\dfrac{x^3y^3}{x^2+y^2}$; $f(0, 0)=0$.

(4) $f(x, y) = \frac{1}{x^2 + y^2}$; $f(0, 0) = 0$.

(5) $f(x, y) = \frac{x^4 - y^4}{x^4 + y^4}$; $f(0, 0) = 0$.

(6) $f(x, y) = \frac{x^2y^2}{x^4 + y^4}$; $f(0, 0) = 0$.

Level Lines and Surfaces

3016. Given the function $z = f(x, y) = \frac{1}{x^2 + y^2}$, construct a uniform net of curves of it for $z = 1, 2, 3, 4$.

3017. The function $z = f(x, y)$ is given as follows: at the point $P(x, y)$ its value is equal to the angle subtended at this point by a segment AB given in the xOy plane. Find the level lines of function $f(x, y)$.

Trace the level lines of the functions of problems 3018–3021 by assigning to z values 1 apart from -5 to $+5$:

3018. $z = xy$.

3019. (2762). $z = x^2y + x$.

3020. $z = y(x^2 + 1)$.

3021. (2764). $z = \frac{xy - 1}{x^2}$.

3022. Draw the level lines of the function $z = (x^2 + y^2)^2 - 2(x^2 - y^2)$, by assigning to z values every $\frac{1}{2}$ from -1 to $\frac{3}{2}$.

3023. Draw the level lines of the function z given implicitly by the equation $\left(\frac{3}{2}\right)^z [(x - 5)^2 + y^2] = \left(\frac{2}{3}\right)^z [(x + 5)^2 + y^2]$, by giving z values unity apart from -4 to 4.

3024. Draw the level lines of the function z given implicitly by the equation $y^2 = 2^{-z}(x - z)$, by giving z values $\frac{1}{2}$ apart from -3 to 3.

3025. Find the level lines of the function z given implicitly by the equation $z + x \ln z + y = 0$.

3026. A point A is given in space. The distance of a variable point M from point A is a function of the coordinates of M. Find the level surfaces of this function, corresponding to distances equal to 1, 2, 3, 4.

3027. A function $u = f(x, y, z)$ is specified as follows: its value at the point $P(x, y, z)$ is equal to the sum of the distances of this point from two given points: $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$. Find the level surfaces of function $f(x, y, z)$.

3028. Find the level surfaces of the function

$$u = \ln \frac{1 + \sqrt{x^2 + y^2 + z^2}}{1 - \sqrt{x^2 + y^2 + z^2}}.$$

3029. Find the level surfaces of the function

$$u = \frac{x^2 + y^2}{z}.$$

3030. Find the level surfaces of the functions:

(1) $u = 5^{2x+3y-z}$, (2) $u = \tan(x^2 + y^2 - 2z^2)$.

3031. Figure 58 illustrates the level lines of a function $z = f(x, y)$. Construct the graph of the functions:

(1) $z = f(x, 0)$; (2) $z = f(x, 4)$; (3) $z = f(1, y)$;
(4) $z = f(-5, y)$; (5) $z = f(x, 3x)$; (6) $z = f(x, x^2)$.

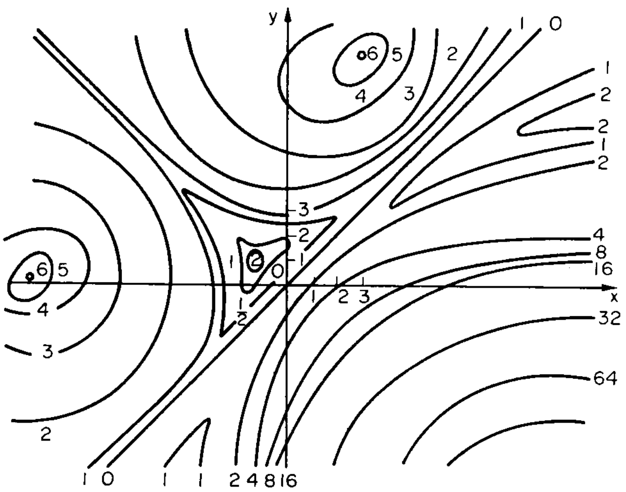

FIG. 58.

3. Derivatives and Differentials of Functions of Several Variables

Partial Derivatives

3032. The volume v of a gas is a function of its temperature and pressure: $v = f(p, T)$. The mean coefficient of expansion of the gas at constant pressure, for a temperature variation from T_1 to T_2, is defined as $\frac{v_2 - v_1}{v_1(T_2 - T_1)}$. How should we define the coefficient of expansion at constant pressure at a given temperature T_0?

3033. The temperature of a given point A of a rod Ox is a function of the abscissa x of A and time t: $\theta = f(x, t)$. What are the physical significances of partial derivatives $\frac{\partial \theta}{\partial t}$ and $\frac{\partial \theta}{\partial x}$?

3034. The area S of a rectangle is given in terms of the base b and height h by the formula $S = bh$. Find $\frac{\partial S}{\partial h}, \frac{\partial S}{\partial b}$ and explain the geometrical meaning of the results.

3035. Given the two functions $u = \sqrt{a^2 - x^2}$ (a is constant) and $z = \sqrt{y^2 - x^2}$, find $\frac{du}{dx}$ and $\frac{\partial z}{\partial x}$. Compare the results.

Find the partial derivatives with respect to each of the independent variables of the functions of problems 3036–3084 ($x, y, z, u, v, t, \varphi$ and ψ are variables):

3036. $z = x - y$. **3037.** $z = x^3y - y^3x$.

3038. $\theta = axe^{-t} + bt$ (a, b constants).

3039. $z = \frac{u}{v} + \frac{v}{u}$. **3040.** $z = \frac{x^3 + y^3}{x^2 + y^2}$.

3041. $z = (5x^2y - y^3 + 7)^3$. **3042.** $z = x\sqrt{y} + \frac{y}{\sqrt[3]{x}}$.

3043. $z = \ln (x + \sqrt{x^2 + y^2})$.

3044. $z = \arctan \frac{x}{y}$. **3045.** $z = \frac{1}{\arctan \frac{y}{x}}$.

3046. $z = x^y$. **3047.** $z = \ln (x^2 + y^2)$.

3048. $z = \ln \dfrac{\sqrt{x^2 + y^2} - x}{\sqrt{x^2 + y^2} + x}$. **3049.** $z = \arcsin \dfrac{\sqrt{x^2 - y^2}}{\sqrt{x^2 + y^2}}$.

3050. $z = \ln \tan \dfrac{x}{y}$. **3051.** $z = e^{-\frac{x}{y}}$.

3052. $z = \ln (x + \ln y)$. **3053.** $u = \arctan \dfrac{v + w}{v - w}$.

3054. $z = \sin \dfrac{x}{y} \cos \dfrac{y}{x}$. **3055.** $z = \left(\dfrac{1}{3}\right)^{\frac{y}{x}}$.

3056. $z = (1 + xy)^y$. **3057.** $z = xy \ln (x + y)$.

3058. $z = x^{x^y}$. **3059.** $u = xyz$.

3060. $u = xy + yz + zx$. **3061.** $u = \sqrt{x^2 + y^2 + z^2}$.

3062. $u = x^3 + yz^2 + 3yx - x + z$.

3063. $w = xyz + yzv + zvx + vxy$.

3064. $u = e^{x(x^2+y^2+z^2)}$. **3065.** $u = \sin (x^2 + y^2 + z^2)$.

3066. $u = \ln (x + y + z)$. **3067.** $u = x^{\frac{y}{z}}$.

3068. $u = x^{y^z}$.

3069. $f(x, y) = x + y - \sqrt{x^2 + y^2}$.

3070. $z = \ln \left(x + \dfrac{y}{2x}\right)$. **3071.** $z = (2x + y)^{2x+y}$.

3072. $z = (1 + \log_y x)^3$. **3073.** $z = xye^{\sin \pi xy}$.

3074. $z = (x^2 + y^2) \dfrac{1 - \sqrt{x^2 + y^2}}{1 + \sqrt{x^2 + y^2}}$.

3075. $z = \arctan \sqrt{x^y}$. **3076.** $z = 2\sqrt{\dfrac{1 - \sqrt{xy}}{1 + \sqrt{xy}}}$.

3077. $z = \ln [xy^2 + yx^2 + \sqrt{1 + (xy^2 + yx^2)^2}]$.

3078. $z = \sqrt{1 - \left(\dfrac{x + y}{xy}\right)^2} + \arcsin \dfrac{x + y}{xy}$.

3079. $z = \arctan \left(\arctan \dfrac{y}{x}\right) - \dfrac{1}{2} \dfrac{\arctan \dfrac{y}{x} - 1}{\arctan \dfrac{y}{x} + 1} - \arctan \dfrac{y}{x}$.

3080. $u = \dfrac{k}{(x^2 + y^2 + z^2)^2}$. **3081.** $u = \operatorname{arc\,tan}\,(x - y)^z$.

3082. $u = x^{\frac{y}{z}}$.

3083. $u = \ln \dfrac{1 - \sqrt{x^2 + y^2 + z^2}}{1 + \sqrt{x^2 + y^2 + z^2}}$.

3084. $w = \dfrac{1}{2}\tan^2 (x^2y^2 + z^2v^2 - xyzv) + \ln\cos (x^2y^2 +$
$+ z^2v^2 - xyvz)$.

3085. $u = \dfrac{\cos(\varphi - 2\psi)}{\cos(\varphi + 2\psi)}$. Find $\left(\dfrac{\partial u}{\partial \psi}\right)_{\substack{\varphi=\frac{\pi}{4}\\ \psi=\pi}}$.

3086. $u = \sqrt{az^3 - bt^3}$. Find $\dfrac{\partial u}{\partial z}$ and $\dfrac{\partial u}{\partial t}$ at $z = b$, $t = a$.

3087. $z = \dfrac{x\cos y - y\cos x}{1 + \sin x + \sin y}$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at $x = y = 0$.

3088. $u = \sqrt{\sin^2 x + \sin^2 y + \sin^2 z}$. Find $\left(\dfrac{\partial u}{\partial z}\right)_{\substack{x=0\\ y=0\\ z=\frac{\pi}{4}}}$

3089. $u = \ln(1 + x + y^2 + z^3)$. Find $u'_x + u'_y + u'_z$ at $x = y = z = 1$.

3090. $f(x, y) = x^3y - y^3x$. Find $\left(\dfrac{\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial f}{\partial y}}\right)_{\substack{x=1\\ y=2}}$.

3091. What angle does the tangent to the curve

$$\begin{cases} z = \dfrac{x^2 + y^2}{4}, \\ y = 4 \end{cases}$$

at the point (2, 4, 5) form with the positive direction of the axis of abscissae?

3092. What angle does the tangent to the curve

$$\begin{cases} z = \sqrt{1 + x^2 + y^2} \\ x = 1 \end{cases}$$

at the point $(1,\ 1,\ \sqrt{3})$ form with the positive direction of the axis of ordinates?

3093. What is the angle of intersection of the plane curves obtained as a result of the intersection of the surfaces

$$z = x^2 + \frac{y^2}{6} \text{ and } z = \frac{x^2 + y^2}{3}$$

with the plane $y = 2$?

Differentials. Approximations

Find the partial differentials with respect to each of the independent variables of the functions of problems 3094–3097:

3094. $z = xy^3 - 3x^2y^2 + 2y^4$.

3095. $z = \sqrt{x^2 + y^2}$.

3096. $z = \dfrac{xy}{x^2 + y^2}$.

3097. $u = \ln (x^3 + 2y^3 - z^3)$.

3098. $z = \sqrt[3]{x + y^2}$. Find $d_y z$ for $x = 2$, $y = 5$, $\Delta y = 0{\cdot}01$.

3099. $z = \sqrt{\ln xy}$. Find $d_x z$ for $x = 1$, $y = 1{\cdot}2$, $\Delta x = 0{\cdot}016$.

3100. $u = p - \dfrac{qr}{p} + \sqrt{p + q + r}$. Find $d_p u$ for $p = 1$, $q = 3$, $r = 5$, $\Delta p = 0{\cdot}01$.

Find the total differentials of the functions of problems 3101–3109:

3101. $z = x^2y^4 - x^3y^3 + x^4y^2$.

3102. $z = \dfrac{1}{2} \ln (x^2 + y^2)$.

3103. $z = \dfrac{x + y}{x - y}$.

3104. $z = \arcsin \dfrac{x}{y}$.

3105. $z = \sin (xy)$.

3106. $z = \arctan \dfrac{x + y}{1 - xy}$.

3107. $z = \dfrac{x^2 + y^2}{x^2 - y^2}$.

3108. $z = \arctan (xy)$.

3109. $u = x^{yz}$.

Applications to Computations

3110. Find the value of the total differential of the function $z = x + y - \sqrt{x^2 + y^2}$ for $x = 3$, $y = 4$, $\Delta x = 0{\cdot}1$, $\Delta y = 0{\cdot}2$.

3111. Find the value of the total differential of the function $z = e^{xy}$ for $x = 1$, $y = 1$, $\Delta x = 0{\cdot}15$, $\Delta y = 0{\cdot}1$.

3112. Find the value of the total differential of the function $z = \dfrac{xy}{x^2 - y^2}$ for $x = 2$, $y = 1$, $\Delta x = 0{\cdot}01$, $\Delta y = 0{\cdot}03$.

3113. Work out approximately the variation of the function $z = \dfrac{x + 3y}{y - 3x}$ when x varies from $x_1 = 2$ to $x_2 = 2{\cdot}5$ and y from $y_1 = 4$ to $y_2 = 3{\cdot}5$.

3114. Evaluate approximately

$$\ln\left(\sqrt[3]{1{\cdot}03} + \sqrt[4]{0{\cdot}98} - 1\right).$$

3115. Work out approximately $1{\cdot}04^{2{\cdot}02}$.

3116. Find the length of the segment of the straight line $x = 2$, $y = 3$ lying between the surface $z = x^2 + y^2$ and its tangent plane at the point (1, 1, 2).

3117. A body weighs $(4{\cdot}1 \pm 0{\cdot}1)$ g in air and $(1{\cdot}8 \pm 0{\cdot}2)$ g in water. Find the specific weight of the body and indicate the error in the working.

3118. The base radius of a cone is equal to $10{\cdot}2 \pm 0{\cdot}1$ cm, the generator is equal to $44{\cdot}6 \pm 0{\cdot}1$ cm. Find the volume of the cone and indicate the error in the working.

3119. The formula

$$S = \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin (B + C)}$$

is used for calculating the area S of a triangle with side a and angles B, C. Find the relative error δ'_S in calculating S if the relative errors of the given elements are respectively δ'_a, δ'_B, δ'_C.

3120. One side of a triangle has a length of $2{\cdot}4$ m and increases at a rate of 10 cm/sec; the second side, of length

1·5 m, diminishes at a rate of 5 cm/sec. The angle included between these sides is equal to 60° and increases at a rate of 2° per sec. How, and at what rate, does the area of the triangle vary?

3121. The frustum of a cone has base radii $R = 30$ cm, $r = 20$ cm, and height $h = 40$ cm. How does the volume of the frustum vary if R increases by 3 mm, r by 4 mm, h by 2 mm?

3122. Show that, when calculating the period T of vibration of a pendulum in accordance with the formula $T = \pi\sqrt{\frac{l}{g}}$ (l is the length of the pendulum, g the acceleration due to gravity), the relative error is equal to half the sum of the relative errors in the determination of l and g (all the errors are assumed sufficiently small).

3123. Express the maximum error when evaluating the radius r of arc AB (Fig. 59) of a circle in terms of the errors ds and dp in measuring chord $2s$ and length p. Work out dr when $2s = 19{\cdot}45$ cm $\pm 0{\cdot}5$ mm, $p = 3{\cdot}62$ cm $\pm 0{\cdot}3$ mm.

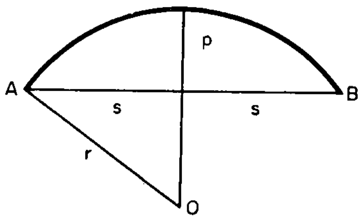

FIG. 59.

4. Differentiation of Functions

Functions of a Function†

3124. $u = e^{x-2y}$, where $x = \sin t$, $y = t^3$; $\frac{du}{dt} = ?$

3125. $u = z^2 + y^2 + zy$, $z = \sin t$, $y = e^t$; $\frac{du}{dt} = ?$

† The numbering of the problems in this edition differs from that of the previous editions as from this article to the end of Chap. X.

3126. $z = \arcsin(x - y),\ x = 3t,\ y = 4t^3;\ \dfrac{dz}{dt} = ?$

3127. $z = x^2y - y^2x$, where $x = u\cos v,\ y = u\sin v;\ \dfrac{\partial z}{\partial z} = ?$ $\dfrac{\partial z}{\partial v} = ?$

3128. $z = x^2 \ln y,\ x = \dfrac{u}{v},\ y = 3u - 2v;\ \dfrac{\partial z}{\partial u} = ?\ \dfrac{\partial z}{\partial v} = ?$

3129. $u = \ln(e^x + e^y);\ \dfrac{\partial u}{\partial x} = ?$ Find $\dfrac{du}{dx}$, if $y = x^3$.

3130. $z = \arctan(xy)$; find $\dfrac{dz}{dx}$, if $y = e^x$.

3131. $u = \arcsin\dfrac{x}{z}$, where $z = \sqrt{x^2 + 1};\ \dfrac{du}{dx} = ?$

3132. $z = \tan(3t + 2x^2 - y),\ x = \dfrac{1}{t},\ y = \sqrt{t};\ \dfrac{dz}{dt} = ?$

3133. $u = \dfrac{e^{ax}(y - z)}{a^2 + 1},\ y = a\sin x,\ z = \cos x;\ \dfrac{du}{dx} = ?$

3134. $z = \dfrac{xy \arctan(xy + x + y)}{x + y};\ \dfrac{\partial z}{\partial x} = ?\ \dfrac{\partial z}{\partial y} = ?\ dz = ?$

3135. $z = (x^2 + y^2)\, e^{\frac{x^2+y^2}{xy}};\ \dfrac{\partial z}{\partial x} = ?\ \dfrac{\partial z}{\partial y} = ?\ dz = ?$

3136. $z = f(x^2 - y^2, e^{xy});\ \dfrac{\partial z}{\partial x} = ?\ \dfrac{\partial z}{\partial y} = ?$

3137. Verify that the function $z = \arctan\dfrac{x}{y}$, where $x = u + v,\ y = u - v$, satisfies the relationship

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{u - v}{v^2 + u^2}.$$

3138. Verify that the function $z = \varphi(x^2 + y^2)$, where φ is a differentiable function, satisfies the relationship

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0.$$

3139. $u = \sin x + F(\sin y - \sin x)$; verify that $\dfrac{\partial u}{\partial y}\cos x + \dfrac{\partial u}{\partial x}\cos y = \cos x \cos y$, whatever the differentiable function F.

3140. $z = \dfrac{y}{f(x^2 - y^2)}$; verify that $\left(\dfrac{1}{x}\right)\left(\dfrac{\partial z}{\partial x}\right) + \left(\dfrac{1}{y}\right)\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{z}{y^2}$, whatever the differentiable function f.

3141. Show that a homogeneous differentiable function of zero degree $z = F\left(\dfrac{y}{x}\right)$ (see problem 2961) satisfies the relationship $\left(x\dfrac{\partial z}{\partial x}\right) + \left(y\dfrac{\partial z}{\partial y}\right) = 0.$

3142. Show that the homogeneous function of the kth degree $u = x^k F\left(\dfrac{z}{x}, \dfrac{y}{x}\right)$, where F is a differentiable function, satisfies the relationship

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = ku.$$

3143. Verify the proposition of problem 3142 for the function

$$u = x^5 \sin\frac{z^2 + y^2}{x^2}.$$

3144. Given the differentiable function $f(x, y)$, prove that, if variables x, y are replaced by linear homogeneous functions of X, Y, the function $F(X, Y)$ obtained is connected with the given function by the relationship

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = X\frac{\partial F}{\partial X} + Y\frac{\partial F}{\partial Y}.$$

Functions Given Implicitly and Parametrically

In problems 3145–3155, find the derivative $\dfrac{dy}{dx}$ of the functions given implicitly by the equations indicated:

3145. $x^3y - y^3x = a^4$. **3146.** $x^2y^2 - x^4 - y^4 = a^4$.

3147. $xe^y + ye^x - e^{xy} = 0$.

3148. $(x^2 + y^2)^2 - a^2(x^2 - y^2) = 0$.

3149. $\sin(xy) - e^{xy} - x^2y = 0$.

3150. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. **3151.** $xy - \ln y = a$.

3152. $\arctan\dfrac{x+y}{a} - \dfrac{y}{a} = 0.$ **3153.** $yx^2 = e^y.$

3154. $ye^x + e^y = 0.$ **3155.** $y^x = x^y.$

3156. $F(x, y) = F(y, x)$. Show that the derivative of y with respect to x can be expressed with the aid of a fraction whose numerator is obtained from the denominator by interchanging the letters y and x.

3157. $x^2 + y^2 - 4x - 10y + 4 = 0$; find $\dfrac{dy}{dx}$ for $x = 6$, $y = 2$ and for $x = 6$, $y = 8$. Give a geometrical interpretation of the results.

3158. $x^4y + xy^4 - ax^2y^2 = a^5$; find $\dfrac{dy}{dx}$ for $x = y = a$.

3159. Show that it follows from $x^2y^2 + x^2 + y^2 - 1 = 0$ that

$$\frac{dx}{\sqrt{1-x^4}} + \frac{dy}{\sqrt{1-y^4}} = 0.$$

3160. Show that it follows from $a + b(x+y) + cxy = m(x-y)$ that

$$\frac{dx}{a + 2bx + cx^2} = \frac{dy}{a + 2by + cy^2}.$$

3161. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$; $\dfrac{\partial z}{\partial x} = ?$ $\dfrac{\partial z}{\partial y} = ?$

3162. $x^2 - 2y^2 + z^2 - 4x + 2z - 5 = 0$; $\dfrac{\partial z}{\partial x} = ?$ $\dfrac{\partial z}{\partial y} = ?$

3163. $z^3 + 3xyz = a^3$; $\dfrac{\partial z}{\partial x} = ?$ $\dfrac{\partial z}{\partial y} = ?$

3164. $e^z - xyz = 0$; $\dfrac{\partial z}{\partial x} = ?$ $\dfrac{\partial z}{\partial y} = ?$

3165. Show that, whatever the differentiable function φ, it follows from $\varphi(cx - az, cy - bz) = 0$ that

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = c.$$

3166. $F(x, y, z) = 0$. Prove that

$$\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial x} = 1; \quad \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial y} = -1$$

(see *Course*, sec. 151).

3167. Find the total differential of the function z defined by the equation $\cos^2 x + \cos^2 y + \cos^2 z = 1$.

3168. The function z is given parametrically: $x = u + v$, $y = u - v$, $z = uv$. Express z as an explicit function of x and y.

3169. $x = u + v$, $y = u^2 + v^2$, $z = u^3 + v^3$. Express z as an explicit function of x and y.

3170. $x = u \cos v$, $y = u \sin v$, $z = kv$. Express z as an explicit function of x and y.

Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ and dz for the functions of problems 3171–3174:

3171. $x = \frac{u^2 + v^2}{2}$, $y = \frac{u^2 - v^2}{2}$, $z = uv$.

3172. $x = \sqrt{a}(\sin u + \cos v)$, $y = \sqrt{a}(\cos u - \sin v)$,
$z = 1 + \sin(u - v)$.

3173. $x = u + v$, $y = u - v$, $z = u^2 v^2$.

3174. $x = e^u \cos v$, $y = e^u \sin v$, $z = uv$.

3175. The relationships $u = f(x, y)$, $v = F(x, y)$, where f and F are differentiable functions of x and y, define x and y as differentiable functions of u and v. Show that

$$\left(\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}\right)\left(\frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}\right) = 1.$$

Find the total differentials of the functions of problems 3176–3177:

3176. $x = u \cos v$, $y = u \sin v$, $z = u^2$.

3177. $x = v \cos u - u \cos u + \sin u$, $y = v \sin u - u \sin u - \cos u$, $z = (u - v)^2$.

3178. u and v are functions of x, y, z satisfying the relationships $uv = 3x - 2y + z$, $v^2 = x^2 + y^2 + z^2$. Show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

3179. Let $y = f(x, t)$, $F(x, y, t) = 0$. Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial F}{\partial t} - \dfrac{\partial f}{\partial t}\cdot\dfrac{\partial F}{\partial x}}{\dfrac{\partial f}{\partial t}\cdot\dfrac{\partial F}{\partial y} + \dfrac{\partial F}{\partial t}}.$$

3180. Let $f(x, y, z) = 0$, $F(x, y, z) = 0$. Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial F}{\partial z} - \dfrac{\partial F}{\partial x}\cdot\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial F}{\partial z} - \dfrac{\partial F}{\partial y}\cdot\dfrac{\partial f}{\partial z}}.$$

5. Repeated Differentiation

3181. $z = x^3 + xy^2 - 5xy^3 + y^5$. Verify that

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}.$$

3182. $z = x^y$. Verify that $\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{\partial^2 z}{\partial y\partial x}$.

3183. $z = \mathrm{e}^x(\cos y + x\sin y)$. Verify that

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x}.$$

3184. $z = \operatorname{arc\,tan}\dfrac{y}{x}$. Verify that $\dfrac{\partial^3 z}{\partial y^2\partial x} = \dfrac{\partial^3 z}{\partial x\partial y^2}$.

Find $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial z\partial y}$ and $\dfrac{\partial^2 z}{\partial y^2}$ for the functions of problems 3185–3192:

3185. $z = \dfrac{1}{3}\sqrt{(x^2+y^2)^3}$.

3186. $z = \ln(x + \sqrt{x^2+y^2})$.

3187. $z = \arctan \dfrac{x+y}{1-xy}$. **3188.** $z = \sin^2 (ax + by)$.

3189. $z = e^{xe^y}$. **3190.** $z = \dfrac{x-y}{x+y}$.

3191. $z = y^{\ln x}$. **3192.** $z = \arcsin (xy)$.

3193. $u = \sqrt{x^2 + y^2 + z^2 - 2xz}$; $\dfrac{\partial^2 u}{\partial y\, \partial z} = ?$

3194. $z = e^{xy^2}$; $\dfrac{\partial^3 z}{\partial x^2\, \partial y} = ?$

3195. $z = \ln (x^2 + y^3)$; $\dfrac{\partial^3 z}{\partial x\, \partial y^2} = ?$

3196. $z = \sin xy$; $\dfrac{\partial^3 z}{\partial x\, \partial y^2} = ?$

3197. $w = e^{xyz}$; $\dfrac{\partial^3 w}{\partial x\, \partial y\, \partial z} = ?$

3198. $v = x^m y^n z^p$; $\dfrac{\partial^6 v}{\partial x\, \partial y^3 \partial z^2} = ?$

3199. $z = \ln (e^x + e^y)$; verify that $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 1$ and that

$$\frac{\partial^2 z}{\partial x^2} \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2 = 0.$$

3200. $u = e^x(x \cos y - y \sin y)$. Show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

3201. $u = \ln \dfrac{1}{\sqrt{x^2 + y^2}}$; show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

3202. $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$; show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.
(see *Course,* sec. 153).

3203. $r = \sqrt{x^2 + y^2 + z^2}$; show that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}, \quad \frac{\partial^2 (\ln r)}{\partial x^2} + \frac{\partial^2 (\ln r)}{\partial y^2} + \frac{\partial^2 (\ln r)}{\partial z^2} = \frac{1}{r^2}.$$

3204. For what value of the constant a does the function $v = x^3 + axy^2$ satisfy the equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0?$$

3205. $z = \dfrac{y}{y^2 - a^2x^2}$; show that $\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial y^2}$.

3206. $v = \dfrac{1}{x-y} + \dfrac{1}{y-z} + \dfrac{1}{z-x}$; show that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + 2\left(\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial^2 v}{\partial y\,\partial z} + \frac{\partial^2 v}{\partial z\,\partial x}\right) = 0.$$

3207. $z = f(x, y)$, $\xi = z + y$, $\eta = x - y$. Verify that

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 4\frac{\partial^2 z}{\partial \xi\,\partial \eta}.$$

3208. $v = x \ln(x + r) - r$, where $r^2 = x^2 + y^2$. Show that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{x+r}.$$

3209. Find the expression for the second derivative $\dfrac{d^2y}{dx^2}$ of the function y given implicitly by the equation $f(x, y) = 0$.

3210. $y = \varphi(x - at) + \psi(x + at)$. Show that

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2},$$

whatever the twice differentiable functions φ and ψ.

3211. $u = \varphi(x) + \psi(y) + (x - y)\,\psi'(y)$. Show that

$$(x - y)\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial u}{\partial y}$$

(φ and ψ are twice differentiable functions).

3212. $z = y\varphi(x^2 - y^2)$. Show that

$$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{z}{y^2}$$

(φ is a differentiable function).

3213. $r = x\varphi(x+y) + y\psi(x+y)$. Show that

$$\frac{\partial^2 r}{\partial x^2} - 2\frac{\partial^2 r}{\partial x\,\partial y} + \frac{\partial^2 r}{\partial y^2} = 0$$

(φ and ψ are twice differentiable functions).

3214. $u = \frac{1}{y}[\varphi(ax+y) + \psi(ax-y)]$. Show that

$$\frac{\partial^2 u}{\partial x^2} = \frac{a^2}{y^2}\cdot\frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right).$$

3215. $u = x^n\varphi\left(\frac{y}{x}\right) + x^{n-1}\psi\left(\frac{y}{x}\right)$. Show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)\,u.$$

3216. $u = x\mathrm{e}^y + y\mathrm{e}^x$. Show that

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} = x\frac{\partial^3 u}{\partial x\,\partial y^2} + y\frac{\partial^3 u}{\partial x^2\,\partial y}.$$

3217. $u = \mathrm{e}^{xyz}$. Show that

$$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \mathrm{e}^{xyz}(x^2y^2z^2 + 3xyz + 1).$$

3218. $u = \ln\frac{x^2 - y^2}{xy}$. Show that

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x^2\,\partial y} - \frac{\partial^3 u}{\partial x\,\partial y^2} - \frac{\partial^3 u}{\partial y^3} = 2\left(\frac{1}{y^3} - \frac{1}{x^3}\right).$$

Find the second-order differentials of the functions of problems 3219–3223:

3219. $z = xy^2 - x^2y$.

3220. $z = \ln(x-y)$.

3221. $z = \frac{1}{2(x^2+y^2)}$.

3222. $z = x\sin^2 y$.

3223. $z = \mathrm{e}^{xy}$.

3224. $u = xyz$; $\mathrm{d}^2u = ?$

3225. $z = \sin(2x+y)$. Find d^3z at the points $(0, \pi)$; $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

3226. $u = \sin(x + y + z)$; $d^2u = ?$

3227. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; $d^2z = ?$

3228. $z^3 - 3xyz = a^3$; $d^2z = ?$

3229. $3x^2y^2 + 2z^2xy - 2zx^3 + 4zy^3 - 4 = 0$. Find d^2z at the point $(2, 1, 2)$.

Change of Variables

3230. Transform the differential expression

$$x^4 \frac{d^2y}{dx^2} + 2x^3 \frac{dy}{dx} + y$$

to a new independent variable by putting $x = \frac{1}{t}$.

3231. Transform the differential expression

$$x^2y'' - 4xy' + y$$

to a new independent variable by putting $x = e^z$.

3232. Transform the differential expression

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + ay,$$

by putting $x = \sin t$.

3233. Transform the differential expression

$$\frac{y''}{y'^3} + y,$$

by regarding y as the independent variable and x as a function of y.

3234. Transform the expression

$$y'y''' - 3y''^2,$$

by taking y as the independent variable.

3235. Transform the expression

$$yy'' = 2(y^2 + y'^2)$$

to a new function v by putting $y = \frac{1}{v}$.

3236. Transform to polar coordinates the equation

$$\frac{dy}{dx} = \frac{x+y}{x-y}.$$

3237. Transform the expression

$$k = \frac{y''}{(1+y'^2)^{\frac{3}{2}}}$$

to polar coordinates ϱ, φ, bearing in mind that $x = \varrho \cos \varphi$, $y = \varrho \sin \varphi$.

3238. The function z depends on x, y. Carry out a change of independent variables in the expression

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y}$$

with the aid of the formulae $x = u \cos v$, $y = u \sin v$.

3239. Transform Laplace's operator

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

to polar coordinates.

3240. Transform the expression

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + kz$$

to a new function w on condition that $z = w(\sqrt{x^2+y^2})$ or $z = w(r)$, where $r = \sqrt{x^2+y^2}$.

3241. In the expression

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2}$$

replace the independent variables x and y by variables u and v, and the function z by the variable w, taking these variables to be connected by the relationships $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$; $z = \frac{u^2-v^2}{4} - w$.

CHAPTER XI

APPLICATIONS OF THE DIFFERENTIAL CALCULUS FOR FUNCTIONS OF SEVERAL VARIABLES

1. Taylor's Formula. Extrema of Functions of Several Variables

Taylor's Formula

3242. $f(x, y) = x^3 + 2y^3 - xy$; expand the function $f(x + h, y + k)$ in powers of h and k.

3243. $f(x, y) = x^3 + y^2 - 6xy - 39x + 18y + 4$; find the increment received by the function when the independent variables change from the values $x = 5$, $y = 6$ to the values $x = 5 + h$, $y = 6 + k$.

3244. $f(x, y) = \frac{xy^3}{4} - yx^3 + \frac{x^2y^2}{2} - 2x + 3y - 4$; find the increment taken by the function when the independent variables pass from the values $x = 1$, $y = 2$ to the values $x = 1 + h$, $y = 2 + k$. Evaluate $f(10{\cdot}2, 2{\cdot}03)$, taking into account terms up to and including the second order.

3245. $f(x, y, z) = Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx$; expand $f(x + h, y + k, z + l)$ in powers of h, k and l.

3246. Expand $z = \sin x \sin y$ in powers of $\left(x - \frac{\pi}{4}\right)$ and $\left(y - \frac{\pi}{4}\right)$. Find the terms of the first and second order and R_2 (the second-order remainder term).

3247. Expand the function $z = x^y$ in powers of $(x - 1)$ and $(y - 1)$, finding the terms up to and including the third order. Use the result to evaluate (without tables!) $1{\cdot}1^{1{\cdot}02}$.

3248. $f(x, y) = e^x \sin y$; expand $f(x + h, y + k)$ in powers of h and k, taking terms up to and including the third order in h and k. Use the result to evaluate $e^{0\cdot 1} \sin 0\cdot 49\pi$.

3249. Find the first few terms of the expansion of the function $e^x \sin y$ in a Taylor series in the neighbourhood of of the point $(0, 0)$.

3250. Find the first few terms of the expansion of the function $e^x \ln(1 + y)$ in a Taylor series in the neighbourhood of the point $(0, 0)$.

Expand the functions of problems 3251–3256 in Taylor series for $x_0 = 0$, $y_0 = 0$:

3251. $z = \dfrac{1}{1 - x - y + xy}$. **3252*.** $z = \arctan \dfrac{x - y}{1 + xy}$.

3253. $z = \ln (1 - x) \ln (1 - y)$.

3254. $z = \ln \dfrac{1 - x - y + xy}{1 - x - y}$.

3255. $z = \sin (x^2 + y^2)$.

3256. $z = e^x \cos y$.

3257. Find the first few terms of the expansion on powers of $x - 1$, $y - 1$ of the function z, given implicitly by the equation

$$z^3 + yz - xy^2 - x^3 = 0$$

and equal to unity for $x = 1$, $y = 1$.

3258. Obtain the approximation

$$\frac{\cos x}{\cos y} \approx 1 - \frac{1}{2}(x^2 - y^2)$$

for sufficiently small values of $|x|$, $|y|$.

Extremals

Find the stationary points of the functions of problems 3259–3267:

3259. $z = 2x^3 + xy^2 + 5x^2 + y^2$.

3260. $z = e^{2x}(x + y^2 + 2y)$. **3261.** $z = xy(a - x - y)$.

3262. $z = (2ax - x^2)(2by - y^2)$.

3263. $z = \sin x + \sin y + \cos(x + y)$ $\left(0 \leqq x \leqq \frac{\pi}{4}, \quad 0 \leqq y \leqq \frac{\pi}{4}\right)$.

3264. $z = \dfrac{a + bx + cy}{\sqrt{1 + x^2 + y^2}}$.

3265. $z = y\sqrt{1 + x} + x\sqrt{1 + y}$.

3266. $u = 2x^2 + y^2 + 2z - xy - xz$.

3267. $u = 3\ln x + 2\ln y + 5\ln z + \ln(22 - x - y - z)$.

3268. Figure 60 illustrates the level lines of the function $z = f(x, y)$. What special features has the function at the points A, B, C, D and on the curve EF?

3269. A function z is given implicitly:

$$2x^2 + 2y^2 + z^2 + 8xz - z + 8 = 0.$$

Find its stationary points.

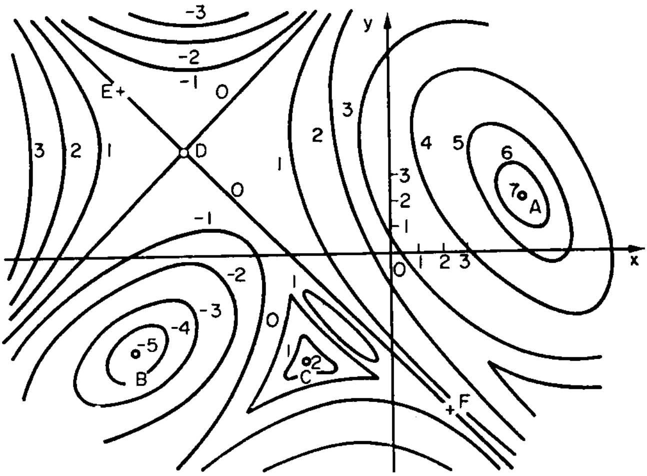

FIG. 60.

3270. A function z is given implicitly:

$$5x^2 + 5y^2 + 5z^2 - 2xy - 2xz - 2yz - 72 = 0.$$

Find its stationary points.

3271*. Find the extremal points of the function

$$z = 2xy - 3x^2 - 2y^2 + 10.$$

3272. Find the extremal points of the function

$$z = 4(x - y) - x^2 - y^2.$$

3273. Find the extremal points of the function

$$z = x^2 + xy + y^2 + x - y + 1.$$

3274. Show that the function $z = x^2 + xy + y^2 + \frac{a^3}{x} + \frac{a^3}{y}$ has a minimum at the point $x = y = \frac{a}{\sqrt[3]{3}}$.

3275. Show that the function $z = x^4 + y^4 - 2x^2 - 4xy - 2y^2$ has a minimum for $x = \sqrt{2}$, $y = \sqrt{2}$ and for $x = -\sqrt{2}$, $y = -\sqrt{2}$.

3276. Show that the function $z = x^3 + y^2 - 6xy - 39x + 18y + 20$ has a minimum for $x = 5$, $y = 6$.

3277. Find the stationary points of the function

$$z = x^3y^2(12 - x - y),$$

satisfying the condition $x > 0$, $y > 0$, and examine their nature.

3278. Find the stationary points of the function

$$z = x^3 + y^3 - 3xy$$

and examine their nature.

Greatest and Least Values

3279. Find the greatest and least values of the function $z = x^2 - y^2$ in the circular domain $x^2 + y^2 \leqq 4$.

3280. Find the greatest and least values of the function $z = x^2 + 2xy - 4x + 8y$ in the rectangle bounded by the straight lines

$$x = 0, \ y = 0, \ x = 1, \ y = 2.$$

3281. Find the greatest value of the function

$$z = x^2y(4 - x - y)$$

in the triangle bounded by the straight lines $x = 0$, $y = 0$, $x + y = 6$.

3282. Find the greatest and least values of the function

$$z = e^{-x^2-y^2}(2x^2 + 3y^2)$$

in the circle $x^2 + y^2 \leqq 4$.

3283. Find the greatest and least values of the function

$$z = \sin x + \sin y + \sin (x + y)$$

in the rectangle $0 \leqq x \leqq \frac{\pi}{2}$; $0 \leqq y \leqq \frac{\pi}{2}$.

3284. Write the positive number a as the sum of three positive terms such that their product is a maximum.

3285. Express the positive number a as the product of four positive factors such that their sum is a minimum.

3286. Find the point of the xOy plane such that the sum of the squares of its distances from the straight lines $x = 0$, $y = 0$, $x + 2y - 16 = 0$ is a minimum.

3287. Draw the plane through the point (a, b, c) such that the volume of the tetrahedron, cut out by it from the coordinate trihedral, is a minimum.

3288. Given the n points $A_1(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$, find the point of the xOy plane such that the sum of the squares of its distances from all the given points is a minimum.

3289. Given the three points $A(0, 0, 12)$, $B(0, 0, 4)$ and $C(8, 0, 8)$, find the point D on the xOy plane such that the sphere passing through A, B, C, and D has minimum radius.

3290. Inscribe the rectangular parallelepiped of maximum volume in a given sphere of diameter $2R$.

Conditional Extrema

Investigate the extrema of the functions of problems 3291–3296:

3291. $z = x^m + y^m$ $(m > 1)$ for $x + y = 2$ $(x \geqq 0, y \geqq 0)$.

3292. $z = xy$ for $x^2 + y^2 = 2a^2$.

3293. $z = \frac{1}{x} + \frac{1}{y}$ for $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{a^2}$.

3294. $z = a\cos^2 x + b\cos^2 y$ for $y - x = \frac{\pi}{4}$.

3295. $u = x + y + z$ for $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

3296. $u = xyz$ for $\begin{cases} (1)\ x + y + z = 5, \\ (2)\ xy + xz + yz = 8. \end{cases}$

3297*. Establish the relationship

$$\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n} \geqq \left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)^2.$$

3298. $f(x, y) = x^3 - 3xy^2 + 18y$, where $3x^2y - y^3 - 6x = 0$. Prove that $f(x, y)$ has extrema at the points $x = y = \pm\sqrt{3}$.

3299. Find the minimum of the function $ax^2 + by^2 + cz^2$, where a, b, c are positive constants, and x, y, z are connected by the relationship

$$x + y + z = 1.$$

3300. Find the extrema of the function

$$u = y^2 + 4z^2 - 4yz - 2xz - 2xy$$

on condition that

$$2x^2 + 3y^2 + 6z^2 = 1.$$

3301. Find the point on the plane $3x - 2z = 0$ such that the sum of the squares of its distances from the points $A(1, 1, 1)$ and $B(2, 3, 4)$ is a minimum.

3302. Find the point on the plane $x + y - 2z = 0$ such that the sum of the squares of its distances from the planes $x + 3z = 6$ and $y + 3z = 2$ is a minimum.

3303. Given the points $A(4, 0, 4)$, $B(4, 4, 4)$, $C(4, 4, 0)$, find the point S on the surface of the sphere $x^2 + y^2 + z^2 = 4$ such that the volume of pyramid $SABC$ is (a) a maximum, (b) a minimum. Check the answer by elementary geometry.

3304. Find the rectangular parallelepiped of given volume V having the least surface area.

3305. Find the rectangular parallelepiped of given surface area S having the maximum volume.

3306. Find the volume of the greatest rectangular parallelepiped that can be inscribed in an ellipsoid with semi-axes a, b and c.

3307. A marquee is in the form of a cylinder with a conical top over it. What are the relationships between the linear dimensions of the marquee for the manufacture of it to require the least amount of material for a given volume?

3308. The section of a channel is an isosceles trapezoid of given area. How must its dimensions be chosen for the wetted area of the channel to be a minimum (Fig. 61)?

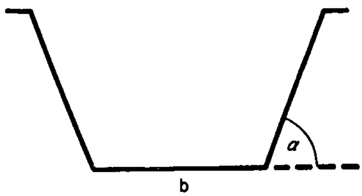

FIG. 61.

3309. Of all the rectangular parallelepipeds having a given diagonal, find the one whose volume is a maximum.

3310. Find the external dimensions of an open box (without a lid) in the form of a rectangular parallelepiped with given wall-thickness α and volume V, such that the least amount of material goes into it.

3311. Find the maximum volume of a parallelepiped, given that the sum of all its ribs is equal to $12a$.

3312. Circumscribe about a given ellipse the triangle with base parallel to the major axis of the ellipse, such that the area of the triangle is a minimum.

3313. Find the points on the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ nearest to and furthest away from the straight line

$$3x - y - 9 = 0.$$

3314. Find the point on the parabola $x^2 + 2xy + y^2 + 4y = 0$ closest to the straight line $3x - 6y + 4 = 0$.

3315. Find the point on the parabola $2x^2 - 4xy + 2y^2 - x - y = 0$ closest to the straight line $9x - 7y + 16 = 0$.

3316. Find the greatest distance of points of the surface

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 1$$

from the plane $z = 0$.

3317. Find the sides of the right-angled triangle having the least perimeter for a given area S.

3318. A prism with a rectangular base is inscribed in a right elliptic cone of height H cm, the semi-axes of the base of which are a and b cm; the prism is such that the sides of its base are parallel to the axes, whilst the intersection of the base diagonals lies at the centre of the ellipse. What must be the sides of the base and the height of the prism for its volume to be a maximum? What is this maximum volume?

3319. Find the equilateral triangular pyramid of given volume such that the sum of its ribs is a minimum.

3320. Given two points on an ellipse, find the third point on the ellipse such that the triangle, the vertices of which are at the given points, has the maximum area.

3321. Draw the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which is furthest from the origin.

3322. Find the points on the ellipsoid of revolution $\frac{x^2}{96} + y^2 + z^2 = 1$ closest to and furthest away from the plane $3x + 4y + 12z = 288$.

3323. Given the plane curves $f(x, y) = 0$ and $\varphi(x, y) = 0$, show that the distance between points (α, β) and (ξ, η), lying on the respective curves, has an extremum when the following condition is fulfilled:

$$\frac{\alpha - \xi}{\beta - \eta} = \frac{\left(\frac{\partial f}{\partial x}\right)_{\substack{x=\alpha \\ y=\beta}}}{\left(\frac{\partial f}{\partial y}\right)_{\substack{x=\alpha \\ y=\beta}}} = \frac{\left(\frac{\partial \varphi}{\partial x}\right)_{\substack{x=\xi \\ y=\eta}}}{\left(\frac{\partial \varphi}{\partial y}\right)_{\substack{x=\xi \\ y=\eta}}}.$$

Use this to find the shortest distance between the ellipse $x^2 + 2xy + 5y^2 - 16y = 0$ and the straight line $x + y - 8 = 0$.

2. Plane Curves

Tangents and Normals

Write down the equations of the tangent and normal at the point indicated to the curves of problems 3324–3327:

3324. $x^3y + y^3x = 3 - x^2y^2$ at the point (1, 1).

3325. $a^2(x^4 + y^4) - x^3y^3 = 9a^6$ at the point $(a, 2a)$.

3326. $\cos xy = x + 2y$ at the point (1, 0).

3327. $2x^3 - x^2y + 3x^2 + 4xy - 5x - 3y + 6 = 0$ at its point of intersection with the Oy axis.

Singular Points

Find the singular points of the curves of problems 3328–3340:

3328. $y^2 = x^2(x - 1)$.

3329. $a^2x^2 = (x^2 + y^2)y^2$.

3330. $y^2 = ax^2 + bx^5$.

3331. $y^2 = x(x - a)^2$.

3332. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

3333. $x^4 + y^4 - 8x^2 - 10y^2 + 16 = 0$.

3334. $x^4 + 12x^3 - 6y^3 + 36x^2 + 27y^2 - 81 = 0$.

3335. $x^3 + y^3 + 3axy = 0$.

3336. $x^2 + y^2 = x^4 + y^4$.

3337. $y = x \ln x$.

3338. $y^2 = \sin^3 x$.

3339. $y^2 = (x - a)^3$.

3340. $x^5 = (y - x^2)^2$.

Envelopes

3341. Find the equation of the envelope of the family of straight lines $y = ax + f(a)$. In particular, put $f(a) = \cos a$.

3342. Find the envelope of the family of straight lines $y = 2mx + m^4$.

3343. A pencil of straight lines is drawn through the point $A(a, 0)$. Find the envelope of the family of normals to the lines of the pencil, drawn at the points of intersection of the lines with Oy (see *Course*, sec. 45).

3344. Find the envelope of the family of parabolas $y^2 = a(x - a)$.

3345. Find the envelope of the family of parabolas

$$ax^2 + a^2y = 1.$$

3346. Find the envelope of the family of parabolas

$$y = a^2 (x - a)^2.$$

3347. Find the envelope of the family of semicubical parabolas

$$(y - a)^2 = (x - a)^3.$$

3348. Find the envelope of the family of curves

$$x^2 + ay^2 = a^3.$$

3349. Find the envelope of the family of ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with the condition that the sum of the semi-axes of each ellipse is equal to d.

3350. The radii of a circle are projected on to two mutually perpendicular diameters of the circle, and ellipses are constructed with the projections as semi-axes. Find the envelope of this family of ellipses.

3351. Find the envelope of the family of circles having their centres on the parabola $y = bx^2$ and passing through its vertex.

3352. A straight line moves so that the sum of the lengths of the segments that it cuts out on the coordinate axes remains constant and equal to a. Find the envelope of the family of straight lines thus obtained.

3353. Find the envelope of the diameters of a circle rolling without slip on a given straight line (radius of circle $= R$).

3354. Circles are drawn by taking as diameters the chords of a circle (of radius R) parallel to a given direction. Find the envelope of this family of circles.

3355. A straight line moves so that the product of the segments that it cuts off on the coordinate axes is equal to a constant a. Find the envelope of these straight lines.

3356. Show that every curve is the envelope of its tangents.

3357. Show that the evolute of a curve is the envelope of the family of its normals. Find the evolute of the parabola $y^2 = 2px$ as the locus of the centres of curvature and as the envelope of the family of normals. Compare the results.

3358. Prove the theorem: if curve (A) is the envelope of the family of straight lines $x \cos t + y \sin t - f(t) = 0$, the evolute of curve (A) is the envelope of the family of straight lines $-x \sin t + y \cos t - f'(t) = 0$.

3359. The radius vector $\overline{OM}$ of an arbitrary point M of the rectangular hyperbola $xy = 1$ is projected on to the asymptotes of the hyperbola. Find the envelope of the family of ellipses constructed by taking the projections of $\overline{OM}$ as semi-axes.

3. Vector Functions of a Scalar Argument. Curves in Space. Surfaces

Vector Functions of a Scalar Argument

3360. Prove the differentiation formulae

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}\boldsymbol{v}) = \boldsymbol{u}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \boldsymbol{v}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}, \quad \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{u}\times\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\times\boldsymbol{v}.$$

Here, $\boldsymbol{u}$ and $\boldsymbol{v}$ are vector functions of the scalar argument t.

3361. Given $\boldsymbol{r} = \boldsymbol{r}(t)$, find the derivatives:

$$\text{(a)}\ \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}^2); \quad \text{(b)}\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right); \quad \text{(c)}\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}\times\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right); \quad \text{(d)}\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right).$$

3362. Given that vectors $\boldsymbol{r}(t)$ and $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$ are collinear for all values of t, prove that vectors $\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}, \frac{\mathrm{d}^3\boldsymbol{r}}{\mathrm{d}t^3}, \ldots, \frac{\mathrm{d}^n\boldsymbol{r}}{\mathrm{d}t^n}$ are also collinear with $\boldsymbol{r}(t)$.

3363. Show that, if the modulus $|\boldsymbol{r}|$ of function $\boldsymbol{r}(t)$ remains constant for all values of t, than $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \perp \boldsymbol{r}$. (What is the geometrical meaning of this fact?) Does the converse theorem hold?

3364. Given: $\boldsymbol{r} = \boldsymbol{a}\cos\omega t + \boldsymbol{b}\sin\omega t$, where ω, $\boldsymbol{a}$, $\boldsymbol{b}$ are constants, prove that

$$(1)\ \boldsymbol{r}\times\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \omega\boldsymbol{a}\times\boldsymbol{b} \text{ and } (2)\ \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} + \omega^2\boldsymbol{r} = 0.$$

3365. Prove that, if $\boldsymbol{e}$ is the unit vector in the direction of vector $\boldsymbol{E}$, then $\boldsymbol{e}\times\mathrm{d}\boldsymbol{e} = \frac{\boldsymbol{E}\times\mathrm{d}\boldsymbol{E}}{E^2}$.

3366. Prove that, if $\boldsymbol{r} = \boldsymbol{a}\mathrm{e}^{\omega t} + \boldsymbol{b}\mathrm{e}^{-\omega t}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors, then $\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} - \omega^2\boldsymbol{r} = 0$.

3367. $\boldsymbol{u} = \alpha(x, y, z, t)\,\boldsymbol{i} + \beta(x, y, z, t)\,\boldsymbol{j} + \gamma(x, y, z, t)\,\boldsymbol{k}$, where x, y, z are functions of t. Prove that

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \frac{\partial\boldsymbol{u}}{\partial t} + \frac{\partial\boldsymbol{u}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial\boldsymbol{u}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial\boldsymbol{u}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}.$$

3368. Given: $\boldsymbol{r} = \boldsymbol{r}(u)$, $u = \varphi(x)$. Express the derivatives $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}x}, \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}x^2}, \frac{\mathrm{d}^3\boldsymbol{r}}{\mathrm{d}x^3}$ in terms of $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}u}, \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}u^2}, \frac{\mathrm{d}^3\boldsymbol{r}}{\mathrm{d}u^3}$.

3369. Show that, if the relationship $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \alpha\boldsymbol{r}$, where $\alpha = \text{const}$, holds for the vector function $\boldsymbol{r} = \boldsymbol{r}(t)$, the hodograph of the function $\boldsymbol{r}(u)$ is a straight line through the pole.

3370. Let the function $\boldsymbol{r}(t)$ be defined, continuous and differentiable in the interval (t_1, t_2), whilst $\boldsymbol{r}(t_1) = \boldsymbol{r}(t_2)$. Apply

Rolle's theorem to the function $\boldsymbol{a} \cdot \boldsymbol{r}$, where $\boldsymbol{a}$ is an arbitrary constant vector. Explain the result geometrically.

3371. Given the radius vector $\boldsymbol{r}\{a \sin t, -a \cos t, bt^2\}$ (t is time, a and b are constants) of a particle moving in space, find the hodographs of the velocity and acceleration.

3372. Find the trajectory of the motion for which the radius vector of a moving point satisfies the condition

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{a} \times \boldsymbol{r},$$

where $\boldsymbol{a}$ is a constant vector.

3373. A material particle moves in accordance with the law

$$\boldsymbol{r} = \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2$$

($\boldsymbol{r}$ is the radius vector of the particle at the instant t, $\boldsymbol{v}_0$ and $\boldsymbol{g}$ are given vectors). Show that (1) the momentum of the particle is a quadratic function of time; (2) $\boldsymbol{v}_0$ is the initial velocity (i.e. the value of the velocity vector at the instant $t = 0$); (3) the motion proceeds with constant acceleration, equal to the vector $\boldsymbol{g}$; (4) the motion proceeds along a parabola (provided that vectors $\boldsymbol{v}_0$ and $\boldsymbol{g}$ are not collinear), the axis of which is parallel to vector $\boldsymbol{g}$.

3374. The law of motion of a material particle is given by

$$\boldsymbol{r} = \boldsymbol{a} \cos t + \boldsymbol{b} \sin t + \boldsymbol{c},$$

where vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to each other. Find the trajectory of the motion. At what instants is the velocity extremal? At what instants is the acceleration extremal?

3375. The formulae for transforming from Cartesian to spherical coordinates are $x = \varrho \sin \theta \cos \varphi$, $y = \varrho \sin \theta \sin \varphi$, $z = \varrho \cos \theta$, where ϱ is the distance of the given point from the pole, θ is its latitude, and φ its azimuth or longitude (see *Course*, sec. 152). Find the components of the velocity of a moving particle in the directions of the orthogonal unit vectors $\boldsymbol{e}_\varrho$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\varphi$.

Curves in Space

Form the equations of the tangent line and normal plane to the curves of problems 3376–3383 at the points indicated:

3376. $\boldsymbol{r}\left(\frac{t^4}{4}, \frac{t^3}{3}, \frac{t^2}{2}\right)$, i.e. $x = \frac{t^4}{4}$, $y = \frac{t^2}{3}$, $z = \frac{t^2}{2}$, at an arbitrary point.

3377. $x = a \cos \varphi$, $y = a \sin \varphi$, $z = \frac{k}{2\pi}\varphi$, at the point $\left(\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{2}, \frac{k}{8}\right)$. Prove that the tangent at every point of the curve forms the same angle with Oz (see *Course*, sec. 164).

3378. $x = at$, $y = \frac{1}{2}at^2$, $z = \frac{1}{3}at^3$ at the point $(6a, 18a, 72a)$.

3379. $x = t - \sin t$, $y = 1 - \cos t$, $z = 4 \sin \frac{t}{2}$ at the point $\left(\frac{\pi}{2} - 1, 1, 2\sqrt{2}\right)$.

3380. $y^2 + z^2 = 25$, $x^2 + y^2 = 10$ at the point $(1, 3, 4)$.

3381. $2x^2 + 3y^2 + z^2 = 47$, $x^2 + 2y^2 = z$ at the point $(-2, 1, 6)$.

3382. $x^2 + y^2 = z^2$, $x = y$ at the point (x_0, y_0, z_0).

3383. $x^3 + z^3 = a^3$, $y^3 + z^3 = b^3$ at an arbitrary point.

3384. Find the point of the curve $\boldsymbol{r}\{\cos t, \sin t, e^t\}$ at which the tangent is parallel to the plane

$$\sqrt{3}x + y - 4 = 0.$$

Form the equations of the osculating plane, the principal normal and the binormal to the curves of problems 3385–3387 at the points indicated:

3385. $y^2 = x$, $x^2 = z$ at the point $(1, 1, 1)$.

3386. $x^2 = 2az$, $y^2 = 2bz$ at an arbitrary point.

3387. $\boldsymbol{r}\{e^t, e^{-t}, t\sqrt{2}\}$ at the point $(e, e^{-1}, \sqrt{2})$.

3388. Prove that the tangents, principal normals and binormals of the curve $\boldsymbol{r}\{e^t \cos t,\ e^t \sin t,\ e^t\}$ form constant angles with Oz.

Form the equations of the tangent line, normal plane, binormal, osculating plane, principal normal and rectifying plane to the curves of problems 3389–3392 at the points indicated:

3389. $x = t^2,\ y = 1 - t,\ z = t^3$ at the point $(1, 0, 1)$.

3390. $x^2 + y^2 + z^2 = 3,\ x^2 + y^2 = 2$ at the point $(1, 1, 1)$.

3391. $\boldsymbol{r}\{\sin t,\ \cos t,\ \tan t\}$ at the point $\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2},\ 1\right)$.

3392. $\boldsymbol{r}\{t^3 - t^2 - 5,\ 3t^2 + 1,\ 2t^3 - 16\}$ at the point corresponding to the value of the parameter $t = 2$.

3393. Prove that the curve

$$\boldsymbol{r}\ \{2t + 3,\ 3t - 1,\ t^2\}$$

has the same osculating plane at every point. Interpret this fact geometrically.

3394. Prove that the curve

$$\boldsymbol{r}\{a_1t^2 + b_1t + c_1,\ a_2t^2 + b_2t + c_2,\ a_3t^2 + b_3t + c_3\}$$

is plane, and form the equation of the plane in which it is situated.

3395. Find the radius of torsion of the curve $\boldsymbol{r}\{\cos t,\ \sin t,\ \cosh t\}$.

3396. Find the radius of curvature of the curve $\boldsymbol{r}\{\ln \cos t,\ \ln \sin t,\ \sqrt{2}\, t\}$. Prove that the torsion at any point of it is equal to the curvature at that point.

3397. Prove that the ratio of curvature to torsion remains constant at every point of the curve $\boldsymbol{r}\{e^t \cos t,\ e^t \sin t,\ e^t\}$ (see problem 3388).

3398. How can we express the curvature of a spatial curve given by the equations $y = \varphi(x),\ z = \psi(x)$?

3399. Express the vectors τ_1, ν_1, β_1 in terms of the derivatives of the radius vector of a point on the curve $\boldsymbol{r} = \boldsymbol{r}(t)$.

3400. Express each of the vectors $\boldsymbol{\tau}_1$, $\boldsymbol{\nu}_1$, $\boldsymbol{\beta}_1$ in terms of the other two.

3401. Find the vector $\boldsymbol{\omega}(s)$ (Darboux vector) satisfying the conditions

$$\frac{d\boldsymbol{\tau}_1}{ds} = \boldsymbol{\omega}\times\boldsymbol{\tau}_1; \quad \frac{d\boldsymbol{\nu}_1}{ds} = \boldsymbol{\omega}\times\boldsymbol{\nu}_1; \quad \frac{d\boldsymbol{\beta}_1}{ds} = \boldsymbol{\omega}\times\boldsymbol{\beta}_1.$$

Length of Arc of a Spatial Curve

Find the length of arc of the curves of problems 3402–3409:

3402. $\boldsymbol{r}\{2t, \ln t, t^2\}$, from $t = 1$ to $t = 10$.

3403. $\boldsymbol{r}\{a\cos t, a\sin t, a\ln\cos t\}$ from the point $(a, 0, 0)$ to the point $\left(\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{2}, -\frac{a}{2}\ln 2\right)$

3404. $\boldsymbol{r}\{e^t\cos t, e^t\sin t, e^t\}$ from the point $(1, 0, 1)$ to the point corresponding to the parameter t.

3405. $x^2 = 3y$, $2xy = 9z$ from the point $(0, 0, 0)$ to the point $(3, 3, 2)$.

3406. $z^2 = 2ax$, $9y^2 = 16xz$ from the point $(0, 0, 0)$ to the point $\left(2a, \frac{8a}{3}, 2a\right)$.

3407. $4ax = (y+z)^2$, $4x^2 + 3y^2 = 3z^2$ from the origin to the point (x, y, z).

3408. $y = \sqrt{2ax - x^2}$, $z = a\ln\frac{2a}{2a - x}$ from the origin to the point (x, y, z).

3409. $y = a\arcsin\frac{x}{a}$, $z = \frac{1}{4}a\ln\frac{a+x}{a-x}$ from the origin to the point $\left(\frac{a}{2}, \frac{a\pi}{6}, \frac{a}{4}\ln 3\right)$.

Surfaces

Find the equations of the tangent planes and normals to the surfaces of problems 3410–3419 at the points indicated:

3410. $z = 2x^2 - 4y^2$ at the point $(2, 1, 4)$.

3411. $z = xy$ at the point $(1, 1, 1)$.

3412. $z = \dfrac{x^3 - 3axy + y^3}{a^2}$ at the point $(a, a, -a)$.

3413. $z = \sqrt{x^2 + y^2} - xy$ at the point $(3, 4, -7)$.

3414. $z = \arctan \dfrac{y}{x}$ at the point $\left(1, 1, \dfrac{\pi}{4}\right)$.

3415. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ at the point $\left(\dfrac{a\sqrt{3}}{3}, \dfrac{b\sqrt{3}}{3}, \dfrac{c\sqrt{3}}{3}\right)$.

3416. $x^3 + y^3 + z^3 + xyz - 6 = 0$ at the point $(1, 2, -1)$.

3417. $3x^4 - 4y^3z + 4z^2xy - 4z^3x + 1 = 0$ at the point $(1, 1, 1)$.

3418. $(z^2 - x^2)\, xyz - y^5 = 5$ at the point $(1, 1, 2)$.

3419. $4 + \sqrt{x^2 + y^2 + z^2} = x + y + z$ at the point $(2, 3, 6)$.

3420. Prove that the equation of the tangent plane at any point $M_0(x_0, y_0, z_0)$ of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ has the form

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} = 1.$$

3421. Draw the tangent plane to the ellipsoid $x^2 + 2y^2 + z^2 = 1$ parallel to the plane

$$x - y + 2z = 0.$$

3422. Draw the tangent plane to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ such that it cuts off equal segments on the positive coordinate semi-axes.

3423. Prove that the surfaces $x + 2y - \ln z + 4 = 0$ and $x^2 - xy - 8x + z + 5 = 0$ touch each other (i.e. have a common tangent plane) at the point $(2, -3, 1)$.

3424. Prove that all the planes tangential to the surface $z = xf\left(\dfrac{y}{x}\right)$ intersect in a single point.

3425. Write the equations of the tangent plane and normal to the sphere $\boldsymbol{r}\{u \cos v,\ u \sin v,\ \sqrt{a^2 - u^2}\}$ at the point $\boldsymbol{r}_0\{x_0, y_0, z_0\}$.

3426. Write the equations of the tangent plane and normal to the hyperbolic paraboloid $\boldsymbol{r}\{a(u+v),\ b(u-v),\ uv\}$ at the arbitrary point $\boldsymbol{r}_0\{x_0, y_0, z_0\}$.

3427. Prove that the surfaces $x^2+y^2+z^2=ax$ and $x^2+y^2+z^2=by$ are orthogonal to each other.

3428. Prove that the tangent plane to the surface $xyz=a^3$ at any point of it forms a tetrahedron of constant volume with the coordinate planes. Find this volume.

3429. Prove that the tangent planes to the surface $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}$ cut off segments on the coordinate axes such that the sum of the segments is equal to a.

3430. Write the equation of the tangent plane to the surface $z=xy$ which is perpendicular to the straight line

$$\frac{x+2}{2}=\frac{y+2}{1}=\frac{z-1}{-1}.$$

3431. Prove that, for the surface $x^2+y^2+z^2=y$, the length of the segment of normal between the surface and the xOy plane is equal to the distance from the origin to the trace of the normal on this plane.

3432. Prove that the normal to the surface of the ellipsoid of revolution

$$\frac{x^2+z^2}{9}+\frac{y^2}{25}=1$$

at any point of it $P(x, y, z)$ forms equal angles with the straight lines OA and OB, if $A(0, -4, 0)$ and $B(0, 4, 0)$.

3433. Prove that all the normals to the surface of revolution

$$z=f\left(\sqrt{x^2+y^2}\right)$$

cut the axis of revolution.

3434. Draw the tangent plane to the surface $x^2-y^2-3z=0$ that passes through the point $A(0, 0, -1)$ and is parallel to the straight line $\frac{x}{2}=\frac{y}{1}=\frac{z}{2}$.

3435. Find the points on the surface $x^2 + y^2 + z^2 - 6y + 4z = 12$ at which the tangent planes are parallel to the coordinate planes.

3436. Find the tangent plane to the surface

$$x = u + v,\ y = u^2 + v^2,\ z = u^3 + v^3$$

at an arbitrary point:

(a) by taking the equation of the surface in the parametric form;

(b) by writing the equation of the surface in the form $z = f(x, y)$.

3437. Find the locus of the feet of the perpendiculars dropped from the origin on to the tangent planes to the paraboloid of revolution $2pz = x^2 + y^2$.

3438. Find the locus of the feet of the perpendiculars dropped from the origin on to the tangent planes to the surface $xyz = a^3$.

4. Scalar Field. Gradient. Directional Derivative

Gradient

3439. (1) $\psi(x, y) = x^2 - 2xy + 3y - 1$. Find the components of the gradient at the point (1, 2).

(2) $u = 5x^2y - 3xy^3 + y^4$. Find the components of the gradient at an arbitrary point.

3440. (1) $z = x^2 + y^2$. Find grad z at the point (3, 2).
(2) $z = \sqrt{4 + x^2 + y^2}$. Find grad z at the point (2, 1).

(3) $z = \arctan \dfrac{y}{x}$. Find grad z at the point (x_0, y_0).

3441. (1) Find the maximum steepness of ascent of the surface $z = \ln (x^2 + 4y^2)$ at the point (6, 4, ln 100).

(2) Find the maximum steepness of ascent of the surface $z = x^y$ at the point (2, 2, 4).

3442. What is the direction of maximum variation of the function $\varphi(x, y, z) = x \sin z - y \cos z$ at the origin?

3443. (1) $z = \arcsin \dfrac{x}{x+y}$. Find the angle between the gradients of this function at the points (1, 1) and (3, 4).

(2) Given the functions $z = \sqrt{x^2 + y^2}$ and $z = x - 3y + \sqrt{3xy}$, find the angle between the gradients of these functions at the point (3, 4).

3444. (1) Find the point at which the gradient of the function $z = \ln\left(x + \dfrac{1}{y}\right)$ is equal to $\boldsymbol{i} - \dfrac{16\boldsymbol{j}}{9}$.

(2) Find the points at which the modulus of the gradient of the function $z = (x^2 + y^2)^{\frac{3}{2}}$ is equal to 2.

3445. Prove the following relationships (φ and ψ are differentiable functions, c is a constant):

$$\operatorname{grad}(\varphi + \psi) = \operatorname{grad}\varphi + \operatorname{grad}\psi; \qquad \operatorname{grad}(c + \varphi) = \operatorname{grad}\varphi;$$

$$\operatorname{grad}(c\varphi) = c\operatorname{grad}\varphi; \qquad \operatorname{grad}(\varphi\psi) = \varphi\operatorname{grad}\psi + \psi\operatorname{grad}\varphi;$$

$$\operatorname{grad}(\varphi^n) = n\varphi^{n-1}\operatorname{grad}\varphi; \qquad \operatorname{grad}[\varphi(\psi)] = \varphi'(\psi)\operatorname{grad}\psi$$

3446. $z = \varphi(u, v)$, $u = \psi(x, y)$, $v = \zeta)(x, y)$. Prove that

$$\operatorname{grad} z = \frac{\partial\varphi}{\partial u}\operatorname{grad} u + \frac{\partial\varphi}{\partial v}\operatorname{grad} v.$$

3447. (1) $u(x, y, z) = x^3y^2z$. Find the components of grad u at the point (x_0, y_0, z_0).

(2) $u(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Find grad u.

3448. Show that the function $u = \ln(x^2 + y^2 + z^2)$ satisfies the relationship $u = 2\ln 2 - \ln(\operatorname{grad} u)^2$.

3449. Prove that, if x, y, z are functions of t, then

$$\frac{d}{dt} f(x, y, z) = \operatorname{grad} f \cdot \frac{d\boldsymbol{r}}{dt},$$

where

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}.$$

3450. Use the relationship proved in the previous problem for finding the gradient of the function:

$$(1)\ f = \boldsymbol{r}^2;\quad (2)\ f = |\boldsymbol{r}|;\quad (3)\ f = F(\boldsymbol{r}^2);\quad (4)\ f = (\boldsymbol{ar})(\boldsymbol{br});$$
$$(5)\ f = (\boldsymbol{abr});$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors.

Directional Derivatives

3451. (1) Find the derivative of the function $z = x^3 - 3x^2y + 3xy^2 + 1$ at the point $M(3, 1)$ in the direction from this point to the point $(6, 5)$.

(2) Find the derivative of the function $z = \arctan xy$ at the point $(1, 1)$ with respect to the direction of the bisector of the first quadrant.

(3) Find the derivative of the function $z = x^2y^2 - xy^3 - 3y - 1$ at the point $(2, 1)$ in the direction from this point to the origin.

(4) Find the derivative of the function $z = \ln(e^x + e^y)$ at the origin in the direction α.

3452. Find the derivative of the function $z = \ln(x + y)$ at the point $(1, 2)$ of the parabola $y^2 = 4x$ with respect to the direction of this parabola.

3453. Find the derivative of $z = \arctan\frac{y}{x}$ at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, of the circle $x^2 + y^2 - 2x = 0$ with respect to the direction of this circle.

3454. Show that the derivative of $z = \frac{y^2}{x}$ at any point of the ellipse $2x^2 + y^2 = 1$ with respect to the normal to the ellipse is zero.

3455. (1) Find the derivative of $u = xy^2 + z^3 - xyz$ at the point $M(1, 1, 2)$ in the direction forming angles of 60°, 45°, 60° respectively with the coordinate axes.

(2) Find the derivative of $w = xyz$ at the point $A(5, 1, 2)$ in the direction from this point to the point $B(9, 4, 14)$.

3456. Find the derivative of $u = x^2y^2z^2$ at the point $A(1, -1, 3)$ in the direction from this point to the point $B(0, 1, 1)$.

3457. Show that the derivative of $u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ at any point $M(x, y, z)$ in the direction from this point to the origin is equal to $\frac{2u}{r}$, where $r = \sqrt{x^2 + y^2 + z^2}$.

3458. Prove that the derivative of $u = f(x, y, z)$ in the direction of its gradient is equal to the modulus of the gradient (see *Course*, sec. 161).

3459. Find the derivative of the function

$$u = \frac{1}{r}, \text{ where } r^2 = x^2 + y^2 + z^2,$$

in the direction of its gradient.

CHAPTER XII

MULTIPLE INTEGRALS AND ITERATED INTEGRATION

1. Double and Triple Integrals

3460. A thin lamina (its thickness is neglected) lies in the xOy plane, occupying a domain D. The density of the lamina is a function of a point: $\gamma = \gamma(P) = \gamma(x, y)$. Find the mass of the lamina.

3461. An electric charge with a surface density of $\sigma = \sigma(P) = \sigma(x, y)$ is distributed over the lamina of problem 360. Form the expression for the total charge on the lamina.

3462. The lamina of problem 3460 revolves about the Ox axis with angular velocity ω. Form the expression for the kinetic energy of the lamina.

3463. The specific heat of the lamina of problem 3460 varies according to the law $c = c(P) = c(x, y)$. Find the amount of heat absorbed by the lamina when it is heated from a temperature t_1 to a temperature t_2.

3464. A body occupies a spatial domain Ω; its density is a function of a point: $\gamma = \gamma(P) = \gamma(x, y, z)$. Find the mass of the body.

3465. Electric charge is distributed non-uniformly in the body of problem 3464; the charge density is a function of a point: $\delta = \delta(x, y, z)$. Find the total charge of the body.

Estimate the integrals of problems 3466–3476:

3466. $\iint\limits_D (x + y + 10)\, d\sigma$, where D is the circular domain $x^2 + y^2 \leqq 4$.

3467. $\iint\limits_D (x^2 + 4y^2 + 9)\, d\sigma$, where D is the circular domain $x^2 + y^2 \leqq 4$.

3468. $\iint\limits_D (x+y+1)\,d\sigma$, where D is the rectangle $0 \leqq x \leqq 1$, $0 \leqq y \leqq 2$.

3469. $\iint\limits_D (x+xy-x^2-y^2)\,d\sigma$, where D is the rectangle $0 \leqq x \leqq 1$, $0 \leqq y \leqq 2$.

3470. $\iint\limits_D xy(x+y)\,d\sigma$, where D is the square $0 \leqq x \leqq 2$, $0 \leqq y \leqq 2$.

3471. $\iint\limits_D (x+1)^y\,d\sigma$, where D is the square $0 \leqq x \leqq 2$, $0 \leqq y \leqq 2$.

3472. $\iint\limits_D (x^2+y^2-2\sqrt{x^2+y^2}+2)\,d\sigma$, where D is the square $0 \leqq x \leqq 2$, $0 \leqq y \leqq 2$.

3473. $\iint\limits_D (x^2+y^2-4x-4y+10)\,d\sigma$, where D is the domain bounded by the ellipse $x^2+4y^2-2x-16y+13=0$ (including the boundary).

3474. $\iiint\limits_\Omega (x^2+y^2+z^2)\,dv$, where Ω is the sphere $x^2+y^2+z^2 \leqq R^2$.

3475. $\iiint\limits_\Omega (x+y+z)\,dv$, where Ω is the cube $x \geqq 1$, $y \geqq 1$, $z \geqq 1$, $x \leqq 3$, $y \leqq 3$, $z \leqq 3$.

3476. $\iiint\limits_\Omega (x+y-z+10)\,dv$, where Ω is the sphere $x^2+y^2+z^2 \leqq 3$.

2. Iterated Integration

The Double Integral. Rectangular Domain

Evaluate the double integrals of problems 3477–3484, over the rectangular domains specified by the inequalities in brackets:

3477. $\iint\limits_D xy\,dx\,dy \quad (0 \leqq x \leqq 1,\ 0 \leqq y \leqq 2)$;

3478. $\iint\limits_D e^{x+y}\,dx\,dy \quad (0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1).$

3479. $\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy \quad (0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1).$

3480. $\iint\limits_D \frac{dx\,dy}{(x+y+1)^2} \quad (0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1).$

3481. $\iint\limits_D \frac{y\,dx\,dy}{(1+x^2+y^2)^{\frac{3}{2}}} \quad (0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1).$

3482. $\iint\limits_D x \sin(x+y)\,dx\,dy \quad \left(0 \leqq x \leqq \pi,\ 0 \leqq y \leqq \frac{\pi}{2}\right).$

3483. $\iint\limits_D x^2 y e^{xy}\,dx\,dy \quad (0 \leqq x \leqq 1,\ 0 \leqq y \leqq 2).$

3484. $\iint\limits_D x^2 y \cos(xy^2)\,dx\,dy \quad \left(0 \leqq x \leqq \frac{\pi}{2},\ 0 \leqq y \leqq 2\right).$

The Double Integral. Arbitrary Domain

Find the limits of the iterated integral $\iint\limits_D f(x, y)\,dx\,dy$ for the given (finite) domains of integration D.

3485. The parallelogram with sides

$$x = 3,\ x = 5,\ 3x - 2y + 4 = 0,\ 3x - 2y + 1 = 0.$$

3486. The triangle with sides $x = 0,\ y = 0,\ x + y = 2$.

3487. $x^2 + y^2 \leqq 1,\ x \geqq 0,\ y \geqq 0$.

3488. $x + y \leqq 1,\ x - y \leqq 1,\ x \geqq 0$.

3489. $y \geqq x^2,\ y \leqq 4 - x^2$.

3490. $\frac{x^2}{4} + \frac{y^2}{9} \leqq 1$.

3491. $(x-2)^2 + (y-3)^2 \leqq 4$.

3492. D is bounded by the parabolas $y = x^2$ and $y = \sqrt{x}$.

3493. The triangle with sides

$$y = x,\ y = 2x \text{ and } x + y = 6.$$

3494. The parallelogram with sides

$$y = x,\ y = x + 3,\ y = -2 + 1,\ y = -2x + 5.$$

3495. $y - 2x \leqq 0,\ 2y - x \geqq 0,\ xy \leqq 2.$

3496. $y^2 \leqq 8x,\ y \leqq 2x,\ y + 4x - 24 \leqq 0.$

3497. D is bounded by the hyperbola $y^2 - x^2 = 1$ and the circle $x^2 + y^2 = 9$ (the domain we have in mind contains the origin).

Change the order of integration in the integrals of problems 3498–3503:

3498. $\int\limits_0^1 dy \int\limits_y^{\sqrt{y}} f(x, y)\, dx.$ **3499.** $\int\limits_{-1}^1 dx \int\limits_0^{\sqrt{1-x^2}} f(x, y)\, dy.$

3500. $\int\limits_0^r dx \int\limits_x^{\sqrt{2rx-x^2}} f(x, y)\, dy.$ **3501.** $\int\limits_{-2}^2 dx \int\limits_{-\frac{1}{\sqrt{2}}\sqrt{4-x^2}}^{\frac{1}{\sqrt{2}}\sqrt{4-x^2}} f(x, y)\, dy.$

3502. $\int\limits_1^2 dx \int\limits_x^{2x} f(x, y)\, dy.$ **3503.** $\int\limits_0^2 dx \int\limits_{2x}^{6-x} f(x, y)\, dy.$

3504. By changing the order of integration, write each expression as one iterated integral:

(1) $\int\limits_0^1 dx \int\limits_0^x f(x, y)\, dy + \int\limits_1^2 dx \int\limits_0^{2-x} f(x, y)\, dy;$

(2) $\int\limits_0^1 dx \int\limits_0^{x^2} f(x, y)\, dy + \int\limits_1^3 dx \int\limits_0^{\frac{1}{2}(3-x)} f(x, y)\, dy;$

(3) $\int\limits_0^1 dx \int\limits_0^{x^{\frac{2}{3}}} f(x, y)\, dy + \int\limits_1^2 dx \int\limits_0^{1-\sqrt{4x-x^2-3}} f(x, y)\, dy.$

3505. Write the double integral $\iint\limits_D f(x, y)\, dx\ dy$, where D is the domain indicated in Fig. 62, 63, 64 and 65, as a sum of iterated integrals (with the least number of terms). The shapes shown in Fig. 64 and 65 are composed of straight lines and arcs of circles.

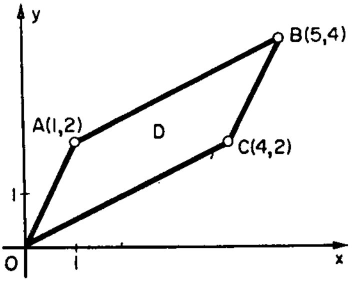

Fig. 62.

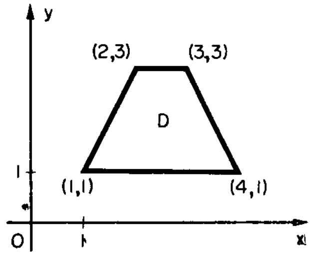

Fig. 63.

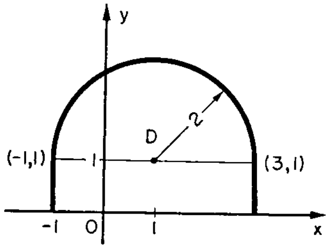

Fig. 64.

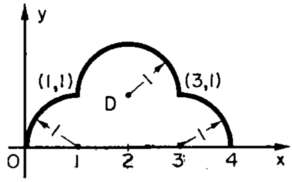

Fig. 65.

Evaluate the integrals of problems 3506–3512:

3506. (1) $\int\limits_0^a dx \int\limits_0^{\sqrt{x}} dy$; (2) $\int\limits_2^4 dx \int\limits_x^{2x} \frac{y}{x}\, dy$; (3) $\int\limits_1^2 dy \int\limits_0^{\ln y} e^x\, dx$.

3507. $\iint\limits_D x^3y^2\, dx\, dy$, where D is the circle $x^2 + y^2 \leqq R^2$.

3508. $\iint\limits_D (x^2 + y)\, dx\, dy$, D is the domain bounded by the parabolas $y = x^2$ and $y^2 = x$.

3509. $\iint\limits_D \frac{x^2}{y^2}\, dx\, dy$, D is the domain bounded by the straight lines $x = 2$, $y = x$ and the hyperbola $xy = 1$.

3510. $\iint\limits_D \cos(x + y)\, dx\, dy$, D is the domain bounded by the straight lines $x = 0$, $y = \pi$ and $y = x$.

3511. $\iint\limits_D \sqrt{1 - x^2 - y^2}\, dx\, dy$, D is the portion of the circle $x^2 + y^2 \leqq 1$ lying in the first quadrant.

3512. $\iint\limits_D x^2y^2\sqrt{1-x^3-y^3}\,dx\,dy$, D is the domain bounded by the curve $x^3+y^3=1$ and the coordinate axes.

3513. Find the mean value of the function $z = 12 - 2x - 3y$ in the domain bounded by the straight lines $12 - 2x - 3y = 0$, $x = 0$, $y = 0$.

3514. Find the mean value of the function $z = 2x + y$ in the triangle bounded by the axes and the straight line $x + y = 3$.

3515. Find the mean value of the function $z = x + 6y$ in the triangle bounded by the straight lines $y = x$, $y = 5x$ and $x = 1$.

3516. Find the mean value of the function

$$z = \sqrt{R^2 - x^2 - y^2}$$

in the circular domain $x^2 + y^2 \leqq R^2$.

Triple Integrals

Evaluate the thrice iterated integrals of problems 3517–3524:

3517. $\int\limits_0^1 dx \int\limits_0^2 dy \int\limits_0^3 dz.$

3518. $\int\limits_0^a dx \int\limits_0^b dy \int\limits_0^c (x+y+z)dz.$

3519. $\int\limits_0^a dx \int\limits_0^x dy \int\limits_0^y xyz\,dz.$

3520. $\int\limits_0^a dx \int\limits_0^x dy \int\limits_0^{xy} x^3y^2z\,dz.$

3521. $\int\limits_0^{e-1} dx \int\limits_0^{e-x-1} dy \int\limits_e^{x+y+e} \frac{\ln(z-x-y)}{(x-e)(x+y-e)}\,dz.$

3522. $\iiint\limits_\Omega \frac{dx\,dy\,dz}{(x+y+z+1)^3}$, Ω is the domain bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

3523. $\iiint\limits_\Omega xy\,dx\,dy\,dz$, Ω is the domain bounded by the hyperbolic paraboloid $z = xy$ and the planes $x + y = 1$ and $z = 0$ $(z \geqq 0)$.

3524. $\iiint\limits_{\Omega} y \cos(z + x)\, dx\, dy\, dz$, Ω is the domain bounded by the cylinder $y = \sqrt{x}$ and the planes $y = 0$, $z = 0$ and $z + x = \frac{\pi}{2}$.

3. Integrals in Polar, Cylindrical and Spherical Coordinates

Double Integrals

In problems 3525–3531, transform the double integral $\iint\limits_{D} f(x, y)\, dx\, dy$ to polar coordinates ϱ and $\varphi (x = \varrho \cos \varphi$, $y = \varrho \sin \varphi)$, and fix the limits of integration.

3525. D is a circular domain: (1) $x^2 + y^2 \leqq R^2$; (2) $x^2 + y^2 \leqq ax$; (3) $x^2 + y^2 \leqq by$.

3526. D is the domain bounded by the circles $x^2 + y^2 = 4x$, $x^2 + y^2 = 8x$ and the straight lines $y = x$ and $y = 2x$.

3527. D is the domain consisting of the common part of the two circular regions $x^2 + y^2 \leqq ax$ and $x^2 + y^2 \leqq by$.

3528. D is the domain bounded by the straight lines

$$y = x,\ y = 0 \text{ and } x = 1.$$

3529. D is the segment cut from the circle $x^2 + y^2 = 4$ by the straight line $x + y = 2$.

3530. D is the interior of the right-hand loop of the Bernoulli lemniscate $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$.

3531. D is the domain given by the inequalities $x \geqq 0$, $y \geqq 0$, $(x^2 + y^2)^3 \leqq 4a^2x^2y^2$.

Transform the double integrals of problems 3532–3535 to polar coordinates:

3532. $\int\limits_0^R dx \int\limits_0^{\sqrt{R^2-x^2}} f(x, y)\, dy.$ **3533.** $\int\limits_{R/2}^{2R} dy \int\limits_0^{\sqrt{2Ry-y^2}} f(x, y)\, dx.$

3534. $\int\limits_0^R dx \int\limits_0^{\sqrt{R^2-x^2}} f(x^2 + y^2)\, dy.$

3535. $\int\limits_0^{\frac{R}{\sqrt{1+R^2}}} dx \int\limits_0^{Rx} f\left(\frac{y}{x}\right) dy + \int\limits_{\frac{R}{\sqrt{1+R^2}}}^{R} dx \int\limits_0^{\sqrt{R^2-x^2}} f\left(\frac{y}{x}\right) dy.$

Evaluate the double integrals of problems 3536–3540 by transforming to polar coordinates:

3536. $\int\limits_0^R dx \int\limits_0^{\sqrt{R^2-x^2}} \ln(1+x^2+y^2)\, dy.$

3537. $\iint\limits_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\, dx\, dy$, where the domain D is given by the inequalities $x^2+y^2 \leqq 1$, $x \geqq 0$, $y \geqq 0$.

3538. $\iint\limits_D (h-2x-3y)\, dx\, dy$, where D is the circular domain $x^2+y^2 \leqq R^2$.

3539. $\iint\limits_D \sqrt{R^2-x^2-y^2}\, dx\, dy$, where D is the circular domain $x^2+y^2 \leqq Rx$.

3540. $\iint\limits_D \arctan\frac{y}{x}\, dx\, dy$, where D is part of an annular region given by

$$x^2+y^2 \geqq 1, \quad x^2+y^2 \leqq 9, \quad y \geqq \frac{x}{\sqrt{3}}, \quad y \leqq \sqrt{3}x.$$

3541. By starting from geometrical considerations, show that, if Cartesian coordinates are transformed in accordance with the formulae $x = a\varrho\cos\varphi$, $y = b\varrho\sin\varphi$ (a and b are constants), the elementary area becomes

$$d\sigma = ab\varrho\, d\varrho\, d\varphi.$$

By using the result of this last problem and choosing a and b in the best way, transform the double integrals of problems 3542–3544:

3542. $\iint\limits_D f(x, y)\,dx\,dy$, where domain D is bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

3543. $\iint\limits_D f(x, y)\,dx\,dy$, where domain D is bounded by the curve $\left(x^2 + \frac{y^2}{3}\right)^2 = x^2y$.

3544. $\iint\limits_D f\left(\sqrt{4 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\right) dx\,dy$, where domain D is part of the elliptic ring bounded by the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\frac{x^2}{4a^2} + \frac{y^2}{4b^2} = 1$ and lying in the first quadrant.

3545. Evaluate $\iint\limits_D xy\,dx\,dy$, where D is the domain bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and lying in the first quadrant.

3546. Evaluate $\iint\limits_D \sqrt{xy}\,dx\,dy$, where D is the domain bounded by the curve $\left(\frac{x^2}{2} + \frac{y^2}{3}\right)^4 = \frac{xy}{\sqrt{6}}$ and lying in the first quadrant.

Triple Integrals

In problems 3547–3551, pass to cylindrical coordinates ϱ, φ, z ($x = \varrho\cos\varphi$, $y = \varrho\sin\varphi$, $z = z$) or spherical coordinates ϱ, φ, θ ($x = \varrho\cos\varphi\sin\theta$, $y = \varrho\sin\varphi\sin\theta$, $z = \varrho\cos\theta$) in the triple integral $\iiint\limits_\Omega f(x, y, z)\,dx\,dy\,dz$, and fix the limits of integration.

3547. Ω is the domain lying in the first octant and bounded by the cylinder $x^2 + y^2 = R^2$ and the planes $z = 0$, $z = 1$, $y = x$ and $y = x\sqrt{3}$.

3548. Ω is the domain bounded by the cylinder $x^2 + y^2 = 2x$, the plane $z = 0$ and the paraboloid $z = x^2 + y^2$.

3549. Ω is the part of the sphere $x^2 + y^2 + z^2 \leqq R^2$ lying in the first octant.

3550. Ω is the part of the sphere $x^2 + y^2 + z^2 \leqq R^2$ lying inside the cylinder $(x^2 + y^2)^2 = R^2(x^2 - y^2)$ $(x \geqq 0)$.

3551. Ω is the common part of the two spheres

$$x^2 + y^2 + z^2 \leqq R^2 \quad \text{and} \quad x^2 + y^2 + (z - R)^2 \leqq R^2.$$

Evaluate the integrals of problems 3552–3556 by passing to cylindrical or spherical coordinates:

3552. $\int\limits_0^1 dx \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int\limits_0^a dz.$

3553. $\int\limits_0^2 dx \int\limits_0^{\sqrt{2x-x^2}} dy \int\limits_0^a z\sqrt{x^2 + y^2}\, dz.$

3554. $\int\limits_{-R}^{+R} dx \int\limits_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int\limits_0^{\sqrt{R^2-x^2-y^2}} (x^2 + y^2)\, dz.$

3555. $\int\limits_0^1 dx \int\limits_0^{\sqrt{1-x^2}} dy \int\limits_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2 + y^2 + z^2}\, dz.$

3556. $\iiint\limits_{\Omega} (x^2 + y^2)\, dx\, dy\, dz$, where the domain Ω is given by the inequalities $z \geqq 0$, $r^2 \leqq x^2 + y^2 + z^2 \leqq R^2$.

3557. $\iiint\limits_{\Omega} \frac{dx\, dy\, dz}{x^2 + y^2 + (z - 2)^2}$, where the domain Ω is the sphere $x^2 + y^2 + z^2 \leqq 1$.

3558. $\iiint\limits_{\Omega} \frac{dx\, dy\, dz}{x^2 + y^2 + (z - 2)^2}$, where the domain Ω is the cylinder $x^2 + y^2 \leqq 1$, $-1 \leqq z \leqq 1$.

4. Applications of Double and Triple Integrals

The Volume of a Solid. I

In problems 3559–3596, find by double integration the volumes of the solids bounded by the surfaces indicated (the parameters that appear in the conditions of the problems are assumed positive):

3559. By the coordinate planes, the planes $x = 4$ and $y = 4$ and the paraboloid of revolution $z = x^2 + y^2 + 1$.

3560. By the coordinate planes, the planes $x = a$, $y = b$ and the elliptic paraboloid $z = \frac{x^2}{2p} + \frac{y^2}{2q}$.

3561. By the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, and the coordinate planes (pyramid).

3562. By the planes $y = 0$, $z = 0$, $3x + y = 6$, $3x + 2y = 12$ and $x + y + z = 6$.

3563. By the paraboloid of revolution $z = x^2 + y^2$, the coordinate planes and the plane $x + y = 1$.

3564. By the paraboloid of revolution $z = x^2 + y^2$ and the planes $z = 0$, $y = 1$, $y = 2x$ and $y = 6 - x$.

3565. By the cylinder $y = \sqrt{x}$, $y = 2\sqrt{x}$ and the planes $z = 0$ and $x + z = 6$.

3566. By the coordinate planes, the plane $2x + 3y - 12 = 0$ and the cylinder $z = \frac{1}{2}y^2$.

3567. By the cylinder $z = 9 - y^2$, the coordinate planes and the plane $3x + 4y = 12$ $(y \geqq 0)$.

3568. By the cylinder $z = 4 - x^2$, the coordinate planes and the plane $2x + y = 4$ $(x \geqq 0)$.

3569. By the cylinder $2y^2 = x$, the planes $\frac{x}{4} + \frac{y}{2} + \frac{z}{4} = 1$ and $z = 0$.

3570. By the circular cylinder of radius r whose axis is the axis of ordinates, the coordinate planes and the plane $\frac{x}{r}+\frac{y}{a}=1$.

3571. By the elliptic cylinder $\frac{x^2}{4}+y^2=1$, the planes $z=12-3x-4y$ and $z=1$.

3572. By the cylinders $x^2+y^2=R^2$ and $x^2+z^2=R^2$.

3573. By the cylinders $z=4-y^2$, $y=\frac{x^2}{2}$ and the plane $z=0$.

3574. By the cylinders $x^2+y^2=R^2$, $z=\frac{x^3}{a^2}$ and the plane $z=0$ $(x \geqq 0)$.

3575. By the hyperbolic paraboloid $z=x^2-y^2$ and the planes $z=0$, $x=3$.

3576. By the hyperbolic paraboloid $z=xy$, the cylinder $y=\sqrt{x}$ and the planes $x+y=2$, $y=0$ and $z=0$.

3577. By the paraboloid $z=x^2+y^2$, the cylinder $y=x^2$ and the planes $y=1$ and $z=0$.

3578. By the elliptic cylinder $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$ and the planes $y=\frac{b}{a}x$, $y=0$ and $z=0$ $(x \geqq 0)$.

3579. By the paraboloid $z=\dfrac{a^2-x^2-4y^2}{a}$ and the plane $z=0$.

3580. By the cylinders $y=e^x$, $y=e^{-x}$, $z=e^2-y^2$ and the plane $z=0$.

3581. By the cylinders $y=\ln x$ and $y=\ln^2 x$ and the planes $z=0$ and $y+z=1$.

3582*. By the cylinders $z=\ln x$ and $z=\ln y$ and the planes $z=0$ and $x+y=2e$ $(x \geqq 1)$.

3583. By the cylinders $y=x+\sin x$, $y=x-\sin x$ and $z=\frac{(x+y)^2}{4}$ (the parabolic cylinder whose generators are parallel to the straight line $x - y = 0$, $z = 0$) and the plane $z = 0$ $(0 \leqq x \leqq \pi,\ y \geqq 0)$.

3584. By the conical surface $z^2 = xy$ (Fig. 66), the cylinder $\sqrt{x} + \sqrt{y} = 1$ and the plane $z = 0$.

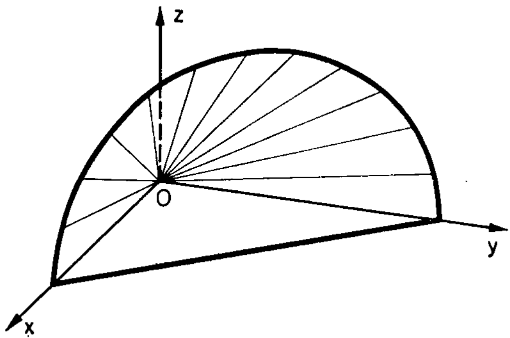

FIG. 66.

3585. By the conical surface $4y^2 = x(2 - z)$ (the parabolic cone, Fig. 67) and the planes $z = 0$ and $x + z = 2$.

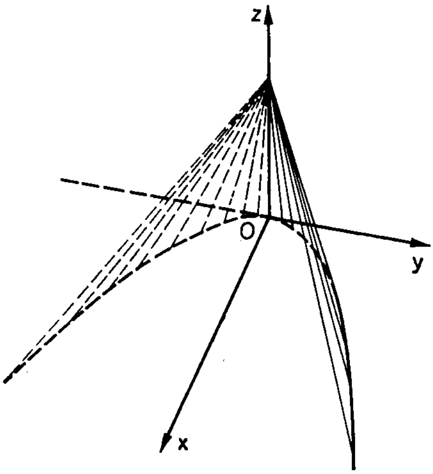

FIG. 67.

3586. By the surface $z = \cos x \cos y$ and the planes $x = 0$, $y = 0$, $z = 0$ and $x + y = \dfrac{\pi}{2}$.

3587. By the cylinder $x^2 + y^2 = 4$, the planes $z = 0$ and $z = x + y + 10$.

3588. By the cylinder $x^2 + y^2 = 2x$, the planes $2x - z = 0$ and $4x - z = 0$.

3589. By the cylinder $x^2 + y^2 = R^2$, the paraboloid $Rz = 2R^2 + x^2 + y^2$ and the plane $z = 0$.

3590. By the cylinder $x^2 + y^2 = 2ax$, the paraboloid $z = \frac{x^2 + y^2}{a}$ and the plane $z = 0$.

3591. By the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$. (Viviani's problem. See *Course*, sec. 174).

3592. By the hyperbolic paraboloid $z = \frac{xy}{a}$, the cylinder $x^2 + y^2 = ax$ and the plane $z = 0$ $(x \geqq 0, y \geqq 0)$.

3593. By the cylinders $x^2 + y^2 = x$ and $x^2 + y^2 = 2x$, the paraboloid $z = x^2 + y^2$ and the planes $x + y = 0$, $x - y = 0$ and $z = 0$.

3594. By the cylinders $x^2 + y^2 = 2x$, $x^2 + y^2 = 2y$ and the planes $z = x + 2y$ and $z = 0$.

3595. By the conical surface $z^2 = xy$ and the cylinder $(x^2 + y^2)^2 = 2xy$ $(x \geqq 0,\ y \geqq 0, z \geqq 0)$.

3596. By the helicoid $z = h \arctan \frac{y}{x}$, the cylinder $x^2 + y^2 = R^2$ and the planes $x = 0$ and $z = 0$ $(x \geqq 0, y \geqq 0)$.

Area of a Plane Figure

Find by double integration the areas of problems 3597–3608:

3597. The domain bounded by the straight lines

$$x = 0, \ \ y = 0, \ \ x + y = 1.$$

3598. The domain bounded by the straight lines $y = x$, $y = 5x$, $x = 1$.

3599. The domain bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

3600. The domain lying between the parabola $y^2 = \frac{b^2}{a} x$ and the straight line $y = \frac{b}{a} x$.

3601. The domain bounded by the parabolas $y = \sqrt{x}$, $y = 2\sqrt{x}$ and the straight line $x = 4$.

3602*. The domain bounded by the curve $(x^2 + y^2)^2 = 2ax^3$.

3603. The domain bounded by the curve $(x^2 + y^2)^3 = x^4 + y^4$.

3604. The domain bounded by the curve $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$ (Bernoulli's lemniscate).

3605. The domain bounded by the curve $x^3 + y^3 = 2xy$, lying in the first quadrant (loop).

3606. The domain bounded by the curve $(x + y)^3 = xy$, lying in the first quadrant (loop).

3607. The domain bounded by the curve $(x + y)^5 = x^2y^2$, lying in the first quadrant (loop).

3608*. The domain bounded by the curve:

$$(1)\quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{xy}{c^2}\,; \quad (2)\quad \left(\frac{x^2}{4} + \frac{y^2}{9}\right) = \frac{x^2 + y^2}{25}\,.$$

Volume of a Solid. II

Evaluate by triple integration the volumes of the solids bounded by the surfaces given in problems 3509–3625 (the parameters appearing in the conditions of the problems are assumed positive):

3609. By the cylinders $z = 4 - y^2$ and $z = y^2 + 2$ and the planes $x = -1$ and $x = 2$.

3610. By the paraboloid $z = x^2 + y^2$ and $z = x^2 + 2y^2$ and the planes $y = x$, $y = 2x$ and $x = 1$.

3611. By the paraboloids $z = x^2 + y^2$ and $z = 2x^2 + 2y^2$, the cylinder $y = x^2$ and the plane $y = x$.

3612. By the cylinders $z = \ln(x + 2)$ and $z = \ln(6 - x)$ and the planes $x = 0$, $x + y = 2$ and $x - y = 2$.

3613*. By the paraboloid $(x-1)^2+y^2=z$ and the plane $2x+z=2$.

3614*. By the paraboloid $z=x^2+y^2$ and the plane $z=x+y$.

3615*. By the sphere $x^2+y^2+z^2=4$ and the paraboloid $x^2+y^2=3z$.

3616. By the sphere $x^2+y^2+z^2=R^2$ and the paraboloid $x^2+y^2=R(R-2z)$ $(z \geqq 0)$.

3617. By the paraboloid $z=x^2+y^2$ and the cone $z^2=xy$.

3618. By the sphere $x^2+y^2+z^2=4Rz-3R^2$ and the cone $z^2=4(x^2+y^2)$ (we have in mind the part of the sphere lying inside the cone).

3619*. $(x^2+y^2+z^2)^2=a^3x$.

3620. $(x^2+y^2+z^2)^2=axyz$.

3621. $(x^2+y^2+z^2)^3=a^2z^4$.

3622. $(x^2+y^2+z^2)^3=\dfrac{a^6z^2}{x^2+y^2}$.

3623. $(x^2+y^2+z^2)^3=a^2(x^2+y^2)^2$.

3624. $(x^2+y^2)^2+z^4=a^3z$.

3625. $x^2+y^2+z^2=1$, $x^2+y^2+z^2=16$, $z^2=x^2+y^2$, $x=0$, $y=0$, $z=0$ $(x \geqq 0,\ y \geqq 0,\ z \geqq 0)$.

Surface Areas

3626. Find the area of the part of the plane $6x+3y+2z=12$ lying in the first octant.

3627. Find the area of the part of the surface $z^2=2xy$, which is situated above the rectangle lying in the plane $z=0$ and bounded by the straight lines $x=0$, $y=0$, $x=3$, $y=6$.

3628. Find the area of the part of the cone $z^2=x^2+y^2$, lying above the xOy plane and cut out by the plane $z=\sqrt{2}\left(\dfrac{x}{2}+1\right)$.

Find the areas of the indicated parts of the surfaces of problems 3629–3639.

3629. $z^2 = x^2 + y^2$, cut out by the cylinder $z^2 = 2py$.

3630*. $y^2 + z^2 = x^2$, lying inside the cylinder $x^2 + y^2 = R^2$.

3631. $y^2 + z^2 = x^2$, cut out by the cylinder $x^2 - y^2 = a^2$ and the planes $y = b$ and $y = -b$.

3632. $z^2 = 4x$, cut out by the cylinder $y^2 = 4x$ and the plane $x = 1$.

3633. $z = xy$, cut out by the cylinder $x^2 + y^2 = R^2$.

3634. $2z = x^2 + y^2$, cut out by the cylinder $x^2 + y^2 = 1$.

3635. $x^2 + y^2 + z^2 = a^2$, cut out by the cylinder $x^2 + y^2 = R^2$ $(R \leqq a)$.

3636. $x^2 + y^2 + z^2 = R^2$, cut out by the cylinder $x^2 + y^2 = Rx$.

3637. $x^2 + y^2 + z^2 = R^2$, cut out by the surface $(x^2 + y^2)^2 = R^2(x^2 - y^2)$.

3638. $z = \dfrac{x + y}{x^2 + y^2}$, cut out by the surfaces $x^2 + y^2 = 1$, $x^2 + y^2 = 4$ and lying in the first octant.

3639. $(x \cos \alpha + y \sin \alpha)^2 + z^2 = a^2$, lying in the first octant $\left(\alpha < \dfrac{\pi}{2}\right)$.

3640*. Find the area of the part of the earth's surface (assuming it to be spherical, with a radius $R \approx 6400$ km), lying between the meridians $\varphi = 30°$, $\varphi = 60°$ and the parallels $\theta = 45°$ and $\theta = 60°$.

3641. Find the total surface area of the body bounded by the sphere $x^2 + y^2 + z^2 = 3a^2$ and the paraboloid $x^2 + y^2 = 2az$ $(z \geqq 0)$.

3642. The axes of two equal cylinders of radius R intersect at right angles. Find the area of the part of the surface of one of the cylinders, lying in the other.

Moments and Centres of Gravity

Find by double integration the statical moments of the homogeneous plane figures (density $\gamma = 1$) of problems 3643–3646:

3643. The rectangle with sides a and b with respect to side a.

3644. A semi-circle with respect to its diameter.

3645. A circle with respect to a tangent.

3646. A regular hexagon with respect to one side.

3647. Prove that the statical moment of a triangle with base a with respect to this base depends only on the height of the triangle.

Find by double integration the centres of gravity of the homogeneous plane figures of problems 3648–3652:

3648. The figure bounded by the upper half of an ellipse, based on the major axis.

3649. The figure bounded by the sine wave $y = \sin x$, the axis and the straight line $x = \frac{\pi}{4}$.

3650. The circular sector corresponding to an angle α at the centre (radius of circle R).

3651. The circular segment corresponding to angle α at the centre (radius of circle R).

3652. The figure bounded by the closed curve $y^2 = x^2 - x^4$ $(x \geqq 0)$.

Find the moments of inertia of the homogeneous plane figures (density $\gamma = 1$) of problems 3653–3659.

3653. The circle of radius R with respect to a tangent.

3654. The square with side a with respect to one corner.

3655. The ellipse with respect to its centre.

3656. The rectangle of sides a and b with respect to the point of intersection of the diagonals.

3657. The isosceles triangle with base a and height h with respect to the vertex.

3658. The circle of radius R with respect to a point lying on the circumference.

3659. The segment of the parabola bounded by a chord perpendicular to the axis, with respect to the vertex of the parabola (length of chord $= a$, "height" $= h$).

3660. Prove that the moment of inertia of a circular annulus with respect to the centre is twice the moment of inertia with respect to any axis passing through the centre (lying in the plane of the annulus).

3661. Prove that the sum of the moments of inertia of a plane figure F with respect to any pair of mutually perpendicular axes, lying in the plane of the figure and passing through a fixed point O, is a constant.

3662*. Prove that the moment of inertia of a plane figure with respect to any axis is equal to $Md^2 + I_c$, where M is the mass distributed over the figure, d is the distance from the axis to the centre of gravity of the figure, and I_c is the moment of inertia with respect to an axis parallel to the given axis and passing through the centre of gravity (Steiner's theorem).

Find the statical moments of the homogeneous bodies of problems 3663–3665 (density $\gamma = 1$):

3663. The rectangular parallelepiped with ribs a, b and c with respect to its faces.

3664. The right circular cone (base radius R, height H) with respect to a plane through the vertex parallel to the base.

3665. The body bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the xOy plane with respect to this plane.

Find the centres of gravity of the homogeneous bodies bounded by the surfaces indicated in problems 3666–3672:

3666. By the planes $x = 0$, $y = 0$, $z = 0$, $x = 2$, $y = 4$ and $x + y + z = 8$ (truncated parallelepiped).

3667. By the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and the coordinate planes (we have in mind the body located in the first octant).

3668. By the cylinder $z=\frac{y^2}{2}$ and the planes $x=0$, $y=0$, $z=0$ and $2x-3y-12=0$.

3669. By the cylinders $y=\sqrt{x}$, $y=2\sqrt{x}$ and the planes $z=0$ and $x+z=6$.

3670. By the paraboloid $z=\frac{x^2+y^2}{2a}$ and the sphere x^2+ $+y^2+z^2=3a^2$ $(z\geqq 0)$.

3671. By the sphere $x^2+y^2+z^2=R^2$ and the cone $z\tan\alpha=\sqrt{x^2+y^2}$ (spherical sector).

3672. $(x^2+y^2+z^2)^2=a^3z$.

Find the centres of gravity of the homogeneous surfaces of problems 3673–3674:

3673. The part of the sphere lying in the first octant.

3674. The part of the paraboloid $x^2+y^2=2z$, cut off by the plane $z=1$.

In problems 3675–3680, find the moments of inertia of the homogeneous bodies with mass equal to M.

3675. The rectangular parallelepiped with ribs a, b and c with respect to each of the ribs and with respect to the centre of gravity.

3676. A sphere with respect to a tangent line.

3677. The ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ with respect to each of its three axes.

3678. The right circular cylinder (base radius R, height H) with respect to a base diameter and with respect to a diameter of its central section.

3679. The hollow sphere of external radius R, internal radius r, with respect to a diameter.

3680. The paraboloid of revolution (base radius R, height H) with respect to the axis through its centre of gravity and perpendicular to the axis of revolution (equatorial moment).

Find the moments of inertia of the parts indicated of the homogeneous surfaces of problems 3681–3683 (the mass of each part $= M$):

3681. The lateral surface of a cylinder (base radius R, height H) with respect to the axis through its centre of gravity and perpendicular to the cylinder axis.

3682. The part of the paraboloid $x^2 + y^2 = 2cx$, cut off by the plane $z = c$, with respect to the Oz axis.

3683. The lateral surface of the frustrum of a cone (base radii R and r, height H) with respect to its axis.

Miscellaneous Problems

3684. Find the mass of a square lamina of side $2a$, if the density of the material of the lamina is proportional to the square of the distance from the point of intersection of the diagonals and is equal to unity at the corners.

3685. A plane annular ring is bounded by two concentric circles of radii R and r ($R > r$). Find the mass of the ring, given that the density is inversely proportional to the distance from its centre. The density on the inner circumference is equal to unity.

3686. Mass is distributed over the area bounded by an ellipse with semi-axes a and b in such a way that its density is proportional to the distance from the major axis, the density at unit distance from this axis being γ. Find the total mass.

3687. A body is bounded by two concentric spherical surfaces, the radii of which are r and R ($R > r$). Knowing that the density of the material is inversely proportional to the distance from the centre of the sphere and is equal to γ at unit distance, find the total mass of the body.

3688. Find the mass of the body bounded by a right circular cylinder of radius R and height H, if the density at any point is numerically equal to the square of the distance of the point from the centre of the cylinder base.

3689*. Find the mass of a body bounded by a circular cone, the height of which is equal to h, and the angle between the axis and generator of which is equal to α; the density is proportional to the nth power of the distance from the plane through the vertex of the cone parallel to the base, the density at unit distance being γ ($n > 0$).

3690. Find the mass of a sphere of radius R, if the density is proportional to the cube of the distance from the centre and is equal to γ at unit distance.

3691. Find the mass of a body bounded by the paraboloid $x^2 + y^2 = 2az$ and the sphere $x^2 + y^2 + z^2 = 3a^2$ ($z > 0$), if the density at any given point is equal to the square of the sum of the coordinates of the point.

3692*. The density at any point of the sphere $x^2 + y^2 + z^2 \leqq 2Rz$ is numerically equal to the square of the distance of the point from the origin. Find the coordinates of the centre of gravity of the sphere.

3693*. Find the statical moment of the common part of the spheres $x^2 + y^2 + z^2 \leqq R^2$ and $x^2 + y^2 + z^2 \leqq 2Rz$ with respect to the xOy plane. The density at any point of the body is numerically equal to the distance of the point from the xOy plane.

3694*. Prove that the moment of inertia of a body with respect to any axis is equal to $Md^2 + I_c$, where M is the mass of the body, d is the distance from the axis to the centre of gravity of the body, and I_c is the moment of inertia with respect to the axis parallel to the given axis and passing through the centre of gravity (Steiner's theorem; cf. problem 3662).

Solve problems 3695–3698 on the basis of Newton's law of universal gravitation (see the remark on problem 2670):

3695. Given a homogeneous sphere of radius R and density γ, find the force with which it attracts a material particle of mass m at a distance a from its centre ($a > R$). Show that the force of interaction is the same as when the total mass of the sphere is concentrated at its centre.

3696*. Prove that the Newton force of interaction between two homogeneous spheres is the same as when the masses of the spheres are concentrated at their centres.

3697. Given the non-homogeneous continuous sphere $x^2 + y^2 + z^2 \leqq R^2$ with density varying in accordance with the law $\gamma = \lambda z^2$, find the force with which it attracts a material particle of mass m, located on the z axis at a distance $2R$ from the centre of the sphere.

3698. Given a homogeneous body bounded by two concentric spheres (spherical layer), prove that the force of attraction by the layer on a point situated in the interior cavity is equal to zero.

The centre of pressure is defined as the point of application of the resultant of all the pressures on the given plane figure (all the pressure forces are perpendicular to the plane of the figure). When determining the coordinates of the centre of pressure, we start from the fact that the statical moment of the resultant (i.e. the pressure on the entire area) with respect to any axis is equal to the sum of the statical moments of the separate forces with respect to the same axis. On the basis of this, solve problems 3699–3701.

3699. Find the centre of pressure of a rectangle with sides a and $b (a > b)$, a greater side of which is located along the free surface of the fluid, whilst the plane of the rectangle is perpendicular to this surface.

Show that the position of the centre of pressure does not vary if the plane of the rectangle is inclined at an angle $\alpha (\alpha \neq 0)$ to the surface of the fluid. How are the above results affected if the greater side a is at a depth h (remaining parallel to the surface) instead of lying on the surface?

3700. A triangle of height h is in a plane inclined at an angle α to the free surface of a fluid. What is the depth of the centre of pressure of the triangle, if:

(a) its base lies on the fluid surface?

(b) its vertex lies on the surface, whilst its base is parallel to the surface?

3701. Find the centre of pressure of the figure bounded by an ellipse with semi-axes a and b $(a > b)$, given that the major axis is perpendicular to the fluid surface and the upper end of this axis lies at a distance h from the surface.

3702*. Prove that the fluid pressure on a plane area, arbitrarily submerged in fluid, is equal to the weight of a cylindrical column of the fluid, situated above the area, when the area lies horizontally at the depth of its centre of pressure.

5. Improper Integrals. Integrals Depending on a Parameter

Improper Double and Triple Integrals

Evaluate or establish the divergence of the improper integrals of problems 3703–3711:

3703. $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{dx\,dy}{1+x^2+y^2}$. **3704.** $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{dx\,dy}{(1+x^2+y^2)^{\frac{3}{2}}}$.

3705. $\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\frac{dx\,dy}{(x^2+y^2+a^2)^2}$. **3706.** $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}e^{-|x|-|y|}\,dx\,dy$.

3707. $\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}(x+y)\,e^{-(x+y)}\,dx\,dy$.

3708. $\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}xy\,e^{-x^2-y^2}\,dx\,dy$.

3709*. $\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}e^{-(x^2+2xy\cos\alpha+y^2)}\,dx\,dy$.

3710*. $\int\limits_{0}^{\infty}dx\int\limits_{x}^{\infty}e^{-y^2}\,dy$.

3711*. $\int\limits_{0}^{\infty}dx\int\limits_{2x}^{\infty}xe^{-y}\frac{\sin y}{y^2}\,dy$.

Discover which of the improper integrals of problems 3712–3715 over the circular domain of radius R with centre at the origin is convergent:

3712. $\iint\limits_{D}\ln\sqrt{x^2+y^2}\,dx\,dy$ (see *Course*, sec. 179).

3713. $\iint\limits_D \frac{e^{-x^2-y^2}}{x^2+y^2}\,dx\,dy.$ **3714.** $\iint\limits_D \frac{\sin(x^2+y^2)}{\sqrt{(x^2+y^2)^3}}\,dx\,dy.$

3715. $\iint\limits_D \frac{\cos(x^2+y^2)}{x^2+y^2}\,dx\,dy.$

3716. Can a number m be chosen such that the improper integral $\iint \frac{dx\,dy}{\sqrt{(x^2+y^2)^m}}$, taken over the whole of the plane, is convergent?

Evaluate the improper integrals of problems 3717–3719:

3717. $\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty \frac{dx\,dy\,dz}{\sqrt{(1+x+y+z)^7}}.$

3718. $\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty \frac{xy\,dx\,dy\,dz}{(1+x^2+y^2+z^2)^3}.$

3719*. $\int\limits_{-\infty}^\infty\int\limits_{-\infty}^\infty\int\limits_{-\infty}^\infty e^{-x^2-y^2-z^2}\,dx\,dy\,dz.$

Examine the convergence of the improper integrals of problems 3720–3722, taken over the sphere Ω of radius R with centre at the origin:

3720. $\iiint\limits_\Omega \frac{dx\,dy\,dz}{\sqrt{(x^2+y^2+z^2)^3}\ln\sqrt[3]{x^2+y^2+z^2}}.$

3721. $\iiint\limits_\Omega \frac{\ln\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2}\,dx\,dy\,dz.$

3722. $\iiint\limits_\Omega \frac{xyz}{(x^2+y^2+z^2)^3}\,dx\,dy\,dz.$

3723. Evaluate $\iiint\limits_\Omega \ln(x^2+y^2+z^2)\,dx\,dy\,dz$, where the domain Ω is the sphere of radius R with centre at the origin.

3724*. Find the volume of the body bounded by the surface $z=(x^2+y^2)\,e^{-(x^2+y^2)}$ and the plane $z=0$.

3725. Evaluate the volume of the body bounded by the surface $z = x^2y^2e^{-(x^2+y^2)}$ and the plane $z = 0$.

3726. Find the volume of the body bounded by the plane $z = 0$ and the part of the surface $z = xe^{-(x^2+y^2)}$ lying above this plane.

3727. Given a homogeneous body bounded by the right circular cylinder of base radius R (height H, density γ), find the force acting on a particle of mass m situated at the centre of the base.

3728. Given a homogeneous body bounded by the right circular cone of base radius R and height H (density γ), find the force with which the body attracts a particle of mass m located at the vertex of the cone.

3729. Given a non-homogeneous continuous sphere of radius R, the density γ of which is connected with the distance from the centre r by the relationship $\gamma = a - br$ ($a > 0$, $b > 0$):

(a) find the constants a and b, if it is known that the density at the centre of the sphere is equal to γ_c, whilst the density on the surface of the sphere is equal to γ_0.

(b) Find the force of attraction by the sphere on a particle of mass m, located on the surface of the sphere.

Integrals Depending on a Parameter. Leibniz's Rule

3730. Find the domain of definition of the function

$$f(x) = \int_0^1 \frac{dz}{\sqrt{x^2 + z^2}}.$$

3731. Find the curvature of the curve $y = \int_{\pi}^{2\pi} \frac{\sin \alpha x}{\alpha}\, d\alpha$ at the point with abscissa $x = 1$.

3732. Using the equation $\int_0^b \frac{dx}{1 + ax} = \frac{1}{a} \ln (1 + ab)$, obtain by differentiation with respect to the parameter the formula:

$$\int_0^b \frac{x\,dx}{(1+ax)^2} = \frac{1}{a^2}\ln(1+ab) - \frac{b}{a(1+ab)}\,.$$

3733. Starting from the equation $\int_0^\infty \frac{dx}{x^2+a^2} = \frac{1}{a}\arc\tan\frac{b}{a}$, evaluate $\int_0^b \frac{dx}{(x^2+a^2)^3}$.

3734. Starting from the equation $\int_0^\infty \frac{dx}{a^2+x^2} = \frac{\pi}{2a}$, evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^n}$ (n is a positive integer).

3735. Evaluate the integral $\int_0^\infty e^{-ax}x^{n-1}\,dx$ (n is a positive integer) with $a > 0$, by finding as a preliminary $\int_0^\infty e^{-ax}\,dx$.

3736*. By starting from the equation (see problem 2318):

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2\,|ab|}\,,$$

find

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}\,.$$

Evaluate the integrals of problems 3737–3749 with the aid of differentiation with respect to a parameter:

3737. $\int_0^\infty \frac{1-e^{-ax}}{xe^x}\,dx$ $(a > -1)$.

3738. $\int\limits_0^\infty \frac{1 - e^{-ax^2}}{xe^{x^2}}\, dx \quad (a > -1).$

3739. $\int\limits_0^1 \frac{\arctan ax}{x\sqrt{1 - x^2}}\, dx.$

3740. $\int\limits_0^1 \frac{\ln(1 - a^2x^2)}{x^2\sqrt{1 - x^2}}\, dx \quad (a^2 < 1).$

3741. $\int\limits_0^\infty \frac{\arctan ax}{x(1 + x^2)}\, dx.$

3742. $\int\limits_0^1 \frac{\ln(1 - a^2x^2)}{\sqrt{1 - x^2}}\, dx \quad (a^2 < 1).$

3743. $\int\limits_0^\pi \frac{\ln(1 + a\cos x)}{\cos x}\, dx \quad (a^2 < 1).$

3744. $\int\limits_0^{\frac{\pi}{2}} \ln\left(\frac{1 + a\sin x}{1 - a\sin x}\right) \frac{dx}{\sin x} \quad (a^2 < 1).$

3745. $\int\limits_0^\infty \frac{1 - e^{-ax^2}}{x^2}\, dx \quad (a > 0)$ knowing that

$$\int\limits_0^\infty e^{-ax^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \quad (a > 0) \text{ (see problem 2439).}$$

3746*. $\int\limits_0^\infty \frac{e^{-ax^2} - e^{-bx^2}}{x^2}\, dx \quad (a > 0,\ b > 0).$

3747*. $\int\limits_0^\infty e^{-ax} \frac{\sin bx - \sin cx}{x}\, dx \quad (a > 0).$

3748. $\int\limits_0^\infty e^{-ax}\frac{\cos bx - \cos cx}{x}\,dx \ (a > 0)$.

3749*. $\int\limits_0^{\frac{\pi}{2}} \ln(a^2\cos^2 x + b^2\sin^2 x)\,dx$.

3750. Having evaluated $\int\limits_0^{\frac{\pi}{2}}\frac{\arctan(a\tan x)}{\tan x}\,dx$, find $\int\limits_0^{\frac{\pi}{2}}\frac{x}{\tan x}\,dx$.

3751*. By using the equation $\int\limits_0^1 x^n\,dx = \frac{1}{n+1}$, evaluate

$$\int\limits_0^1\frac{x^\beta - x^\alpha}{\ln x}\,dx \quad (\alpha > -1,\ \beta > -1).$$

3752. By using the equation $2a\int\limits_0^\infty e^{-a^2x^2}dx = \sqrt{\pi}$, (see problem 2439), evaluate $\int\limits_0^\infty\left(e^{-\frac{a^2}{x^2}} - e^{-\frac{b^2}{x^2}}\right)dx$.

3753. By starting from the relationship $\int\limits_0^\infty e^{-z^2}\,dz = \frac{\sqrt{\pi}}{2}$ (Poisson's integral), deduce the equation $\frac{1}{\sqrt{x}} = \frac{2}{\sqrt{\pi}}\int\limits_0^\infty e^{-z^2x}\,dz$ $(x > 0)$, and use this to evaluate the integrals (Fresnel or diffraction integrals):

$$\text{(a)}\ \int\limits_0^\infty\frac{\cos x\,dx}{\sqrt{x}};\quad \text{(b)}\ \int\limits_0^\infty\frac{\sin x\,dx}{\sqrt{x}}.$$

Miscellaneous Problems

3754. Let $f(x)$ be continuous for $x \geqq 0$ and tend to a finite limit $f(\infty)$ as $x \to \infty$. Further, let $\int\limits_0^\infty f'(\alpha x)\,dx$ be uniformly convergent for $0 < a \leqq \alpha \leqq b$. Prove that, with these conditions,

$$\int\limits_0^\infty \frac{f(ax) - f(bx)}{x}\,dx = [f(\infty) - f(0)] \ln \frac{a}{b}.$$

Evaluate the integrals of problems 3755–3756 by using the results of problem 3754:

3755. $\displaystyle\int\limits_0^\infty \frac{\operatorname{arc\,tan} ax - \operatorname{arc\,tan} bx}{x}\,dx.$

3756. $\displaystyle\int\limits_0^\infty \frac{e^{-ax^n} - e^{-bx^n}}{x}\,dx \quad (n > 0).$

3757*. Let $f(x)$ be continuous for $x \geqq 0$ and let $\int\limits_A^\infty \frac{f(x)}{x}\,dx$ be convergent for any $A > 0$.

Prove that, with these conditions, if $a > 0$ and $b > 0$, we have $\displaystyle\int\limits_0^\infty \frac{f(ax) - f(bx)}{x}\,dx = f(0) \ln \frac{b}{a}$ (cf. problem 3754).

Evaluate the integrals of problems 3758–3762 by using the result of problem 3757 ($a > 0$, $b > 0$):

3758. $\displaystyle\int\limits_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx.$

3759. $\displaystyle\int\limits_0^\infty \frac{\cos ax - \cos bx}{x}\,dx.$

3760. $\displaystyle\int\limits_0^\infty \frac{\sin ax \sin bx}{x}\,dx.$

3761. $\displaystyle\int\limits^\infty \frac{b \sin ax - a \sin bx}{x^2}\,dx.$

3762*. $\displaystyle\int\limits_0^\infty \frac{\sin^3 x}{x^2}\,dx.$

3763*. The Laplace function $\Phi(x)$ is defined thus: $\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$ (this function plays a large part in the theory of probability). Prove the relationships:

(1) $\int_0^x \Phi(az)\, dz = \frac{e^{-a^2x^2} - 1}{a\sqrt{\pi}} + x\Phi(ax)$;

(2) $\int_0^\infty [1 - \Phi(x)]\, dx = \frac{1}{\sqrt{\pi}}$.

3764*. Functions $\mathrm{si}(x)$ and $\mathrm{ci}(x)$ are usually defined thus: $\mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t}\, dt$ ("integral sine") and $\mathrm{ci}(x) = -\int_x^\infty \frac{\cos t}{t}\, dt$ ("integral cosine"). Prove that

$$\int_0^\infty \sin x\, \mathrm{si}(x)\, dx = \int_0^\infty \cos x\, \mathrm{ci}(x)\, dx = -\frac{\pi}{4}.$$

3765*. The function $J_0(x)$, defined as

$$J_0(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(x \sin\theta)\, d\theta,$$

is termed the Bessel function of zero order. Prove that:

(1) $\int_0^\infty e^{-ax} J_0(x)\, dx = \frac{1}{\sqrt{1 + a^2}}$ $(a > 0)$;

(2) $$\int_0^\infty \frac{\sin ax}{x} J_0(x)\, dx = \begin{cases} \frac{\pi}{2}, & \text{if } a \geqq 1; \\ \arcsin a, & \text{if } |a| \leqq 1; \\ -\frac{\pi}{2}, & \text{if } a \leqq -1. \end{cases}$$

3766. Prove that the function

$$y = \int_0^\infty \frac{e^{-xz}}{1+z^2}\, dz$$

satisfies the differential equation

$$y'' + y = \frac{1}{x}.$$

3767*. Prove that the function

$$y = \int_{-1}^{-1} (z^2 - 1)^{n-1} e^{xz}\, dz$$

satisfies the differential equation

$$xy'' + 2ny' - xy = 0.$$

3768*. Prove that the function

$$y = \int_0^\infty \frac{e^{-xz}}{(1+z^2)^{n+1}}\, dz$$

satisfies the differential equation

$$xy'' - 2ny' + xy = 1.$$

3769*. Prove that the zero order Bessel function $J_0(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(x \sin\theta)\, d\theta$ satisfies the differential equation

$$J_0''(x) + \frac{J_0'(x)}{x} + J_0(x) = 0.$$

CHAPTER XIII

LINE AND SURFACE INTEGRALS

1. Line Integrals

Evaluation of Integrals

Evaluate the line integrals of problems 3770–3775:

3770. $\int\limits_L \frac{ds}{x-y}$, where L is the segment of the straight line $y = \frac{1}{2}x - 2$ lying between the points $A(0, -2)$ and $B(4, 0)$.

3771. $\int\limits_L xy\, ds$, where L is the rectangular contour with corners $A(0, 0)$, $B(4, 0)$, $C(4, 2)$ and $D(0, 2)$.

3772. $\int\limits_L y\, ds$, where L is the arc of the parabola $y^2 = 2px$ cut off by the parabola $x^2 = 2py$.

3773. $\int\limits_L (x^2 + y^2)^n\, ds$, where L is the circle $x = a\cos t$, $y = a\sin t$.

3774. $\int\limits_L xy\, ds$, where L is the quarter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ lying in the first quadrant.

3775. $\int\limits_L \sqrt{2y}\, ds$, where L is the first arc of the cycloid

$$x = a(t - \sin t), \qquad y = a(1 - \cos t).$$

3776. Obtain a formula for evaluating $\int\limits_L F(x, y)\, ds$, if curve L is given by the equation $\varrho = \varrho(\varphi)$ $(\varphi_1 \leqq \varphi \leqq \varphi_2)$ in polar coordinates.

3777*. Evaluate $\int\limits_L (x - y)\,ds$, where L is the circle

$$x^2 + y^2 = ax.$$

3778. Evaluate $\int\limits_L x\sqrt{x^2 - y^2}\,ds$, where L is the curve given by the equation $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ $(x \geqq 0)$ (half a lemniscate).

3779. Evaluate $\int\limits_L \arctan\frac{y}{x}\,ds$, where L is the part of the spiral of Archimedes $\varrho = 2\varphi$ lying inside the circle of radius R with centre at the origin (pole).

3780. Evaluate $\int \frac{z^2\,ds}{x^2 + y^2}$, taken along the first turn of the helix $x = a\cos t$, $y = a\sin t$, $z = at$.

3781. Evaluate $\int\limits_L xyz\,ds$, where L is the quadrant of the circle $x^2 + y^2 + z^2 = R^2$, $x^2 + y^2 = \frac{R^2}{4}$ lying in the first octant.

3782. Evaluate $\int\limits_L (2z - \sqrt{x^2 + y^2})\,ds$, where L is the first turn of the conical helix

$$x = t\cos t,\ \ y = t\sin t,\ \ z = t.$$

3783. Evaluate $\int\limits_L (x + y)\,ds$, where L is the quadrant of the circle $x^2 + y^2 + z^2 = R^2$, $y = x$, lying in the first octant.

Applications of Integrals

3784. Find the mass of the portion of the curve $y = \ln x$ between the points with abscissae x_1 and x_2, if the density at any point of the curve is equal to the square of the abscissa of that point.

3785. Find the mass of the portion of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

between the points with abscissae $x_1 = 0$, $x_2 = a$, if the density at any point of the curve is inversely proportional to the ordinate of the point, the density at the point $(0, a)$ being equal to δ.

3786. Find the mass of the quarter of the ellipse $x = a \cos t$, $y = b \sin t$, situated in the first quadrant, if the density at any point is equal to the ordinate of the point.

3787. Find the mass of the first turn of the helix $x = a \cos t$, $y = a \sin t$, $z = bt$, the density at any point of which is equal to the square of the radius vector of the point.

3788. Find the mass of the arc of the curve $x = e^t \cos t$, $y = e^t \sin t$, $z = e^t$ from the point corresponding to $t = 0$ to an arbitrary point, if the density at any point of the arc is inversely proportional to the square of the radius vector of the point and is equal to unity at the point $(1, 0, 1)$.

3789. Find the coordinates of the centre of gravity of the first half-turn of the helix $x = a \cos t$, $y = a \sin t$, $z = bt$, the density being assumed constant.

3790. Find the statical moment of the first turn of the conical helix $x = t \cos t$, $y = t \sin t$, $z = t$ with respect to the xOy plane, the density being assumed proportional to the square of the distance from this plane: $\varrho = kz^2$.

3791. Find the moment of inertia with respect to the coordinate axes of the first turn of the helix $x = a \cos t$, $y = a \sin t$, $z = \frac{h}{2\pi} t$.

Find the areas of the parts of the cylindrical surfaces of problems 3792–3797 lying between the xOy plane and the surfaces indicated:

3792. $x^2 + y^2 = R^2,\ z = R + \dfrac{x^2}{R}$.

3793. $y^2 = 2px,\ z = \sqrt{2px - 4x^2}$.

3794. $y^2 = \dfrac{4}{9}(x-1)^3,\ \ z = 2 - \sqrt{x}$.

3795. $x^2 + y^2 = R^2,\ 2Rz = xy$.

3796. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ \ z = kx$ and $z = 0$ $(z \geqq 0)$ ("cylindrical horse-shoe").

3797. $y = \sqrt{2px},\ \ z = y$ and $x = \dfrac{8}{9}p$.

3798. Find the area of the surface, which is cut out from a circular cylinder of radius R by an equal cylinder, when their axes intersect at right angles (cf. the solution of problem 3642).

3799. Find the area of the part of the surface of the cylinder $x^2 + y^2 = Rx$ lying inside the sphere $x^2 + y^2 + z^2 = R^2$.

According to the Biot–Savart law, the force acting on a magnetic dipole of pole-strength m due to a current flowing in a wire is equal in magnitude to $\dfrac{mI \sin \alpha \, \mathrm{d}s}{r^2}$, where I is the current, ds an element of length of the wire, r the distance from the element to the magnetic dipole, α the angle between the straight line joining the magnetic dipole and the element and the direction of the element. This force is directed along the normal to the plane containing the element and the point at which the magnetic dipole is located; the direction of the force is established by the "screw" rule. On the basis of this law, solve problems 3800–3805.

3800. Find the force with which a current I in an infinite straight wire acts on a magnetic dipole m, situated at a distance a from the wire.

3801. A current I flows along a circuit in the form of a square of side α. What is the force exerted by the current on a magnetic dipole m at the centre of the square?

3802. Prove that the current I flowing along the arc of a curve whose equation is given in polar coordinates by $\varrho = \varrho(\varphi)$, exerts a force $f = mI \int\limits_{\varphi_1}^{\varphi_2} \frac{d\varphi}{\varrho}$ on a magnetic dipole situated at the pole.

3803. What is the force exerted by a current I flowing in a closed elliptic circuit on a magnetic dipole m situated at a focus of the ellipse?

3804. What is the force exerted by the current I flowing in an infinite parabolic circuit on a magnetic dipole m located at the focus of the parabola? The distance from the vertex to the focus is equal to $\frac{p}{2}$.

3805. What is the force exerted by a current I flowing in a circular circuit of radius R on a magnetic dipole m located at a point P, lying on the perpendicular to the plane of the circle through its centre and distant h from the centre?

Given a fixed h, for what value of R is this force a maximum?

2. Coordinate Line Integrals

Evaluation of Integrals

Evaluate the line integrals of problems 3806–3821:

3806. $\int\limits_L x\, dy$, where L is the triangle formed by the coordinate axes and the straight line $\frac{x}{2} + \frac{y}{3} = 1$, taken in a positive direction (i.e. anticlockwise).

3807. $\int\limits_L x\, dy$, where L is the segment of the straight line $\frac{x}{a} + \frac{y}{b} = 1$ from its point of intersection with the axis of abscissae to its point of intersection with the axis of ordinates.

3808. $\int\limits_L (x^2 - y^2)\,dx$, where L is the arc of the parabola $y = x^2$ from the point $(0, 0)$ to the point $(2, 4)$.

3809. $\int\limits_L (x^2 + y^2)\,dy$, where L is the rectangle with corners (given in order of the circuit round them) at $A(0, 0)$, $B(2, 0)$, $C(4, 4)$ and $D(0, 4)$.

3810. $\int\limits_{(0,0)}^{(\pi, 2\pi)} -x\cos y\,dx + y\sin x\,dy$ along the segment joining the points $(0, 0)$ and $(\pi, 2\pi)$.

3811. $\int\limits_{(0,0)}^{(1,1)} xy\,dx + (y - x)\,dy$ along the curves (1) $y = x$, (2) $y = x^2$, (3) $y^2 = x$, (4) $y = x^3$.

3812. $\int\limits_{(0,0)}^{(1,1)} 2xy\,dx + x^2\,dy$ along the curve (1) $y = x$, (2) $y = x^2$, (3) $y = x^3$, (4) $y^2 = x$.

3813. $\int\limits_L y\,dx + x\,dy$, where L is the quadrant of the circle $x = R\cos t$, $y = R\sin t$, from $t_1 = 0$ to $t_2 = \dfrac{\pi}{2}$.

3814. $\int\limits_L y\,dx - x\,dy$, where L is the ellipse $x = a\cos t$, $y = b\sin t$, taken in the positive direction.

3815. $\int\limits_L \dfrac{(y^2\,dx - x^2\,dy)}{(x^2 + y^2)}$, where L is the semi-circle $x = a\cos t$, $y = a\sin t$ from $t_1 = 0$ to $t_2 = \pi$.

3816. $\int\limits_L (2a - y)\,dx - (a - y)\,dy$, where L is the first (from the origin) arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

3817. $\int\limits_L \dfrac{(x^2\,dy - y^2\,dx)}{x^{\frac{5}{3}} + y^{\frac{5}{3}}}$, where L is the quadrant of the astroid $x = R\cos^3 t$, $y = R\sin^3 t$, from the point $(R, 0)$ to the point $(0, R)$.

3818. $\int\limits_L x\,dx + y\,dy + (x + y - 1)\,dz$, where L is the segment of a straight line from the point $(1, 1, 1)$ to the point $(2, 3, 4)$.

3819. $\int\limits_L yz\,dx + z\sqrt{R^2 - y^2}\,dy + xy\,dz$, where L is the arc of the helix $x = R\cos t$, $y = R\sin t$, $z = \frac{at}{2\pi}$, from the point of intersection of the curve with the plane $z = 0$ to its point of intersection with the plane $z = a$.

3820. $\int\limits_{(1,1,1)}^{(4,4,4)} \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2 + y^2 + z^2 - x - y + 2z}}$ along a straight line.

3821. $\int\limits_L y^2\,dx + z^2\,dy + x^2\,dz$, where L is the curve of intersection of the sphere $x^2 + y^2 + z^2 = R^2$ and the cylinder $x^2 + y^2 = Rx$ $(R > 0,\ z \geqq 0)$, the integration circuit being anticlockwise as seen from the origin.

Green's Formula

In problems 3822–3823, transform the line integrals over a closed contour L, taken in the positive direction, to double integrals over the domain bounded by the contour:

3822. $\int\limits_L (1 - x^2)\,y\,dx + x\,(1 + y^2)\,dy$.

3823. $\int\limits_L (e^{xy} + 2x\cos y)\,dx + (e^{xy} - x^2\sin y)\,dy$.

3824. If the contour of integration L is the circle $x^2 + y^2 = R^2$, evaluate the integral of problem 3822 by the following two methods:

(1) directly,

(2) by using Green's formula.

3825. Evaluate $\int\limits_L (xy + x + y)\,dx + (xy + x - y)\,dy$, where L is: (1) the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; (2) the circle $x^2 + y^2 = ax$.

The integration is carried out in a positive direction. (Carry out the evaluation by two methods: (1) directly, (2) with the aid of Green's formula).

3826. Prove that the integral

$$\int\limits_L (yx^3 + e^y)\,dx + (xy^3 + xe^y - 2y)\,dy$$

is equal to zero, if L is a closed curve, symmetrical with respect to the origin or to both coordinate axes.

3827. Evaluate with the aid of Green's formula the difference between the integrals

$$I_1 = \int\limits_{AmB} (x + y)^2\,dx - (x - y)^2\,dy$$

and

$$I_2 = \int\limits_{AnB} (x + y)^2\,dx - (x - y)^2\,dy,$$

where AmB is the straight segment joining points $A(0, 0)$ and $B(1, 1)$, whilst AnB is the arc of the parabola $y = x^2$.

3828. Prove that the integral

$$\int\limits_L \{x \cos(N, x) + y \sin(N, x)\}\,dS,$$

where (N, x) is the angle between the outward normal to the curve and the positive direction of the axis of abscissae, taken over the closed contour L in the positive direction, is equal to twice the area of the figure bounded by contour L.

3829. Prove that the value of $\int\limits_L (2xy - y)\,dx + x^2\,dy$, where L is a closed contour, expresses the area of the domain bounded by the contour.

3830. Show that $\int\limits_L \varphi(y)\,dx + [x\varphi'(y) + x^3]\,dy$ is equal to three times the moment of inertia of the homogeneous plane figure bounded by contour L, with respect to the axis of ordinates.

Independence of the Integral on the Contour of Integration. Determination of the Primitive

In problems 3831–3835, show that the integrals, taken over a closed contour, vanish independently of the form of the functions appearing in the integrand:

3831. $\int\limits_L \varphi(x)\,\mathrm{d}x + \psi(y)\,\mathrm{d}y.$ **3832.** $\int\limits_L f(xy)\,(y\,\mathrm{d}x + x\,\mathrm{d}y).$

3833. $\int\limits_L f\left(\frac{y}{x}\right)\frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2}.$

3834. $\int\limits_L [f(x+y) + f(x-y)]\,\mathrm{d}x + [f(x+y) - f(x-y)]\,\mathrm{d}y.$

3835. $\int\limits_L f(x^2 + y^2 + z^2)\,(x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z).$

3836*. Prove that $\int\limits_L \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}$, taken in the positive direction round any closed contour, which contains the origin, is equal to 2π (see *Course*, sec. 186).

3837. Evaluate $\int\limits_L \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + 4y^2}$ round the circle $x^2 + y^2 = 1$ in the positive direction.

In problems 3838–3844, evaluate the line integrals of total differentials:

3838. $\int\limits_{(-1,\,2)}^{(2,\,3)} y\,\mathrm{d}x + x\,\mathrm{d}y.$ **3839.** $\int\limits_{(0,\,0)}^{(2,\,1)} 2xy\,\mathrm{d}x + x^2\,\mathrm{d}y.$

3840. $\int\limits_{(3,\,4)}^{(5,\,12)} \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2 + y^2}$ (the origin does not lie on the contour of integration).

3841. $\int\limits_{(P_1)}^{(P_2)} \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{\sqrt{x^2 + y^2}}$, where points P_1 and P_2 are situated on concentric circles with centres at the origin and radii R_1

and R_2 respectively (the origin does not lie on the contour of integration).

3842. $\int\limits_{(1,-1,2)}^{(2,1,3)} x\,dx - y^2\,dy + z\,dz.$

3843. $\int\limits_{(1,2,3)}^{(3,2,1)} yz\,dx + zx\,dy + xy\,dz.$

3844. $\int\limits_{(7,2,3)}^{(5,3,1)} \frac{zx\,dy + xy\,dz - yz\,dx}{(x - yz)^2}$ (the contour of integration does not cut the surface $z = \frac{x}{y}$).

Find the functions having total differentials as given in problems 3845–3852:

3845. $du = x^2\,dx + y^2\,dy.$

3846. $du + 4(x^2 - y^2)(x\,dx - y\,dy).$

3847. $du = \frac{(x + 2y)\,dx + y\,dy}{(x + y)^2}.$

3848. $du = \frac{x}{y\sqrt{x^2 + y^2}}\,dx - \left(\frac{x^2 + \sqrt{x^2 + y^2}}{y^2\sqrt{x^2 + y^2}}\right) dy.$

3849. $du = \left[\frac{x - 2y}{(y - x)^2} + x\right] dx + \left[\frac{y}{(y - x)^2} - y^2\right] dy.$

3850. $du = (2x\cos y - y^2\sin x)\,dx + (2y\cos x - x^2\sin y)\,dy.$

3851. $du = \frac{2x(1 - e^y)}{(1 + x^2)^2}\,dx + \left(\frac{e^y}{1 + x^2} + 1\right) dy.$

3852. $du = \frac{(3y - x)\,dx + (y - 3x)\,dy}{(x + y)^3}.$

3853. Chose a number n such that the expression $\frac{(x - y)\,dx + (x + y)\,dy}{(x^2 + y^2)^n}$ is a total differential; find the corresponding function.

3854. Select constants a and b such that the expression $\dfrac{(y^2 + 2xy + ax^2)\,dx - (x^2 + 2xy + by^2)\,dy}{(x^2 + y^2)^2}$ is a total differential; find the corresponding function.

In problems 3855–3860, find the function, given its total differential:

3855. $du = \dfrac{dx + dy + dz}{x + y + z}$.

3856. $du = \dfrac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2 + y^2 + z^2}}$.

3857. $du = \dfrac{yz\,dx + xz\,dy + xy\,dz}{1 + x^2y^2z^2}$.

3858. $du = \dfrac{2(zx\,dy + xy\,dz - yz\,dx}{(x - yz)^2}$.

3859. $du = \dfrac{dx - 3dy}{z} + \dfrac{3y - x + z^3}{z^2}\,dz$.

3860. $$du = e^{\frac{y}{z}}\,dx + \left(\frac{e^{\frac{y}{z}}(x + 1)}{z} + ze^{yz}\right)dy + \left(-\frac{e^{\frac{y}{z}}(x + 1)}{z^2} + ye^{yz} + e^{-z}\right)dz.$$

Applications of Integrals

Find with the aid of the line integral the areas of the figures bounded by the closed curves of problems 3861–3868:

3861. The ellipse $x = a\cos t$, $y = b\sin t$.

3862. The astroid $x = a\cos^3 t$, $y = a\sin^3 t$.

3863. The cardioid $x = 2a\cos t - a\cos 2t$, $y = 2a\sin t - a\sin 2t$.

3864*. The loop of the folium of Descartes $x^3 + y^3 - 3axy = 0$.

3865. The loop of the curve $(x + y)^3 = xy$.

3866. The loop of the curve $(x + y)^4 = x^2y$.

3867*. The lemniscate of Bernoulli $(x^2+y^2)^2 = 2a^2(x^2-y^2)$.

3868*. The loop of the curve $(\sqrt{x}+\sqrt{y})^{12} = xy$.

Work

3869. A force acts at every point of a plane, having a constant magnitude F and in the direction of the positive axis of abscissae. Find the work done by the force when a particle of mass m moves along the arc of the circle $x^2 + y^2 = R^2$ lying in the first quadrant.

3870. A force F, the projections of which on the axes are $X = xy$, $Y = x + y$, acts at every point of the plane. Find the work done by the force F when a particle of mass m moves from the origin to the point $(1, 1)$:

(1) along the straight line $y = x$; (2) along the parabola $y = x^2$; (3) along the two-link step line, the sides of which are parallel to the axes (two cases).

3871. A force F acts at every point M of the ellipse $x = a\cos t$, $y = b\sin t$, equal in magnitude to the distance of M from the centre of the ellipse and directed towards the centre. (a) Find the work done by force F when a material particle P of mass m is displaced along the arc of the ellipse lying in the first quadrant; (b) find the work done, if the point P circuits the entire ellipse.

3872. The projections of a force on the coordinate axes are given by $X = 2xy$ and $Y = x^2$. Show that the work done by the force when a particle of mass m is displaced depends only on its initial and final positions, and not on the form of the path. Calculate the work done when the particle moves from the point $(1, 0)$ to the point $(0, 3)$.

3873. A force has a magnitude inversely proportional to the distance of its point of application from the xOy plane, whilst it is directed towards the origin. Find the work done when a particle of mass m moves under the action of the force along the straight line $x = at$, $y = bt$, $z = ct$ from the point $M(a, b, c)$ to the point $N(2a, 2b, 2c)$.

3874. A force has a magnitude inversely proportional to the perpendicular distance of its point of application from Oz, whilst it is directed perpendicularly towards this axis. Find the work done by the force when a particle of mass m is displaced along the circle $x = \cos t$, $y = 1$, $z = \sin t$ from the point $M(1, 1, 0)$ to the point $N(0, 1, 1)$.

3875. Prove that the work done by the force of attraction between two particles when one of them moves is independent of the form of the path. The magnitude of the force of attraction F is given by Newton's law $F = \frac{km_1m_2}{r^2}$, where r is the distance between the particles of masses m_1 and m_2, and k is the gravitational constant.

3. Surface Integrals

Integrals over a Surface Area

Evaluate the integrals of problems 3876–3884:

3876. $\iint\limits_S \left(z + 2x + \frac{4}{3}y\right) dq$, where S is the part of the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$ lying in the first octant.

3877. $\iint\limits_S xyz\, dq$, where S is the part of the plane $x + y + z = 1$ lying in the first octant.

3878. $\iint\limits_S x\, dq$, where S is the part of the sphere $x^2 + y^2 + z^2 = R^2$ lying in the first octant.

3879. $\iint\limits_S y\, dq$, where S is the hemisphere $z = \sqrt{R^2 - x^2 - y^2}$.

3880. $\iint\limits_S \sqrt{R^2 - x^2 - y^2}\, dq$, where S is the hemisphere $z = \sqrt{R^2 - x^2 - y^2}$.

3881. $\iint\limits_S x^2y^2\,dq$, where S is the hemisphere $z = \sqrt{R^2 - x^2 - y^2}$.

3882. $\iint\limits_S \frac{dq}{r^2}$, where S is the cylinder $x^2 + y^2 = R^2$ bounded by the planes $z = 0$ and $z = H$, whilst r is the distance of a point of the surface from the origin.

3883. $\iint\limits_S \frac{dq}{r^n}$, where S is the sphere $x^2 + y^2 + z^2 = R^2$, and r is the distance of a point of the sphere from the fixed point $P(0, 0, c)$ $(c > R)$.

3884. $\iint\limits_S \frac{dq}{r}$, where S is the part of the surface of the hyperbolic paraboloid $z = xy$ cut off by the cylinder $x^2 + y^2 = R^2$, and r is the distance of a point of the surface from Oz.

3885*. Find the mass of a sphere, if the surface density at any given point is equal to the distance of the point from some fixed diameter of the sphere.

3886. Find the mass of a sphere, if the surface density at any given point is equal to the square of the distance of the point from some fixed diameter of the sphere.

Coordinate Surface Integrals

Evaluate the surface integrals of problems 3887–3893:

3887. $\iint\limits_S x\,dy\,dz + y\,dx\,dz + z\,dx\,dy$, where S is the positive side of the cube made up of the planes $x = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 1$, $z = 1$.

3888. $\iint\limits_S x^2y^2z\,dx\,dy$, where S is the positive side of the lower half of the sphere $x^2 + y^2 + z^2 = R^2$.

3889. $\iint\limits_S z\,dx\,dy$, where S is the outside of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

3890. $\iint\limits_S z^2\,dx\,dy$, where S is the outside of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

3891. $\iint\limits_S xz\,dx\,dy+xy\,dy\,dz+yz\,dx\,dz$, where S is the outside of the pyramid composed of the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

3892. $\iint\limits_S yz\,dx\,dy+xz\,dy\,dz+xy\,dx\,dz$, where S is the outside of the surface lying in the first octant and consisting of the cylinder $x^2+y^2=R^2$ and the planes $x=0$, $y=0$, $z=0$ and $z=H$.

3893. $\iint\limits_S y^2z\,dx\,dy+xz\,dy\,dz+x^2y\,dx\,dz$, where S is the outside of the surface lying in the first octant and consisting of the paraboloid of revolution $z=x^2+y^2$, the cylinder $x^2+y^2=1$ and the coordinate planes (Fig. 68).

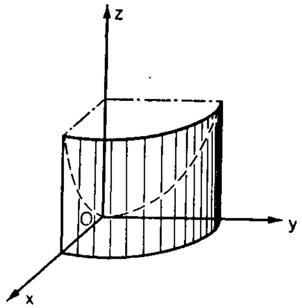

FIG. 68.

Stokes's Formula

3894. Transform $\int\limits_L (y^2+z^2)\,dx+(x^2+z^2)\,dy+(x^2+y^2)\,dz$, taken over a closed contour, to an integral over the surface "stretched" over this contour with the aid of Stokes' formula.

3895. Evaluate $\int\limits_L x^2y^3\,dx + dy + z\,dz$, where the contour L is the circle $x^2 + y^2 = R^2$, $z = 0$: (a) directly, and (b) by using Stokes' formula, taking as the surface the hemisphere $z = +\sqrt{R^2 - x^2 - y^2}$. The integration is in a positive direction round the circle in the xOy plane.

Ostrogradskii's Formula

3896. With the aid of Ostrogradskii's formula, transform the following surface integral over a closed surface to a triple integral over the volume of the body bounded by the surface: $\iint\limits_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy$. The integration is carried out over the outside of surface S.

3897. With the aid of Ostrogradskii's formula, transform the following surface integral over a closed surface to a triple integral over the volume of the body bounded by this surface:

$$\iint\limits_S \sqrt{x^2 + y^2 + z^2}\,\{\cos(N, x) + \cos(N, y) + \cos(N, z)\}\,d\sigma,$$

where N is the outward normal to surface S.

3898. Evaluate the integral of problem 3897, if S is the sphere of radius R with centre at the origin.

3899. Evaluate the integral

$$\iint\limits_S [x^3\cos(N, x) + y^3\cos(N, y) + z^3\cos(N, z)]\,d\sigma,$$

where S is the sphere of radius R with centre at the origin, and N is the outward normal.

3900. Evaluate the integrals of problems 3891–3893 by using Ostrogradskii's formula.

CHAPTER XIV

DIFFERENTIAL EQUATIONS

1. Equations of the First Order

Equations with Separable Variables

Find the general solutions of the differential equations of problems 3901–3910:

3901. $(xy^2 + x)\,\mathrm{d}x + (y - x^2y)\,\mathrm{d}y = 0.$

3902. $xyy' = 1 - x^2.$ **3903.** $yy' = \dfrac{1 - 2x}{y}.$

3904. $y' \tan x - y = a.$ **3905.** $xy' + y = y^2.$

3906. $y' + \sqrt{\dfrac{1 - y^2}{1 - x^2}} = 0.$

3907. $\sqrt{1 - y^2}\,\mathrm{d}x + y\sqrt{1 - x^2}\,\mathrm{d}y = 0.$

3908. $\mathrm{e}^{-s}\left(1 + \dfrac{\mathrm{d}s}{\mathrm{d}t}\right) = 1.$ **3909.** $y' = 10^{x+y}.$

3910. $y' + \sin\dfrac{x + y}{2} = \sin\dfrac{x - y}{2}.$

3911. The relationship between the velocity v and the path traversed l in the bore of the gun is given in ballistics by the following: $v = \dfrac{al^n}{b + l^n}$, where $v = \dfrac{\mathrm{d}l}{\mathrm{d}t}$ and $n < 1$. Find the relationship between the time t of motion of the projectile and the distance l travelled along the bore.

3912. If x is the amount of hydriodic acid HI, decomposed at the instant t, the speed of the decomposition $\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)$ is given by the differential equation $\dfrac{\mathrm{d}x}{\mathrm{d}t} = k_1\left(\dfrac{1 - x}{v}\right)^2 - k_2\left(\dfrac{x}{v}\right)^2$

where k_1, k_2 and v are constants. Integrate this equation.

In problems 3913–3916, find the particular solutions of the differential equations satisfying the given initial conditions:

3913. $y' \sin x = y \ln y; \quad y \big|_{x=\frac{\pi}{2}} = \mathrm{e}.$

3914. $y' = \dfrac{1+y^2}{1+x^2}; \quad y \big|_{x=0} = 1.$

3915. $\sin y \cos x \, dy = \cos y \sin x \, dy; \quad y \big|_{x=0} = \dfrac{\pi}{4}.$

3916. $y - xy' = b(1 + x^2y'); \quad y \big|_{x=1} = 1.$

3917. Find the curve passing through the point $(2, 3)$ and having the property that the segment of any tangent to it, lying between the coordinate axes, is bisected by the point of contact.

3918. Find the curve passing through the point $(2, 0)$ and having the property that the segment of any tangent, between the point of contact and the axis of ordinates, has a constant length equal to 2.

3919. Find all the curves such that the segment of any tangent, lying between the point of contact and the axis of abscissae, is bisected at its point of intersection with the axis of ordinates.

3920. Find all the curves, for which the subtangent is proportional to the abscissa of the point of contact (the coefficient of proportionality is equal to k).

3921. Find the curve passing through the point $(a, 1)$ and having constant subtangent $(= a)$.

3922. Find the curve for which the length of the normal (measured from the point on the curve to its intersection with the axis of abscissae) is a constant a.

3923. Find the curve, for which the sum of the lengths of the tangent and subtangent at any point is proportional to the product of the coordinates of the point of contact (coefficient of proportionality k).

3924. Find the curve $y = f(x)$ ($f(x) \geqq 0$, $f(0) = 0$), bounding the curvilinear trapezium with base $[0, x]$, the area of which is proportional to the $(n + 1)$th power of $f(x)$. We are given that $f(1) = 1$.

3925. A material particle weighing 1 g moves along a straight line, under the action of a force which is directly proportional to time, measured from the instant $t = 0$, and inversely proportional to the velocity of the particle. The velocity is 50 cm/sec at $t = 10$ sec., whilst the force is 4 dynes. What is the velocity a minute after the start of the motion?

3926. A material particle moves along a straight line, its kinetic energy at an instant t being directly proportional to its average speed in the interval of time from zero to t. Given that the path $s = 0$ at $t = 0$, prove that the motion is uniform.

3927. A motor-boat moves in quiet water at a speed $v = 10$ km/hr. Its motor is switched off at full speed, and its speed reduces to $v_1 = 6$ km/hr after time $t = 20$ sec. Assuming that the resistance of the water to the boat is proportional to its speed, find the speed 2 min after stopping the motor; find also the distance travelled by the boat during 1 min after stopping the motor.

3928. The bottom of a cylindrical vessel with a cross-section S cm^2 and a vertical axis contains a small circular hole of area q cm^2, which can be closed by a diaphragm (as in a camera lens). Fluid is poured into the vessel to a height h. The diaphragm starts to open at time $t = 0$, the area of the opening being proportional to time and the aperture being completely open after T sec. What is the height H of the fluid in the vessel T sec after the start of the experiment? (See problems 2701–2706; also *Course*, sec. 116).

3929. The rate of cooling of a body is proportional to the difference in temperature between the body and the environment. We assumed that the coefficient of proportionality is constant in problems 2710–2711. It is assumed in certain calculations to be linearly dependent on time: $k = k_0(1 + \alpha t)$. On this assumption, find the relationship between the body temperature θ and time t, putting $\theta = \theta_0$ at $t = 0$, the temperature of the environment being θ_1.

3930*. The rate of growth of the area of a young leaf of Queen Victoria, which is well-known to have a circular shape, is proportional to the circumference of the leaf and the amount of sunlight incident on it. The latter is in turn proportional to the area of the leaf and the cosine of the angle between the incident ray and the vertical. Find the relationship between the area S of the leaf and time t, if we know that the area is 1600 cm^2 at 6 a.m. and is 2500 cm^2 at 6 p.m. on the same day. (Assume that the observation is carried out at the equator on the day of the equinox, when the angle between the incident sunlight and the vertical can be taken as 90° at 6 a.m. and at 6 p.m. and 0° at noon.)

By substituting for the required function, reduce the equations of problems 3931–3933 to equations with separable variables, and solve them:

3931. $y' = \cos(x - y)$ (put $u = x - y$).

3932. $y' = 3x - 2y + 5$.

3933. $y' \sqrt{1 + x + y} = x + y - 1$.

Homogeneous Equations

Find the general solutions of the equations of problems 3934–3944:

3934. $y' = \dfrac{y^2}{x^2} - 2$.

3935. $y' = \dfrac{x + y}{x - y}$.

3936. $x\,dy - y\,dx = y\,dy$.

3937. $y' = \dfrac{2xy}{x^2 - y^2}$.

3938. $y' = \dfrac{x}{y} + \dfrac{y}{x}$.

3939. $xy' - y = \sqrt{x^2 + y^2}$.

3940. $y^2 + x^2y' = xyy'$.

3941. $y' = e^{\frac{y}{x}} + \dfrac{y}{x}$.

3942. $xy' = y \ln \dfrac{y}{x}$.

3943. $(3y^2 + 3xy + x^2)\,\mathrm{d}x = (x^2 + 2xy)\,\mathrm{d}y.$

3944. $y' = \dfrac{y}{x} + \dfrac{\varphi\left(\dfrac{y}{x}\right)}{\varphi'\left(\dfrac{y}{x}\right)}.$

In problems 3945–3948, find the particular solutions of the differential equations, satisfying the given initial conditions:

3945. $(xy' - y)\arctan\dfrac{y}{x} = x;\quad y\,|_{x=1} = 0.$

3946. $(y^2 - 3x^2)\,\mathrm{d}y + 2xy\,\mathrm{d}x = 0;\quad y\,|_{x=0} = 1.$

3947. $y' = \dfrac{y^2 - 2xy - x^2}{y^2 + 2xy - x^2};\quad y|_{x=1} = -1.$

3948. $y\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2x\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = 0;\quad y|_{x=0} = \sqrt{5}.$

3949. Reduce the equation $y' = \dfrac{y}{x} + \varphi\left(\dfrac{x}{y}\right)$ to a quadrature. What must be the function $\varphi\left(\dfrac{x}{y}\right)$ for the general solution of the equation to be $y = \dfrac{x}{\ln|Cx|}$?

3950. Find the curve, such that the square of the length of the segment, cut off by any tangent from the axis of ordinates, is equal to the product of the coordinates of the point of contact.

3951. Find the curve, such that the initial ordinate of any tangent is equal to the corresponding subnormal.

3952. Find the curve for which the length of the radius vector of any point M is equal to the distance between the origin and the point of intersection of the tangent at M with Oy.

3953*. What surface of revolution is represented by the mirror of a projector, if light rays starting from a point source are directed in a parallel pencil after reflexion?

Linear Equations

Find the general solutions of the equations of problems 3954–3964:

3954. $y' + 2y = 4x$. **3955.** $y' + 2xy = xe^{-x^2}$.

3956. $y' + \dfrac{1-2x}{x^2}\,y = 1$.

3957. $(1+x^2)\,y' - 2xy = (1+x^2)^2$.

3958. $y' + y = \cos x$. **3959.** $y' + ay = e^{mx}$.

3960. $2y\,dx + (y^2 - 6x)\,dy = 0$.

3961. $y' = \dfrac{1}{2x - y^2}$. **3962.** $y' = \dfrac{y}{2y\ln y + y - x}$.

3963. $x(y' - y) = (1+x^2)\,e^x$.

3964. $y' + y\Phi'(x) - \Phi(x)\,\Phi'(x) = 0$, where $\Phi(x)$ is a given function.

In problems 3965–3968, find the particular solutions of the equations satisfying the indicated initial conditions:

3965. $y' - y\tan x = \sec x$; $y|_{x=0} = 0$.

3966. $xy' + y - e^x = 0$; $y|_{x=a} = b$.

3967. $xy' - \dfrac{y}{x+1} = x$; $y|_{x=1} = 0$.

3968. $t(1+t^2)\,dx = (x + xt^2 - t^2)\,dt$; $x|_{t=1} = -\dfrac{\pi}{4}$.

3969. Let y_1 and y_2 be two distinct solutions of the equation

$$y' + P(x)\,y = Q(x).$$

(a) Prove that $y = y_1 + C(y_2 - y_1)$ is the general solution of the equation (C is a constant).

(b) For what ratio of constants α and β is the linear combination $\alpha y_1 + \beta y_2$ a solution of the equation?

(c) Prove that, if y_3 is a third particular solution, different from y_1 and y_2, the ratio $\dfrac{y_2 - y_1}{y_3 - y_1}$ is constant.

3970. Prove the identity (see problem 2345): $\int\limits_0^x e^{zx - z^2}\,dz =$

$= e^{\frac{x^2}{4}} \int\limits_0^x e^{-\frac{z^2}{4}}\, dz$, by composing a differential equation for the function $I(x) = \int\limits_0^x e^{zx-z^2}\, dz$ and solving it.

3971. Find the curve for which the initial ordinate of any tangent is less than the abscissa of the point of contact by two units of scale.

3972*. Find the curve for which the area of the rectangle, constructed on the abscissa of any point and the initial ordinate of the tangent at this point, is constant ($= a^2$).

3973*. Find the curve for which the area of the triangle, formed by the axis of abscissae, the tangent and the radius vector of the point of contact, is constant ($= a^2$).

3974. A particle of mass m moves along a straight line; a force acts on it, proportional to time (coefficient of proportionality k_1) measured from the instant when the velocity is zero. In addition, the particle is subject to the resistance of the medium, proportional to the velocity (coefficient of proportionality k). Find the relationship between velocity and time.

3975. A particle of mass m moves along a straight line; a force acts on it, proportional to the cube of time, measured from the instant when the velocity is v_0 (coefficient of proportionality k). In addition, the particle is subject to the resistance of the medium, proportional to the product of velocity and time (coefficient of proportionality k_1). Find the relationship between velocity and time.

3976. The initial temperature $\theta°$ C of a body is equal to the temperature of the environment. The body receives heat from a heating device (the rate of heat supply is a given function of time: $c\varphi(t)$, where c is the constant specific heat of the body). In addition, the body loses heat to the environment (the rate of cooling is proportional to the temperature difference between the body and the surrounding medium).

Find the relationship between the temperature of the body and time, measured from the start of the experiment.

Solve problems 3977–3978, by assuming that, if a variable electric current $I = I(t)$ flows along a conductor with self-inductance L and resistance R, the voltage drop along the conductor is equal to $L\frac{dI}{dt} + RI$ (see *Course*, sec. 194).

3977. The potential difference across the terminals of a coil falls uniformly from $E_0 = 2V$ to $E_1 = 1V$ in the course of 10 sec. What is the current at the end of the tenth second, if it is $16\frac{2}{3}$ amp at the start of the experiment? The coil resistance is 0·12 ohm, its inductance 0·1 Henry.

3978. Find the current in a coil at the instant t, if its resistance is R, inductance L, initial current $I_0 = 0$, and the electromotive force varies in accordance with the law $E = E_0 \sin \omega t$.

Miscellaneous Problems (Equations with Separable Variables, Homogeneous and Linear Equations)

Find the general solutions of the equations of problems 3979–3997;

3979. $y' = \dfrac{x^2 + xy + y^2}{x^2}$.

3980. $x^2\,dy + (3 - 2xy)\,dx = 0$.

3981. $x(x^2 + 1)\,y' + y = x(1 + x^2)^2$.

3982. $y' = \dfrac{y+1}{x}$.

3983. $y' = \dfrac{1 + y^2}{xy\,(1 + x^2)}$.

3984. $(8y + 10x)\,dx + (5y + 7x)\,dy = 0$.

3985. $x^3y' = y(y^2 + x^2)$.

3986. $\dfrac{xy' - y}{x} = \tan\dfrac{y}{x}$.

3987. $\left(x - y\cos\frac{y}{x}\right)dx + x\cos\frac{y}{x}\,dy = 0.$

3988. $y' = e^{2x} - e^{x}y.$

3989. $\dfrac{dx}{x^2 - xy + y^2} = \dfrac{dy}{2y^2 - xy}.$

3990. $\dfrac{dy}{dx} = \dfrac{1}{x\cos y + \sin 2y}.$

3991. $(x - 2xy - y^2)\,dy + y^2\,dx = 0.$

3992. $y' + y\cos x = \sin x\cos x.$

3993. $(x + 1)\,y' - ny = e^{x}(x + 1)^{n+1}.$

3994. $y\,dx = (y^3 - x)\,dy.$

3995. $\left(\dfrac{dy}{dx}\right)^2 - (x + y)\dfrac{dy}{dx} + xy = 0.$

3996*. $yy'\sin x = \cos x(\sin x - y^2).$

3997. $y' = (x + y)^2.$

3998. Show that the integral curves of the equation $(1 - x^2)\,y' + xy = ax$ are ellipses and hyperbolas with centres at the point $(0, a)$ and axes parallel to the coordinate axes, each curve having one constant axis equal to the number 2.

In problems 3999–4002, find the particular solutions of the equations satisfying the given initial conditions:

3999. $\dfrac{y - xy'}{x + yy'} = 2;\quad y|_{x=1} = 1.$

4000. $y' - \dfrac{y}{1 - x^2} = 1 + x;\quad y|_{x=0} = 1.$

4001. $(1 + e^{x})\,yy' = e^{y};\quad y|_{x=0} = 0.$

4002. $y' = 3x^2y + x^5 + x^2;\quad y|_{x=0} = 1.$

4003. Show that the following property is possessed only by the straight lines $y = kx$ and the hyperbolas $xy = m$: the length of the radius vector of any point of the curve is equal to the length of the tangent drawn at that point.

4004. Find the curve for which the length of the normal is proportional to the square of the ordinate. The coefficient of proportionality is equal to k.

4005. Find the curve, for which any tangent cuts the axis of ordinates at a point equidistant from the origin and the point of contact.

4006. Find the equation of the curve cutting the axis of abscissae at the point $x = 1$ and having this property: the length of the subnormal at any point of the curve is equal to the arithmetic mean of the coordinates of the point.

4007. Find the curve for which the area of the trapezium, formed by the coordinate axes, the ordinate of any given point of the curve and the tangent at this point, is equal to half the square of the abscissa.

4008. Find the curve, for which the area, included between the axis of abscissae, the curve and two ordinates, one of which is constant whilst the other is variable, is equal to the ratio of the cube of the variable ordinate to the variable abscissa.

4009. Find the curve for which the area of the figure, bounded by the axis of abscissae, two ordinates and the arc MM' of the curve, is proportional to arc MM' for any choice of points M, M'.

4010. Find the curve for which the abscissa of the centre of gravity of the curvilinear trapezium, formed by the coordinate axes, the straight line $x = a$ and the curve, is equal to $\frac{3a}{4}$ for any a.

4011*. Find the curve, all the tangents to which pass through a given point (x_0, y_0).

4012. Find the curve through the origin, all the normals to which pass through a given point (x_0, y_0).

4013. What curve has the following property: the angle formed by the tangent at any point with Ox is twice the angle which the radius vector of the point of contact forms with Ox?

4014. A force acts on a body, proportional to time. Moreover, the body is subject to the resistance of the medium, proportional to its velocity. Find the law of motion of the body (the path as a function of time).

4015. A particle falls in a medium, of which the resistance is proportional to the square of the particle velocity. Show that the equation of motion is $\frac{dv}{dt} = g - kv^2$, where k is a constant, g is the acceleration due to gravity. Integrate this equation and show that v tends to $\sqrt{\frac{g}{k}}$ as $t \to \infty$.

4016. The braking action due to friction on a disc rotating in a fluid is proportional to the angular velocity:

(1) The disc starts to rotate with an angular velocity of 3 revolutions per sec, its angular velocity after 1 min being 2 revolutions per sec. What is its angular velocity 3 min after the start of the rotation?

(2) The disc starts to revolve with an angular velocity of 5 revolutions per sec, its angular velocity after 2 min being 3 revolutions per sec. At what instant, measured from the start of the rotation, will its angular velocity be 1 revolution per sec?

4017. A bullet enters a board of thickness $h = 10$ cm with a speed $v_0 = 200$ m/sec, and leaves after penetrating the board with a speed $v_1 = 80$ m/sec. If the resistance of the board to the motion of the bullet is proportional to the square of the latter's velocity, find how long it takes the bullet to pass through the board.

4018*. A drop of water, of initial mass M_0 g and evaporating uniformly at a rate of m g/sec, moves under the action of inertia with an initial velocity v_0 cm/sec. The resistance of the medium is proportional to the velocity of the drop and to its radius. The initial resistance (at $t = 0$) is f_0 dynes. Find the velocity of the drop as a function of time.

4019*. A drop of water, with an initial mass M_0 g and evaporating uniformly at a rate of m g/sec, falls freely in air.

The air resistance is proportional to the speed of the drop (coefficient of proportionality k).

Find the speed of the drop as a function of time, as measured from the initial instant, at which the speed of the drop is given as zero. Assume that $k \neq 2m$.

4020*. Solve the previous problem for a drop of spherical shape, assuming that the air resistance is proportional to the product of the speed and surface area of the drop. The density of the fluid is γ. (Reduce to quadratures.)

4021. The natural growth in the population of a large town is proportional to the present number of inhabitants and the interval of time. Furthermore, the town population increases due to immigration: the rate of growth of the population by this means is proportional to time, measured from the instant when the population was equal to A_0. Find the number of inhabitants as a function of time (assuming that the process is continuous).

4022. A pickle, containing 10 kg of dissolved salt, is placed in a reservoir whose volume is 100 l. Water flows into the reservoir at a rate of 3 l./min, whilst the mixture is pumped at the same rate into a second reservoir whose capacity is also 100 l.; the second reservoir is originally filled with pure water, and the excess fluid pours out. How much salt will the second reservoir contain after an hour? What is the maximum amount of salt in the second reservoir? When is the maximum amount reached? (The salt concentration in each of the reservoirs is kept uniform by mixing.) (See *Course*, sec. 192).

4023. The voltage and resistance in a circuit vary uniformly during 1 min, from zero to 120 V, and from zero to 120 ohms, respectively (see problems 3977–3978). The inductance is constant (1 Henry). The initial current is I_0. Find the current as a function of time during the first minute.

4024*. Gas is contained in a narrow horizontal cylindrical tube AB, which is hermetically sealed. The tube revolves uniformly about a vertical axis OO_1 (Fig. 69), passing through one end, with an angular velocity ω. The length of the tube is l cm, its cross-sectional area S cm^2, the mass of the enclosed gas M g, the pressure p_0 when the tube is at rest (constant throughout the tube). Find the pressure distribution along the tube, i.e. p as a function of x, when the tube is revolving.

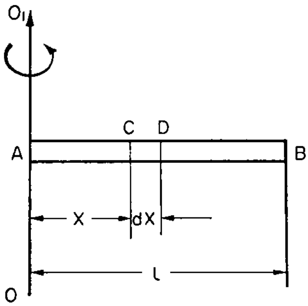

FIG. 69.

Further Examples of First-Order Equations

Use substitution of the variables to reduce the equations of problems 4025–4037 to linear or homogeneous equations:

4025. $y' = \dfrac{2y - x - 5}{2x - y + 4}$. **4026.** $y' = \dfrac{2x - y + 1}{x - 2y + 1}$.

4027. $(x + y + 1)\,dx = (2x + 2y - 1)\,dy$.

4028. $y' = \dfrac{2(y + 2)^2}{(x + y - 1)^2}$. **4029.** $y' = \dfrac{y^2 - x}{2y(x + 1)}$.

4030. $y' = \dfrac{y^3}{2(xy^2 - x^2)}$. **4031.** $(1 + y^2)\,dx = x\,dy$.

4032. $(x^2y^2 - 1)\,y' + 2xy^3 = 0$.

4033. $yy' + x = \dfrac{1}{2}\left(\dfrac{x^2 + y^2}{x}\right)^2$. **4034.** $xy' + 1 = e^y$.

4035. $(x^2 + y^2 + 1)\,dy + xy\,dx = 0$.

4036. $x\,dx + y\,dy + x\,(x\,dy - y\,dx) = 0$.

4037. $(x^2 + y^2 + y)\,dx = x\,dy$.

Solve the Bernoulli equations of problems 4038–4047:

4038. $y' + 2xy = 2x^3y^3.$ **4039.** $y' + \dfrac{y}{x+1} + y^2 = 0.$

4040. $y^{n-1}(ay' + y) = x.$ **4041.** $x\,\mathrm{d}x = \left(\dfrac{x^2}{y} - y^3\right)\mathrm{d}y.$

4042. $xy' + y = y^2 \ln x.$

4043. $y' - y \tan x + y^2 \cos x = 0.$

4044. $y' + \dfrac{2y}{x} = \dfrac{2\sqrt{y}}{\cos^2 x}.$ **4045.** $xy' - 4y - x^2\sqrt{y} = 0.$

4046. $y\,\mathrm{d}y - \dfrac{ay^2}{x^2}\mathrm{d}x = \dfrac{b\,\mathrm{d}x}{x^2}.$

4047. $y' = \dfrac{y\varphi'(x) - y^2}{\varphi(x)}$, where $\varphi(x)$ is a given function.

4048. Find the curve, such that the segment cut off the axis of ordinates (or of abscissae) by the tangent at any point is:

(1) proportional to the square of the ordinate (or abscissa) of the point of contact;

(2) proportional to the cube of the ordinate (or abscissa) of the point of contact.

4049. Find the curves, specified by equations of the form $\varrho = f(\varphi)$, for which the area of the sector, bounded by the curve and the radius vectors of a fixed point (ϱ_0, φ_0) and of a variable point (ϱ, φ), is proportional to the product of the polar coordinates ϱ and φ of the variable point. The coefficient of proportionality is k.

Exact Differential Equations

Find the general solutions of the equations of problems 4050–4057:

4050. $(2x^3 - xy^2)\,\mathrm{d}x + (2y^3 - x^2y)\,\mathrm{d}y = 0.$

4051. $\dfrac{x\,\mathrm{d}y}{x^2 + y^2} = \left(\dfrac{y}{x^2 + y^2} - 1\right)\mathrm{d}x.$

4052. $\mathrm{e}^y\,\mathrm{d}x + (x\mathrm{e}^y - 2y)\,\mathrm{d}y = 0.$

4053. $yx^{y-1}\,\mathrm{d}x + x^y \ln x\,\mathrm{d}y = 0.$

4054. $\dfrac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = \dfrac{y\,dx - x\,dy}{x^2}$.

4055. $\dfrac{y + \sin x \cos^2(xy)}{\cos^2(xy)}\,dx + \dfrac{x}{\cos^2(xy)}\,dy + \sin y\,dy = 0.$

4056. $(1 + x\sqrt{x^2+y^2})\,dx + (-1 + \sqrt{x^2+y^2})\,y\,dy = 0.$

4057. $\left(\dfrac{1}{y}\sin\dfrac{x}{y} - \dfrac{y}{x^2}\cos\dfrac{y}{x} + 1\right)dx +$

$$+\left(\frac{1}{x}\cos\frac{y}{x} - \frac{x}{y^2}\sin\frac{x}{y} + \frac{1}{y^2}\right)dy = 0.$$

Integrating Factors

Find the integrating factor and general solution of the equations of problems 4058–4062:

4058. $(x^2 + y)\,dx - x\,dy = 0.$

4059*. $y(1 + xy)\,dx - x\,dy = 0.$

4060. $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0.$

4061. $\dfrac{y}{x}\,dx + (y^3 - \ln x)\,dy = 0.$

4062. $(x\cos y - y\sin y)\,dy + (x\sin y + y\cos y)\,dx = 0.$

4063. Show that $e^{\int P(x)dx}$ is the integrating factor of the linear equation $\dfrac{dy}{dx} + P(x)y = Q(x)$.

4064. Find the integrating factor of Bernoulli's equation

$$y' + P(x)\,y = y^nQ(x).$$

4065. Find the conditions for which the equation

$$X(x, y)\,dx + Y(x, y)\,dy = 0$$

admits of an integrating factor of the form $M = F(x + y)$.

4066. Find the conditions for which the equation

$$X(x, y)\,dx + Y(x, y)\,dy = 0$$

admits of an integrating factor of the form $M = F(xy)$.

Various Problems

Find the general solutions of the equations of problems 4067–4088:

4067. $y' = ax + by + c.$ **4068.** $ay' + by + cy^m = 0.$

4069. $y' = \dfrac{x + y - 2}{y - x - 4}.$ **4070.** $y' = \dfrac{y^2 + xy - x^2}{y^2}.$

4071. $y' = \dfrac{a^2}{(x + y)^2}.$ **4072.** $y'(y^2 - x) = y.$

4073. $\dfrac{2x\,dx}{y^3} + \dfrac{y^2 - 3x^2}{y^4}\,dy = 0.$

4074. $(2y + xy^3)\,dx + (x + x^2y^2)\,dy = 0.$

4075. $\left(2xy + x^2y + \dfrac{y^3}{3}\right)dx + (x^2 + y^2)\,dy = 0.$

4076. $y' = \dfrac{(1 + y)^2}{x(y + 1) - x^2}.$

4077. $x\,dy + y\,dx + y^2(x\,dy - y\,dx) = 0.$

4078. $\left[\dfrac{1}{x} - \dfrac{y^2}{(x - y)^2}\right]dx + \left[\dfrac{x^2}{(x - y)^2} - \dfrac{1}{y}\right]dy = 0.$

4079. $y' = x\sqrt{y} + \dfrac{xy}{x^2 - 1}.$ **4080.** $y \sin x + y' \cos x = 1.$

4081. $y' - y + y^2 \cos x = 0.$

4082. $y' = \dfrac{\cos x \sin y + \tan^2 x}{\sin x \cos y}.$

4083. $xy' \cos \dfrac{y}{x} = y \cos \dfrac{y}{x} - x.$

4084. $\left(x \cos \dfrac{y}{x} + y \sin \dfrac{y}{x}\right) y\,dx + \left(x \cos \dfrac{y}{x} - y \sin \dfrac{y}{x}\right) x\,dy = 0.$

4085. $y' = \dfrac{x}{\cos y} - \tan y$

4086. $y - y' \cos x = y^2 \cos x\,(1 - \sin x).$

4087. $2yy' = e^{\frac{x^2+y^2}{x}} + \dfrac{x^2 + y^2}{x} - 2x.$

4088. $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0.$

4089. Find the curve, such that the ratio of the subnormal at any point to the sum of the abscissa and ordinate of the point is equal to the ratio of the ordinate to the abscissa of the point.

4090. Find the curve with the property, that the segment of the tangent at any point, contained between Ox and the straight line $y = ax + b$, is bisected by the point contact.

4091. Find the curve for which the ratio between the distance from the normal at any point to the origin, and the distance from the same normal to the point (a, b), is equal to a constant k.

4092. Find the curve for which the distance from the origin to the tangent at any point is equal to the distance from the origin to the normal at that point.

4093*. Find the curve with the following property: the ordinate of any point of it is the mean proportional between the abscissa and the sum of abscissa and subnormal drawn to the curve at that point.

4094. A voltage is introduced uniformly (from zero to 120 V) during the course of two minutes into an electrical circuit with a resistance $R = \frac{3}{2}$ ohms. In addition, inductance is automatically introduced, so that the number of Henries in the circuit is equal to the current expressed in Amperes. Find the current as a function of time during the first two minutes of the experiment.

2. Equations of the First Order (Continued)

Tangent Field. Isoclines

4095. Given the differential equation $y' = -\frac{x}{y}$, (a) draw the tangent field determined by the equation, (b) examine the disposition of a field vector with respect to the radius vector of any given point of the field, (c) examine on the basis of the tangent field the form of the integral curves of the equation, (d) find the integral curves by solving the equation by the usual method (separation of the variables), (e) obtain the family of isoclines of the equation (see *Course*, sec. 196).

4096. Write down the differential equations whose isoclines are:

(1) rectangular hyperbolas $xy = a$;
(2) parabolas $y^2 = 2px$;
(3) circles $x^2 + y^2 = R^2$.

4097. Find the isoclines of the differential equation of the family of parabolas $y = ax^2$. Draw a figure. Interpret the result geometrically.

4098. Show that the isoclines of a homogeneous equation (and only of a homogeneous equation) are straight lines through the origin.

4099. Indicate the linear equations whose isoclines are straight lines.

4100. Let y_1, y_2, y_3 be the ordinates of any three isoclines of a linear equation, corresponding to the same abscissa. Show that the ratio $\dfrac{y_2 - y_1}{y_3 - y_1}$ retains the same value, whatever this abscissa.

The Approximate Integration of Differential Equations

4101. Given the equation $y' = \dfrac{x^2 + y^2}{10}$, draw approximately the integral curve through the point $M(1, 1)$ corresponding to the interval $1 \leqq x \leqq 5$.

4102. Given the equation $y' = \dfrac{1}{(x^2 + y^2)}$ draw approximately the integral curve through the point $(0{\cdot}5,\ 0{\cdot}5)$ corresponding to the interval $0{\cdot}5 \leqq x \leqq 3{\cdot}5$.

4103. Given the equation $y' = xy^3 + x^2$, use Euler's method to evaluate y for $x = 1$, if y is the particular solution satisfying the initial condition $y|_{x=0} = 0$. Evaluate y to two decimal places.

4104. Given the equation $y' = \sqrt{x}y^2 + 1$, use Euler's method to evaluate y for $x = 2$, if y is the particular solution satisfying the initial condition $y|_{x=0} = 0$. Evaluate y to two decimal places.

4105. Given the equation

$$y' = \frac{xy}{2}$$

and the initial condition $y|_{x=0} = 1$, solve the equation exactly, and find the value of y for $x = 0{\cdot}9$. Furthermore, find the value with the aid of an approximate method, by dividing the interval $[0, 0{\cdot}9]$ into 9 parts. Indicate the relative error in the latter result.

4106. Given the equation

$$y' = \frac{3x^2}{x^3 + y + 1}$$

and the initial condition $y|_{x=1} = 0$, solve the equation exactly, and, by using any of the approximate methods of integration of equations, find the value of x for $y = 1$ (compare with the value of x obtained from the strict solution).

4107. $y' = y^2 + xy + x^2$. Find by the method of successive approximations the second approximation for the solution, satisfying the initial condition $y|_{x=0} = 1$.

4108. $y' = xy^3 - 1$. Find the value at $x = 1$ of the solution of the equation that satisfies the initial condition $y|_{x=0} = 0$. Go as far as the third approximation in the method of successive approximations.

Work to two decimal places.

Find the first few terms of the expansions in power series of the solutions of the equations of problems 4109–4116, with the indicated initial conditions:

4109. $y' = y^3 - x;\ y|_{x=0} = 1$.

4110. $y' = x^2y^2 - 1;\ y|_{x=0} = 1$.

4111. $y' = x^2 - y^2;\ y|_{x=0} = 0$.

4112. $y' = \dfrac{1 - x^2}{y} + 1;\ y|_{x=0} = 1$.

4113. $y' = \frac{xy}{1+x+y}$; $y|_{x=0} = 0$.

4114. $y' = e^y + xy$; $y|_{x=0} = 0$.

4115. $y' = \sin y - \sin x$; $y|_{x=0} = 0$.

4116. $y' = 1 + x + x^2 - 2y^2$; $y|_{x=1} = 1$.

Singular Solutions. Clairaut's Equation and Lagrange's Equation

Find the general and singular solutions of the Clairaut and Lagrange equations of problems 4117–4130:

4117. $y = xy' + y'^2$.

4118. $y = xy' - 3y'^3$.

4119. $y = xy' + \frac{1}{y'}$.

4120. $y = xy' + \sqrt{1 + y'^2}$.

4121. $y = xy' + \sin y'$.

4122. $y = 2xy' + y^2y'^3$.

4123. $y = y'^2(x + 1)$.

4124. $2yy' = x(y'^2 + 4)$.

4125. $y = yy'^2 + 2xy'$.

4126. $y = x(1 + y') + y'^2$.

4127. $y = y' + \frac{e^x}{y'}$.

4128. $y = y'(x + 1) + y'^2$.

4129. $y = px + a\sqrt[3]{1 - p^3}$, where $p = \frac{dy}{dx}$.

4130. $x = y\left(\frac{1}{\sqrt{p}} - \frac{1}{p}\right)$, where $p = \frac{dy}{dx}$.

Find the singular solutions of the equations of problems 4131–4133, by employing the same method as is used in the case of Lagrange and Clairaut equations (see *Course,* sec. 197):

4131. $y'^2 - yy' + e^x = 0$.

4132. $x^2y'^2 - 2(xy - 2) - y' + y^2 = 0$.

4133. $y'(y' - 2x) = 2(y - x^2)$.

4134. Prove the theorem: if a linear differential equation is of Clairaut's type, the family of its integral curves consists of a pencil of straight lines.

4135. The area of a triangle formed by the tangent to a curve and the coordinate axes is constant. Find the curve.

4136. Find the curve, the tangents to which cut out segments on the coordinate axes such that the sum of the segments is $2a$.

4137. Find the curve, such that the product of the distances of any tangent from two given points is constant.

4138. Find the curve, for which the area of the rectangle, whose sides are the tangent and normal at any given point, is equal to the area of the rectangle, whose sides are equal in length to the abscissa and ordinate of the point.

4139. Find the curve, for which the sum of the normal and subnormal is proportional to the abscissa.

4140*. Find the curve, for which the segment of the normal lying between the coordinate axes is of constant length a.

4141. The velocity of a material particle at any instant differs from the average velocity (from the initial to the present instant) by an amount, proportional to the kinetic energy of the particle, and inversely proportional to the time, measured from the initial instant. Find the path as a function of time.

Orthogonal and Isogonal Trajectories; Involutes

Find the trajectories orthogonal to those given in problems 4142–4147:

4142. The ellipses with a common major axis equal to $2a$.

4143. The parabolas $y^2 = 4(x - a)$.

4144. The circles $x^2 + y^2 = 2ax$.

4145. The cissoids $(2a - x)\, y^2 = x^3$.

4146. The equal parabolas touching a given straight line, the point of contact of each parabola being its vertex.

4147. The circles of equal radius whose centres lie on a given straight line.

4148. Find the family of trajectories intersecting at an angle $\alpha = 60°$ the curves $x^2 = 2a(y - x\sqrt{3})$ (a is a parameter).

4149. Find the isogonal trajectories of the family of parabolas $y^2 = 4ax$, the angle of intersection being $\alpha = 45°$.

4150*. Find the plane sound distribution curves from a fixed acoustic source in the plane, if a wind blows with constant velocity a in a direction parallel to a given straight line passing through the source.

Find the involutes of the curves of problems 4151–4154 (see *Course*, sec. 82):

4151. The circle $x^2 + y^2 = R^2$.

4152. The catenary $y = a \cosh \frac{x}{a}$.

4153. The involute of the circle

$$x = a(\cos t + t \sin t),\ y = a(\sin t - t \cos t).$$

4154. The semicubical parabola $y = 3t^2$, $x = -2t^3$.

3. Equations of the Second and Higher Orders

Particular Cases of Second-Order Equations

Find the general solutions of the equations of problems 4155–4182:

4155. $y'' = x + \sin x$.

4156. $y'' = \arctan x$.

4157. $y'' = \ln x$.

4158. $xy'' = y'$.

4159. $y'' = y' + x$.

4160. $y'' = \frac{y'}{x} + x$.

4161. $(1 + x^2)\, y'' + (y')^2 + 1 = 0$.

4162. $xy'' = y' \ln \frac{y'}{x}$.

4163. $(y'')^2 = y'$.

4164. $2xy'y'' = (y')^2 + 1$.

4165. $y'' - 2 \cot x\, y' = \sin^3 x$.

4166. $1 + (y')^2 = 2yy''$.

4167. $(y')^2 + 2yy'' = 0$.

4168. $a^2y'' - y = 0.$ **4169.** $y'' = \dfrac{1}{4\sqrt{y}}.$

4170. $y'' + \dfrac{2}{1-y}(y')^2 = 0.$ **4171.** $yy'' + (y')^2 = 1.$

4172. $yy'' = (y')^2.$ **4173.** $2yy'' - 3(y')^2 = 4y^2.$

4174. $y(1 - \ln y)\,y'' + (1 + \ln y)\,(y')^2 = 0.$

4175. $y'' = 2yy'.$

4176. $\cos y \cdot \dfrac{d^2y}{dx^2} + \sin y\left(\dfrac{dy}{dx}\right)^2 = \dfrac{dy}{dx}.$

4177. $yy'' - (y')^2 = y^2y'.$ **4178.** $yy'' - yy'\ln y = (y')^2.$

4179. $y'' = y'\left(\dfrac{y'}{y} - 2\sqrt{\dfrac{y'}{y} - 4}\right).$

4180. $(x + a)\,y'' + x(y')^2 = y'.$

4181*. $yy'y'' = (y')^3 + (y'')^2.$

4182. $xy'' - \dfrac{1}{4}(y'')^2 - y' = 0.$

Solve the equations of problems 4183–4188 with the aid of the appropriate substitution: $yy' = p$, $(y')^2 = p$, $xy' = p$, $\dfrac{y'}{y} = p$, etc.

4183. $xyy'' + x(y')^2 = 3yy'.$ **4184.** $xy'' = y'(e^y - 1).$

4185. $yy'' + (y')^2 = x.$ **4186.** $y'' + \dfrac{1}{x}y' - \dfrac{y}{x^2} = 0.$

4187. $x^2y\dfrac{d^2y}{dx^2} - \left(x\dfrac{dy}{dx} - y\right)^2 = 0.$

4188. $yy'' = y'(2\sqrt{yy'} - y').$

Find the particular solutions of the equations of problems 4189–4199 with the stated initial conditions:

4189. $y''(x^2 + 1) = 2xy'$; $y|_{x=0} = 1$, $y'|_{x=0} = 3.$

4190. $xy'' + x(y')^2 - y' = 0$; $y|_{x=2} = 2$, $y'|_{x=2} = 1.$

4191. $y'' = \dfrac{y'}{x} + \dfrac{x^2}{y'}$; $y|_{x=2} = 0$, $y'|_{x=2} = 4.$

4192. $2y'' = 3y^2$; $y|_{x=-2} = 1$, $y'|_{x=-2} = -1$.

4193. $yy'' = (y')^2 - (y')^3$; $y|_{x=1} = 1$, $y'|_{x=1} = -1$.

4194. $y^3y'' = -1$; $y|_{x=1} = 1$, $y'|_{x=1} = 0$.

4195. $y^4 - y^3y'' = 1$; $y|_{x=0} = \sqrt{2}$, $y'|_{x=0} = \frac{\sqrt{2}}{2}$.

4196. $y'' = e^{2y}$; $y|_{x=0} = 0$, $y'|_{x=0} = 1$.

4197. $2(y')^2 = y''(y-1)$; $y|_{x=1} = 2$, $y'|_{x=1} = -1$.

4198*. $x^4y'' = (y - xy')^3$; $y|_{x=1} = 1$, $y'|_{x=1} = 1$.

4199. $y'' = xy' + y + 1$; $y|_{x=0} = 1$, $y'|_{x=0} = 0$.

4200*. What curve has the property that the radius of curvature at any point is proportional to the length of the normal? Take as the coefficient of proportionality $k = -1$, $+1$, -2, $+2$.

4201. Find the curve, for which the projection of the radius of curvature on Oy is constant, equal to a.

4202. Find the curve through the origin such that the area of the triangle MTP (Fig. 70), formed by the tangent at any given point M of the curve, the ordinate MP of M and the axis of abscissae, is proportional to the area OMP of the curvilinear triangle.

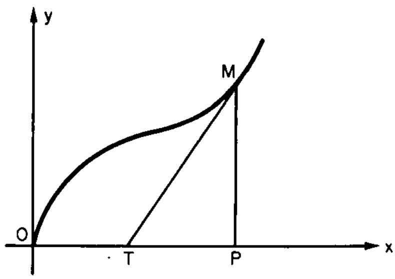

FIG. 70.

4203. Find the curve, the length of arc of which, measured from a given point, is proportional to the slope of the tangent at the final point of the arc.

4204. A particle of mass m is thrown vertically upwards with initial velocity v_0. The air resistance is equal to kv^2.

Thus, if we take the vertical as Oy, we have for the movement upwards:

$$m\frac{d^2y}{dt^2} = -mg - kv^2,$$

and during the fall:

$$m\frac{d^2y}{dt^2} = -mg + kv^2,$$

where $v = \frac{dy}{dt}$. Find the velocity of the particle at the instant when it reaches the ground.

4205. A thin, flexible, inextensible cord is suspended from both ends. What is the equilibrium shape of the cord under the action of a load, uniformly distributed along the projection of the cord on the horizontal plane? (The weight of the cord is neglected.) (See *Course*, sec. 200).

4206. Find the law of rectilinear motion if it is known that the work done by the force is proportional to the time measured from the initial instant of the motion.

4207*. A light ray from air (refractive index m_0) is incident at an angle α_0 from the vertical on a liquid with variable refractive index. The latter depends linearly on the depth and is constant in a plane parallel to the horizontal; it is equal to m_1 on the surface of the liquid, and equal to m_2 at a depth h. Find the shape of the light ray in the liquid. (The refractive index of a medium is inversely proportional to the velocity of propagation of the light.) (See *Course*, sec. 70.)

Particular Cases of Higher Order Equations

Find the general solutions of the equations of problems 4208–4217:

4208. $y''' = \frac{1}{x}$. **4209.** $y''' = \cos 2x$.

4210. $y^{(X)} = e^{ax}$. **4211.** $x^2y''' = (y'')^2$.

4212. $xy^{(V)} = y^{(IV)}$. **4213.** $y''' = (y'')^3$.

4214. $y'y''' = 3(y'')^2$. **4215.** $yy''' - y'y'' = 0$.

4216. $y'''[1 + (y')^2] = 3y'(y'')^2$.

4217. $(y'')^2 - y'y''' = \left(\frac{y'}{x}\right)^2$.

Approximate Solutions

4218. A differential equation of the form $y'' = f_1(x) + f_2(y) + f_3(y')$ is encountered when investigating the vibrations of a material system with one degree of freedom.

Solve this equation graphically, if:

(1) $f_1(x) = 0$, $f_2(y) = -\sqrt{y}$, $f_3(y') = 0{\cdot}5y'$ and $y|_{x=0} = y'|_{x=0} = 0$;

(2) $f_1(x) = -x$, $f_2(y) = 0$, $f_3(y') = -0{\cdot}1y' - 0{\cdot}1y'^3$ and $y|_{x=0} = y'|_{x=0} = 1$.

4219. $y'' = yy' - x^2$; $y|_{x=0} = 1$, $y'|_{x=0} = 1$.

(1) Solve the given equation graphically.

(2) Find the first few terms of the expansion of the solution in a power series.

4220. Find the first six terms of the expansion in a series of the solution of the differential equation $y'' = \frac{y'}{y} - \frac{1}{x}$, satisfying the initial conditions $y|_{x=1} = 1$, $y'|_{x=1} = 0$.

4221. Obtain in the form of a power series the particular solution of the equation $y'' = x \sin y'$, satisfying the initial conditions $y|_{x=1} = 0$, $y'|_{x=1} = \frac{\pi}{2}$. (Take the first six terms.)

4222. Find in the form of a power series the particular solution $y = f(x)$ of the equation $y'' = xyy'$, satisfying the initial conditions $f(0) = 1$, $f'(0) = 1$. If we confine ourselves to the first five terms, will this be sufficient to evaluate $f(-0{\cdot}5)$ to an accuracy of $0{\cdot}001$?

4223. Find the first seven terms of the series expansion of the solution of the differential equation $yy'' + y' + y = 0$, satisfying the initial conditions $y|_{x=0} = 1$, $y'|_{x=0} = 0$. Of what order of smallness is the difference $y - (2 - x - e^{-x})$ as $x \to 0$?

4224. Find the first 12 terms of the series expansion of the solution of the differential equation $y'' + yy' - 2 = 0$, satisfying the initial conditions $y|_{x=0} = 0$, $y'|_{x=0} = 0$. Evaluate $\int\limits_0^1 y\,dx$ to an accuracy of 0·001. Evaluate $y'|_{x=0\cdot 5}$ to an accuracy of 0·00001.

4225*. An electrical circuit is made up of an inductance $L = 0{\cdot}4$ Henry in series with an electrolytic bath. The bath contains a litre of water, acidified with a small quantity of sulphuric acid. The water is decomposed by a current, with the result that the concentration, and hence the resistance of the solution in the bath is variable. The voltage is held constant (20 V) at the terminals. The amount of substance decomposed by electrolysis is proportional to the current, the time and the electrochemical equivalent of the substance (Faraday's law). The electrochemical equivalent of water is equal to 0·000187 g/coulomb. The resistance of the solution at the start of the experiment is $R_0 = 2$ ohm, and the initial current is 10 amp. Find the volume of water in the vessel as a function of time (in the form of a power series).

4226*. An electrical circuit is made up of an inductance $L = 0{\cdot}4$ Henry in series with an electrolytic bath, the initial resistance of which is 2 ohm. The bath contains a litre of water in which 10 g of hydrogen chloride are dissolved. The acid is decomposed by a current, with the result that the concentration of the solution varies (cf. the previous problem, where the amount of dissolved substance does not vary, but the volume of the solution varies). The voltage at the terminals of the circuit is 20 V, the electrochemical equivalent k of hydrogen chloride is equal to 0·000381 g/coulomb, the initial current is 10 amp. Find the amount of hydrochloric acid in the solution as a function of time (in the form of a power series).

4. Linear Equations

4227. The functions x^3 and x^4 satisfy a certain homogeneous linear differential equation of the second order. Show that they form a fundamental system, and form the equation.

4228. The same for the functions e^x and x^2e^x.

4229. The functions x, x^3, e^x form a fundamental system of solutions of a third order linear homogeneous equation. Form this equation.

4230. The functions x^2 and x^3 form a fundamental system of solutions of a second order linear homogeneous equation. Find the solution of this equation satisfying the initial conditions $y|_{x=1} = 1$, $y'|_{x=1} = 0$.

4231. The functions $\cos^2 x$ and $\sin^2 x$ satisfy a certain linear homogeneous equation of the second order:

(a) prove that they form a fundamental system of solutions;

(b) form the equation;

(c) show that the functions 1 and $\cos 2x$ form another fundamental system for this equation.

4232*. If y_1 is a particular solution of the equation

$$y'' + y'P(x) + yQ(x) = 0,$$

then

$$y_2 = Cy_1 \int e^{-\int P(x)dx} \frac{dx}{y_1^2} \quad (C \text{ is a constant})$$

is also a solution. Prove this by three methods:

(1) by direct verification, (2) by substituting $y = y_1 z$, (3) by using Ostrogradskii's formula (see *Course*, sec. 202).

4233. By using the formula of problem 4232, find the general solution of the equation $(1 - x^2)\, y'' - 2xy' + 2y = 0$, knowing its particular solution $y_1 = x$.

4234. Solve the equation $y'' + \frac{2}{x} y' + y = 0$, knowing its particular solution $y_1 = \frac{\sin x}{x}$.

4235. The equation $(2x - x^2)\, y'' + (x^2 - 2)\, y' + 2(1 - x)\, y = 0$ has a solution $y = e^x$. Find the solution satisfying the initial conditions $y|_{x=1} = 0$, $y'|_{x=1} = 1$.

4236*. Find the necessary and sufficient condition for the equation $y'' + y'P(x) - yQ(x) = 0$ to have two linearly independent solutions y_1 and y_2, satisfying the condition $y_1y_2 = 1$.

4237*. Find the general solution of the equation

$$(1 - x^2)\, y'' - xy' + 9y = 0,$$

if its particular solution is a third-degree polynomial.

It is easy to pick out a particular solution of the equations of problems 4238–4240 (excluding the trivial solution $y = 0$). Find the general solutions of these equations:

4238. $y'' - (\tan x)\, y' + 2y = 0.$

4239. $y'' - y' + \dfrac{y}{x} = 0.$

4240. $y'' - \dfrac{2x}{x^2+1}\, y' + \dfrac{2y}{x^2+1} = 0.$

4241. Find the general solution of the equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0,$$

knowing the particular solutions $y_1 = x$ and $y_2 = x^2$.

Find the general solutions of the non-homogeneous equations of problems 4242–4244:

4242. $x^2y'' - xy' + y = 4x^3.$

4243. $y'' - \dfrac{x}{x-1}\, y' + \dfrac{1}{x-1}\, y = x - 1.$

4244. $(3x + 2x^2)\, y'' - 6(1 + x)\, y' + 6y = 6.$

4245. The equation $(1 + x^2)\, y'' + 2xy' - 2y = 4x^2 + 2$ has the particular solution $y = x^2$. Find the solution satisfying the conditions $y|_{x=-1} = 0$, $y'|_{x=-1} = 0$.

4246. Find the first six terms of the expansion in a power series of the solution of the differential equation $y'' - (1 + x^2)\, y = 0$, satisfying the initial conditions $y|_{x=0} = -2$, $y'|_{x=0} = 2$.

4247. Find the first nine terms of the expansion in a power series of the solution of the differential equation $y'' = x^2y - y'$, satisfying the initial conditions $y|_{x=0} = 1$, $y'|_{x=0} = 0$.

4248. Write as a power series the particular solution of the equation $y'' - xy' + y - 1 = 0$; $y|_{x=0} = 0$, $y'|_{x=0} = 0$.

4249. Write as a power series the general solution of the equation $y'' = ye^x$. (Take the first six terms.)

4250. Write as a power series the general solution of the equation $y'' + xy' - x^2y = 0$. (Take the first six terms.)

Equations with Constant Coefficients

Find the general solutions of the equations of problems 4251–4261:

4251. $y'' + y' - 2y = 0$.

4252. $y'' - 9y = 0$.

4253. $y'' - 4y' = 0$.

4254. $y'' - 2y' - y = 0$.

4255. $3y'' - 2y' - 8y = 0$.

4256. $y'' + y = 0$.

4257. $y'' + 6y' + 13y = 0$.

4258. $4y'' - 8y' + 5y = 0$.

4259. $y'' - 2y' + y = 0$.

4260. $4\dfrac{d^2x}{dt^2} - 20\dfrac{dx}{dt} + 25x = 0$.

4261. $2y'' + y' + 2\sin^2 15° \cos^2 15° y = 0$.

Find the solutions of the equations of problems 4262–4264 satisfying the stated initial conditions:

4262. $y'' - 4y' + 3y = 0$; $\quad y|_{x=0} = 6$, $y'|_{x=0} = 10$.

4263. $y'' + 4y' + 29y = 0$; $\quad y|_{x=0} = 0$, $y'|_{x=0} = 15$.

4264. $4y'' + 4y' + y = 0$; $\quad y|_{x=0} = 2$, $y'|_{x=0} = 0$.

4265. Given that $y_1 = e^{mx}$ is a particular solution of a certain linear homogeneous equation of the second order with constant coefficients, and that the discriminant of the corresponding characteristic equation vanishes, find the particular solution of the equation which, together with its derivative, becomes unity for $x = 0$.

4266. Find the integral curve of the equation $y'' + 9y = 0$, passing through the point $M(\pi, -1)$ and touching the straight line $y + 1 = x - \pi$ at this point.

4267. Find the integral curve of the equation $y'' + ky = 0$, passing through the point $M(x_0, y_0)$ and touching the straight line $y - y_0 = a(x - x_0)$ at this point.

Form the general solutions of the non-homogeneous equations of problems 4268–4282, by finding their particular solutions either by inspection (see *Course*, sec. 205), or by the method of variation of the arbitrary constants (see *Course*, sec. 203), or by using the general formula (see *Course*, sec. 206):

4268. $2y'' + y' - y = 2e^x$. **4269.** $y'' + a^2y = e^x$.

4270. $y'' - 7y' + 6y = \sin x$.

4271. $y'' + 2y' + 5y = -\frac{17}{2}\cos 2x$.

4272. $y'' - 6y' + 9y = 2x^2 - x + 3$.

4273. $y'' - 2y' + 2y = 2x$. **4274.** $y'' + 4y' - 5y = 1$.

4275. $y'' - 3y' + 2y = f(x)$, if $f(x)$ is equal to:

(1) $10e^{-x}$; (2) $3e^{2x}$; (3) $2\sin x$; (4) $2x^3 - 30$; (5) $2e^x \cos\frac{x}{2}$; (6) $x - e^{-2x} + 1$; (7) $e^x(3 - 4x)$; (8) $3x + 5\sin 2x$; (9) $2e^x - e^{-2x}$; (10) $\sin x \sin 2x$; (11) $\sinh x$.

4276. $2y'' + 5y' = f(x)$, if $f(x)$ is equal to:

(1) $5x^2 - 2x - 1$; (2) e^x; (3) $29\cos x$; (4) $\cos^2 x$; (5) $0{\cdot}1e^{-2{\cdot}5x} - 25\sin 2{\cdot}5x$; (6) $29x\sin x$; (7) $100xe^{-x}\cos x$; (8) $3\cosh\frac{5}{2}x$.

4277. $y'' - 4y' + 4y = f(x)$, if $f(x)$ equal to:

(1) 1; (2) e^{-x}; (3) $3e^{2x}$; (4) $2(\sin 2x + x)$; (5) $\sin x\cos 2x$; (6) $\sin^3 x$; (7) $8(x^2 + e^{2x} + \sin 2x)$; (8) $\sinh 2x$; (9) $\sinh x + \sin x$; (10) $e^x - \sinh(x - 1)$.

4278. $y'' + y = f(x)$, if $f(x)$ is equal to:

(1) $2x^3 - x + 2$; (2) $-8\cos 3x$; (3) $\cos x$; (4) $\sin x - 2e^{-x}$; (5) $\cos x\cos 2x$; (6) $24\sin^4 x$; (7) $\cosh x$.

4279. $5y'' - 6y' + 5y = f(x)$, if $f(x)$ equal to:

(1) $5e^{\frac{3}{5}x}$; (2) $\sin\frac{4}{5}x$; (3) $e^{2x} + 2x^3 - x + 2$;

(4) $e^{\frac{3}{5}x}\cos x$; (5) $e^{\frac{3}{5}x}\sin\frac{4}{5}x$; (6) $13e^x\cosh x$.

4280. $y''+y+\cot^2 x=0$.

4281. $y''-2y'+y=\dfrac{e^x}{x^2+1}$.

4282*. $y''-y'=f(x)$, if $f(x)$ is equal to:

(1) $\dfrac{e^x}{1+e^x}$; (2) $e^{2x}\sqrt{1-e^{2x}}$; (3) $e^{2x}\cos e^x$.

Find the particular solutions of the equations of problems 4283–4287, satisfying the stated initial conditions:

4283. $4y''+16y'+15y=4e^{-\frac{3}{2}x}$; $y|_{x=0}=3$, $y'|_{x=0}=-5{\cdot}5$.

4284. $y''-2y'+10y=10x^2+18x+6$; $y|_{x=0}=1$, $y'|_{x=0}=3{\cdot}2$.

4285. $y''-y'=2(1-x)$; $y|_{x=0}=1$, $y'|_{x=0}=1$.

4286. $y''-2y'=e^x(x^2+x-3)$; $y|_{x=0}=2$, $y'|_{x=0}=2$.

4287. $y''+y+\sin 2x=0$; $y|_{x=\pi}=y'|_{x=\pi}=1$.

4288*. Show that the particular solution $\bar{y}$ of the second-order equation with constant coefficients and right-hand side Ae^{px} (p and A are real or complex numbers) has the form $\bar{y}=\dfrac{A}{\varphi(p)}e^{px}$, if p is not a root of the characteristic equation $\varphi(r)\equiv a_0r^2+a_1r+a_2=0$; $\bar{y}=\dfrac{Ax}{\varphi'(p)}e^{px}$ if p is a simple root of the characteristic equation; $\bar{y}=\dfrac{Ax^2}{\varphi''(p)}e^{px}$ if p is a double root of the characteristic equation.

Find the general solution of the Euler equations of problems 4289–4292 (see *Course*, sec. 208):

4289. $x^2y''-9xy'+21y=0$. **4290.** $x^2y''+xy'+y=x$

4291. $y''-\dfrac{y'}{x}+\dfrac{y}{x^2}=\dfrac{2}{x}$.

4292. $x^2y''-2xy'+2y+x-2x^3=0$.

4293. If the axis of a turbine shaft is arranged horizontally, and if the centre of gravity of a disc fastened to the shaft does not lie on the axis, the bending y (Fig. 71) of the shaft axis satisfies on rotation the equation

$$\frac{d^2y}{dt^2} + \left(\frac{1}{m\alpha} - \omega^2\right) y = g \cos \omega t + \omega^2 e,$$

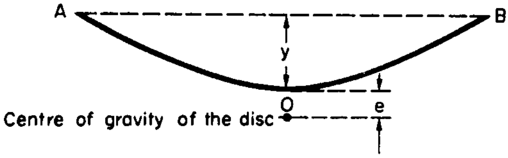

FIG. 71.

where m is the mass of the disc, α is constant number, depending on the type of clamping of the ends A and B, ω is the angular velocity of rotation, and e is the eccentricity of the centre of gravity of the disc. Find the general integral of this equation.

4294. A material particle of mass 1 g is repelled along a straight line from a certain centre with a force, proportional to its distance from this centre (coefficient of proportionality 4). The resistance of the medium is proportional to the velocity (coefficient of proportionality 3). At the initial instant the distance from the centre is 1 cm, and the velocity is zero. Find the law of motion.

4295. A particle of mass 1 g moves along a straight line towards a point A under the action of a force of attraction proportional to its distance from point A. At a distance of 1 cm, the force acting is 0·1 dynes. The resistance of the medium is proportional to the velocity and is equal to 0·4 dynes at a velocity of 1 cm/sec. At the instant $t = 0$ the particle is situated 10 cm to the right of point A and its velocity is zero. Find the distance as a function of time and work out this distance for $t = 3$ sec (to an accuracy of 0·01 cm).

4296. A material particle of mass m moves along a straight line from A to B under the action of a constant force F. The resistance of the medium is proportional to the distance of the particle from B and is equal to f ($f < F$) at the initial instant (at point A). The initial velocity is zero. How long does it take the particle to move from A to B? ($AB = a$).

4297. A body of mass 200 g is suspended from a spring and moved from its position of rest by pulling out the spring 2 cm, after which it is let go (without initial velocity). Find the equation of motion of the body, assuming that the resistance of the medium is proportional to the velocity. If the body moves with a velocity of 1 cm/sec, the medium displays a resistance of 0·1 g; the spring tension when it is extended 2 cm is equal to 10 kg.

The weight of the spring is neglected.

4298. A small cylindrical block of wood ($S = 100$ cm^2, $h = 20$ cm, $\gamma = 0{\cdot}5$ g/cm^3) is completely submerged in water and let go without initial velocity. Assuming that the friction force is proportional to the height of the submerged part, find what the coefficient of proportionality k must be for precisely half the block to appear above the water surface as a result of its first rise.

How long (t_1) does the first rise last?

What is the equation of motion during the first rise?

4299*. A long thin pipe rotates with constant angular velocity ω about a vertical axis perpendicular to it. A small sphere of mass m is situated inside the pipe at a distance a_0 from the axis at the initial instant. Assuming that the velocity of the sphere relative to the pipe is zero at the initial instant, find the law of relative motion of the sphere.

4300. Solve the previous problem on the assumption that the sphere is fixed to a point O via a spring. The spring force acting on the sphere is proportional to the spring deformation, and a force of k dynes produces a 1 cm change in length of the spring. The length of the spring in the free state is a_0.

Equations of Higher Orders

Find the general solutions of the equations of problems 4301–4311:

4301. $y''' + 9y' = 0.$

4302. $y^{(IV)} - 13y'' + 36y = 0.$

4303. $y^{(IV)} = 8y'' - 16y.$ **4304.** $y^{(IV)} = 16y.$

4305. $y''' - 13y' - 12y = 0.$

4306. $y''' - 3y'' + 3y' - y = 0.$

4307. $y^{(IV)} + 2y'' + y''' = 0.$ **4308.** $y^{(n)} = y^{(n-2)}.$

4309. $y^{(IV)} + y = 0.$

4310. $64y^{(VIII)} + 48y^{(VI)} + 12y^{(IV)} + y'' = 0.$

4311. $y^{(n)} + \frac{n}{1} y^{(n-1)} + \frac{n(n-1)}{1\,.\,2} y^{(n-2)} + \ldots + \frac{n}{1} y' +$
$+ y = 0$

4312. $y''' = -y'$; $y|_{x=0} = 2$, $y'|_{x=0} = 0$, $y''|_{x=0} = -1.$

4313. $y^{(V)} = y'$; $y|_{x=0} = 0$, $y'|_{x=0} = 1$, $y''|_{x=0} = 0$,
$y'''|_{x=0} = 1$, $y^{(IV)}|_{x=0} = 2.$

Obtain the general solutions of the non-homogeneous equations of problems 4314–4320, by finding their particular solutions, either by inspection (see *Course*, sec. 205), or by the method of variation of the arbitrary constants (see *Course*, sec. 203), or by using general formula (see *Course*, sec. 206):

4314. $y''' - 4y'' + 5y' - 2y = 2x + 3.$

4315. $y''' - 3y' + 2y = e^{-x}(4x^2 + 4x - 10).$

4316. $y^{(IV)} + 8y'' + 16y = \cos x.$

4217. $y^{(IV)} + 2a^2y'' + a^4y = \cos ax.$

4318. $y^{(V)} + y''' = x^2 - 1.$

4319. $y^{(IV)} - y = xe^x + \cos x.$

4320. $y^{(IV)} - 2y'' + y = 8(e^x + e^{-x}) + 4(\sin x + \cos x).$

4321. $y''' + 2y'' + y' + 2e^{-2x} = 0$; $y|_{x=0} = 2$,
$y'|_{x=0} = 1$, $y''|_{x=0} = 1.$

4322. $y''' - y' = 3(2 - x^2)$; $y|_{x=0} = y'|_{x=0} = y''|_{x=0} = 1.$

4323. Solve Euler's equation $x^3y''' + xy' - y = 0.$

5. Systems of Differential Equations

4324. $\begin{cases} \dfrac{dx}{dt} = y - 7x, \\ \dfrac{dy}{dt} + 2x + 5y = 0. \end{cases}$

4325. $\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = x + e^t + e^{-t}. \end{cases}$

4326. $\begin{cases} \dfrac{dx}{dt} = 2y - 5x + e^t, \\ \dfrac{dy}{dt} = x - 6y + e^{-2t}. \end{cases}$

4327. $\begin{cases} yzy' = x\left(y' = \dfrac{dy}{dx}\right), \\ y^2z' = x\left(z' = \dfrac{dz}{dx}\right). \end{cases}$

4328. $\begin{cases} y' = \dfrac{x+y}{z}, \\ z' = \dfrac{x-y}{y}. \end{cases}$

4329. $\begin{cases} xy' = y, \\ xzz' + x^2 + y^2 = 0. \end{cases}$

4330. $\begin{cases} y' = \dfrac{2xy}{x^2 - y^2 - z^2}, \\ z' = \dfrac{2xz}{x^2 - y^2 - z^2}. \end{cases}$

4331. $\begin{cases} z = y'(z - y)^2 \\ y = z'(z - y)^2 \end{cases}.$

4332. $\begin{cases} 4\dfrac{dx}{dt} - \dfrac{dy}{dt} + 3x = \sin t, \\ \dfrac{dx}{dt} + y = \cos t. \end{cases}$

4333. $\begin{cases} \dfrac{d^2y}{dt^2} = x, \\ \dfrac{d^2x}{dt^2} = y. \end{cases}$

4334. $\begin{cases} \dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} + x = e^t, \\ \dfrac{dx}{dt} + \dfrac{d^2y}{dt^2} = 1. \end{cases}$

4335. $\dfrac{dx}{z - y} = \dfrac{dy}{x - z} = \dfrac{dz}{y - x}.$

Find the particular solutions of the systems of differential equations of problems 4336–4339, satisfying the stated initial conditions:

4336. $\begin{cases} \dfrac{dy}{dx} = \dfrac{y^2 - yz}{x^2 - yz}, & y|_{x=0} = 1; \\ \dfrac{dz}{dx} = \dfrac{z(x+y)}{x^2 - yz}, & z|_{x=0} = -1. \end{cases}$

4337. $\begin{cases} \dfrac{dx}{dt} = 1 - \dfrac{2x}{t}, & x|_{t=1} = \dfrac{1}{3}; \\ \dfrac{dy}{dt} = x + y - 1 + \dfrac{2x}{t}, & y|_{t=1} = -\dfrac{1}{3}. \end{cases}$

4338. $\begin{cases} \dfrac{dx}{dt} = z + y - x, \\ \dfrac{dy}{dt} = z + x - y, \\ \dfrac{dz}{dt} = x + y + z, \end{cases}$ $\begin{array}{l} x|_{t=0} = 1; \\ y|_{t=0} = z|_{t=0} = 0. \end{array}$

4339. $\begin{cases} \dfrac{dx}{dt} = y + z, & x|_{t=0} = -1, \\ \dfrac{dy}{dt} = z + x, & y|_{t=0} = 1; \\ \dfrac{dz}{dt} = x + y & z|_{t=0} = 0. \end{cases}$

4340. Find the pair of curves with the following property: (a) tangents at points with the same abscissae intersect on the axis of ordinates; (b) normals at points with the same absissae intersect on the axis of abscissae; (c) one of the curves passes through the point (1, 1), the other through (1, 2).

4341. Given two curves: $y = f(x)$, through the point (0, 1), and $y = \int\limits_{-\infty}^{x} f(x)\, dx$, through the point $\left(0, \frac{1}{2}\right)$, such that the tangents to both curves at points with the same abscissae intersect on the axis of abscissae, find $y = f(x)$.

4342. Find the spatial curve through the point (0, 1, 1) with the following properties: (a) when the point of contact moves along the curve, the trace of the tangent on the yOx plane describes the bisector of the angle between the positive directions of Ox and Oy; (b) the distance of this trace from the origin is equal to the z coordinate of the point of contact.

4343. Two small spheres, each of mass m, are joined by a very light spring (the extension of which is proportional to the extending force). The length of the unextended spring is l_0. The spring is extended to a length l_1, then at the instant $t = 0$ both spheres, situated one above the other, start to fall (the resistance of the medium is neglected). After T sec the length of the spring is shortened to l_0. Find the law of motion of each sphere.

4344. A horizontal pipe revolves about a vertical axis with angular velocity 2 radians per sec. Two small spheres of mass 300 g and 200 g are situated in the pipe, and joined by a very light spring of length 10 cm, the heavier of the spheres being the further from the axis of rotation. A force of 24000 dynes extends the spring 1 cm, whilst the centre of gravity of the system of spheres is 10 cm from the axis of rotation. The spheres are maintained in position by a certain mechanism. At the instant, which we take as the initial instant, the mechanism is put out of action, and the spheres start to move. Find the law of motion of each sphere relative to the pipe. (Friction is neglected.)

4345. The rate of growth of a culture of micro-organisms is proportional to the quantity of them and the amount of nutrient (coefficient of proportionality k). The rate of decrease of the nutrient is proportional to the initial quantity of micro-organisms and time (coefficient of proportionality k_1). At the start of the experiment there are A_0 g of micro-organisms and B_0 g of nutrient in the vessel. Find the amount A of micro-organisms and the amount B of nutrient as functions of time.

4346*. Suppose that bacteria multiply at a rate proportional to the initial amount (coefficient of proportionality a), but that poisons are at work at the same time, destroying them at a rate proportional to the amount of poison and the amount of bacteria (coefficient of proportionality b). Further, suppose that the rate of working of the poisons is proportional to the initial amount of bacteria (coefficient of proportionality c). The number of bacteria increases at first up to a certain maximum value, then decreases, tending to zero. Show that, for any instant t, the number N of bacteria is given by

$$N = \frac{4M}{(e^{kt} + e^{-kt})^2},$$

where M is the greatest number of bacteria and time t is measured from the instant when $N = M$, k being a constant.

4347. Two cylinders, the bases of which lie in the same plane, are joined at the bottoms by a capillary tube, and are filled with liquid to different heights (H_1 and H_2). The volume of liquid flowing through the tube in unit time is proportional to the difference in the heights, i.e. is equal to $\alpha(h_1 - h_2)$, where α is a coefficient of proportionality. Find the law of variation of the heights of the fluid in the vessels above the capillary tube. The cross-sectional areas of the vessels are S_1 and S_2.

6. Numerical Problems

4348. One kilogramme of water, the specific heat of which is reckoned constant (1 cal/deg), and the initial temperature of which is θ_0, is heated by an electrical device submerged in the water, the resistance R of which depends linearly on the temperature θ: $R = R_0(1 + 0{\cdot}004\theta)$, where R_0 is the resistance at 0° C (the law holds for the majority of pure metals). The heat insulation of the vessel is so good that heat transmission may be neglected. Find the temperature θ as a function of time t in the interval $0 \leqq t \leqq T$, if:

(1) The voltage E is introduced uniformly from $E = 0$ to $E = E_1$ in the course of T sec. Calculate to an accuracy of 1° the number of degrees by which the temperature is raised at the end of the 10th minute, if $\theta_0 = 0°$, $E_1 = 110$ V, $R_0 = 10$ ohms and $T = 10$ min.

(2) The current is alternating and the voltage varies in accordance with the law $E = E_0 \sin 100\pi t$. Calculate to an

accuracy of 1° the number of degrees by which the temperature of the water is raised at the end of the 10th minute, if $\theta_0 = 0°$, $E_0 = 110$ V and $R = 10$ ohms.

4349. A litre of water is heated by a spiral with a resistance of 24 ohms. The water gives out heat to the surrounding medium, which has a temperature of 20° C (the rate of cooling is proportional to the temperature difference between body and medium). We also know that, if the current is switched off, the water temperature drops from 40° to 30° in 10 min. The initial water temperature is 20° C. To what temperature is the water heated after 10 min, if:

(1) The voltage is introduced uniformly from $E_0 = 0$ to $E_1 = 120$ V during 10 min? Accuracy: 0·1°.

(2) The current is alternating, and the voltage variation is given by $E = 110 \sin 100\pi t$? Accuracy: 0·1°.

4350. Given the equation $y' = \frac{x}{y} - x^2$, form a table of the values of the solution which satisfies the initial condition $y|_{x=1} = 1$, by giving x values from 1 to 1·5 every 0·05. Carry out the working to three decimal places.

4351. Calculate the value at $x = 1$ of the particular solution of the differential equation $y' = y + x$, satisfying the initial condition $y|_{x=0} = 1$. Then calculate the first five approximations y_1, y_2, y_3, y_4, y_5 (to four decimal places) by the method of successive approximations. Compare the results.

4352. We know that $\int e^{-x^2}\,dx$ cannot be expressed explicitly in terms of elementary functions. Using the fact that the function $y = e^{x^2}\int_0^x e^{-x^2}\,dx$ is a solution of the equation $y' = 2xy + 1$, evaluate $\int_0^{0{\cdot}5} e^{-x^2}\,dx$. Use the method of successive approximations taking the first five approximations. Compare the result with the approximate value calculated from Simpson's rule.

4353. $y = f(x)$ is a solution of the differential equation $y' = y^2 - x$ with the initial condition $y|_{x=0} = 1$. Find by the method of successive approximations the fourth approximation (y_4), the number of terms being limited to that required for evaluating y_4 (0·3) to three decimal places. Then find the first few terms of the expansion of $f(x)$ in a power series; evaluate $f(0{\cdot}3)$ also to three figures after the point and, assuming $f(0{\cdot}3)$ to be the more accurate result, estimate the error in the value of y_4 (0·3).

4354. $y = f(x)$ is a solution of the differential equation $y'' = \frac{y'}{y} - \frac{1}{x}$ with the initial conditions $y|_{x=1} = 1, y'|_{x=1} = 0$. Find $f(1{\cdot}6)$ to an accuracy of 0·001.

4355*. $y = f(x)$ is a solution of the differential equation $y'' = y' - y + x$ with the initial conditions $y|_{x=1} = 1$, $y'|_{x=1} = 0$. Find $f(1{\cdot}21)$ to an accuracy of 0·000001.

4356*. $y = f(x)$ is a solution of the differential equation $y'' = xy' - y + e^x$ with the initial conditions $y|_{x=0} = 1$, $y'|_{x=0} = 0$. Find $f\left(\frac{1}{2}\right)$ to an accuracy of 0·0001.

4357. A curve is given by the equation $y = f(x)$. Find the series expansion of $f(x)$, knowing that it satisfies the differential equation $y'' = xy$ and the initial conditions $y|_{x=0} = 0$, $y'|_{x=0} = 1$. Evaluate the curvature of the curve at the point with abscissa 1 to an accuracy of 0·0001.

CHAPTER XV

TRIGONOMETRIC SERIES

1. Trigonometric Polynomials

4358. By using Euler's formulae $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$, show that the functions $\sin^n x$ and $\cos^n x$ can be written as nth order trigonometric polynomials.

4359. Prove that the relationships hold:

$$\int_0^{2\pi} \sin^n x \cos mx \, dx = \int_0^{2\pi} \sin^n x \sin mx \, dx =$$

$$= \int_0^{2\pi} \cos^n x \cos mx \, dx = \int_0^{2\pi} \cos^n x \sin mx \, dx = 0,$$

if $m > n$ (m and n are integers).

4360. Prove that every nth order trigonometric polynomial consisting of cosines only can be written as $P(\cos \varphi)$, where $P(x)$ is an nth degree polynomial in x.

4361. Prove with the aid of Euler's formulae (see problem 4358) the relationship

$$\cos \varphi + \cos 2\varphi + \ldots + \cos n\varphi = \frac{\sin \dfrac{n\varphi}{2} \cos \dfrac{(n+1)\varphi}{2}}{\sin \dfrac{\varphi}{2}}.$$

4362. Prove the relationships:

(1) $\cos \varphi + \cos 3\varphi + \ldots + \cos (2n-1)\varphi = \dfrac{\sin 2n\varphi}{2 \sin \varphi}$;

(2) $\sin \varphi + \sin 2\varphi + \ldots + \sin n\varphi = \dfrac{\sin \dfrac{n}{2}\varphi \sin \dfrac{n+1}{2}\varphi}{\sin \dfrac{\varphi}{2}}$.

4363. Find the zeros of the trigonometric polynomials

$$\sin\varphi + \sin 2\varphi + \ldots + \sin n\varphi$$

and

$$\cos\varphi + \cos 2\varphi + \ldots + \cos n\varphi$$

in the interval $[0, 2\pi]$.

4364. Show that the trigonometric polynomial

$$\sin\varphi + \frac{\sin 2\varphi}{2} + \ldots + \frac{\sin n\varphi}{n}$$

has a maximum in the interval $[0, \pi]$ at the points $\frac{\pi}{n+1}$, $3\frac{\pi}{n+1}, \ldots, (2q-1)\frac{\pi}{n+1}$ and a minimum at the points $\frac{2\pi}{n}$, $2\cdot\frac{2\pi}{n}, \ldots, (q-1)\frac{2\pi}{n}$, where $q = \frac{n}{2}$ if n is even, and $q = \frac{n+1}{2}$ if n is odd.

4365*. Prove that a trigonometric polynomial with no constant term:

$$\Phi_n(\varphi) = a_1\cos\varphi + b_1\sin\varphi + \ldots + a_n\cos n\varphi + b_n\sin n\varphi,$$

and not identically zero, cannot retain a constant sign for all φ.

2. Fourier Series

4366. Show that the function $y = x^3\sin\frac{1}{x}$ for $x \neq 0$, and $y = 0$ for $x = 0$, is continuous along with its first derivative in the interval $[-\pi, \pi]$, but does not satisfy the conditions of Dirichlet's theorem. Can it be expanded in a Fourier series in the interval $[-\pi, \pi]$?

Solve problems 4367–4371 on the assumption that $f(x)$ is a continuous function.

4367. Function $f(x)$ satisfies the condition

$$f(x+\pi) = -f(x).$$

Show that all its even Fourier coefficients are zero ($a_0 = a_2 = b_2 = a_4 = b_4 = \ldots = 0$).

4368. Function $f(x)$ satisfies the condition

$$f(x + \pi) = f(x).$$

Show that all its odd Fourier coefficients vanish.

4369. Function $f(x)$ satisfies the conditions $f(-x) = f(x)$ and $f(x + \pi) = -f(x)$.

Show that $b_1 = b_2 = b_3 = \ldots = 0$ and $a_0 = a_2 = a_4 = \ldots = 0$.

4370. Function $f(x)$ satisfies the conditions

$$f(-x) = -f(x) \text{ and } f(x + \pi) = -f(x).$$

Prove that $a_0 = a_1 = a_2 = \ldots = 0$ and $b_2 = b_4 = b_6 = \ldots = 0$.

4371. Function $f(x)$ satisfies the conditions:

(a) $f(-x) = f(x)$ and $f(x + \pi) = f(x)$;

(b) $f(-x) = -f(x)$ and $f(x + \pi) = f(x)$.

Which of its Fourier coefficients vanish?

4372. Expand in a Fourier series the function equal to -1 in the interval $[-\pi, 0]$ and equal to 1 in the interval $(0, \pi)$.

4373. Expand in a sine series the function $y = \frac{\pi}{4} - \frac{x}{2}$ in the interval $(0, \pi)$.

4374. By using the results of problems 4372 and 4373, obtain the expansions of functions $y = x$ and $y = \frac{\pi - x}{2}$. Indicate the intervals in which the formulae obtained are valid.

4375. Expand the function $y = \frac{\pi}{4} - \frac{x}{2}$ in the interval $(0, \pi)$ in a cosine series.

4376. Expand the function $y = x^2$ in a Fourier series: (1) in the interval $(-\pi, \pi)$, (2) in the interval $(0, 2\pi)$ (Figs. 72 and 73).

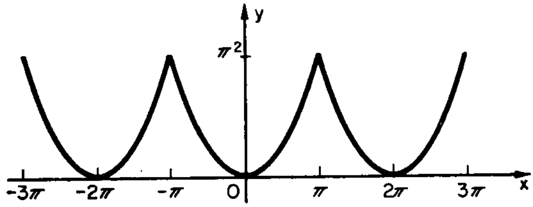

Fig. 72.

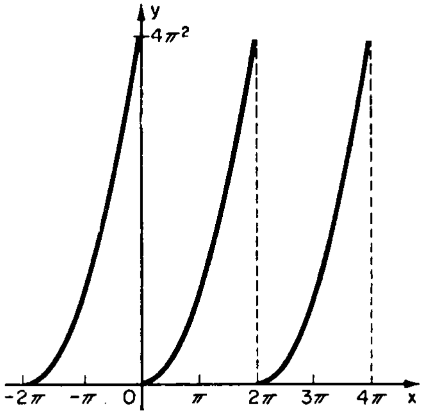

Fig. 73.

Calculate with the aid of the series obtained the sums of the numerical series:

$$S_1 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \ldots,$$

$$S_2 = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \ldots + (-1)^{n-1}\frac{1}{n^2} + \ldots,$$

$$S_3 = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots + \frac{1}{(2n-1)^2} + \ldots$$

Expand the functions of problems 4377–4390 in Fourier series in the indicated intervals:

4377. The function $y = x^2$ in the interval $(0, \pi)$ in a cosine series.

4378. Function $y = x^3$ in the interval $(-\pi, \pi)$.

4379. The function $f(x)$, equal to 1 for $-\pi < x < 0$ and equal to 3 for $0 < x < \pi$.

4380. The function $f(x)$, equal to 1 in the interval $(0, h)$ and equal to 0 in the interval (h, π), in a cosine series $(0 < h < \pi)$.

4381. The continuous function $f(x)$, equal to 1 for $x = 0$, equal to 0 in the interval $(2h, \pi)$ and linear in the interval $(0, 2h)$, in a cosine series $\left(0 < h < \frac{\pi}{2}\right)$.

4382. The function $y = |x|$ in the interval $(-l, l)$.

4383. The function $y = e^x - 1$ in the interval $(0, 2\pi)$.

4384. The function $y = e^x$ in the interval $(-l, l)$.

4385. The function $y = \cos ax$ in the interval $(-\pi, \pi)$ (a is not an integer).

4386. The function $y = \sin ax$ in the interval $(-\pi, \pi)$ (a is not an integer).

4387. The function $y = \sin ax$ (a is an integer) in the interval $(0, \pi)$ in a cosine series.

4388. The function $y = \cos ax$ (a is an integer) in the interval $(0, \pi)$ in a sine series.

4389. The function $y = \sinh x$ in the interval $(-\pi, \pi)$.

4390. The function $y = \cosh x$ in the interval $(0, \pi)$ in a cosine series and a sine series.

4391. Expand in a Fourier series the function whose graph is illustrated in Fig. 74.

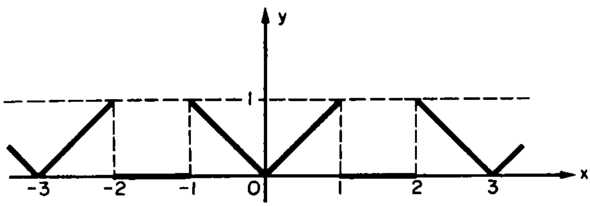

Fig. 74.

4392*. Expand in a Fourier series the function whose graph is illustrated in Fig. 75.

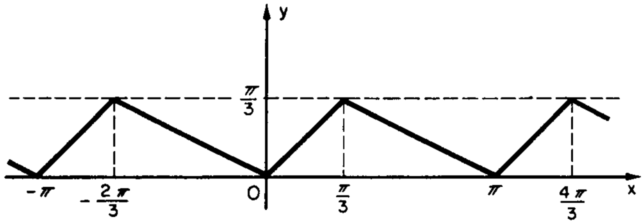

FIG. 75.

4393*. Expand in Fourier series the functions whose graphs are illustrated in Figs. 76 and 77.

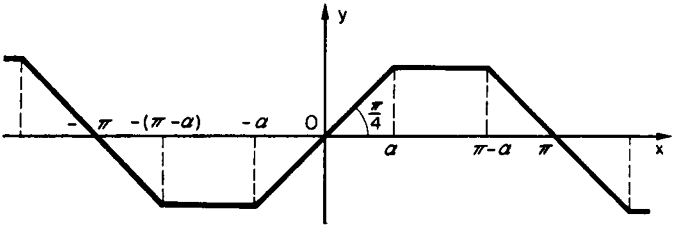

FIG. 76.

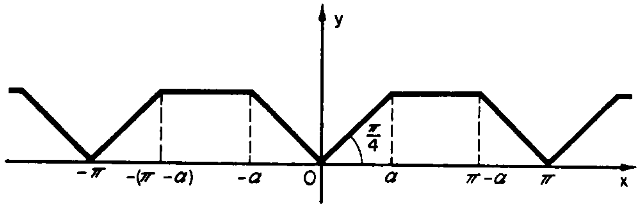

FIG. 77.

4394. Expand the function $y = x(\pi - x)$ in a sine series in the interval $(0, \pi)$. Use the result to find the sum of the series

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots + \frac{(-1)^{n-1}}{(2n-1)^3} + \ldots$$

4395. Given the function $\varphi(x) = (\pi^2 - x^2)^2$.

(a) show that the equalities hold:

$$\varphi(-\pi) = \varphi(\pi),\ \varphi'(-\pi) = \varphi'(\pi) \text{ and } \varphi''(-\pi) = \varphi''(\pi)$$
$$[\text{but } \varphi'''(-\pi) \neq \varphi'''(\pi)].$$

(b) by using these equalities, expand $\varphi(x)$ in a Fourier series in the interval $(-\pi, \pi)$ (see *Course*, sec. 214);

(c) calculate the sum of the series

$$1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \ldots + \frac{(-1)^{n-1}}{n^4} + \ldots$$

3. Krylov's Method. Harmonic Analysis

Improve the convergence of the trigonometric series of problems 4396–4400 by bringing the coefficients of the series up to the order indicated in brackets (k).

4396*. $\sum\limits_{n=1}^{\infty} \frac{n^2}{n^3+1} \sin nx \quad (k = 4).$

4397. $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1} \sin nx \quad (k = 2).$

4398*. $\sum\limits_{n=0}^{\infty} \frac{n^2+1}{n^4+1} \cos nx \quad (k = 4).$

4399*. $\sum\limits_{n=2}^{\infty} \frac{n \sin \frac{n\pi}{2}}{n^2-1} \cos nx \quad (k = 5).$

4400. Functions $f_i(x)$ $(i = 1, 2, 3)$ are given in the interval $[0, 2\pi]$ by the following table:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
$f_1(x)$	27	32	35	30	26	20	18	22	26	30	32	36
$f_2(x)$	0·43	0·87	0·64	0·57	0·28	0	−0·30	−0·64	−0·25	0·04	0·42	0·84
$f_3(x)$	2·3	3·2	2·1	1·6	−0·4	−0·2	0·4	0·3	0·7	0·9	1·2	1·6

Find the approximate expression for these functions as a second-order trigonometric polynomial. (See *Course*, sec. 220.)

CHAPTER XVI

ELEMENTS OF THE THEORY OF FIELDS[†]

Vector Field, Divergence and Curl

4401. Find the vector lines of the homogeneous field $\boldsymbol{A}(P) = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$, where a, b and c are constants.

4402. Find the vector lines of the plane field $\boldsymbol{A}(P) = -\omega y\boldsymbol{i} + \omega x\boldsymbol{j}$, where ω is constant.

4403. Find the vector lines of the field $\boldsymbol{A}(P) = -\omega y\boldsymbol{i} + \omega x\boldsymbol{j} + h\boldsymbol{k}$, where ω and h are constants.

4404. Find the vector lines of the fields:

(1) $\boldsymbol{A}(P) = (y + z)\,\boldsymbol{i} - x\boldsymbol{j} - x\boldsymbol{k}$;
(2) $\boldsymbol{A}(P) = (z - y)\,\boldsymbol{i} + (x - z)\,\boldsymbol{j} + (y - x)\,\boldsymbol{k}$;
(3) $\boldsymbol{A}(P) = x(y^2 - z^2)\,\boldsymbol{i} - y(z^2 + x^2)\,\boldsymbol{j} + z(x^2 + y^2)\,\boldsymbol{k}$.

Evaluate the divergence and curl of the vector fields of problems 4405–4408:

4405. $\boldsymbol{A}(P) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

4406. $\boldsymbol{A}(P) = (y^2 + z^2)\,\boldsymbol{i} + (z^2 + x^2)\,\boldsymbol{j} + (x^2 + y^2)\,\boldsymbol{k}$.

4407. $\boldsymbol{A}(P) = x^2yz\boldsymbol{i} + xy^2z\boldsymbol{j} + xyz^2\boldsymbol{k}$.

4408. $\boldsymbol{A}(P) = \operatorname{grad}(x^2 + y^2 + z^2)$.

4409. A vector field is formed by a force having a constant magnitude F and the direction of the positive axis of abscissae. Find the divergence and curl of this field.

4410. A plane vector field is formed by a force, inversely proportional to the square of the distance of its point of application from the origin and directed to the origin. (For example, the plane electrostatic field produced by a point charge.) Find the divergence and curl of this field.

[†] Problems on the properties of a scalar field and its gradient are located in the section 4 of chapter XI.

4411. Find the divergence and curl of the spatial field in which the forces are subject to the same conditions as in problem 4410.

4412. A vector field is formed by a force, inversely proportional to the distance of its point of application from Oz, perpendicular to this axis and directed towards it. Find the divergence and curl of the field.

4413. A vector field is formed by a force, inversely proportional to the distance of its point of application to the xOy plane and directed towards the origin. Find the divergence of the field.

4414. Find div $(a\boldsymbol{r})$, where a is a constant scalar.

4415. Prove that

$$\operatorname{div}(\varphi \boldsymbol{A}) = \varphi \operatorname{div} \boldsymbol{A} + (\boldsymbol{A} \operatorname{grad} \varphi),$$

where $\varphi = \varphi(x, y, z)$ is a scalar function.

4416. Evaluate div $\boldsymbol{b}(\boldsymbol{r}\boldsymbol{a})$ and div $\boldsymbol{r}(\boldsymbol{r}\boldsymbol{a})$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors.

4417. Evaluate div $(\boldsymbol{a}\times\boldsymbol{r})$, where $\boldsymbol{a}$ is a constant vector.

4418. Without passing to coordinates, evaluate the divergence of the vector field:

$$\text{(1)}\ \boldsymbol{A}(P) = \boldsymbol{r}(\boldsymbol{a}\cdot\boldsymbol{r}) - 2\boldsymbol{a}r^2, \quad \text{(2)}\ \boldsymbol{A}(P) = \frac{\boldsymbol{r}-\boldsymbol{r}_0}{|\boldsymbol{r}-\boldsymbol{r}_0|^3},$$

$$\text{(3)}\ \operatorname{grad}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}.$$

4419. Work out the divergence of the vector field

$$\boldsymbol{A}(P) = f(|\boldsymbol{r}|)\frac{\boldsymbol{r}}{|\boldsymbol{r}|}.$$

Show that the divergence of the field is zero only when $f(|\boldsymbol{r}|) = \frac{C}{r^2}$, if the field is in space, and $f(|\boldsymbol{r}|) = \frac{C}{|\boldsymbol{r}|}$, if the field is plane, where C is an arbitrary constant number.

4420. Prove that

$$\operatorname{curl}[\boldsymbol{A}_1(P) + \boldsymbol{A}_2(P)] = \operatorname{curl}\boldsymbol{A}_1(P) + \operatorname{curl}\boldsymbol{A}_2(P).$$

4421. Evaluate curl $\varphi\boldsymbol{A}(P)$, where $\varphi = \varphi(x, y, z)$ is a scalar function.

4422. Evaluate curl $r\boldsymbol{a}$, where r is the distance of a point from the origin, and $\boldsymbol{a}$ is a constant vector.

4423. Evaluate curl $(\boldsymbol{a}\times\boldsymbol{r})$, where $\boldsymbol{a}$ is a constant vector.

4424. A rigid body rotates with constant angular velocity ω about an axis. Find the divergence and curl of the field of the linear velocities.

4425. Prove that

$$\boldsymbol{n}(\operatorname{grad}(\boldsymbol{A}\boldsymbol{n}) - \operatorname{curl}(\boldsymbol{A}\times\boldsymbol{n})) = \operatorname{div}\boldsymbol{A},$$

if $\boldsymbol{n}$ is a unit constant vector.

The differential operations of vector analysis (grad, div, curl) are conveniently represented with the aid of the symbolic vector ∇ (Hamilton's operator — Nabla):

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}.$$

The application of this operator to a (scalar or vector) quantity is to be understood as implying: the operation of multiplying this vector by the given quantity is to be carried out in accordance with the rules of vector algebra, then the multiplication of the symbol $\frac{\partial}{\partial x}$ and so on by the quantity S is to be regarded as finding the corresponding derivative. Thus grad $u = \nabla u$; div $\boldsymbol{A} = \nabla\boldsymbol{A}$; curl $\boldsymbol{A} = \nabla\times\boldsymbol{A}$.

The second order differential operations can also be written with the aid of Hamilton's operator:

$$\nabla\nabla u = \operatorname{div}\operatorname{grad} u;\ \ \nabla\times\nabla u = \operatorname{curl}\operatorname{grad} u;$$
$$\nabla(\nabla\boldsymbol{A}) = \operatorname{grad}\operatorname{div}\boldsymbol{A};\ \ \nabla(\nabla\times\boldsymbol{A}) = \operatorname{div}\operatorname{curl}\boldsymbol{A};$$
$$\nabla\times(\nabla\times\boldsymbol{A}) = \operatorname{curl}\operatorname{curl}\boldsymbol{A}.$$

4426. Show that $(\boldsymbol{r}\nabla)\,\boldsymbol{r}^n = n\boldsymbol{r}^n$, where $\boldsymbol{r}$ is the radius vector.

4427. Prove the relationships:

$$(1)\ \operatorname{curl}\operatorname{grad} u = 0;\ \ (2)\ \operatorname{div}\operatorname{curl}\boldsymbol{A} = 0.$$

4428. Prove that

$$\operatorname{div}\operatorname{grad} u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

(This expression is called Laplace's operator and is usually written as $\triangle u$. By using Hamilton's operator, it can be written in the form $\triangle u = (\nabla \nabla)\, u = \nabla^2 u$.)

4429. Prove that

$$\operatorname{curl}\operatorname{curl} \boldsymbol{A}(P) = \operatorname{grad}\operatorname{div} \boldsymbol{A}(P) - \triangle \boldsymbol{A}(P),$$

where $\triangle \boldsymbol{A}(P) = \triangle A_x \boldsymbol{i} + \triangle A_y \boldsymbol{j} + \triangle A_z \boldsymbol{k}$.

Potential

4430. A vector field is formed by a constant vector $\boldsymbol{A}$. Show that this field has a potential, and find it.

4431. A vector field is formed by a force, proportional to the distance of its point of application from the origin and directed towards the origin. Show that this field is conservative, and find its potential.

4432. The forces of a field are inversely proportional to the distances of their points of application from the Oxy plane and are directed towards the origin. Is this field conservative?

4433. The forces of a field are proportional to the square of the distances of their points of application from the Oz axis and are directed to the origin. Is this field conservative?

4434. A vector field is formed by a force inversely proportional to the distance of its point of application from Oz, perpendicular to this axis and directed towards it. Show that this field is conservative, and find its potential.

4435. A vector field is formed by the linear velocities of points of a rigid body, rotating about its axis. Has this field a potential?

4436. The forces of a field are given as: $\boldsymbol{A}(P) = f(r)\dfrac{\boldsymbol{r}}{r}$ (called a centred field). Show that the potential of the field is equal to

$$u(x, y, z) = \int_a^r f(r)\, dr \quad (r = \sqrt{x^2 + y^2 + z^2}).$$

Hence obtain, as a particular case, the potential of the field of gravitational force of a point mass and the potential of the field of problem 4431.

4437. Find the work done by the forces of the field $\boldsymbol{A}(p) = xy\boldsymbol{i} + yz\boldsymbol{j} + xz\boldsymbol{k}$ on displacement of a point mass m round a closed curve, consisting of the segment of the straight line $x + z = 1$, $y = 0$, the quadrant of the circle $x^2 + y^2 = 1$, $z = 0$ and the segment of the straight line $y + z = 1$, $x = 0$

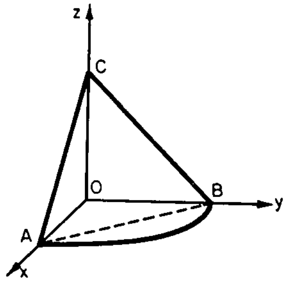

FIG. 78.

(Fig. 78) in the direction indicated on the figure. How does the amount of work change, if the arc BA is replaced by the step-line BOA or the straight line BA?

Potential of Force of Attraction†

4438. Given in the $O\xi\eta$ plane a homogeneous rod AB of length $2l$ with linear density δ, disposed on the $O\xi$ axis, symmetrically with respect to the origin (Fig. 79):

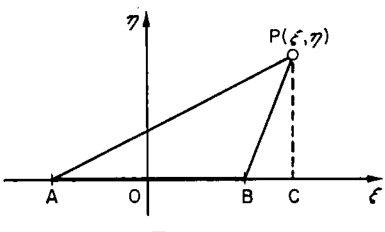

FIG. 79.

† Here (in problems 4438—4449) we have in mind a force of attraction acting in accordance with Newton's law. Instead of referring to the "potential of a mass", distributed over (or in) a given geometrical entity, we speak for brevity of the "potential of the given entity".

(a) Find the potential $u(x, y)$ of the rod.

(b) Show that the projections X and Y of the force of attraction acting on a point P of mass m with coordinates $\xi = x$, $\eta = y$, is equal to

$$X = mk\delta\left(\frac{1}{PA} - \frac{1}{PB}\right), \quad Y = -\frac{mk\delta}{y}\left(\frac{CB}{PB} + \frac{AC}{PA}\right),$$

whilst the resultant force R is equal in magnitude to $R = \frac{2mk\delta}{y} \sin\frac{1}{2}(\alpha + \beta)$, where k is the gravitational constant (C is the projection of point P on the $O\xi$ axis, α is the angle APC, β the angle BPC).

4439. Find the potential of the circumference of the circle $x^2 + y^2 = R^2$, $z = 0$ at the point $(R, 0, 2R)$, if the density at every point is equal to the absolute value of the sine of the angle between the radius vector of the point and the axis of abscissae.

4440. Find the potential of the first turn of the homogeneous (density δ) helix $x = a\cos t$, $y = a\sin t$, $z = bt$ at the origin.

4441. Find the potential of the homogeneous square with side a (surface density δ) at one of its corners.

4442. Mass is distributed on the Oxy plane with density δ, decreasing with the distance ϱ from the origin in accordance with the law $\delta = \dfrac{1}{1 + \varrho^2}$. Find the potential at the point $(0, 0, h)$. (Consider the three cases: $h < 1$, $h = 1$ and $h > 1$.)

4443*. Find the potential of the homogeneous lateral surface of a right circular cylinder:

(1) at the centre of its base,

(2) at the mid-point of its axis (radius of cylinder R, height H, surface density δ).

4444. Find the potential of the homogeneous lateral surface of a right circular cone (base radius R, height H) at its vertex.

4445. Given a homogeneous right circular cylinder (base radius R, height H, density δ):

(1) Find the potential at the centre of its base.

(2) Find the potential at the mid-point of its axis.

4446. Given a homogeneous right circular cone (base radius R, height H, density δ). Find the potential of the cone at its vertex.

4447. Find the potential of the homogeneous hemisphere $x^2 + y^2 + z^2 \leqq R^2 (z \geqq 0)$ with density δ at the point $A(0, 0, a)$. (Consider the two cases: $a \geqq R$ and $a \leqq R$).

4448. Find the potential of the homogeneous body bounded by two concentric spheres with radii R and r $(R > r)$ and density δ at the point at a distance a from the centre of the spheres. (Consider the three cases: $a \geqq R$, $a \leqq r$ and $r \leqq a \leqq R$.) Show that, if the point is situated in the interior cavity of the body, the force of attraction acting on this point is zero.

4449. Find the potential of the non-homogeneous continuous sphere $x^2 + y^2 + z^2 \leqq R^2$ at the point $A(0, 0, a)$ $(a > R)$ if the density $\delta = kz^2$, i.e. it is proportional to the square of the distance of the point from the $0xy$ plane.

Flux and Circulation (Plane Case)

4450. Find the flux and circulation of a constant vector $\boldsymbol{A}$ round an arbitrary closed curve L.

4451. Find the flux and circulation of the vector $\boldsymbol{A}(P) = a\boldsymbol{r}$, where a is a constant scalar, and $\boldsymbol{r}$ is the radius vector of the point P, round an arbitrary closed curve L.

4452. Find the flux and circulation of the vector $\boldsymbol{A}(P) = x\boldsymbol{i} - y\boldsymbol{j}$ round an arbitrary closed curve L.

4453. Find the flux and circulation of the vector $\boldsymbol{A}(P) = (x^3 - y)\,\boldsymbol{i} + (y^3 + x)\,\boldsymbol{j}$ round a circle of radius R with centre at the origin.

4454. The potential of the velocity field of particles in a fluid flow is equal to $u = \ln r$, where $r = \sqrt{x^2 + y^2}$. Find the quantity of fluid flowing out of a closed contour L surrounding the origin per unit time (the flux) and the quantity of fluid flowing per unit time round the contour (the circulation). How is the result changed if the origin lies outside the contour (and not in it)?

4455. The potential of the velocity field of the particles in a fluid flow is equal to $u = \varphi$, where $\varphi = \arctan \frac{y}{x}$. Find the flux and circulation of the vector round the closed contour L.

4456. The potential of the velocity field of particles in a fluid flow is equal to $u(x, y) = x(x^2 - 3y^2)$. Find the quantity of fluid flowing per unit time through the straight segment joining the origin to the point $(1, 1)$.

Flux and Circulation (Spatial Case)

4457. Prove that the flux of the radius vector $\boldsymbol{r}$ through any closed surface is equal to three times the volume bounded by this surface.

4458. Find the flux of the radius vector through the lateral surface of a circular cylinder (base radius R, height H), if the cylinder axis passes through the origin.

4459. By using the results of problems 4457 and 4458, find the flux through both bases of the cylinder of the previous problem.

4460. Find the flux of the radius vector through the lateral surface of a circular cone, the base of which lies on the xOy plane, whilst its axis is Oz. (The height of the cone is 1, the base radius 2.)

4461. Find the flux of the vector $\boldsymbol{A}(P) = xy\boldsymbol{i} + yz\boldsymbol{j} + zx\boldsymbol{k}$ through the boundary of the piece of the sphere $x^2 + y^2 + z^2 = 1$ lying in the first octant.

4462*. Find the flux of the vector $\boldsymbol{A}(P) = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$ through the lateral surface of the pyramid with vertex at the point $S(0, 0, 2)$ the base of which is the triangle with vertices $O(0, 0, 0)$, $A(2, 0, 0)$ and $B(0, 1, 0)$.

4463. Find the circulation of the radius vector along one turn AB of the helix $x = a\cos t$, $y = a\sin t$, $z = bt$, where

A and B are the points corresponding to values 0 and 2π of the parameter.

4464. A rigid body rotates with constant angular velocity ω about the Oz axis. Find the circulation of the field of the linear velocities along the circle of radius R whose centre lies on the axis of rotation, whilst the plane of the circle is perpendicular to the axis of rotation in the direction of rotation.

4465*. Find the flux of the vorticity of the vector field $\boldsymbol{A}(P) = y\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k}$ through the surface of the paraboloid of revolution $z = 2(1 - x^2 - y^2)$, cut off by the plane $z = 0$.

ANSWERS

Chapter I

1. All the positive integers except $n = 1$ and $n = 2$. If S is the sum of the angles, and n the number of sides, then $S = \pi(n - 2)$.

4. (a) The function vanishes for $x = -2$, $x = 1$, $x = 6$;
(b) the function is positive for $x < -2$, $-2 < x < 1$, $x > 6$;
(c) the function is negative for $1 < x < 6$.

6. $r = \dfrac{1}{\sqrt{\pi h}}$. **7.** $S = \dfrac{a^2 - b^2}{4} \tan \alpha$. **8.** $b = \sqrt{25 - a^2}$.

9. $f(0) = -2$; $f(1) = -0{\cdot}5$; $f(2) = 0$; $f(-2) = 4$; $f\left(-\dfrac{1}{2}\right) = -5$; $f(\sqrt{2}) = -0{\cdot}242, \ldots$, $\left|f\left(\dfrac{1}{2}\right)\right| = 1$; $\varphi(0) = 2$; $\varphi(1) = 0{\cdot}5$; $\varphi(2) = 0$; $\varphi(-2) = -4$; $\varphi(4) = 0{\cdot}4$; $f(-1)$ does not exist; $\varphi(-1)$ does not exist.

10. $f(1) = 0$; $f(a) = a^3 - 1$; $f(a + 1) = a^3 + 3a^2 + 3a$; $f(a - 1) = a^3 - 3a^2 + 3a - 2$; $2f(2a) = 16a^3 - 2$.

11. $F(0) = \dfrac{1}{4}$; $F(2) = 1$; $F(3) = 2$; $F(-1) = \dfrac{1}{8}$; $F(2{\cdot}5) = \sqrt{2}$; $F(-1{\cdot}5) = \dfrac{1}{\sqrt{128}}$; $\varphi(0) = \dfrac{1}{4}$; $\varphi(2) = 1$; $\varphi(-1) = \dfrac{1}{2}$; $\varphi(x) = 2^{x-2}$ for $x > 0$ and $\varphi(x) = 2^{-x-2}$ for $x < 0$; $\varphi(-1) + F(1) = 1$.

12. $\psi(0) = 0$; $\psi(1) = a$; $\psi(-1) = -\dfrac{1}{a}$; $\psi\left(\dfrac{1}{a}\right) = a^{\frac{1-a}{a}}$; $\psi(a) = a^{a+1}$; $\psi(-a) = -a^{1-a}$.

13. $\varphi(t^2) = t^6 + 1$; $[\varphi(t)]^2 = t^6 + 2t^3 + 1$.

20. $\dfrac{f(b) - f(a)}{b - a}$ is equal to the tangent of the angle between the secant through the points $(a, f(a))$ and $(b, f(b))$, and the positive direction of Ox.

22. (a) $x_1 = 0$, $x_2 = 2$; (b) $x_1 = -1$, $x_2 = 3$.

23. $x_1 = -2$, $x_2 = 5$, $x_3 = -\dfrac{1}{2}$.

24. One root will always be $x = a$. **25.** 4 and -2; -2, 2, 4, 10.

26. $x_1 = -3,\ x_2 = -2,\ x_3 = 2,\ x_4 = 3.$ **27.** $x \leqq -1$ and $x \geqq 2.$
28. $a = 4,\ \ b = -1.$

29. $a = -\dfrac{1}{2\sin 0{\cdot}5}$ or, since $\sin 0{\cdot}5 \approx 0{\cdot}48$, we have $a \approx -1{\cdot}04$, $b = 1$ and $c = -\dfrac{1}{2} + 2k\pi$. (Alternatively, $a = \dfrac{1}{2\sin 0{\cdot}5} \approx 1{\cdot}04$, $b = -1,\ c = \dfrac{1}{2} + (2k+1)\pi,\ k = 0,\ \pm 1,\ \pm 2,\ \ldots$).

30. $y = (x+1)^2$. **31.** $y = \left|\dfrac{1}{\cos x}\right|$. **32.** $y = \sqrt[3]{(a^t+1)^2}$.

33. $u = \sqrt{1 + (\log \sin x)^2}$. **34.** $v = \sin(1+x)$.

35. (1) $y = v^3,\quad v = \sin x$; (2) $y = \sqrt[3]{v},\quad v = u^2,\quad u = x+1$; (3) $y = \log v,\quad v = \tan x$; (4) $y = u^3,\quad u = \sin v,\quad v = 2x+1$; (5) $y = 5^u,\ u = v^2,\ v = 3x+1$.

36. (a) $-\dfrac{3}{8}$; (b) 0; (c) $\sin 12$; (d) $-\sin 2x\cos^2 2x$; (e) $x^9 - 3x^7 + 3x^5 - 2x^3 + x$; (f) 0; (g) $\sin(2\sin 2x)$.

38. (1) $y = \pm\sqrt{1-x^2}$; (2) $y = \pm\dfrac{b}{a}\sqrt{x^2-a^2}$; (3) $y = \sqrt[3]{a^3-x^3}$; (4) $y = \dfrac{C}{x}$; (5) $y = \dfrac{\log_2 5}{x}$; (6) $y = \dfrac{10000}{x} - 1$;
(7) $y = \log_2(x^3+7) - \log_2(x^2-2) - x$; (8) $y = \arccos\dfrac{x^2}{1+x}$.

39*. Let $x > 0$ and $y > 0$, then $y + y - x - x = 0$; $y = x$ (the graph is the bisector of the first quadrant). Let $x > 0$ and $y < 0$, then $y - y - x - x = 0$; $x = 0$ (the graph is the negative half of Oy). Let $x < 0$ and $y > 0$, then $y + y - x + x = 0$; $y = 0$ (the graph is the negative half of Ox). Let $x < 0$ and $y < 0$, then $y - y - x + x = 0$ (an identity; the "graph" is the aggregate of points of the third quadrant).

40.

x	1	2	3	4	5	6
y	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{120}$	$\frac{1}{720}$

41.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
u	0	1	2	2	3	3	4	4	4	4	5	5	6	6	6	6	7	7	8	8

42.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
u	0	0	0	1	0	2	0	2	1	2	0	4	0	2	2	3	0	4	0	4

43. If $f(x)$ is the weight of segment AM, we have: $f(x) = 2x$ for $0 \leqq x \leqq 1$, $f(x) = 2 + \frac{3}{2}(x - 1)$ for $1 < x \leqq 3$, $f(x) = x + 2$ for $3 < x \leqq 4$. The function is defined for $0 \leqq x \leqq 4$.

44. $S = \pi(2R - x)^2$ for $0 \leqq x \leqq R$; $S = \pi R^2$ for $R \leqq x \leqq 3R$; $S = \pi(6Rx - x^2 - 8R^2)$ for $3R \leqq x \leqq 4R$. The function $S = f(x)$ is not defined outside the interval $[0, 4R]$.

45. $V = \pi x\left(R^2 - \frac{x^2}{4}\right)$; $0 < x < 2R$; $-\infty < x < +\infty$.

46. $S = \frac{\pi x^2}{2R}\sqrt{4R^2 - x^2}$; $0 < x < 2R$; $-2R \leqq x \leqq 2R$.

47. (1) $x > 0$; (2) $x > -3$; (3) $x \leqq \frac{5}{2}$; (4) $-\infty < x \leqq 0$; (5) the whole of the real axis except for the points $x = \pm 1$; (6) the whole of the real axis; (7) defined everywhere except for $x = 0$, $x = -1$, $x = 1$; (8) the whole of the real axis except for the points $x = 1$ and $x = 2$; (9) $-1 \leqq x \leqq 1$; (10) $-\infty < x < 0$ and $4 < x < \infty$; (11) $-\infty < x \leqq 1$ and $3 \leqq x < \infty$; the function is not defined in the interval $(1, 3)$; (12) $-\infty < x < 1$ and $2 < x < \infty$; the function is not defined in the interval $[1, 2]$; (13) $-4 \leqq x \leqq 4$; (14) $1 \leqq x \leqq 3$; (15) $0 \leqq x \leqq 1$; (16) $-\frac{3}{2} \leqq x \leqq \frac{5}{2}$; (17) $0 \leqq x \leqq \frac{1}{2}$; (18) $-1 \leqq x \leqq 1$; (19) $-\infty < x < 0$; (20) meaningless; (21) $1 \leqq x \leqq 4$; (22) $2k\pi < x < (2k + 1)\pi$, where k is an integer; (23) $2k\pi \leqq x \leqq (2k + 1)\pi$, where k is an integer; (24) $0 < x < 1$ and $1 < x < \infty$.

48. (1) $-2 \leqq x < 0$ and $0 < x < 1$; (2) $-1 \leqq x \leqq 3$; (3) $1 \leqq x < 4$; (4) $\frac{3}{2} < x < 2$ and $2 < x < \infty$; (5) the domain of definition only consists of the single point $x = 1$; (6) $-1 < x < 0$ and $1 < x < 2$; $2 < x < \infty$; (7) $3 - 2\pi < x < 3 - \pi$; $3 < x \leqq 4$; (8) $-4 \leqq x \leqq -\pi$ and $0 \leqq x \leqq \pi$; (9) $2k\pi < x < (2k + 1)\pi$, where k is an integer; (10) $4 < x < 5$ and $6 < x < \infty$; (11) nowhere defined; (12) $-1 < x \leqq 1$ and $2 \leqq x < 3$; (13) the whole of the real axis; (14) $4 \leqq x \leqq 6$; (15) $2 < x < 3$.

49. (1) Yes; (2) they are identical in any interval not containing the point $x = 0$; (3) identical in the interval $[0, \infty)$; (4) identical in the interval $(0, \infty)$.

50. (1) e.g. $y = \sqrt{4 - x^2}$; (2) e.g. $y = \dfrac{1}{x\sqrt{4 - x^2}}$; (3) e.g. $y = \dfrac{1}{x-2} + \dfrac{1}{x-3} + \dfrac{1}{x-4}$.

51. (1) $1 < x \leqq 3$; (2) $0 \leqq x < +\infty$ for two branches and $1 \leqq x < +\infty$ for the other two branches.

52. $-\infty < x < \infty$.

53. (1) $y > 0$ for $x > 2$; $y < 0$ for $x < 2$; $y = 0$ for $x = 2$; (2) $y > 0$ for $x < 2$ and $x > 3$; $y < 0$ for $2 < x < 3$; $y = 0$ for $x_1 = 2$ and $x_2 = 3$; (3) $y > 0$ in the interval $(-\infty, \infty)$, the function has no zeros; (4) $y > 0$ in the intervals $(0, 1)$, $(2, +\infty)$; $y < 0$ in the intervals $(-\infty, 0)$ and $(1, 2)$; $y = 0$ for $x_1 = 0$, $x_2 = 1$, $x_3 = 2$; (5) $y > 0$ for $x \neq 0$; $y = 0$ for $x = 0$.

54. (1), (3), (8), (10), (11), (15) are even; (5), (6), (9), (12), (14) (17) are odd; (2), (4), (7), (13), (16) are neither even nor odd.

55. (1) $y = (x^2 + 2) + 3x$; (2) $y = (1 - x^4) + (-x^3 - 2x^5)$; (3) $y = (\sin 2x + \tan x) + \cos\dfrac{x}{2}$.

57. (1) $y = \dfrac{a^x + a^{-x}}{2} + \dfrac{a^x - a^{-x}}{2}$;

(2) $y = \dfrac{(1 + x)^{100} + (1 - x)^{100}}{2} + \dfrac{(1 + x)^{100} - (1 - x)^{100}}{2}$.

59. Functions (1), (5), (6), (8).

60. For the graphs see Fig. 80 and 81.

61. (1) Decreasing in the interval $(-\infty, 0)$, increasing in $(0, +\infty)$; (2) decreasing in the interval $(-\infty, 0)$, retaining the constant value zero in the interval $(0, +\infty)$.

62. (1) Maximum $= 1$, minimum $= 0$; (2) maximum $= 1$, minimum $= -1$; (3) maximum $= 2$, minimum $= 0$; (4) there is no maximum value, the minimum $= 1$.

65. $I = \dfrac{E}{3}$. **66.** (a) $p = 0{\cdot}727h$; (b) $10{\cdot}5$ g/cm^2; (c) $36{\cdot}4$ cm.

67. $F = \dfrac{8}{45}\omega$.

68. (1) $y = \dfrac{2}{3}x + 4$; (2) $y = 1{\cdot}195x + 1{\cdot}910$; (3) $y = -0{\cdot}57x + 8{\cdot}63$.

69. (a) $V = 100 + 0{\cdot}35t$; (b) 100 cm³. **70.** $S = 16{\cdot}6 + 1{\cdot}34t$.

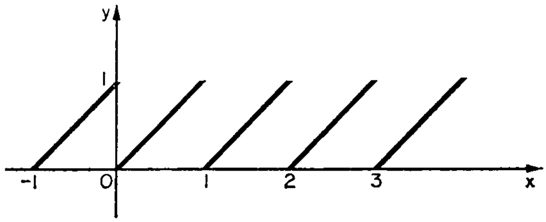

FIG. 80.

71. $V = 12 - 0{\cdot}7t$.

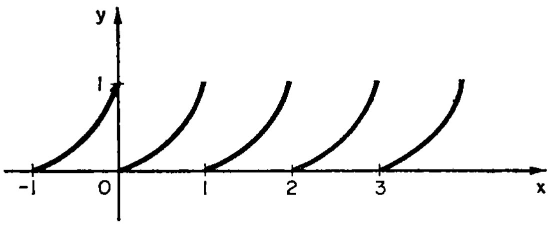

FIG. 81.

72. $\Delta y = 6$. **73.** $\Delta y = -6$. **74.** $\Delta x = 4$.

75. For the finite value of the argument $x_2 = 2a$.

76. $x = 3$; the solution is found graphically by seeking the point of intersection of the graph of $y = \varphi(x)$ and the straight line $y = 2x - 4$.

78*. It should be remarked that the sign of equality is excluded by hypothesis from the relationship $|f(x) + \varphi(x)| \leqq |f(x)| + |\varphi(x)|$ which always holds: $f(x)$ and $\varphi(x)$ must have opposite signs by hypothesis; by considering the two possible cases, we get $x < 3$ and $x > 4$. The problem can be solved by drawing the graphs of functions $\Phi(x) = |f(x) + \varphi(x)|$ and $\psi(x) = |f(x)| + |\varphi(x)|$.

79. $x < 2$. See the hint on the solution of problem 78*.

82. $$y = \begin{cases} 0 & \text{in the interval } (-\infty;\ -3), \\ -\frac{5}{9}x^2 + 5 & \text{in the interval } [-3;\ 3], \\ \frac{2}{3}x - 2 & \text{in the interval } [3;\ 6]. \end{cases}$$

83. (1) $y = -\frac{7}{8}$ for $x = \frac{1}{4}$; (2) $y = \frac{17}{4}$ for $x = -\frac{3}{2}$;

(3) $y = 5$ for $x = 0$; (4) $y = -\frac{7a^2}{8}$ for $x = \frac{a}{4}$; (5) $y = \frac{a^4}{4b^2}$ for $x = \frac{a^2}{2b^2}$.

84. (1) $y = -6$ for $x = -2$; (2) $y = 0{\cdot}31875$ for $x = \frac{3}{8}$; (3) $y = \frac{5}{8}$ for $x = \frac{1}{4}$; (4) $y = -a^4$ for $x = 0$; (5) $y = -\frac{9}{4}b^2$ for $x = \frac{b}{2a}$.

85. $a = \frac{a}{2} + \frac{a}{2}$. **86.** $a = \frac{a}{2} + \frac{a}{2}$. **87.** 4 m. **88.** 50 cm.

89. The one for which the axial section is square.

90. The smaller the height of the cone, the greater its lateral surface; the function has a maximum for a base radius equal to $\frac{P}{4}$, i.e. when the cone degenerates to a plane disc.

91. 12·5 cm.

92. The height of the rectangle must be equal to half the height of the triangle.

93. The radius of the cylinder must be equal to half the radius of the cone.

94. For $H > 2R$, the radius of the cylinder must be equal to $\frac{RH}{2(H-R)}$; for $H \leqq 2R$ the total surface of the inscribed cylinder will be the greater, the greater the radius of its base.

95. $\frac{P}{2}$. **96.** $a = \frac{P}{6 - \sqrt{3}}$. **97.** $\frac{4}{\pi + 4}$.

98. The side must be equal to 10 cm.

99. The side of the base and the lateral ribs must be 10 cm.

100. The side of the triangle must be equal to $\frac{3a}{9 + 4\sqrt{3}}$ cm.

101. The point is $\left(\frac{b}{6}, \frac{b}{6}\right)$.

102. The point is $\left(\frac{15}{11}, \frac{37}{11}\right)$.

104. (1) $x_1 \approx -1{\cdot}1$, $x_2 \approx 2{\cdot}1$; (2) $x_1 = -1$, $x_2 = \frac{5}{2}$; (3) $x_1 \approx 0{\cdot}5$, $x_2 \approx 4{\cdot}1$; (4) $x_1 = x_2 = \frac{3}{2}$; (5) there are no real roots.

105. $x_1 = -3$, $x_2 = 8$. To find the solution graphically we seek the point of intersection of the graph of $y = \varphi(x)$ and the parabola $y^2 = 7x + 25$.

106. If $b^2 - 4ac > 0$ and $a > 0$, the function is defined throughout the real axis except for the interval $x_1 \leqq x \leqq x_2$, where x_1 and x_2 are the roots of the trinomial. If $b^2 - 4ac > 0$ and $a < 0$, the function is defined only for $x_1 < x < x_2$. If $b^2 - 4ac < 0$ and $a > 0$, the function is defined throughout the real axis. If $b^2 - 4ac < 0$ and $a < 0$, the function is defined nowhere. Finally, if $b^2 - 4ac = 0$, the function will be defined throughout the real axis except for the single point $x = -\dfrac{b}{2a}$, if $a > 0$, and is nowhere defined if $a < 0$.

107. $f(x + 1) = 2x^2 + 5x + 3$.

108*. Let $\dfrac{x^2 + 2x + c}{x^2 + 4x + 3c} = m$, where m is an arbitrary real number; then $(m - 1)x^2 + 2(2m - 1)x + c(3m - 1) = 0$. The argument x must be a real number, consequently $(2m - 1)^2 - (m - 1) -(3mc - c) \geqq 0$ or $(4 - 3c)m^2 + 4(c - 1)m - (c - 1) \geqq 0$; but since m is real, this inequality is in turn only valid for

$$\begin{cases} 4 - 3c > 0, \\ 4(c - 1)^2 + (4 - 3c)(c - 1) \leqq 0; \end{cases}$$

hence $0 \leqq c \leqq 1$; but $c \neq 0$ by hypothesis, so that $0 < c \leqq 1$.

109. $pv = 1748$.

110. x is inversely proportional to v.

111. x is directly proportional to v.

112. The amount of material separated out is inversely proportional to the volume of the solvent.

114. (1) for $x = 1$, $y = 4$ is the maximum value;
for $x = 5$, $y = \dfrac{4}{5}$ is the minimum value:
(2) for $x = -1$, $y = \dfrac{1}{7}$ is the maximum value;
for $x = 2$, $y = -2$ is the minimum value;
(3) for $x = 0$, $y = 1$ is the maximum value;
for $x = 4$, $y = -\dfrac{3}{5}$ is the minimum value.

117. (1) $y = x$; (2) $y = \dfrac{x}{2}$; (3) $y = \dfrac{1 - x}{3}$; (4) $y = \pm\sqrt{x - 1}$;
(5) $y = \dfrac{1}{x}$; (6) $y = \dfrac{x - 1}{x}$; (7) $y = 1 \pm \sqrt{x + 1}$;

(8) $y = \pm\sqrt{x^3 - 1}$; (9) $y = \log\frac{x}{10}$; (10) $y = -2+10^{x-1}$;

(11) $y = 2^{\frac{1}{x}}$; (12) $y = \log_2\frac{x}{1-x}$; (13) $y = \frac{1}{2}\log\frac{x}{2-x}$;

(14) $y = \frac{1}{3}\arcsin\frac{x}{2}$;

(15) $y = \dfrac{1 + \arcsin\frac{x-1}{2}}{1 - \arcsin\frac{x-1}{2}}$; (16) $y = \pm\cos\frac{x}{4}$ $(0 \leqq x \leqq 2\pi)$.

122. $1 < x \leqq 3$; $y = 1 + 2^{1-x^2}$. **123.** $y = \arcsin\sqrt[3]{x - x^2 - 2}$.

125. $x_1 \approx -0{\cdot}5$, $x_2 = 1$, $x_3 \approx 54{\cdot}5$.

126*. (1) $x_1 \approx 1{\cdot}4$, the remaining roots are imaginary; x_1 is the abscissa of the point of intersection of the graphs of the cubical and linear functions $y = x^3$ and $y = -x + 4$.

(2) $x_1 = 1$, $x_2 = -1$, $x_3 = 3$; the substitution $x = x' + \alpha$ should be made, and α chosen so that the coefficient of x'^2 vanishes; further details as in (1); (3) $x_1 = 4$, $x_2 = x_3 = 1$; see the hint on (2); (4) $x_1 = -1$, the remaining roots are imaginary; see hint on (2).

127. (1) $1{\cdot}465\ldots$; (2) $\approx 14{\cdot}26$ cm; (3) almost $6{\cdot}8$ cm.

128. If $y_1 = x^n$, $y_2 = \sqrt[n]{x}$, then

for	$n > 1$	and $0 < x < 1$	$y_1 < y_2$, but for $1 < x < \infty$	$y_1 > y_2$,
for	$0 < n < 1$	and $0 < x < 1$	$y_1 > y_2$, but for $1 < x > \infty$	$y_1 < y_2$,
for	$-1 < n < 0$	and $0 < x < 1$	$y_1 < y_2$, but for $1 < x < \infty$	$y_1 > y_2$,
for	$n < -1$	and $0 < x < 1$	$y_1 > y_2$, but for $1 < x < \infty$	$y_1 < y_2$.

133. $x_1 = 1$, $x_2 = 2$.

134. The points of intersection are: (1, 2); (3, 8); $\left(3, \frac{4}{3}\right)$; $(-1{\cdot}5,\ 0{\cdot}3)$. **135.** $n = 15$.

136. It can be shown by starting from the definition of the hyperbolic functions that $\sinh(-x) = -\sinh x$, $\tanh(-x) = -\tanh x$, $\cosh(-x) = \cosh x$. These functions are not periodic.

140. $y_{\min} \approx 0{\cdot}8$ for $x \approx 0{\cdot}4$. **141.** $y = \dfrac{a^x - a^{-x}}{2}$.

143. (1) $A = 1$, $T = \frac{2}{3}\pi$; (2) $A = 5$, $T = \pi$; (3) $A = 4$, $T = 2$; (4) $A = 2, T = 4\pi$; (5) $A = 1, T = \frac{8}{3}$; (6) $A = 3, T = \frac{16}{5}\pi$.

144. (1) 2; $\frac{2\pi}{3}$; $\frac{3}{2\pi}$; 5; (2) 1; 4π; $\frac{1}{4\pi}$; $\frac{3\pi-1}{2}$; (3) $\frac{1}{3}$; 1; 1; $-\frac{\pi}{3}$; (4) 1; $6\pi^2$; $\frac{1}{6\pi^2}$; $\frac{1}{2\pi}$.

146. The domain of definition is $(0, \pi)$. The area is a minimum for $x = \frac{\pi}{2}$.

147. $x = R \sin \left(\frac{vt}{R} + \frac{\pi}{2} + \arccos \frac{a}{R}\right)$.

148. $y = \sin \left[\frac{t-t_0}{t_1-t_0} (\arcsin y_1 - \arcsin y_0) + \arcsin y_0\right]$;

$$T = \frac{2\pi(t_1-t_0)}{\arcsin y_1 - \arcsin y_0}; \quad \varphi_{\text{init}} = \frac{t_1 \arcsin y_0 - t_0 \arcsin y_1}{t_1 - t_0}.$$

149. $x = R(1 - \cos \varphi) + a - \sqrt{a^2 - R^2 \sin^2 \varphi}$, where $\varphi = 2\pi n t$.

151. (1) $x_1 = 0$, $x_{2,3} \approx \pm 1{\cdot}9$; (2) $x = 0$; $\pm 4{\cdot}5$; $\pm 7{\cdot}72$; further, we can take fairly accurately $x \approx \pm \frac{(2n+1)\pi}{2}$ $(n > 3)$; (3) $x \approx 0{\cdot}74$; (4) $x_1 = 0{\cdot}9$, $x_2 = 2{\cdot}85$, $x_3 = 5{\cdot}8$; (5) there is an infinite set of roots; $x_1 = 0$, x_2 slightly under $\frac{\pi}{2}$, x_3 slightly over $\frac{3\pi}{2}$, and so on.

152. (1) 2π; (2) 2π; (3) 24; (4) 2. **153.** (1) $y = \sqrt{2} \sin \left(x + \frac{\pi}{4}\right)$; (2) $y = \sqrt{5 + 2\sqrt{3}} \sin (x + \varphi_0)$, where $\varphi_0 = \arcsin \frac{1}{\sqrt{5 + 2\sqrt{3}}}$.

155*. (1) Period $\frac{\pi}{2}$. The function can be written in the interval $[0, 2\pi]$ as;

$$y = \sin x + \cos x \text{ in the interval } \left[0, \frac{\pi}{2}\right],$$

$$y = \sin x - \cos x \text{ in the interval } \left[\frac{\pi}{2}, \pi\right],$$

$$y = -\sin x - \cos x \text{ in the interval } \left[\pi, \frac{3\pi}{2}\right],$$

$$y = -\sin x + \cos x \text{ in the interval } \left[\frac{3\pi}{2}, 2\pi\right].$$

(2) Period 2π. The function can be written in the interval $[0, 2\pi]$ as:

$$y = \tan x \quad \text{in the interval } \left[0, \frac{\pi}{2}\right),$$

$$y = 0 \quad \text{in the interval } \left(\frac{\pi}{2}, \pi\right],$$

$$y = -\tan x \quad \text{in the interval } \left[\pi, \frac{3}{2}\pi\right),$$

$$y = 0 \quad \text{in the interval } \left(\frac{3}{2}\pi, 2\pi\right].$$

156. (1) The domain of definition consists of the infinite set of intervals of the form $(2n\pi, (2n+1)\pi)$, where $n = 0, \pm 1, \pm 2 \ldots$; neither even nor odd; periodic, of period 2π. In the interval $\left(0, \frac{\pi}{2}\right)$, the sine increases from 0 to 1, so that $\log \sin x$ increases to 0 whilst remaining negative. The sine decreases from 1 to 0 in the interval $\left(\frac{\pi}{2}, \pi\right)$, so that $\log \sin x$ also decreases. The sine has negative values in the interval $(\pi, 2\pi)$, so that $\log \sin x$ is undefined. (2) The domain of definition consists of the individual points of the form $x = \frac{\pi}{2} + 2\pi n$, where $n = 0, \pm 1, \pm 2, \ldots$ At these points $y = 0$. The graph consists of individual points of the axis of abscissae. (3) The function is defined throughout the real axis, except for the points $x = \pi n$, where $n = 0, \pm 1, \pm 2, \ldots$

158. $\omega = 2 \arcsin \dfrac{\alpha}{2\pi}$.

159. $\gamma = \arctan \dfrac{a(l \cos \varphi + b \sin \varphi)}{b^2 + l^2 + a(b \cos \varphi - l \sin \varphi)}$.

160. $\alpha = \arccos \left[1 - \dfrac{x(2a - x)}{2R(a + R - x)}\right]$.

161. (1) $-1 \leqq x \leqq 1$;
(2) $0 \leqq x \leqq 1$; (3) $0 \leqq x \leqq 1$; (4) $-1 \leqq x \leqq 0$;
(5) $0 < x < \infty$; (6) $-\infty < x < 0$; (7) $0 \leqq x < \infty$;
(8) $-\infty < x \leqq 0$; (9) $-\infty < x < 1$; (10) $1 < x < \infty$.

162. (1) $-1 \leqq x \leqq 1$; (2) $0 \leqq x \leqq 1$; (3) $-\infty < x < \infty$; (4) defined everywhere except for $x = 0$.

163*. Period 2π. For the graph see Fig. 82.

Hint. In the interval $-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}$, $y = \arcsin(\sin x) \equiv x$ by definition of the function $\arcsin x$. To obtain the graph of the function

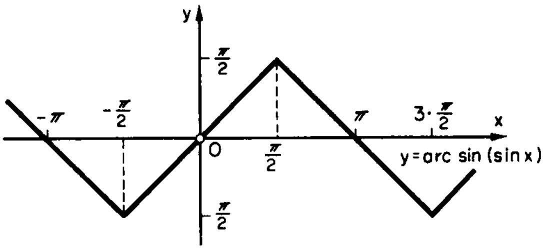

FIG. 82.

in the interval $\frac{\pi}{2} \leqq x \leqq 3\frac{\pi}{2}$, we put $z = x - \pi$, so that $x = \pi + z$, $-\frac{\pi}{2} \leqq z \leqq \frac{\pi}{2}$,

$$y = \text{arc sin}\,(\sin x) = \text{arc sin sin}\,(z + \pi) = -\text{arc sin}\,(\sin z) = -z;$$
$$y = \pi - x \text{ and so on.}$$

167. $y_{\max} \approx 15$, $y_{\min} \approx 5{\cdot}5$; the function passes from increase to decrease at $x = -2$. The zero of the function is at $x \approx -3{\cdot}6$.

169. $y = \frac{1}{32}(267 - 10x - x^2)$ or $y = -0{\cdot}0312x^2 - 0{\cdot}3125x + 8{\cdot}344$; zeros of the function: $x_1 \approx -22{\cdot}09$, $x_2 \approx 12{\cdot}09$. To obtain the roots to an accuracy of 0·01, the coefficients must be taken to an accuracy of 0·0001.

170. $x_1 \approx 2{\cdot}60$ cm, $x_2 \approx 7{\cdot}87$ cm.

171. $x_1 \approx -2{\cdot}3$, $x_2 \approx 3$; the remaining roots are imaginary.

172*. Choose α so that the coefficient of x'^3 vanishes; $x_1 \approx -3{\cdot}6$, $x_2 \approx -2{\cdot}9$, $x_3 \approx 0{\cdot}6$, $x_4 \approx 4{\cdot}8$.

173. $x_1 \approx 0{\cdot}59$, $x_2 \approx 3{\cdot}10$, $x_3 \approx 6{\cdot}29$, $x_4 \approx 9{\cdot}43$; in general $x \approx \pi n$ $(n > 2)$.

174. $x_1 \approx -0{\cdot}57$, $y_1 \approx -1{\cdot}26$; $x_2 \approx -0{\cdot}42$, $y_2 \approx 1{\cdot}19$; $x_3 \approx 0{\cdot}46$, $y_3 \approx 0{\cdot}74$; $x_4 \approx 0{\cdot}54$, $y_4 \approx -0{\cdot}68$.

Chapter II

176. $\lim\limits_{n\to\infty} u_n = 1$, $n \geqq 4$. **177.** $\lim\limits_{n\to\infty} u_n = 0$; $n > \frac{1}{\sqrt{\varepsilon}}$. **178.** $n = 19{,}999$.

179. $\lim\limits_{n\to\infty} v_n = 0$; $n \geqq 1000$. v_n is sometimes greater than, sometimes less than, and sometimes equal to its limit (the last when $n = 2k + 1$, where $k = 0, 1, 2, \ldots$).

180. $\lim\limits_{n\to\infty} u_n = 1$; $n \geqq 14$; $n \geqq \log_2 \dfrac{1}{\varepsilon}$.

181. $n \geqq \dfrac{1}{3}\sqrt{\dfrac{5-6\varepsilon}{\varepsilon}}$, if $\varepsilon \leqq \dfrac{5}{6}$; $n = 0$, if $\varepsilon > \dfrac{5}{6}$.

182. $n \geqq \dfrac{a}{\sqrt{\varepsilon(2+\varepsilon)}}$; the sequence is decreasing.

183. $\lim\limits_{n\to\infty} v_n = 0$; v_n reaches its limit with $n = m + 1$, since, as from this value, $v_n = 0$.

185. 0. **186.** (1) No. (2) Yes.

189. With $a = 0$ this limit can equal any number or be nonexistent.

190. $\delta < \sqrt{4+\varepsilon} - 2$; $\delta < 0{\cdot}00025$. **191.** $\delta < 2 - \sqrt{3}$.

192. $\delta < \dfrac{2}{13}$.

193. $\left|x - \dfrac{\pi}{2}\right| < \dfrac{\pi}{2} - \arcsin 0{\cdot}99 \approx 0{\cdot}136$.

194. $N \geqq \sqrt{\dfrac{1}{\varepsilon} - 1}$, if $\varepsilon \leqq 1$; $N = 0$, if $\varepsilon > 1$.

195. $N \geqq \sqrt{\dfrac{4}{\varepsilon} - 3}$, if $\varepsilon \leqq \dfrac{4}{3}$; $N = 0$, if $\varepsilon > \dfrac{4}{3}$.

196. $n > \dfrac{N-1}{2}$.

197. u_n is a positive large order magnitude if the difference of the progression $d > 0$, and is negative if $d < 0$. The statement holds for a geometric progression only when the denominator of the progression has an absolute value greater than 1.

198. $-\dfrac{1}{10^4+2} < x < \dfrac{1}{10^4-2}$. **199.** $\dfrac{3000}{1001} < x < \dfrac{3000}{999}$.

200. $\delta < \dfrac{1}{\sqrt{N}} = 0{\cdot}01$. **201.** $\log_2 0{\cdot}99 < x < \log_2 1{\cdot}01$.

202. $M \geqq 10^N = 10^{100}$.

203. $\sin x$, $\cos x$ and all the inverse trigonometric functions.

205. No; yes. **206.** No.

207. For example, $x_n = \dfrac{\pi}{2} + 2n\pi$, $x_n = 2\pi n$; no.

209. If $a > 1$, the function is unbounded (but not infinitely large) as $x \to +\infty$; it tends to zero as $x \to -\infty$. If $0 < a < 1$, the function is unbounded as $x \to -\infty$ (but is not infinitely large); it tends to zero as $x \to +\infty$. With $a = 1$ the function is bounded throughout the real axis.

210. (1), (3) and (5) — No; (2) and (4) — Yes.

213. $\dfrac{-1}{10{,}001} < x < \dfrac{2}{9999}$.

214. $N \geqq \left(\dfrac{1-\varepsilon^2}{2\varepsilon}\right)^2$.

215. (1) $y = 1 + \dfrac{1}{x^3 - 1}$; (2) $y = \dfrac{1}{2} + \dfrac{-1}{2(2x^2+1)}$;

(3) $y = -1 + \dfrac{2}{1+x^2}$.

216*. Compare u_n with the sum of the terms of the geometric progression $\dfrac{1}{3}, \dfrac{1}{9}, \dfrac{1}{27}, \ldots, \dfrac{1}{3^n}$.

220. 3. **221.** Yes.

222. $f(x) = 9\pi$ for $0 \leqq x \leqq 5$; $f(x) = 4\pi$ for $5 < x \leqq 10$; $f(x) = \pi$ for $10 < x \leqq 15$. The function is discontinuous at $x = 5$ and $x = 10$.

223. $a = 1$. **224.** $A = -1$, $B = 1$. **225.** $x = 2$; $x = -2$. **226.** $\dfrac{2}{3}$.

227. The function $y = \dfrac{\sin x}{x}$ has a removable discontinuity at the point $x = 0$, $y = \dfrac{\cos x}{x}$ has a discontinuity of the second kind (infinite).

228. The function is discontinuous for $x = 0$.

229. The function has three points of discontinuity. At $x = 0$ the discontinuity is removable, whilst it is of the second kind (infinite) at $x = \pm 1$.

230. No. As $x \to 0$ from the right, $f(x) \to \dfrac{\pi}{2}$, whilst as $x \to 0$ from the left, $f(x) \to -\dfrac{\pi}{2}$.

231. The function is discontinuous at $x = 0$.

232. 0.

234. No. As $x \to 1$ from the right, $y \to 1$, whilst as $x \to 1$ from the left, $y \to 0$.

235. As $x \to 0$ from the right, $y \to 1$, whilst as $x \to 0$ from the left, $y \to -1$.

236. The function is discontinuous at $x = 0$ (discontinuity of the first kind).

237. The function has discontinuities of the first kind at the points $x = \frac{\pi}{2}(2k + 1)$.

238. The function is continuous at $x = 0$, and is discontinuous for $x \neq 0$.

239. All three functions are discontinuous when x is equal to an integer (positive or negative) or zero.

241*. Write the polynomial in the form $x^n \left(a_0 + \frac{a_1}{x} + \ldots + \frac{a_n}{x^n}\right)$ and investigate its behaviour as $x \to \pm\infty$.

244*. Draw schematically the graph of the function $y = \frac{a_1}{x - \lambda_1} + \frac{a_2}{x - \lambda_2} + \frac{a_3}{x - \lambda_3}$, and investigate its behaviour in the neighbourhood of points λ_1, λ_2 and λ_3.

245. 1. **246.** $\frac{1}{2}$. **247.** 3. **248.** ∞. **249.** 0. **250.** 0. **251.** $\frac{15}{17}$. **252.** 1. **253.** 0. **254.** 4. **255.** 1. **256.** 0. **257.** 0. **258.** 0. **259.** 1. **260.** $\frac{4}{3}$. **261.** $\frac{1}{2}$. **262.** $-\frac{1}{2}$. **263.** -1. **264*.** 1. Notice that $\frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n}$. **265.** $\frac{1}{2}$. **266.** 1. **267.** 0. **268.** 9. **269.** $\frac{3}{4}$. **270.** ∞. **271.** 0. **272.** 0. **273.** $-\frac{2}{5}$. **274.** $\frac{1}{2}$. **275.** 6. **276.** ∞. **277.** -1. **278.** ∞. **279.** 0. **280.** $\frac{m}{n}$. **281.** 0. **282.** ∞. **283.** $\frac{1}{2}$. **284.** -1. **285.** 0. **286.** $\frac{1}{4}$. **287.** $-\frac{1}{2}$. **288.** 100. **289.** -1. **290.** 1. **291.** ∞. **292.** 0. **293.** 0. **294.** ∞. **295.** 4. **296.** $\frac{1}{4}$. **297.** 3. **298.** $\frac{1}{2\sqrt{x}}$ if $x > 0$; ∞ if $x = 0$. **299.** $\frac{1}{3}$. **300.** $\frac{2}{3}$. **301.** $\frac{1}{4a\sqrt{a-b}}$. **302.** $\frac{m}{n}$. **303*.** $\frac{1}{2}$. Add and subtract unity from the numerator. **304.** $-\frac{1}{4}$. **305.** One root rends to $-\frac{c}{b}$, the other to ∞. **306.** 0. **307.** 0. **308.** 0 if $x \to +\infty$; ∞ if $x \to -\infty$. **309.** $\frac{1}{2}$ if $x \to +\infty$; $-\infty$ if $x \to -\infty$. **310.** $\frac{a+b}{2}$ if $x \to +\infty$; ∞ if $x \to -\infty$. **311.** $\pm\frac{5}{2}$. **312.** 0. **313.** 1. **314.** 3. **315.** k. **316.** $\frac{\alpha}{\beta}$. **317.** $\frac{2}{5}$.

318. 0 if $n > m$; 1 if $n = m$; ∞ if $n < m$; **319.** $\frac{2}{3}$.

320. $\frac{1}{3}$. **321.** $\frac{1}{2}$. **322.** $\frac{3}{4}$. **323.** ∞. **324.** -1. **325.** $\frac{1}{2}$. **326.** ∞.

327. 0. **328.** $\frac{1}{2}$. **329.** ∞. **330.** $-\frac{3}{2}$. **331.** 1. **332.** $\frac{\pi}{2}$. **333.** $\frac{2}{\pi}$.

334. $-\frac{a}{\pi}$. **335.** $\frac{\sqrt{2}}{2}$. **336.** 2. **337.** $\frac{\sqrt{2}}{2}$. **338.** -2. **339.** $-2\sin a$.

340. $\frac{\beta^2-\alpha^2}{2}$. **341.** $\cos^3\alpha$. **342.** $\frac{\sin 2\beta}{2\beta}$. **343.** $-\sin a$.

344. $\frac{2\sin a}{\cos^3 a}$. **345.** $\frac{\sqrt{2}}{8}$. **346.** 1. **347.** 6. **348.** $\frac{3}{2}$. **349.** -1.

350. $\frac{1}{\sqrt{2\pi}}$. Put $\arccos x = y$. **351.** $\frac{1}{e}$. **352.** $\frac{1}{e}$. **353.** 1. **354.** e^{mk}.

355. e^6. **356.** $e^{-\frac{2}{3}}$ **357.** e^2. **358.** 0 if $x \to +\infty$; ∞ if $x \to -\infty$.

359. ∞ if $x \to +\infty$; 0 if $x \to -\infty$. **360.** 1.

361. ∞, if $x \to +\infty$; 0, if $x \to -\infty$. **362.** e^2. **363.** e. **364.** $\sqrt{e}$.

365. k. **366.** $\frac{1}{a}$. **367.** a. **368.** $\frac{1}{e}$. **369.** $\ln a$. **370.** $\frac{2}{3}$. **371.** e.

372. $\frac{3}{2}$; add and subtract unity from the numerator.

373. 2. **374.** 1. **375.** $a - b$. **376.** 1.

377. 0 if $x \to +\infty$; ∞ if $x \to -\infty$.

378. 1 if $x \to +\infty$; -1 if $x \to -\infty$.

379. (1) a^n; (2) 0 if $A \neq 0$, a^n if $A = 0$ and $a \neq 0$, and ∞ if $A = a = 0$; (3) $\frac{1}{1+A}$.

380. 0 if $x \to +\infty$; $-\infty$ if $x \to -\infty$.

381. With $a > 1$, the limit is equal to 1 if $x \to +\infty$, and 0 if $x \to -\infty$. With $a < 1$, the limit is equal to 0 if $x \to +\infty$, and 1 if $x \to -\infty$. With $a = 1$ the limit is equal to $\frac{1}{2}$.

382. With $a > 1$ the limit is equal to 1 if $x \to +\infty$, and -1 if $x \to -\infty$. With $a < 1$, vice versa. With $a = 1$ the limit is 0.

383. 0. **384.** 0. **385.** 1. **386.** 0. **387.** $-\cos a$. **388.** $\frac{1}{12}$. **389.** $\frac{1}{8}$.

390*. $\frac{\sin x}{x}$. Multiply and divide by $\sin\frac{x}{2^n}$.

391. $\frac{1}{2}$. **392.** 0. **393*.** $-\frac{1}{2}$. Use the formula $\operatorname{arc\,tan} b - \operatorname{arc\,tan} a = \operatorname{arc\,tan}\frac{b-a}{1+ab}$. **394.** $\frac{1}{2}$. **395*.** $\frac{1}{2}$. Replace $\operatorname{arc\,sin} x$ by $\operatorname{arc\,tan}\frac{x}{\sqrt{1-x^2}}$ and use the hint on problem 393.

396. ∞ if $n < 1$; e if $n = 1$; 1 if $n > 1$.

397*. 1. Take the expression $1 - (1 - \cos x)$ instead of $\cos x$.

398. $-\frac{1}{2}$. **399.** $\frac{1}{e}$. **400.** e. **401.** e^{ab}.

402. v_n is of the higher order of smallness.

403. u_n and v_n are equivalent infinitesimals.

405. Of the same order.

406. The order of smallness is different at $x = 0$. Δy and Δx are equivalent for $x = \pm\frac{\sqrt{3}}{3}$. **407.** No. **408.** Of the third order.

409. (1) 2; (2) $\frac{1}{2}$; (3) 1; (4) 10. **410.** $x = \frac{1}{2}\sqrt[3]{\frac{a^2}{2b^2}}$.

411. $a = k$. **412.** No. **414.** (1) $\frac{1}{3}$; (2) $\frac{1}{2}$; (3) $\frac{1}{2}$; (4) an equivalent infinitesimal; (5) an equivalent infinitesimal; (6) 1; (7) an equivalent infinitesimal; (8) 2; (9) 2; (10) 1; (11) $\frac{2}{3}$; (12) 2.

415. $a^2\sqrt{3}$. **416.** $2\pi R^2$; $4R^2$.

418. It does not follow from the fact that the step line tends to merge with the straight line (in the sense of their points approximating) that the length of the step line tends to the length of the segment.

419. a. **420.** a, $\frac{\pi a}{2}$. **421.** $2\pi(R + r)$.

422. Both the segment and the angle are of order $\frac{1}{2}$.

425. (1) 10·25; (2) 30·2; (3) 16·125; (4) 40·4; (5) 0·558; (6) 0·145.

426. (1) 10·16; (2) 20·12; (3) 1·02; (4) 4·04.

427. ln 1·01 $\approx$ 0·01, ln 1·02 $\approx$ 0·02, ln 1·1 $\approx$ 0·1, ln 1·2 $\approx$ 0·2.

Chapter III

428. (a) 5; (b) 5. **429.** (a) $v = 15\ \frac{\text{m}}{\text{min}}$; (b) $v = 33\ \frac{\text{m}}{\text{min}}$; (c) $3\,(t_1 + t_2)\ \frac{\text{m}}{\text{min}}$. **430.** 75·88; 60·85; 49·03; 48·05.

431. $53{\cdot}9\ \frac{\text{m}}{\text{sec}}$; $49{\cdot}49\ \frac{\text{m}}{\text{sec}}$; $49{\cdot}25\ \frac{\text{m}}{\text{sec}}$; $49{\cdot}005\ \frac{\text{m}}{\text{sec}}$; $v_5 = 49{\cdot}0\ \frac{\text{m}}{\text{sec}}$; $v_{10} = 98{\cdot}0\ \frac{\text{m}}{\text{sec}}$; $v = 9{\cdot}8t\ \frac{\text{m}}{\text{sec}}$.

432. (a) $4\ \frac{\text{g}}{\text{cm}}$; (b) $40\ \frac{\text{g}}{\text{cm}}$; (c) $4l\ \frac{\text{g}}{\text{cm}}$, where l is the length of segment AM.

433. (1) $95\ \frac{\text{g}}{\text{cm}}$; (2) (a) $35\ \frac{\text{g}}{\text{cm}}$; (b) $5\ \frac{\text{g}}{\text{cm}}$; (c) $185\ \frac{\text{g}}{\text{cm}}$.

434. (1) 1·00201; (2) 1·013.

435*. Introduce the mean angular velocity, then obtain the required quantity by passage to the limit. (See *Course*, sec. 50.)

438. $k = \frac{f'(t)}{f(t)}$, where k is the coefficient of linear expansion.

439. $K = S\,\frac{\varphi'(P)}{\varphi(P)}$. **440.** (1) 56; (2) 19; (3) 7·625 (4) 1·261.

441. (1) 4·52; (2) −0·249; (3) 0·245. **442.** (a) 6·5; (b) 6·1; (c) 6·01; (d) 6·001.

443. $f'(5) = 10$; $f'(-2) = -4$; $f'\left(-\frac{3}{2}\right) = -3$.

444. 3; 0; 6; $\frac{1}{3}$. **445.** $x_1 = 0$, $x_2 = 2$.

446. Does not hold for $f(x) = x^3$. **447.** 1. **448.** 0·4343. **449.** 2·303.

450. The limit is equal to $f'(0)$.

453. (1) $5x^4$; (2) $10x^9$; (3) $\frac{3}{7}x^{-\frac{4}{7}}$; (4) $\frac{2}{3\sqrt[3]{x}}$; (5) $\frac{1}{2\sqrt{x}}$; (6) $-\frac{3}{x^4}$;

(7) $-\frac{1}{x^2}$; (8) $-\frac{3}{5x\sqrt[5]{x^3}}$; (9) $\frac{5x^{\frac{1}{4}}}{4}$; (10) $3{\cdot}5x^4$; (11) x^{11}; (12) $-\frac{7a}{x^8}$;

(13) $\frac{1}{n\sqrt[n]{x^{n-1}}}$; (14) $-\frac{p}{x^2}$; (15) $-\frac{2}{3}ax^{-\frac{5}{3}}$.

454. (1) 0; (2) 6; (3 -4; (4) $k_1 = 2$, $k_2 = 4$.

455. (1, 1); $(-1, -1)$. **456.** (1) (0, 0); (2) $\left(\frac{1}{2}, \frac{1}{4}\right)$.

457. It cannot. **458.** $\alpha_1 = \arctan\frac{1}{7}$, $\alpha_2 = \arctan\frac{1}{13}$.

459. $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = \arctan\frac{3}{4}$. **460.** arc tan 3.

461. $y = 12x - 16$; $x + 12y - 98 = 0$; the subtangent is equal to $\frac{2}{3}$, the subnormal to 96.

462. For $x = 0$ and for $x = \frac{2}{3}$.

463. (1) (2, 4); (2) $\left(-\frac{3}{2}, \frac{9}{4}\right)$; (3) $(-1, 1)$ and $\left(\frac{1}{4}, \frac{1}{16}\right)$.

466. (1) $6x - 5$; (2) $4x^3 - x^2 + 5x - 0{\cdot}3$; (3) $2ax + b$; (4) $\frac{1}{3\sqrt[3]{x^2}}$.

(5) $\frac{1}{\sqrt{x}} + \frac{1}{x^2}$; (6) $\frac{0{\cdot}2}{\sqrt[4]{y^3}} - 10y^2 - \frac{0{\cdot}4}{y^3}$; (7) $\frac{1}{n} - \frac{n}{x^2} + \frac{2x}{m^2} - \frac{2m^2}{x^3}$;

(8) $\frac{3}{2}m\sqrt{x} + \frac{7}{6}n\sqrt[6]{x} + \frac{1}{2}p\frac{1}{\sqrt{x^3}}$; (9) $\frac{2mz + n}{p + q}$;

(10) $-\frac{1}{15}t^{-\frac{5}{3}} + 7{\cdot}28t^{-2{\cdot}4} - \frac{0{\cdot}5}{t\sqrt[5]{t}}$; (11) $2x - 1$;

(12) $3{\cdot}5x^2\sqrt{x} - 1 + \frac{1}{2\sqrt{x}}$; (13) $3v^2 + 2v - 1$; (14) $6(a - x)$;

(15) $\frac{2ax}{a+b} + \frac{b}{a+b} - \frac{c}{(a+b)x^2}$; (16) $\frac{3m(mu+n)^2}{p^3}$.

467. $f(1) = 1$; $f'(1) = 2$; $f(4) = 8$; $f'(4) = 2{\cdot}5$; $f(a^2) = 3a^2 - 2|a|$; $f'(a^2) = 3 - \frac{1}{|a|}$.

468. $f(-1) = -5$; $f'(-1) = -8$; $f'(2) = \frac{19}{16}$;
$f'\left(\frac{1}{a}\right) = 3a^4 + 10a^3 - a^2$.

469. 13. 471. (1) $4x^3 - 3x^2 - 8x + 9$; (2) $7x^6 - 10x^4 + 8x^3 - 12x^2 + 4x + 3$; (3) $-\frac{1}{2\sqrt{x}}\left(1 + \frac{1}{x}\right)$;

(4) $\frac{1}{9}\left(\frac{60}{\sqrt[6]{x}} - \frac{5}{x\sqrt[6]{x^5}} + \frac{\sqrt{3}}{x\sqrt[3]{x}} - 48\sqrt[6]{27x^2}\right)$;

(5) $\frac{1+12x}{3\sqrt[3]{x^2}} + \frac{9\sqrt[3]{x^2} + 10x\sqrt[3]{x} + 36x\sqrt[3]{x^2}}{3\sqrt[3]{x^2}}$; (6) $2x(3x^4 - 28x^2 + 49)$;

(7) $\frac{1 + \sqrt{2} + \sqrt{3} + 2\sqrt{2x} + 2\sqrt{3x} + 2\sqrt{6x} + 3x\sqrt{6}}{2\sqrt{x}}$.

472. $-\frac{2}{(x-1)^2}$. 473. $\frac{1-x^2}{(1+x^2)^2}$. 474. $\frac{3t^2 - 6t - 1}{(t-1)^2}$.

475. $\frac{v^4 + 2v^3 + 5v^2 - 2}{(v^2 + v - 1)^2}$. 476. $\frac{ad - bc}{(cx+d)^2}$.

477. $-\frac{4x}{3(x^2-1)^2} + 1 + 2x - 3x^2$. 478. $\frac{2v^4(v^3 - 5)}{(v^3 - 2)^2}$.

479. $-\frac{6x^2}{(x^3+1)^2}$. 480. $-\frac{6x^2}{(x^2-1)^2}$. 481. $\frac{2v-1}{a^2-3}$.

482. $-\frac{3x^2}{\sqrt{\pi}}$. 483. $-\frac{2t+1}{(t^2+t+1)^2}$.

484. $\frac{3-2t}{(t^2-3t+6)^2}$. 485. $\frac{4x^3(2b^2 - x^2)}{(b^2-x^2)^2}$.

486. $\frac{1 + 2x + 3x^2 - 2x^3 - x^4}{(1+x^3)^2}$.

487. $\frac{6x(1 + 3x - 5x^3)}{(1-x^2)^2(1-2x^3)^2}$. 488. $\frac{a+2bx}{m(a+bm)}$.

489. $-\frac{a^2b^2c^2\left[(x-b)(x-c) + (x-c)(x-a) + (x-a)(x-b)\right]}{(x-a)^2(x-b)^2(x-c)^2}$.

490. $f'(0) = 0$; $f'(1) = 6$. 491. $F'(0) = 11$; $F'(1) = 2$; $F'(2) = -1$.

492. $F'(0) = -\frac{1}{4}$; $F'(-1) = \frac{1}{2}$. 493. $s'(0) = \frac{3}{25}$; $s'(2) = \frac{17}{15}$.

494. $y'(1) = 16$; $y'(a) = 15a^2 + \frac{2}{a^3} - 1$. **495.** $\varrho'(2) = \frac{5}{9}$; $\varrho'(0) = 1$.

496. $\varphi'(1) = -\frac{a+1}{4}$. **497.** $z'(0) = 1$.

498. (1) $4x^3 - 3x^2(a+b+c+d) + 2x(ab+ac+ad+bc+ + bd+cd) - (abc+abd+acd+bcd)$; (2) $8x(x^2+1)^3$;
(3) $-20(1-x)^{19}$; (4) $60(1+2x)^{29}$; (5) $-20x(1-x^2)^9$;
(6) $5(15x^2+2x)(5x^3+x^2+4)^4$; (7) $6(3x^2-1)(x^3-x)^5$;
(8) $6\left(14x+\frac{4}{x^2}\right)\left(7x^2-\frac{4}{x}+6\right)^5$; (9) $4\left(3t^2+\frac{3}{t^4}\right)\left(t^3-\frac{1}{t^3}+3\right)^3$;
(10) $-\frac{4(x+1)}{(x-1)^3}$; (11) $\frac{5(x^2+2x-1)(1+x^2)^4}{(1+x)^6}$;
(12) $24(x^2+x+1)(2x^3+3x^2+6x+1)^3$.

499. $\frac{(s+2)(s+4)}{(s+3)^2}$. **500.** $\frac{(3-t)t^2}{(1-t)^3}$.

501. $\frac{1-\sqrt{2}}{2\sqrt{x}(1+\sqrt{2x})^2}$. **502.** $-\frac{4}{3\sqrt[3]{4x^2}(1+\sqrt[3]{2x})^2}$.

503. $-\frac{x}{\sqrt{1-x^2}}$. **504.** $-\frac{4(1-2\sqrt{x})^3}{\sqrt{x}}$.

505. $\frac{mv^{m-1}}{(1-v)^{m+1}}$. **506.** $-\frac{4(2x-1)}{(x^2-x+1)^3}$.

507. $\frac{x}{\sqrt{(a^2-x^2)^3}}$. **508.** $-\frac{2x}{3\sqrt[3]{(1+x^2)^4}}$.

509. $\frac{2x^3+4x^7}{\sqrt{(1-x^4-x^8)^3}}$. **510.** $\frac{3-x}{2\sqrt{(1-x)^3}}$.

511. $\frac{x(x^2+2a^2)}{\sqrt{(x^2+a^2)^3}}$. **512.** $-\frac{v+\sqrt{a^2+v^2}}{a^2\sqrt{a^2+v^2}}$.

513. $-\frac{2}{3\sqrt[3]{(2x-1)^4}} - \frac{15x}{2\sqrt[4]{(x^2+2)^7}}$. **514.** $u'(1) = 9$.

515. $y'(2) = -\frac{\sqrt{3}}{3}$. **517.** $\cos x - \sin x$.

518. $\frac{1-\cos x - x\sin x}{(1-\cos x)^2}$. **519.** $\frac{x - \sin x\cos x}{x^2\cos^2 x}$.

520. $\varphi\cos\varphi$. **521.** $(\alpha\cos\alpha - \sin\alpha)\left(\frac{1}{\alpha^2} - \frac{1}{\sin^2\alpha}\right)$.

522. $\dfrac{1}{1+\cos t}$. **523.** $\dfrac{\sin x+\cos x+x(\sin x-\cos x)}{1+\sin 2x}$.

524. $\dfrac{(1+\tan x)(\sin x+x\cos x)-x\sin x\sec^2 x}{(1+\tan x)^2}$.

525. $-\sin 2x$. **526.** $\tan^3 x\sec^2 x$. **527.** $-\sin^3 x$.

528. $\dfrac{3}{2}\sin 2x(2-\sin x)$. **529.** $\tan^4 x$. **530.** $2x\dfrac{\sin x}{\cos^3 x}$.

531. $-\dfrac{16\cos 2x}{\sin^3 2x}$. **532.** $3\cos 3x$. **533.** $-\dfrac{a}{3}\sin\dfrac{x}{3}$.

534. $9\cos(3x+5)$. **535.** $\dfrac{1}{2\cos^2\dfrac{x+1}{2}}$. **536.** $\dfrac{1}{\sqrt{1+2\tan x}\cos^2 x}$.

537. $-\dfrac{\cos\dfrac{1}{x}}{x^2}$. **538.** $\cos(\sin x)\cos x$. **539.** $-12\cos^2 4x\sin 4x$.

540. $\dfrac{1}{4\sqrt{\tan\dfrac{x}{2}}\cos^2\dfrac{x}{2}}$. **541.** $\dfrac{x\cos\sqrt{1+x^2}}{\sqrt{1+x^2}}$.

542. $-\dfrac{2x}{3\sin^2\sqrt[3]{1+x^2}\sqrt[3]{(1+x^2)^2}}$. **543.** $4(1+\sin^2 x)^3\sin 2x$.

544. $\dfrac{x^2-1}{2x^2\cos^2\left(x+\dfrac{1}{x}\right)\sqrt{1+\tan\left(x+\dfrac{1}{x}\right)}}$.

545. $\dfrac{\sin\left(2\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)}{\sqrt{x}(1+\sqrt{x})^2}$. **546.** $-3\sin 3x\sin(2\cos 3x)$.

548. $\arcsin x+\dfrac{x}{\sqrt{1-x^2}}$. **549.** $\dfrac{\pi}{2(\arccos x)^2\sqrt{1-x^2}}$.

550. $\dfrac{2\arcsin x}{\sqrt{1-x^2}}$. **551.** $\arcsin x$. **552.** $-\dfrac{1}{(\arcsin x)^2\sqrt{1-x^2}}$.

553. $(\sin x)\arctan x+(x\cos x)\arctan x+\dfrac{x\sin x}{1+x^2}$.

554. $-\dfrac{x+\arccos x\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}$. **555.** $\dfrac{\arctan x}{2\sqrt{x}}+\dfrac{\sqrt{x}}{1+x^2}$. **556.** 0.

557. $\dfrac{1}{x\sqrt{x^2-1}}$. **558.** $-\dfrac{2x^2}{(1+x^2)^2}$. **559.** $\dfrac{\sqrt{1-x^2}+x\arcsin x}{\sqrt{(1-x^2)^3}}$.

560. $\dfrac{2x}{\arctan x}-\dfrac{x^2}{(1+x^2)(\arctan x)^2}$. **561.** $\dfrac{1}{\sqrt{2x-x^2}}$.

562. $-\dfrac{\sqrt{2}}{\sqrt{1+2x-2x^2}}$. **563.** $\dfrac{2x}{1+x^4}$. **564.** $-\dfrac{2}{|x|\sqrt{x^2-4}}$.

565. $\dfrac{\cos x}{|\cos x|}$. **566.** $-\dfrac{2\arctan\frac{1}{x}}{1+x^2}$.

567. $\dfrac{\arccos x}{\sqrt{1-x^2}\sqrt{1-(\arccos x)^2}}$. **568.** $-\dfrac{1}{(1+x)\sqrt{2x(1-x)}}$.

569. $\dfrac{x+1}{8\sqrt[4]{(\arcsin\sqrt{x^2+2x})^3}\sqrt{(1-2x-x^2)(x^2+2x)}}$.

570. $\dfrac{\sin\alpha}{1-\cos\alpha\cos x}$. **571.** $\dfrac{\sqrt{a^2-b^2}}{a+b\cos x}$. **572.** $\dfrac{1}{2(1+x^2)}$.

573. $2x\log_3 x+\dfrac{x}{\ln 3}$. **574.** $\dfrac{2\ln x}{x}$. **575.** $\dfrac{\ln x+1}{\ln 10}$.

576. $\dfrac{1}{2x\sqrt{\ln x}}$. **577.** $\dfrac{x\ln x-x+1}{x\ln^2 x}\ln 2$.

578. $\sin x\ln x+x\cos x\ln x+\sin x$.

579. $-\dfrac{1}{x\ln^2 x}$. **580.** $\dfrac{1-n\ln x}{x^{n+1}}$. **581.** $-\dfrac{2}{x(1+\ln x)^2}$.

582. $\dfrac{1+x^2-2x^2\ln x}{x(1+x^2)^2}$. **583.** $x^{n-1}(n\ln x+1)$.

584. $\dfrac{\ln x}{x\sqrt{1+\ln^2 x}}$. **585.** $-\dfrac{2}{1-2x}$. **586.** $\dfrac{2x-4}{x^2-4x}$.

587. $\cot x$. **588.** $\dfrac{2x}{(x^2-1)\ln 3}$. **589.** $\dfrac{2}{\sin 2x}$.

590. $-\dfrac{2}{\arccos 2x\sqrt{1-4x^2}}$. **591.** $4(\ln^3\sin x)\cot x$.

592. $\dfrac{a}{(ax+b)[1+\ln^2(ax+b)]}$. **593.** $n(1+\ln\sin x)^{n-1}\cot x$.

594. $\dfrac{1}{x\log_5 x\log_3(\log_5 x)\ln 2\ln 3\ln 5}$.

595. $\dfrac{x}{(\operatorname{arc}\tan\sqrt{1+x^2})(2+x^2)\sqrt{1+x^2}}$.

596. $\dfrac{6x^2\arcsin[\ln(a^3+x^3)]}{(a^3+x^3)\sqrt{1-\ln^2(a^3+x^3)}}$. **597.** $\dfrac{\cot\dfrac{x+3}{4}}{12\sqrt[3]{\ln^2\sin\dfrac{x+3}{4}}}$.

598. $2^x\ln 2$. **599.** $10^x\ln 10$. **600.** $-\dfrac{\ln 3}{3^x}$. **601.** $4^{-x}(1-x\ln 4)$.

602. $10^x(1+x\ln 10)$. **603.** $e^x(1+x)$. **604.** $\dfrac{1-x}{e^x}$.

605. $\dfrac{2^x(\ln 2-1)+3x^2-x^3}{e^x}$. **606.** $e^x(\cos x-\sin x)$.

607. $\dfrac{e^x}{\sin^2 x}(\sin x-\cos x)$. **608.** $-\dfrac{\sin x+\cos x}{e^x}$.

609. $\dfrac{(\ln x-1)\ln 2}{\ln^2 x}2^{\frac{x}{\ln x}}$. **610.** $3x^2-3^x\ln 3$. **611.** $\dfrac{e^x}{2\sqrt{1+e^x}}$.

612. $e^x(x^2+1)$. **613.** $\dfrac{2e^x}{(1-e^x)^2}$. **614.** $-\dfrac{2(10^x\ln 10)}{(1+10^x)^2}$.

615. $\dfrac{e^x(x-1)^2}{(x^2+1)^2}$. **616.** $e^x(\cos x+\sin x+2x\cos x)$.

617. $-e^{-x}$. **618.** $2(10^{2x-3}\ln 10)$. **619.** $\dfrac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}$.

620. $(2^x\ln 2)\cos(2^x)$. **621.** $(3^{\sin x}\cos x)\ln 3$.

622. $(3\sin^2 x\cos x)a^{\sin^3 x}\ln a$. **623.** $\dfrac{2e^{\arcsin 2x}}{\sqrt{1-4x^2}}$.

624. $2^{3^x}3^x\ln 2\ln 3$. **625.** $\dfrac{e^{\sqrt{\ln x}}}{2x\sqrt{\ln x}}$.

626. $\cos(e^{x^2+3x-2})e^{x^2+3x-2}(2x+3)$.

627. $-12(10^{1-\sin^4 3x}\ln 10)\sin^3 3x\cos 3x$.

628. $\dfrac{(2ax+b)e^{\sqrt{\ln(ax^2+bx+c)}}}{2(ax^2+bx+c)\sqrt{\ln(ax^2+bx+c)}}$.

629. $\dfrac{\cot\sqrt[3]{\operatorname{arc}\tan(e^{3x})}\,e^{3x}}{(1+e^{6x})\sqrt[3]{[\operatorname{arc}\tan(e^{3x})]^2}}$. **630.** $-2ab^2xe^{-b^2x^2}$.

631. $\frac{2}{a^2}xe^{-\frac{x^2}{a^2}}(a^2-x^2)$. **632.** $Ae^{-k^2x}[\omega\cos(\omega x+\alpha)-k^2\sin(\omega x+\alpha)]$.

633. $a^x x^a\left(\frac{a}{x}+\ln a\right)$. **634.** $3\sinh^2 x\cosh x$. **635.** $\tanh x$.

636. $\frac{1}{\cosh 2x}$. **637.** $-\frac{2x}{\cosh^2(1-x^2)}$. **638.** $2\sinh 2x$.

639. $\sinh(\sinh x)\cosh x$.

640. $\frac{\sinh x}{2\sqrt{\cosh x}}$. **641.** $e^{\cosh^2 x}\sinh 2x$. **642.** $\frac{1}{x\cosh^2(\ln x)}$.

643. $x\cosh x$. **644.** $\frac{3\tanh x}{2\cosh^2 x\sqrt[4]{1+\tanh^2 x}}$. **645.** $\frac{1}{4\cosh^4\frac{x}{2}}$.

646. $\frac{1}{2\sqrt{\cosh x-\sinh x}}$. **647.** $\frac{1}{1-\sinh^4 x}$.

648. $\frac{x(4+\sqrt{x})\sinh 2x+2(2x^2\sqrt{x}-1)\cosh 2x}{2x^2}$.

649. $\frac{xe^{3x}}{\sinh^2 x}[(3x+2)\sinh x-x\cosh x]$.

650. $x^{x^2+1}(2\ln x+1)$. **651.** $x^{x^x}x^x\left(\ln^2 x+\ln x+\frac{1}{x}\right)$.

652. $(\sin x)^{\cos x}\left(\frac{\cos^2 x}{\sin x}-\sin x\ln\sin x\right)$.

653. $(\ln x)^x\left(\frac{1}{\ln x}+\ln\ln x\right)$.

654. $2\sqrt[x]{(x+1)^2}\left[\frac{1}{x(x+1)}-\frac{\ln(x+1)}{x^2}\right]$.

655. $x^2e^{x^2}\sin 2x(3+2x^2+2x\cot 2x)$.

656. $-\frac{2(x-2)(x^2+11x+1)}{3(x-5)^4\sqrt[3]{(x+1)^2}}$. **657.** $-\frac{\ln a}{x\ln^2 x}$.

658. $\frac{57x^2-302x+361}{20(x-2)(x-3)}\cdot\frac{(x+1)^2\sqrt[4]{x-2}}{\sqrt[5]{(x-3)^2}}$.

659. $\frac{1}{2}\sqrt{x\sin x\sqrt{1-e^x}}\left(\frac{1}{x}+\cot x-\frac{1}{2}\frac{e^x}{1-e^x}\right)$.

660. $\frac{1}{\sqrt{1-x^2}\,[(\arcsin x)^2-1]}\sqrt{\frac{1-\arcsin x}{1+\arcsin x}}$.

661. $x^{\frac{1}{x}-2}(1-\ln x)$. **662.** $x^{\sin x}\left(\cos x\ln x+\frac{\sin x}{x}\right)$.

663. $\left(\frac{x}{x+1}\right)^x\left(\frac{1}{x+1}+\ln\frac{x}{x+1}\right)$. **664.** $x^{\sqrt{x}-\frac{1}{2}}(2+\ln x)$.

665. $(x^2+1)^{\sin x}\left[\frac{2x\sin x}{x^2+1}+\cos x\ln(x^2+1)\right]$.

666. $\frac{x^4+6x^2+1}{3x(1-x^4)}\sqrt[3]{\frac{x(x^2+1)}{(x^2-1)^2}}$. **667.** $\frac{(1+\sqrt[3]{x})^2}{\sqrt[3]{x^2}}$.

668. $\frac{a}{k\cos^2\left(\frac{x}{k}+b\right)}$. **669.** $\frac{p}{2\sqrt{1+\sqrt{2px}}\sqrt{2px}}$.

670. $\frac{2x-3}{1+(x^2-3x+2)^2}$. **671.** $\frac{1+\sin x}{(x-\cos x)\ln 10}$.

672. $\frac{3}{2}\sin 2x(\cos x-2)$. **673.** $\sec^2\frac{x}{5}$.

674. $-\frac{1+2\sqrt{x}}{6\sqrt{x}\sqrt[3]{(x+\sqrt{x})^4}}$.

675. $2\sin\frac{x}{2}\cos 2x+\frac{1}{2}\cos\frac{x}{2}\sin 2x$. **676.** $e^{\cos x}(\cos x-\sin^2 x)$.

677. $\frac{x^4(7x^6-40)}{\sqrt[3]{(x^6-8)^2}}$. **678.** $e^{-x^2}\left(\frac{1}{x}-2x\ln x\right)$.

679. $\frac{5(x-1)}{x\sqrt{x}}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^9$. **680.** $-\frac{1}{1+x^2}$. **681.** $2x^2e^{2x+3}$.

682. $\frac{2\sin 2x}{\cos^2 2x}$. **683.** $\frac{1+x^2}{1+x^2+x^4}$. **684.** $-\frac{2(x\cos x+\sin x)}{x^2\sin^2 x}$.

685. $\frac{1}{3}\cot\frac{x}{2}\sin\frac{2x}{3}-\frac{1}{2}\sin^2\frac{x}{3}\cos^2\frac{x}{2}$. **686.** $-\frac{4(31x^5+18)}{27x^5\sqrt[9]{(4x^5+2)^8}}$.

687. $\frac{1}{\sqrt{x^2+a^2}}$. **688.** $\arctan\sqrt{x}+\frac{\sqrt{x}}{2(1+x)}$.

689. $\frac{\tan x\,(1+2\tan^2 x)}{\cos^2 x\sqrt{1+\tan^2 x+\tan^4 x}}$. **690.** $\frac{\cos 2x}{x}-2\sin 2x\ln x$.

691. $\dfrac{1+x^4}{1+x^6}$. **692.** $\dfrac{n\cos x}{\sqrt{1-n^2\sin^2 x}}$. **693.** $\dfrac{\cos x}{2\sqrt{\sin x-\sin^2 x}}$.

694. $\sin^5 3x\cos^3 3x$. **695.** $\dfrac{x\arcsin x}{\sqrt{1-x^2}}$. **696.** $-\dfrac{1}{2}\sin\dfrac{\arcsin x}{2}\dfrac{1}{\sqrt{1-x}}$.

697. $\dfrac{1+2\sqrt{x}+4\sqrt{x}\sqrt{x+\sqrt{x}}}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}$. **698.** $\dfrac{3}{2\sqrt{3x-9x^2}}$.

699. $\dfrac{\ln x-2}{x^2}\sin\left[2\left(\dfrac{1-\ln x}{x}\right)\right]$. **700.** $\dfrac{2x-\cos x}{(x^2-\sin x)\ln 3}$.

701. $-\dfrac{1}{2\sqrt{1-x^2}}$. **702.** $-\dfrac{1}{x\sqrt{1-x^2}\,(x+\sqrt{1-x^2})}$.

703. $\arcsin(\ln x)+\dfrac{1}{\sqrt{1-\ln^2 x}}$. **704.** $-\dfrac{2e^x}{(1+e^x)^2}\sec^2\left(\dfrac{1-e^x}{1+e^x}\right)$.

705. $-\dfrac{2\sin^3 x}{\sqrt{1+\sin^2 x}}$.

706. $-0{\cdot}8\left(\cos\dfrac{2x+1}{2}-\sin 0{\cdot}8x\right)\left(\sin\dfrac{2x+1}{2}+0{\cdot}8\cos 0{\cdot}8x\right)$.

707. $10^{\sqrt{x}}\left(1+\dfrac{\sqrt{x}}{2}\ln 10\right)$. **708.** $-\dfrac{4}{\tan 2x\sin^2 2x}$.

709. $-\dfrac{1}{(x^2+2x+2)\arctan\dfrac{1}{1+x}}$. **710.** $-\dfrac{1}{\sqrt{x^2-1}}$.

711. $\dfrac{x+2}{2\sqrt{x+3}\sqrt[3]{(1+x\sqrt{x+3})^2}}$. **712.** $\dfrac{x(8+9\sqrt{x})}{4\sqrt{1+\sqrt{x}}}$.

713. $-\dfrac{\sin 2x}{2\sqrt{(1+\sin^2 x)^3}}$. **714.** $3x^2\arctan x^3+\dfrac{3x^5}{1+x^6}$.

715. $\dfrac{\cot x\ln\cos x+\tan x\ln\sin x}{\ln^2\cos x}$. **716.** $\sqrt{\dfrac{1-x}{1+x}}$.

717. $\dfrac{4}{(1-4x)^2}\left(\sqrt{\dfrac{1-4x}{1+4x}}+\arcsin 4x\right)$. **718.** $-\dfrac{e^{\frac{1}{\ln x}}}{x\ln^2 x}$.

719. $\dfrac{1}{e^x-1}$. **720.** $10^{x\tan x}\ln 10\left(\tan x+\dfrac{x}{\cos^2 x}\right)$.

721. $2\sin x(x\sin x\cos x^2+\cos x\sin x^2)$. 722. $\dfrac{2\sin x}{\cos 2x\sqrt{\cos 2x}}$.

723. $\dfrac{2-3x-x^3}{2(1-x)(1+x^2)}\sqrt{\dfrac{1-x}{1+x^2}}$. 724. $\dfrac{x^2}{1-x^4}$.

725. $2^{\frac{x}{\ln x}}\dfrac{\ln x-1}{\ln^2 x}\ln 2$. 726. $\sqrt{\dfrac{a-x}{x-b}}$. 727. $-\dfrac{2(2\cos^2 x+1)}{\sin^2 2x}$

728. $-\dfrac{1}{(1+x)\sqrt{1-x^2}}e^{\sqrt{\frac{1-x}{1+x}}}$. 729. $\sqrt{\dfrac{a-x}{a+x}}$.

730. $\dfrac{\sqrt{x^2+1}}{x}$. 731. $-\cos 2x$. 732. $\dfrac{x^2}{\sqrt{(x^2-1)^3}}$.

733. $(a^2+1)\sin x e^{ax}$. 734. $e^{1-\cos x}(1+x\sin x)$.

735. $\dfrac{2e^{-2x}}{(1+e^{-4x})(\arctan e^{-2x})^2}$. 736. $10e^x\sin 3x$. 737. $9x^2\arcsin x$.

738. $\dfrac{e^{-\sqrt{x}}}{4\sqrt{x}\sqrt{(1+e^{-\sqrt{x}})^3}}$. 739. $\dfrac{x}{\sqrt{2+4x-x^2}}$.

740. $\dfrac{(\cos x-\sin x)(e^x+e^{-x})}{e^x\cos x+e^{-x}\sin x}$. 741. $\dfrac{\arctan x}{\sqrt{(1+x^2)^3}}$.

742. $\dfrac{\sin(x-\cos x)(1+\sin x)}{\cos^2(x-\cos x)}$.

743. $e^x\sin x\cos^3 x(1+\cot x-3\tan x)$.

744. $\dfrac{54\sqrt[5]{x^4}}{55\sqrt[11]{(9+6\sqrt[5]{x^9})^{10}}}$. 745. $\dfrac{1}{\sqrt{e^{2x}+4e^x+1}}$.

746. $\dfrac{e^{\arctan\sqrt{1+\ln(2x+3)}}}{(2x+3)[2+\ln(2x+3)]\sqrt{1+\ln(2x+3)}}$.

747. $\dfrac{e^{x^2}}{(e^x+e^{-x})^2}[2x(e^x+e^{-x})-(e^x-e^{-x})]$. 748. $\dfrac{\ln(1+\sin x)}{\sin^2 x}$.

749. $\dfrac{40}{2x-3\sqrt{1-4x^2}}$. 750. $\dfrac{x^5+1}{x^4(x^2+1)}$. 751. $\dfrac{x^2}{\sqrt{1-2x-x^2}}$.

752. $\dfrac{1}{x}-\dfrac{x}{1-x^2}+\cot x$. 753. $\dfrac{(1+2x^2)\sin x+x(1+x^2)\cos x}{\sqrt{1+x^2}}$.

754. $\dfrac{(x^2-32x-73)(3-x)^3}{2(x+1)^6\sqrt{x+2}}$. **755.** $\dfrac{3e^{\sqrt{x}}(2+\sqrt{x})}{10\sqrt[5]{(1+xe^{\sqrt{x}})^2}}$.

756. $\left(2x-\dfrac{1}{1+x^2}\right)\dfrac{e^{x^2-\arctan x+\frac{1}{2}\ln x+1}}{\sqrt{x}}$. **757.** $\dfrac{1}{\cos^5 x}$.

758. $\dfrac{e^x\arctan x}{\ln^5 x}\left[1+x+\dfrac{x}{(1+x^2)\arctan x}-\dfrac{5}{\ln x}\right]$.

759. $\dfrac{(1-x^2)\,e^{3x-1}\cos x}{(\arccos x)^3}\left[\dfrac{3-2x-3x^2}{1-x^2}-\tan x+\right.$

$$\left.+\frac{3}{\sqrt{1-x^2}\arccos x}\right].$$

760. $4\sqrt{(x^2+a^2)^3}$. **761.** $(\arcsin x)^2$. **762.** $\dfrac{e^{-x}-e^x}{e^{-x}+e^x}$.

763. $\dfrac{1}{ae^{mx}+be^{-mx}}$. **764.** $\dfrac{1}{x^3+1}$. **765.** $\dfrac{1}{x}\sqrt{\dfrac{1-x}{1+x}}$.

766. $(\tan 2x)^{\cot\frac{x}{2}}\left(\dfrac{4\cot\frac{x}{2}}{\sin 4x}-\dfrac{\ln\tan 2x}{2\sin^2\frac{x}{2}}\right)$.

767. $\dfrac{3x^2+10x+20}{15(x^2+4)\sqrt[3]{(x-5)^2\sqrt[5]{x^2+4}}}$. **768.** $\dfrac{1}{x^4+x^2+1}$.

769. $-\dfrac{2nx^{n-1}}{x^{2n}+1}$, if n is an even number, and $-\dfrac{2nx^n}{|x|\,(x^{2n}+1)}$, if n is an odd number.

770. $\dfrac{24x^3}{(1+8x^3)^2}$. **774*.** (a) $1-\dfrac{(n+1)\,x^n+nx^{n+1}}{(1-x)^2}$;

(b) $-\dfrac{n(n-1)\,x^{n+1}-2(n+1)^2x^n+n(n+1)\,x^{n-1}+2}{(1-x)^3}$. *Hint;* use the value of the sum $x+x^2+\ldots+x^n$.

776. $\sqrt{1-y^2}\,e^{-\arcsin y}$ and $\dfrac{\cos\ln x}{x}$. **777.** $\dfrac{1}{3(s^2-1)}$.

779. $\dfrac{1}{2\sqrt{x-x^2}}$. **780.** $\alpha'(x)=\dfrac{1}{x\,[1+\ln\alpha(x)]}$.

781. $(\operatorname{Arc\,sinh} x)'=\dfrac{1}{\sqrt{1+x^2}}$; $(\operatorname{Arc\,cosh} x)'=\dfrac{1}{\sqrt{x^2-1}}$;

$(\text{Arc tanh } x)' = \dfrac{1}{1-x^2}$. **782.** $\dfrac{e^t}{1-t}$.

783. $\dfrac{-(1+x^4)^2}{8x^3}$; $-2\sqrt[3]{(1-y^2)^2(1-y)^2}$. **784.** $\dfrac{1}{3y^2-4}$.

785. $\dfrac{1}{2^s \ln 2}\sqrt{1-2^{2s}}$, $\dfrac{1}{\ln 2}\cot t$. **789.** $y = f(x) = \sin^2 x$.

790. $y = \pm\sqrt{1-x^2}$. **791.** $-\dfrac{1}{4}$. **792.** $-\dfrac{b^2 x}{a^2 y}$.

793. $-\sqrt{\dfrac{y}{x}}$. **794.** $\dfrac{ay-x^2}{y^2-ax}$. **795.** $\dfrac{3a^2\cos 3x + y^2 \sin x}{2y\cos x}$.

796. $\dfrac{2a}{3(1-y^2)}$. **797.** $\dfrac{y}{y-x}$. **798.** $\dfrac{x}{y}\cdot\dfrac{y^2-2x^2}{2y^2-x^2}$.

799. $-\dfrac{3x^2+2axy+by^2}{ax^2+2bxy+3y^2}$.

800. $-\dfrac{y\cos^2(x+y)(\cos(xy)-\sin(xy))-1}{x\cos^2(x+y)(\cos(xy)-\sin(xy))-1}$.

801. $2^{x-y}\dfrac{2^y-1}{1-2^x}$. **802.** $\dfrac{1}{2(1+\ln y)}$.

803. $\dfrac{\sqrt{1-y^2}(1-\sqrt{1-x^2})}{\sqrt{1-x^2}(1-\sqrt{1-y^2})}$. **804.** $\dfrac{y^2-xy\ln y}{x^2-xy\ln x}$.

805. $-\dfrac{\sin(x+y)}{1+\sin(x+y)}$. **806.** $-\dfrac{1+y\sin(xy)}{x\sin(xy)}$.

807. $-\sqrt[3]{\dfrac{y}{x}}$. **808.** $\dfrac{e^y}{2-y}$. **809.** $\dfrac{\sin y}{2\sin 2y - \sin y - x\cos y}$.

810. $\dfrac{\sqrt{1-k^2}}{1+k\cos x}$. **811.** $\dfrac{y\cos x + \sin(x-y)}{\sin(x-y)-\sin x}$. **812.** $\dfrac{1+y^2}{y^2}$.

814. (2,4). **816.** $y+4x+4=0$; $8y-2x+15=0$; the subtangent is equal to $\dfrac{1}{2}$, the subnormal to -8.

819. (a) $t_1 = 0$, $t_2 = 8$; (b) $t_1 = 0$, $t_2 = 4$, $t_3 = 8$.

820. $181{\cdot}5\times 10^3$ erg. **821.** $\omega = 13\dfrac{\text{rad}}{\text{sec}}$. **822.** $\omega = 2\pi\dfrac{\text{rad}}{\text{sec}}$.

823. $\omega = (2at-b)\dfrac{\text{rad}}{\text{sec}}$; the velocity becomes zero after $t = \dfrac{b}{2a}$ sec.

824. $23A$. **825.** $(0, 0)$; $(1, 1)$; $(2, 0)$.

827. $(1, 0)$; $(-1, -4)$. **828.** $y = 2x - 2$; $y = 2x + 2$.

829. $3x + y + 6 = 0$.

830. The tangent is $y - y_0 = \cos x_0(x - x_0)$; the normal is $y - y_0 = -\sec x_0(x - x_0)$.

831. The tangent is $x_0(y - y_0) = x - x_0$; the normal is $(y - y_0) + + x_0(x - x_0) = 0$.

832. The tangent is $x + 2y = 4a$; the normal is $y = 2x - 3a$.

833. The tangent is $y - y_0 = \dfrac{x_0^2(3a - x_0)}{y_0(2a - x_0)^2}(x - x_0)$;

the normal is $y - y_0 = -\dfrac{y_0(2a - x_0)^2}{x_0^2(3a - x_0)}(x - x_0)$.

835. The subtangents are equal to $\dfrac{x}{3}$, $\dfrac{2x}{3}$ and $-2x$ respectively; the subnormals are $-3x^5$, $-\dfrac{3x^2}{2}$ and $\dfrac{1}{2x^2}$ respectively.

836. $y = \dfrac{x_0}{2a}\left(x - \dfrac{x_0}{2}\right)$; $y - y_0 = -\dfrac{2a}{x_0}(x - x_0)$.

837. $2x - y + 1 = 0$. **838.** $27x - 3y - 79 = 0$.

839. $2x - y - 1 = 0$. **840.** $4x - 4y - 21 = 0$.

842. $3{\cdot}75$. **844.** $x + 25y = 0$; $x + y = 0$.

845. $(0, 1)$. **846.** $y = x$. **848.** $x - y - 3e^{-2} = 0$.

849. $\dfrac{2}{\sqrt{5}}$. **850.** $\left(1 + \dfrac{\sqrt{3}}{2};\ 1\right)$. **857.** $2x - y \pm 1 = 0$.

858. (a) The parabola $y^2 = \dfrac{1}{2}px$; (b) the straight line $y = \dfrac{1}{\ln b}$, parallel to $0x$; (c) the kappa curve $y\sqrt{a^2 - x^2} + x^2 = 0$; (d) the circle $x^2 + y^2 = a$.

859. (1) $\arctan\dfrac{8}{15}$; (2) $\varphi_1 = 0$, $\varphi_2 = \arctan\dfrac{1}{8}$.

860. (1) $\arctan 3$. (2) $45°$.

861. $90°$. **862.** 45 and $90°$. **863.** $\arctan 3$. **864.** $\arctan(2\sqrt{2})$.

865. When n is odd, the tangent is $\dfrac{x}{a} + \dfrac{y}{b} = 2$, the normal $ax - by = a^2 - b^2$. When n is even, the tangents are $\dfrac{x}{a} \pm \dfrac{y}{b} = 2$, the normals $ax \pm by = a^2 - b^2$.

879. $\Delta y = 1{\cdot}461$; $\mathrm{d}y = 1{\cdot}4$. **880.** $\Delta y = 0{\cdot}1012$; $\mathrm{d}y = 0{\cdot}1$; $\frac{\mathrm{d}y}{\Delta y} = 0{\cdot}9880$.

881. 4. **882.** -2. **883.** $\Delta y = 1{\cdot}91$; $\mathrm{d}y = 1{\cdot}9$; $\Delta y - \mathrm{d}y = 0{\cdot}01$; $\frac{\Delta y - \mathrm{d}y}{\Delta y} = 0{\cdot}0052$. **884.** $\Delta y = 0{\cdot}1$; $\mathrm{d}y = 0{\cdot}1025$; $\Delta y - \mathrm{d}y = -0{\cdot}0025$; $\frac{\Delta y - \mathrm{d}y}{\Delta y} = -0{\cdot}025$.

885.

	$\Delta x = 1$,	$0{\cdot}1$,	$0{\cdot}01$,
	$\Delta y = 18$,	$1{\cdot}161$,	$0{\cdot}110601$,
	$\mathrm{d}y = 11$,	$1{\cdot}1$,	$0{\cdot}11$,
	$\Delta y - \mathrm{d}y = 7$,	$0{\cdot}061$,	$0{\cdot}000601$,
	$\delta = \frac{\Delta y - \mathrm{d}y}{\Delta y} = 0{\cdot}39$,	$0{\cdot}0526$,	$0{\cdot}0055$.

886. $\Delta y \approx 1{\cdot}3$; $\mathrm{d}y \approx 1{\cdot}1$; $\Delta y - \mathrm{d}y \approx 0{\cdot}2$; $\delta = \frac{\Delta y - \mathrm{d}y}{\Delta y} = 0{\cdot}15$.

887. (a) $\mathrm{d}y = 16$, $\frac{\Delta y - \mathrm{d}y}{\Delta y}\,\% = 5{\cdot}88\,\%$; (b) $\mathrm{d}y = 8$, $\frac{\Delta y - \mathrm{d}y}{\Delta y}\,\% = 3{\cdot}03\,\%$; (c) $\mathrm{d}y = 1{\cdot}6$, $\frac{\Delta y - \mathrm{d}y}{\Delta y}\,\% = 0{\cdot}62\,\%$.

888. (a) $\mathrm{d}y = 4{\cdot}8\ \mathrm{cm}^2$; (b) $\mathrm{d}y = 6{\cdot}0\ \mathrm{cm}^2$; (c) $\mathrm{d}y = 9{\cdot}6\ \mathrm{cm}^2$.

889. (1) $\frac{0{\cdot}125}{\sqrt{x}}\mathrm{d}x$; (2) $\frac{5\mathrm{d}x}{3\sqrt[3]{x^2}}$; (3) $-\frac{4\mathrm{d}x}{x^3}$; (4) $-\frac{\mathrm{d}x}{x^5}$; (5) $-\frac{\mathrm{d}x}{4x\sqrt{x}}$;
(6) $-\frac{\mathrm{d}x}{3nx\sqrt[3]{x}}$; (7) $\frac{\mathrm{d}x}{2(a+b)\sqrt{x}}$; (8) $-\frac{p\ln q}{q^x}\mathrm{d}x$;
(9) $-\frac{0{\cdot}2(m-n)}{x^{1{\cdot}2}}\mathrm{d}x$; (10) $-\frac{(m+n)\,\mathrm{d}x}{2x\sqrt{x}}$;
(11) $\left[(2x+4)(x^2-\sqrt{x}) + (x^2+4x+1)\left(2x - \frac{1}{2\sqrt{x}}\right)\right]\mathrm{d}x$;
(12) $-\frac{6x^2\,\mathrm{d}x}{(x^3-1)^2}$; (13) $\frac{2t\,\mathrm{d}t}{(1-t^2)^2}$; (14) $3(1+x-x^2)^2(1-2x)\,\mathrm{d}x$.
(15) $\frac{2\tan x}{\cos^2 x}\mathrm{d}x$; (16) $5^{\ln\tan x}\frac{2\ln 5}{\sin 2x}\mathrm{d}x$; (17) $-2^{-\frac{1}{\cos x}}\ln 2\,\frac{\sin x}{\cos^2 x}\mathrm{d}x$;
(18) $-\frac{\mathrm{d}x}{2\sin\frac{x}{2}}$; (19) $\frac{(x^2-1)\sin x + 2x\cos x}{(1-x^2)^2}\mathrm{d}x$;
(20) $\left(\frac{1}{2\sqrt{\arcsin x}\sqrt{1-x^2}} + \frac{2\arctan x}{1+x^2}\right)\mathrm{d}x$;

(21) $\left(\dfrac{5}{\sqrt{1-x^2}} - \dfrac{1}{1+x^2}\right)\dfrac{dx}{2}$; (22) $\left(3^{-\frac{1}{x^2}} \cdot \dfrac{3}{x^3} \ln 3 + 9x^2 - \dfrac{2}{\sqrt{x}}\right) dx$.

890. (1) $-0{\cdot}0059$; (2) $-0{\cdot}0075$; (3) $0{\cdot}0086$; (4) 0; (5) $0{\cdot}00287$.

891. $\Delta y \approx 0{\cdot}00025$; $\sin 30°1' \approx 0{\cdot}50025$. **892.** $0{\cdot}00582$.

893. $-0{\cdot}0693$. **894.** $d\varrho - = \dfrac{k \sin 2\varphi}{\sqrt{\cos 2\varphi}} d\varphi$.

895. $0{\cdot}3466$. **896.** $\sin 60°03' = 0{\cdot}8665$; $\sin 60°18' = 0{\cdot}8686$.

899. $0{\cdot}995$. **900.** $\operatorname{arc\,tan} 1{\cdot}02 \approx 0{\cdot}795$; $\operatorname{arc\,tan} 0{\cdot}97 \approx 0{\cdot}770$.

901. $0{\cdot}782$. **902.** $0{\cdot}52164$.

903. (a) The change in the length of the cord is $2ds = \dfrac{8f}{3l} df$; (b) the change in the sag is $df = \dfrac{3l}{4f} ds$.

904. The error when finding the angle from its sine is $\Delta x_S = \tan x \, \Delta y$; the error when finding the angle from its tangent is $\Delta x_T = \dfrac{1}{2} \sin 2x \, \Delta z$ (where Δy, Δz are the errors with which y and z are given); $\dfrac{\Delta x_S}{\Delta x_T} = \dfrac{1}{\cos^2 x}$; the accuracy is higher when finding the angle from the logarithm of its tangent than from the logarithm of its sine.

905. $0{\cdot}3\%$. **906.** (1) $dy = \dfrac{(2t^3 + 4t + 7)(3t^2 + 2) dt}{3\sqrt[]{[(t^3 + 2t + 1)(t^3 + 2t + 6)]^2}}$;

(2) $ds = -\dfrac{t}{2} \sin \dfrac{t^2 - 1}{2} dt$; (3) $dz = -ds$;

(4) $dv = \dfrac{2 \ln 3}{3^{\frac{1}{\ln \tan s}} \ln^2 \tan s} \dfrac{ds}{\sin 2s}$; (5) $ds = \dfrac{(4u - 3) du}{2\sqrt{2u^2 - 3u + 1}}$;

(6) $dy = -\dfrac{2ds}{\cos 2s}$.

908. Continuous and differentiable.

909. $f(x)$ is continuous everywhere except for the points $x = 0$ and $x = 2$; $f'(x)$ exists and is continuous everywhere except for the points $x = 0, 1, 2$, where it does not exist.

910. At $x = k\pi$, where k is any integer.

911. Continuous, but non-differentiable.

912. $f'(0) = 0$.

913. Continuous, but non-differentiable.

914. Δy and Δx are quantities of different orders of smallness.

915. Continuous, but non-differentiable.

916. Yes; no. **917.** a. **918.** $a\omega e^{a\varphi}$.

919. The abscissa varies at the rate $v_x = -2r\omega \sin 2\varphi$; the ordinate varies at the rate $v_y = 2r\omega \cos 2\varphi$.

920. The rate of change of the abscissa is $v_x = v(1 + \cos\varphi)$; the rate of change of the ordinate is $v_y = v \sin\varphi$ (φ is the angle between the axis of ordinates and the radius vector of the point).

921. $-\dfrac{p \ln 2}{5540} \approx -0{\cdot}000125p$.

922. 2 units/sec at the point (3, 6) and -2 units/sec at the point $(3, -6)$.

923. 2 cm/sec at the point (3, 4) and -2 cm/sec at the point $(-3, 4)$.

924. At the points $\left(3, \dfrac{16}{3}\right)$ and $\left(-3, -\dfrac{16}{3}\right)$.

925. $4v$ cm/sec and $2av$ cm^2/sec.

926. $2\pi v$ and $2\pi rv$. **927.** $4\pi r^2 v$ and $8\pi rv$ cm^2/sec.

928. For $x = 2\pi k \pm \dfrac{\pi}{3}$ and for $x = 2\pi k \pm \dfrac{2\pi}{3}$.

929. At $x = 2\pi k$. **930.** $\dfrac{1}{n^2}$ times. **932.** (a) Yes; (b) No.

934. (1) $x^2 - 18x + 9y = 0$; (2) $y^2 = 4x^2(1 - x^2)$;
(3) $y^3 = (x - 1)^2$; (4) $x = \operatorname{Arc}\cos(1 - y) \mp \sqrt{2y - y^2}$;
(5) $y = \dfrac{2(1 + x - x^2)}{1 + x^2}$.

935. (1) $t = (2k + 1)\pi$; (2) $t = 1$; (3) $t = \dfrac{\pi}{4} + \pi k$;
(4) $t_1 = 1$, $t_2 = -1$.

936. $-\dfrac{b}{a}\cot\varphi$. **937.** $-\dfrac{b}{a}\tan\varphi$. **938.** $\cot\dfrac{\varphi}{2}$. **939.** $\dfrac{3t^2 - 1}{2t}$.

940. -1. **941.** $\dfrac{t}{2}$. **942.** $\dfrac{\cos\varphi - \varphi\sin\varphi}{1 - \sin\varphi - \varphi\cos\varphi}$.

943. $\dfrac{1 + t^2}{t(2 + 3t - t^3)}$. **944.** $\dfrac{1 - \tan t}{1 + \tan t}$. **945.** $\dfrac{t(2 - t^3)}{1 - 2t^3}$.

946. $-\frac{4}{3}$. **947.** 0 and $\frac{1}{3}$. **948.** Does not exist. **949.** $\frac{\sqrt{3}}{6}$.

950. (1) $t = \frac{\pi}{2} + \alpha$; (2) $t = \pi - \alpha$; (3) $t = \frac{\pi}{6} + \frac{\alpha}{3}$, where α is the angle formed by the tangent with Ox.

956. (1) The curves cut at two points at angles $\alpha_1 = \alpha_2 = \arctan\frac{41}{2} \approx 87°12'$; (2) the curves cut at three points at angles $\alpha_1 = \alpha_2 = 30°$ and $\alpha_3 = 0°$.

958. The length of the tangent is $T = \left|\frac{y}{\sin\frac{3}{2}t}\right|$; the length of the normal is $N = \left|\frac{y}{\cos\frac{3}{2}t}\right|$; the length of the subtangent is $S_T = \left|y\cot\frac{3}{2}t\right|$; the length of the subnormal is $S_N = \left|y\tan\frac{3}{2}t\right|$.

959. $\left|\frac{y}{\cos t}\right|$, $\left|\frac{y}{\sin t}\right|$, $|y\tan t|$ and $|y\cot t|$.

961. $\left|\frac{y}{\sin t}\right|$, $\left|\frac{y}{\cos t}\right|$, $|y\cot t|$ and $|y\tan t|$.

963. $x + 2y - 4 = 0$; $2x - y - 3 = 0$. **964.** $4x + 2y - 3 = 0$; $2x - 4y + 1 = 0$.

965. $y = 2$, $x = 1$. **966.** (1) $4x + 3y - 12a = 0$; $3x - 4y + 6a = 0$; (2) $x + y = \frac{\pi^2\sqrt{2}}{16}$; $y - x = \frac{\pi\sqrt{2}}{2}$; (3) $y = 1 + x\ln a$.

969. $p = 2a\cos t$.

970. $\theta = \varphi$, $\alpha = 2\varphi$. **974.** 3; -3. **975.** (1) 0; (2) 0; $\sqrt{3}$; $-\sqrt{3}$.

977. $\frac{f_1(t)\,f_2'(t)}{f_1'(t)} = \tan\theta$. **978.** $\arctan\frac{2}{3}bt^2 = \arctan\frac{2}{3}\varphi$.

979. $\varrho = \sqrt{a^2\cos^2 t + b^2\sin^2 t}$; $\varphi = \arctan\left(\frac{b}{a}\tan t\right)$; the tangent of the angle between the tangent and radius vector is equal to $\frac{2ab}{(b^2 - a^2)\sin 2t}$.

980. The polar subtangent is $S_T = \frac{\varrho^2}{\frac{d\varrho}{d\varphi}}$; the polar subnormal is $S_N = \frac{d\varrho}{d\varphi}$.

983. $\dfrac{\varrho}{\ln a}$. **984.** $\varrho \ln a$. **985.** $\sqrt{1+a^2}$.

986. $\dfrac{r}{\sqrt{r^2-x^2}} = \dfrac{r}{y}$. **987.** $\dfrac{\sqrt{b^4x^2+a^4y^2}}{b^2x}$.

988. $\sqrt{1+\dfrac{p}{2x}}\,dx$ or $\dfrac{\sqrt{y^2+p^2}}{y}\,dx$. **989.** $\sqrt{1+\dfrac{4}{9ax}}$.

990. $\sqrt{1+\cos^2 x}\,dx$. **991.** $\dfrac{e^x+e^{-x}}{2} = y$. **992.** r.

993. $2a \sin\dfrac{t}{2}$. **994.** $3a \cos t \sin t\,dt$. **995.** $a\sqrt{1+t^2}\,dt$.

996. $4a \sin\dfrac{t}{2}\,dt$. **997.** $a \cot t\,dt$. **998.** at.

999. $a\sqrt{\cosh 2t}\,dt$.

1000. $\dfrac{3}{2}$ m/min; the velocity vector is directed vertically downwards.

1001. $10\sqrt{26} \approx 51$ km/hr; the velocity vector is parallel to the hypotenuse of the right-angled triangle, one adjacent side of which is horizontal and equal to 50 km, whilst the other is vertical and equal to 10 km.

1002. 14·63 km/hr.

1003. 40 km/hr.

1004. $R_\omega\left(\sin\alpha + \dfrac{R\sin 2\alpha}{2\sqrt{l^2+R^2\sin^2\alpha}}\right)$.

1005. 9·43 m/sec. **1006.** 2. **1007.** $-24x$. **1008.** 207 360. **1009.** 360.

1010. $6(5x^4+6x^2+1)$. **1011.** $4\sin 2x$. **1012.** $\dfrac{4}{e}$. **1013.** $-\dfrac{1}{2}$.

1014. $\dfrac{5!}{(1-x)^6}$. **1015.** $\dfrac{6}{x}$. **1016.** $\dfrac{an(n+1)}{x^{n+2}}$. **1017.** $16a\sin 2\varphi$.

1018. $\dfrac{2(-1)^n n!}{(1+x)^{n+1}}$. **1019.** $2e^{x^2}(3x+2x^3)$. **1020.** $\dfrac{6x(2x^3-1)}{(x^3+1)^3}$.

1021. $\dfrac{2x}{1+x^2} + 2\arctan x$. **1022.** $-\dfrac{a^2}{\sqrt{(a^2-x^2)^3}}$.

1023. $-\dfrac{x}{\sqrt{(1+x^2)^3}}$. **1024.** $\dfrac{a+3\sqrt{x}}{4x\sqrt{x}\,(a+\sqrt{x})^3}$.

1025. $\dfrac{e^{\sqrt{x}}(\sqrt{x}-1)}{4x\sqrt{x}}$. **1026.** $-\dfrac{\arcsin x + x\sqrt{1-x^2}}{\sqrt{(1-x^2)^3}}$.

1027. $\dfrac{a(a^2-1)\sin x}{\sqrt{(1-a^2\sin^2 x)^3}}$. **1028.** $x^x\left[(\ln x+1)^2+\dfrac{1}{x}\right]$.

1029. $a^n e^{ax}$. **1030.** $(-1)^n e^{-x}$.

1031. $a^n \sin\left(ax + n\dfrac{\pi}{2}\right) + b^n \cos\left(bx + n\dfrac{\pi}{2}\right)$.

1032. $2^{n-1}\sin\left[2x + (n-1)\dfrac{\pi}{2}\right]$. **1033.** $e^x(x+n)$.

1034. $(-1)^n \dfrac{(n-2)!}{x^{n-1}}$ $(n \geqq 2)$. **1035.** $\dfrac{(-1)^n a^n n!}{(ax+b)^{n+1}}$.

1036. $\dfrac{(-1)^{n-1}a^n(n-1)!}{(ax+b)^n}$. **1037.** $(-1)^{n-1}\dfrac{(n-1)!}{x^n \ln a}$.

1038. $(-1)^n \dfrac{n!}{2}\left[\dfrac{1}{(x+1)^{n+1}} + \dfrac{1}{(x-1)^{n+1}}\right]$.

1039. $(-1)^n n!\left[\dfrac{1}{(x-2)^{n+1}} - \dfrac{1}{(x-1)^{n+1}}\right]$.

1040. $4^{n-1}\cos\left(4x + n\dfrac{\pi}{2}\right)$. **1054.** $\dfrac{d^2x}{dy^2} = -\dfrac{\dfrac{d^2y}{dx^2}}{\left(\dfrac{dy}{dx}\right)^3}$.

1056. $-\dfrac{b^4}{a^2y^3}$. **1057.** $-\dfrac{3r^2x}{y^5}$. **1058.** $-\dfrac{2(3y^4+8y^2+5)}{y^8}$.

1059. $\dfrac{(3-s)e^{2s}}{(2-s)^3}$. **1060.** $-\dfrac{2a^3xy}{(y^2-ax)^3}$.

1061. $-\dfrac{y}{[1-\cos(x+y)]^3}$. **1062.** $-\dfrac{y[(x-1)^2+(y-1)^2]}{x^2(y-1)^3}$.

1063. $\dfrac{d^2x}{dy^2} = \dfrac{\dfrac{d^2y}{dx^2}}{\left(\dfrac{dy}{dx}\right)^3}$. **1064.** $\dfrac{1}{e^2}$. **1065.** $-\dfrac{p^2}{\sqrt{(y^2+p^2)^3}}$.

1069. $-\dfrac{2a}{9b^2t^4}$. **1070.** $-\dfrac{a^2}{y^3} = -\dfrac{1}{a\sin^3 t}$. **1071.** $-\dfrac{3b\cos t}{a^3\sin^5 t}$.

1072. $-\dfrac{1}{a(1+\cos\varphi)^2}$. **1073.** (1) $\dfrac{\cos^2 t - 4\sin^2 t}{9a^2\cos^7 t\sin^3 t}$; (2) 0, since $x + y = 0$.

1074. (1) $4t^2$; (2) $-\dfrac{2}{1-t^2}$. **1075.** $\dfrac{2+t^2}{a(\cos t - t\sin t)^3}$.

1080. 16 m/sec². **1081.** $v = 2t - 4$, $a = 2$. **1082.** $-\dfrac{\pi^2}{18}$ cm/sec².

1084. $-0{\cdot}0015$ m/sec². **1085.** $-\dfrac{1}{8}$ m/sec².

1088. (1) $(x^2 - 379)\sin x - 40x\cos x$; (2) $e^x \sum_{k=0}^{n} C_n^k \sin\left(x + k\dfrac{\pi}{2}\right)$

$$(3)\ \alpha^n x^3 \sin\left(\alpha x + n\frac{\pi}{2}\right) + 3n\alpha^{n-1}x^2\sin\left[\alpha x + (n-1)\frac{\pi}{2}\right] + \\ + 3n(n-1)\,\alpha^{n-2}x\sin\left[\alpha x + (n-2)\frac{\pi}{2}\right] + \\ + n(n-1)(n-2)\,\alpha^{n-3}\sin\left[\alpha x + (n-3)\frac{\pi}{2}\right].$$

1093. $y^{(2n)}(0) = 0$; $y^{(2n+1)}(0) = [1\,.\,3\,.\,5 \ldots (2n+1)]^3$.

1095. $y^{(2n-1)}(0) = 0$; $y^{(2n)}(0) = 2\,[2\,.\,4\,.\,6 \ldots (2n-2)]^2$.

1096. $-\dfrac{2dx^2}{9x\sqrt[3]{x}}$. **1097.** $m(m-1)(m-2)\,x^{m-3}\,dx^3$.

1098. $4(x+1)(5x^2 - 2x - 1)\,dx^2$.

1099. $4^{-x^2}(2\ln 4)(2x^2\ln 4 - 1)\,dx^2$.

1100. $\dfrac{ab(a^2-b^2)\sin 2x\,dx^2}{(a^2\cos^2 x + b^2\sin^2 x)^2}$. **1101.** $\dfrac{4\ln x - 4 - \ln^3 x}{x^2\sqrt{(\ln^2 x - 4)^3}}\,dx^2$.

1102. $-4\sin 2x\,dx^3$. **1103.** $\pm\dfrac{3a\sec^2\varphi}{4\sqrt{\tan\varphi}}(1 + 5\tan^2\varphi)\,d\varphi^2$.

1104. $\dfrac{a^{\frac{2}{3}}\,dx^2}{3x^{\frac{4}{3}}\,y^{\frac{1}{3}}}$. **1105.** (1) $d^2y = \dfrac{4x}{x^4-1}\,d^2x - \dfrac{4(1+3x^4)}{(x^4-1)^2}\,dx^2$;

(2) $d^2y = -4\sec^2 2t\,dt^2$.

1106. (1) $d^2y = \cos z\,d^2z - \sin z\,dz^2$;

(2) $d^2y = a^x\cos(a^x)\ln a\,d^2x - a^x\ln^2 a(a^x\sin a^x - \cos a^x)\,dx^2$

(3) $d^2y = a^{t^3}\ln a[\cos a^{t^3}(6t + 9t^4\ln a) - a^{t^3}\sin a^{t^4}\,9t^4\ln a]\,dt^2$.

Chapter IV

1110. (1) The function has a maximum; (2) is decreasing; (3) is increasing; (4) has a minimum; (5) has a maximum; (6) has a minimum; (7) has a minimum; (8) has a maximum; (9) has a minimum.

1112. The function is increasing at $x_1 = 0$, decreasing at $x_2 = 1$, increasing at $x_3 = -\frac{\pi}{2}$ and decreasing at $x_4 = 2$.

1113. The function is decreasing at $x_1 = \frac{1}{2}$, is increasing at $x_2 = 2$ and $x_3 = e$; has a minimum at $x_4 = 1$.

1114. The function is increasing at $x_1 = 1$, decreasing at $x_2 = -1$, has a minimum at $x_3 = 0$.

1115. The function is decreasing at $x_1 = \frac{1}{2}$, increasing at $x_2 = -\frac{1}{2}$, has a maximum at $x_3 = 0$.

1125. Three roots, lying respectively in the intervals (1, 2), (2, 3) and (3, 4).

1127. $\sin 3x_2 - \sin 3x_1 = 3(x_2 - y_1) \cos 3\xi$, where $x_1 < \xi < x_2$.

1128. $a(1 - \ln a) - b(1 - \ln b) = (b - a) \ln \xi$, where $a < \xi < b$.

1129. $\arcsin [2(x_0 + \Delta x)] - \arcsin 2x_0 = \frac{2\,\Delta x}{\sqrt{1 - 4\xi^2}}$, where $x_0 < \xi < x_0 + \Delta x$.

1135. As $x \to 0$, ξ tends to zero, but without taking all intermediate values: it only takes a sequence of these such that $\cos \frac{1}{\xi}$ tends to zero.

1136. 0·833. **1137.** 0·57. **1138.** 1·0414. **1139.** 0·1990. **1140.** 0·8449.

1141. 1·7853.

1149*. The required inequality follows from the increase of the function $y = \frac{\tan x}{x}$ in the interval $\left(0, \frac{\pi}{2}\right)$.

1150. The function is increasing in $(-\infty, -1)$, decreasing in $(-1, 3)$, increasing in $(3, \infty)$.

1151. The function is decreasing in $(-\infty, -1)$, increasing in $(-1, 0)$, decreasing in $(0, 1)$, increasing in $(1, \infty)$.

1152. The function is increasing in $\left(-\infty, -\frac{1}{2}\right)$, decreasing in $\left(-\frac{1}{2}, \frac{11}{18}\right)$, increasing in $\left(\frac{11}{18}, \infty\right)$.

1153. The function is decreasing in $\left(-\infty, \frac{a}{2}\right)$, increasing in $\left(\frac{a}{2}, \frac{2}{3}a\right)$, decreasing in $\left(\frac{2}{3}a, a\right)$, increasing in (a, ∞).

1154. The function is increasing in $(-\infty, -1)$, decreasing in $(-1, 1)$, increasing in $(1, \infty)$.

1155. The function is decreasing in $(-\infty, 0)$, decreasing in $\left(0, \frac{1}{2}\right)$, increasing in $\left(\frac{1}{2}, 1\right)$, decreasing in $(1, \infty)$.

1156. The function is increasing in $(-\infty, 0)$, decreasing in $(0, \infty)$.

1157. The function is decreasing in $(-\infty, 0)$, increasing in $(0, 2)$, decreasing in $(2, \infty)$.

1158. The function is decreasing in $(0, 1)$, decreasing in $(1, e)$, increasing in (e, ∞).

1159. The function is decreasing in $\left(0, \frac{1}{2}\right)$, increasing in $\left(\frac{1}{2}, \infty\right)$.

1160. The function is decreasing in $\left(0, \frac{\pi}{3}\right)$, increasing in $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$, decreasing in $\left(\frac{5\pi}{3}, 2\pi\right)$.

1161. The function is increasing in $\left(0, \frac{\pi}{6}\right)$, decreasing in $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ increasing in $\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$, decreasing in $\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$, increasing in $\left(\frac{3\pi}{2}, 2\pi\right)$.

1162. Increasing monotonically. **1163.** Increasing monotonically.

1164. $\left(0, \frac{3}{4}a\right)$, increasing; $\left(\frac{3}{4}a, a\right)$, decreasing.

1165. $y_{max} = 0$ for $x = 0$, $y_{min} = -1$ for $x = 1$.

1166. $y_{max} = 17$ for $x = -1$, $y_{min} = -47$ for $x = 3$.

1167. $y_{max} = 4$ for $x = 0$, $y_{min} = \frac{8}{3}$ for $x = -2$.

1168. $y_{max} = 2$ for $x = 0$, $y_{min} = \sqrt[3]{4}$ for $x = 2$.

1169. $y_{max} = \frac{1}{\ln 3}$ for $x = -3$. **1170.** $y_{max} = 0$ for $x = 0$.

1171. $y_{max} = 0$ for $x = 0$, $y_{min} = -\frac{2}{3}$ for $x = 1$.

1172. $y_{min} = 2$ for $x = \frac{2}{3}$. **1173.** $y_{max} = \frac{\sqrt{205}}{10}$ for $x = \frac{12}{5}$.

1174. $y_{max} = \sqrt[3]{a^4}$ for $x = 0$. $y_{min} = 0$ for $x = \pm a$.

1175. $y_{min} = 0$ for $x = 0$.

1176. Increasing monotonically.

1177. $y_{max} = \frac{81}{3}\sqrt[3]{18}$ for $x = \frac{1}{2}$, $y_{min} = 0$ for $x = -1$ and $x = 5$.

1178. $y_{max} = 2{\cdot}5$ for $x = 1$, $y_{min} = \frac{e(4-e)}{2} \approx 1{\cdot}76$ for $x = e$.

1179. $y_{max} = \frac{1}{2}$ for $x = 0$, $y_{min} = \frac{\pi}{8}$ for $x = 1$.

1180. $y_{\max} = 0$ for $x = 0$, $y_{\min} = \dfrac{3\sqrt{3} - 2\pi}{48}$ for $x = \dfrac{1}{2}$.

1181. $y_{\max} = \dfrac{6\pi\sqrt{3} - \pi^2 + 18}{36} \approx 1{\cdot}13$ for $x = \pm\dfrac{\pi}{3}$,

$y_{\min} = 1$ for $x = 0$.

1182. $y_{\max} = \sin\dfrac{1}{2} + 16$ for $x = \dfrac{1}{2}$,

$$y_{\min} = \frac{9\sqrt{3} - 3\pi\sqrt{3} + 18 - \pi^2 + 6\pi}{36} \text{ for } x = \frac{\pi}{6}.$$

1183. $y_{\max} = \dfrac{1}{\pi}$ for $x = 1$, $y_{\min} = -\dfrac{1}{\pi}$ for $x = 3$.

1184. If $ab \leqq 0$, there are no extrema. If $ab > 0$ and $a > 0$, $y_{\min} = 2\sqrt{ab}$ for $x = \dfrac{1}{2p}\ln\dfrac{b}{a}$; if $ab > 0$ and $a < 0$, $y_{\max} = -2\sqrt{ab}$ for $x = \dfrac{1}{2p}\ln\dfrac{b}{a}$. **1185.** 13 and 4. **1186.** 8 and 0.

1187. 2 and -10. **1188.** 2 and -12. **1189.** 10 and 6.

1190. 1 and $\dfrac{3}{5}$. **1191.** $\dfrac{3}{5}$ and -1.

1192. The minimum value is $(a + b)^2$, there is no maximum.

1193. $\dfrac{\pi}{2}$ and $-\dfrac{\pi}{2}$.

1194. The maximum value is 1, there is no minimum.

1195. The minimum value is $\left(\dfrac{1}{e}\right)^{\frac{1}{e}}$, there is no maximum.

1196. $\sqrt[3]{9}$ and 0. **1197.** $\dfrac{\pi}{4}$ and 0. **1208.** 4 and 4. **1209.** 1.

1210. 6 and 6. **1211.** 3, 6 and 4 cm. **1212.** 3 cm. **1213.** 1 cm.

1214. $\sqrt[3]{4v}$ **1215.** Base radius = height $= \sqrt[3]{\dfrac{v}{\pi}}$. **1216.** $H = 2R$.

1217. $\dfrac{20\sqrt{3}}{3}$ cm. **1218.** $2\pi\sqrt{\dfrac{2}{3}} \approx 293°56'$.

1219. Lateral side $= \dfrac{3p}{4}$, base $= \dfrac{p}{2}$.

1220. Lateral side $= \dfrac{3p}{5}$, base $= \dfrac{4p}{5}$.

1221. $\frac{2R\sqrt{3}}{3}$. **1222.** $\frac{4}{3}R$. **1223.** $\frac{2m_0}{3k}$ sec, $\frac{2}{27}\frac{m_0^3 g^2}{k^2}\frac{\text{g cm}^2}{\text{sec}^2}$.

1224. $\sqrt{\frac{2aP}{k}}$. **1225.** 20 km/hr, 720 roubles.

1226. After $1\,{}^{27}/_{43}$ hours $\approx$ 1 hour 38 min.

1227. The distance of the chord from point A must be equal to $\frac{3}{4}$ the diameter of the circle.

1228. $\frac{4R\sqrt{5}}{5}$ and $\frac{R\sqrt{5}}{5}$.

1229. The height of the rectangle is equal to $\frac{\sqrt{8R^2+h^2}-3h}{4}$, where h is the distance of the chord subtended by the arc from the centre, and R is the radius of the circle.

1230. The base radius of the cone must be one and a half times the cylinder radius.

1231. $4R$. **1232.** $\approx 49°$. **1233.** $60°$. **1234.** $R\sqrt{3}$. **1235.** $\frac{4}{3}R$.

1237. $\frac{x}{3}+\frac{y}{6}=1$. **1238.** $a\sqrt{2}$ and $b\sqrt{2}$.

1239. The area of the rectangle $=\frac{2}{\pi}\times$ the area of the ellipse.

1240. The point (2, 3).

1241. $C(-\sqrt{6},\ -\sqrt{6})$ for a maximum, $C(\sqrt{6},\ \sqrt{6})$ for a minimum.

1242. $x = a - p$, if $a > p$; $x = 0$ if $a \leqq p$.

1243. The gutter section is a semicircle.

1244. The length of the beam $= 13\,{}^1/_3$ m, the side of the cross-section $=\frac{2\sqrt{2}}{3}$ m.

1245. The required value is equal to the arithmetic mean of the results of the measurements:

$$x=\frac{x_1+x_2+\ldots+x_n}{n}.$$

1246. At 3 km from the camp. **1247.** At height $\frac{R\sqrt{2}}{2}$.

1248. The distance from the source of intensity I_1 is $\frac{l\sqrt[3]{I_1}}{\sqrt[3]{I_1}+\sqrt[3]{I_2}}$; in other words, the distance l is divided by the required point in the ratio $\sqrt[3]{I_1} : \sqrt[3]{I_2}$.

1249. 2·4 m.

1250. $F_{\min} = \frac{kP}{\sqrt{1+k^2}}$ for $\varphi = \text{arc} \tan k$.

1251. $\approx 4{\cdot}5$. **1252.** $2b + \sqrt{\frac{Sb}{a}}$ and $2a + \sqrt{\frac{Sa}{b}}$.

1253*. $\frac{RHL}{(L-R)(L+2R)}$, where L is the generator of the cone. Notice that the difference between the distance from the centre of the sphere to the vertex of the cone, and the radius of the sphere, is equal to the difference between the height of the cone and the height of the submerged segment.

1254. $\frac{R}{4}$. **1255.** $\frac{R}{2}$. **1256.** $P(p, \pm p\sqrt{2})$.

1263*. $\frac{3}{4}$. Since the function is a constant ($y' = 0$), the value of the constant is equal to the value of the function for any x, e.g. for $x = 0$.

1264. π. **1265.** 0.

1267. $y_{\max} = \frac{4}{27}a^3$ at $x = \frac{a}{3}$, $y_{\min} = 0$ for $x = a$.

1268. $y_{\max} = \frac{a^4}{16}$ for $x = \frac{a}{2}$, $y_{\min} = 0$ for $x = 0$ and $x = a$.

1269. $y_{\max} = -2a$ for $x = -a$, $y_{\min} = 2a$ for $x = a$.

1270. $y_{\max} = \frac{5}{4}$ for $x = \frac{3}{4}$.

1271. $y_{\max} = 1$ for $x = 1$, $y_{\min} = -1$ for $x = -1$.

1272. $y_{\min} = 2$ for $x = 0$.

1273. $y_{\max} = \frac{4}{e^2}$ for $x = 2$, $y_{\min} = 0$ for $x = 0$.

1274. $y_{\min} = e$ for $x = e$.

1275. $y_{\max} = \sqrt[e]{e}$ for $x = e$.

1276. Maximum at $a = 2$.

1277. $a = -\frac{2}{3}$, $b = -\frac{1}{6}$.

1278. Convex in the neighbourhood of $(1, 11)$, concave in the neighbourhood of $(3, 3)$.

1279. Convex in the neighbourhood of $\left(1, \frac{\pi}{4}\right)$, concave in the neighbourhood of $\left(-1, -\frac{\pi}{4}\right)$.

1280. Convex in the neighbourhood of $\left(\frac{1}{e^2}, -\frac{2}{e^4}\right)$, concave in the neighbourhood of $(1, 0)$.

1287. Point of inflexion $\left(\frac{5}{3}, -\frac{250}{27}\right)$. Intervals: convex in $\left(-\infty, \frac{5}{3}\right)$, concave in $\left(\frac{5}{3}, \infty\right)$.

1288. No point of inflexion, the graph is concave.

1289. Points of inflexion $(2, 62)$ and $(4, 206)$. Intervals: concave in $(-\infty, 2)$, convex in $(2, 4)$, concave in $(4, \infty)$.

1290. Points of inflexion $(-3, 294)$ and $(2, 114)$. Intervals: convex in $(-\infty, -3)$, concave in $(-3, 2)$, convex in $(2, \infty)$.

1291. Point of inflexion $(1, -1)$. Intervals: convex in $(-\infty, 1)$, concave in $(1, \infty)$.

1292. No point of inflexion. The graph is concave.

1293. Points of inflexion $\left(-3a, -\frac{9a}{4}\right)$, $(0, 0)$, $\left(3a, \frac{9a}{4}\right)$. Intervals: concave in $(-\infty, -3a)$, convex in $(-3a, 0)$, concave in $(0, 3a)$, convex in $(3a, \infty)$.

1294. Point of inflexion (b, a). Intervals: convex in $(-\infty, b)$, concave in (b, ∞).

1295. Point of inflexion $\left(\arcsin\frac{\sqrt{5}-1}{2}, e^{\frac{\sqrt{5}-1}{2}}\right)$. Intervals: concave in $\left(-\frac{\pi}{2}, \arcsin\frac{\sqrt{5}-1}{2}\right)$, convex in $\left(\arcsin\frac{\sqrt{5}-1}{2}, \frac{\pi}{2}\right)$.

1296. Points of inflexion $(\pm 1, \ln 2)$. Intervals: convex in $(-\infty, -1)$, concave in $(-1, 1)$, convex in $(1, \infty)$.

1297. Point of inflexion $\left(a\,e^{\frac{3}{2}},\ \frac{3}{2}\,e^{-\frac{3}{2}}\right)$. Intervals: convex in $\left(0,\ a\,e^{\frac{3}{2}}\right)$, concave in $\left(a\,e^{\frac{3}{2}},\ \infty\right)$.

1298. No point of inflexion. The graph is concave.

1299. Point of inflexion $\left(\frac{1}{2},\ e^{\arctan\frac{1}{2}}\right)$. Intervals: concave in $\left(-\infty,\ \frac{1}{2}\right)$, convex in $\left(\frac{1}{2},\ \infty\right)$.

1300. Point of inflexion $(1, -7)$. Intervals: convex in $(0, 1)$, concave in $(1, \infty)$.

1305. $a = -\frac{3}{2}$, $b = \frac{9}{2}$.

1306. $\alpha = -\frac{20}{3}$, $\beta = \frac{4}{3}$. The points $(-2,\ -2{\cdot}5)$ and $(0, 0)$ are also points of inflexion.

1307. For $a \leqq -\frac{e}{6}$ and for $a > 0$.

1316. Points of inflexion $(1, 4)$ and $(1, -4)$.

1317. Points of inflexion for $t = \frac{3\pi}{4} \pm k\pi$ $(k = 0, 1, 2, \ldots)$.

1318. $\dfrac{\sin b - \sin a}{\ln \dfrac{b}{a}} = \xi \cos \xi$, where $a < \xi < b$.

1319. $e^b + e^a = 2\,e^{\xi}$, where $a < \xi < b$.

1324. $\dfrac{2}{3\sqrt[6]{a}}$. **1325.** 0. **1326.** 1. **1327.** $\dfrac{\alpha}{\beta}$. **1328.** $\dfrac{1}{3}$. **1329.** $\dfrac{a}{\sqrt{b}}$.

1330. $-\dfrac{1}{2}$. **1331.** 2. **1332.** $\dfrac{m}{n}a^{m-n}$. **1333.** $\dfrac{\ln \frac{a}{b}}{\ln \frac{c}{d}}$. **1334.** -2.

1335. 2. **1336.** $\ln\dfrac{a}{b}$. **1337.** $\cos a$. **1338.** 2. **1339.** 1. **1340.** 1.

1341. $\dfrac{1}{128}$. **1342.** 16. **1343.** 1. **1344.** 1. **1345.** -2. **1346.** 0.

1347. 0. **1348.** a. **1349.** $\dfrac{1}{2}$. **1350.** $\dfrac{4a^2}{\pi}$. **1351.** -1. **1352.** 0.

1353. ∞. **1354.** $\dfrac{a+b+c}{3}$. **1355.** 1. **1356.** ∞. **1357.** 1.

1358. 1. **1359.** e. **1360.** 1. **1361.** e^2. **1362.** $e^{\frac{2}{\pi}}$. **1363.** 1. **1364.** $\frac{1}{2}$.

1366. x^x is greater than $a^x x^a$.

1367. $f(x)$ is greater than $\ln f(x)$.

1374. $f(115) = 1{,}520{,}990$; $f(120) = 1{,}728{,}120$; $\delta_{x=100} = 0{\cdot}03$ (absolute error).

1375. $y = \pm \frac{b}{a} x$. **1376.** $x = 0$, $y = 0$. **1377.** $y = 0$.

1378. $x = b$, $y = c$. **1379.** $x = -1$, $y = \frac{1}{2} x - 1$.

1380. $x + y = 0$. **1381.** $y = x + 2$. **1382.** $y = \pm x$.

1383. $x = 0$, $y = 0$; $x + y = 0$.

1384. $x = b$; $x = 2b$. $y = x + 3(b - a)$.

1385. $y + 1 = 0$; $2x + y + 1 = 0$. **1386.** $x = -\frac{1}{e}$, $y = x + \frac{1}{e}$.

1387. $x = 0$, $y = x$. **1388.** $x = 0$, $y = x + 3$. **1389.** $y = \frac{\pi}{2} x - 1$.

1390. $y = 2x \pm \frac{\pi}{2}$.

1391. $y = x$, if $f(x)$ is not identically constant.

1392. If $\lim\limits_{t \to t_0} \varphi(t) = \infty$, whilst $\lim\limits_{t \to t_0} \psi(t) = b$, $y = b$ is an asymptote; if $\lim\limits_{t \to t_0} \psi(t) = \infty$, whilst $\lim\limits_{t \to t_0} \varphi(t) = a$, $x = a$ is an asymptote.

1393. $x = -1$, $y = 0$. **1394.** $y = \frac{1}{2} x + e$. **1395.** $y = \pm \frac{1}{2} x - \frac{1}{2}$.

1396. $x + y + a = 0$. **1397.** $x = 2$; $2x + 8y + 1 = 0$; $6x - 40y + 9 = 0$.

1398. Is defined everywhere. The graph is symmetrical with respect to the origin. $y_{\text{max}} = \frac{1}{2}$ for $x = 1$, $y_{\text{min}} = -\frac{1}{2}$ for $x = -1$. The graph has points of inflexion at $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$, $(0, 0)$ and $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$. The asymptote is $y = 0$.

1399. Defined everywhere except for $x = \pm 1$. The graph is symmetrical with respect to the axis of ordinates. There are no maxima. $y_{\text{min}} = 1$ for $x = 0$. No points of inflexion. Asymptotes $x = \pm 1$, $y = 0$.

1400. Defined everywhere except for $x = \pm 1$. The graph is symmetrical with respect to the origin. There are no extrema. Point of inflexion $(0, 0)$. Asymptotes $x = -1$, $x = 1$, $y = 0$.

1401. Defined everywhere except for $x = 1$, $x = 2$ and $x = 3$. $y_{max} \approx -2{\cdot}60$ for $x \approx 2{\cdot}58$, $y_{min} \approx 2{\cdot}60$ for $x \approx 1{\cdot}42$. No points of inflexion. Asymptotes $x = 1$, $x = 2$, $x = 3$, $y = 0$.

1402. Not defined for $x = \pm 1$. The graph is symmetrical with respect to the axis of ordinates. $y_{max} = 0$ for $x = 0$. There are no minima. Increasing for $x < -1$, decreasing for $x > 1$. No points of inflexion. Asymptotes $x = \pm 1$, $y = 1$.

1403. Defined everywhere; the graph is symmetrical with respect to the axis of ordinates. $y_{max} = -1$ for $x = 0$; $(1, 0)$ and $(-1, 0)$ are points of inflexion where the tangent is horizontal; $\left(\pm\frac{\sqrt{5}}{5}, -\frac{64}{125}\right)$ are points of inflexion. No asymptotes.

1404. Defined everywhere; the graph is symmetrical with respect to the axis of ordinates. $y_{max} = 0$ for $x = 0$, $y_{min} = -\frac{27}{8}$ for $x = \pm\frac{1}{2}$. Points of inflexion with horizontal tangent at $(\pm 1, 0)$ Four further points of inflexion at $x \approx \pm 0{\cdot}7$ and $x \approx \pm 0{\cdot}26$. No asymptotes.

1405. Defined everywhere except at $x = 0$. $y_{min} = 3$ for $x = \frac{1}{2}$. No maxima. Points of inflexion at $\left(-\frac{\sqrt[3]{2}}{2}, 0\right)$. Asymptote $x = 0$.

1406. Defined everywhere except $x = 0$. The graph is symmetrical with respect to the axis of ordinates. $y_{min} = 2$ for $x = \pm 1$. No maxima. No points of inflexion. Asymptote $x = 0$.

1407. Defined everywhere except $x = 1$. $y_{min} = -1$ for $x = 0$. No maxima. Point of inflexion at $\left(-\frac{1}{2}, -\frac{8}{9}\right)$. Asymptotes $x = 1$ and $y = 0$.

1408. Defined everywhere except $x = \pm\sqrt{3}$. The graph is symmetrical with respect to the origin. $y_{max} = -4{\cdot}5$ for $x = 3$, $y_{min} = 4{\cdot}5$ for $x = -3$. Point of inflexion at $(0, 0)$. Asymptotes $x = \pm\sqrt{3}$ and $x + y = 0$.

1409. Defined everywhere except $x = -1$. No minima. $y_{max} = -3\frac{3}{8}$ at $x = -3$. Point of inflexion at $(0, 0)$. Asymptotes $x = -1$ and $y = \frac{1}{2}x - 1$.

1410. Defined everywhere except $x = 1$. No maxima. $y_{\min} = \frac{27}{4}$ for $x = \frac{3}{2}$. Point of inflexion at (0, 0). Asymptote $x = 1$.

1411. Defined everywhere except $x = 1$. $y_{\max} = 0$ for $x = 0$, $y_{\min} = \frac{4}{3}\sqrt[3]{4}$ for $x = \sqrt[3]{4}$. Point of inflexion $\left(-\sqrt[3]{2}, -\frac{2}{3}\sqrt[3]{2}\right)$. Asymptotes $x = 1$ and $y = x$.

1412. Defined everywhere except $x = -1$. $y_{\max} = \frac{2}{27}$ at $x = 5$, $y_{\min} = 0$ at $x = 1$. Abscissae of the points of inflexion are $5 \pm 2\sqrt{3}$. Asymptotes $x = -1$ and $y = 0$.

1413. Defined everywhere except $x = 0$. $y_{\max} = \frac{7}{2}$ for $x = 1$, $y_{\max} = -\frac{11}{6}$ for $x = -3$, $y_{\min} = \frac{27}{8}$ for $x = 2$. Abscissae of point of inflexion is $\frac{9}{7}$. Asymptotes $x = 0$ and $y = \frac{1}{2}x + 1$.

1414. Defined everywhere except $x = 0$. No maxima. $y_{\min} \approx -0{\cdot}28$ at $x \approx 1{\cdot}46$. Abscissa of point of inflexion $-\sqrt[3]{2}$. Asymptote $x = 0$.

1415. Defined everywhere except at $x = 0$. $y_{\max} = -2{\cdot}5$ for $x = -2$; no minima. No points of inflexion. Asymptotes $x = 0$ and $y = x$.

1416. Defined everywhere. $y_{\max} = \frac{1}{e}$ for $x = 1$. No minima. Point of inflexion at $\left(2, \frac{2}{e^2}\right)$. Asymptote $y = 0$.

1417. Defined everywhere. $y_{\max} = \frac{4}{e^2}$ for $x = 2$, $y_{\min} = 0$ for $x = 0$. Abscissae of points of inflexion are $2 \pm \sqrt{2}$. Asymptote $y = 0$.

1418. Defined everywhere except at $x = 0$. $y_{\min} = e$ for $x = 1$. No maxima. No points of inflexion. Asymptotes $x = 0$, $y = 0$.

1419. Defined for $x > -1$. $y_{\min} = 0$ for $x = 0$. No maxima. No points of inflexion. Asymptote $x = -1$.

1420. Defined everywhere. The graph is symmetrical with respect to the axis of ordinates. $y_{\min} = 0$ for $x = 0$. No maxima. Points of inflexion at $(\pm 1, \ln 2)$. No asymptotes.

1421. Defined everywhere. The graph is symmetrical with respect to the axis of ordinates. $y_{\max} = \frac{1}{e}$ for $x = \pm 1$, $y_{\min} = 0$ for $x = 0$.

Abscissae of the points of inflexion are $\pm \dfrac{\sqrt{5 \pm \sqrt{17}}}{2}$. Asymptote $y = 0$.

1422. Defined everywhere. $y_{\max} = \dfrac{27}{e^3}$ at $x = 3$. No minima. Abscissae of points of inflexion are 0 and $3 \pm \sqrt{3}$. Asymptote $y = 0$.

1423. Defined everywhere. The graph is symmetrical with respect to the origin. $y_{\max} = \dfrac{1}{\sqrt{e}}$ for $x = 1$, $y_{\min} = -\dfrac{1}{\sqrt{e}}$ for $x = -1$. Points of inflexion at $(0, 0)$, $\left(\sqrt{3},\ \sqrt{3}\, e^{-\frac{3}{2}}\right)$ and $\left(-\sqrt{3},\ -\sqrt{3}\, e^{-\frac{3}{2}}\right)$. Asymptote $y = 0$.

1424. Defined everywhere except at $x = 0$. No extrema. The graph has no points of inflexion. Asymptotes $x = 0$, $y = 0$ and $y = -1$.

1425. Defined for $x > 0$. No extrema. Point of inflexion at $\left(e^{\frac{3}{2}},\ e^{\frac{3}{2}} + \dfrac{3}{2} e^{-\frac{3}{2}}\right)$. Asymptotes $x = 0$ and $y = x$.

1426. The function is defined for $-\infty < x < -1$ and for $0 < x < \infty$. It increases from e to ∞ in the interval $(-\infty, -1)$; it increases from 1 to e in the interval $(0, +\infty)$. The graph consists of two separate branches. Asymptotes are $y = e$ and $x = -1$.

1427. Defined everywhere. No extrema. Stationary at $x = \pm k\pi$ $(k = 1, 3, 5, \ldots)$. The graph is symmetrical with respect to the origin and has no asymptotes; points of inflexion $(k\pi, k\pi)$ $(k = 0, \pm 1, \pm 2, \ldots)$; the graph cuts the straight line $y = x$ at the points of inflexion.

1428. Defined everywhere. The graph is symmetrical with respect to the axis of ordinates. The extremal points satisfy the equation $\tan x = -x$. The abscissae of the points of inflexion satisfy the equation $x \tan x = 2$. No asymptotes.

1429. Defined in the intervals $\left(-\dfrac{\pi}{2} + 2k\pi,\ \dfrac{\pi}{2} + 2k\pi\right)$, where $k = 0, \pm 1, \pm 2, \ldots$ Period 2π. The graph is symmetrical with respect to the axis of ordinates. $y_{\max} = 0$ for $x = 2k\pi$. No points of inflexion. Asymptotes $x = \dfrac{\pi}{2} + k\pi$.

1430. Defined in the intervals $\left(-\dfrac{\pi}{2} + 2k\pi,\ \dfrac{\pi}{2} + 2k\pi\right)$, where $k = 0, \pm 1, \pm 2, \ldots$ Period 2π. The graph is symmetrical with respect to the axis of ordinates. $y_{\min} = 1$ for $x = 2k\pi$. The graph has no points of inflexion. Asymptotes $x = \dfrac{\pi}{2} + k\pi$.

1431. Defined everywhere. The graph is symmetrical with respect to the origin. $y_{\max} = \frac{\pi}{2} - 1$ for $x = -1$, $y_{\min} = 1 - \frac{\pi}{2}$ for $x = 1$. Point of inflexion at (0, 0). Asymptotes $y = x \pm \pi$.

1432. Defined everywhere except for $x = 1$ and $x = 3$. $y_{\max} = \frac{1}{e}$ for $x = 2$. No minima. No points of inflexion. Asymptotes $x = 1$, $x = 3$ and $y = 1$.

1433. Defined everywhere. Period 2π. $y_{\min} = 1$ for $x = k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$; $y_{\max} = e - 1$ for $x = \frac{\pi}{2} + 2k\pi$ and $y_{\max} = 1 + \frac{1}{e}$ for $x = \frac{3}{2}\pi + 2k\pi$. No asymptotes.

1434. Defined everywhere. $y_{\max} = \frac{4}{27}$ for $x = \frac{8}{27}$, $y_{\min} = 0$ for $x = 0$. No points of inflexion, no asymptotes.

1435. Defined everywhere. The graph is symmetrical with respect to the axis of ordinates. $y_{\max} = 0$ for $x = 0$, $y_{\min} = -3$ for $x = \pm 1$. No points of inflexion, no asymptotes.

1436. Defined everywhere. The graph is symmetrical with respect to the origin. $y_{\max} = \frac{2}{3}$ for $x = 1$, $y_{\min} = -\frac{2}{3}$ for $x = -1$. Point of inflexion at (0, 0). No asymptotes.

1437. Defined everywhere. $y_{\max} = 2$ for $x = 0$, $y_{\min} = 0$ for $x = -1$. Point of inflexion at $\left(-\frac{1}{2}, 1\right)$. Asymptote $y = 1$.

1438. Defined everywhere. $y_{\max} \approx 2{\cdot}2$ at $x = \frac{7}{11}$, $y_{\min} = 0$ for $x = 1$. Abscissae of points of inflexion are -1 and $\frac{7 \pm 3\sqrt{3}}{11}$. No asymptotes.

1439. Defined everywhere. $y_{\max} = 2\sqrt[3]{4}$ for $x = 4$, $y_{\min} = 0$ for $x = 0$. Point of inflexion at (6, 0). Asymptote $x + y = 2$.

1440. The function is defined for $x \geqq 0$ and is two-valued. The function $y = x + \sqrt{x^5}$ (upper branch of graph) increases monotonically. The function $y = x - \sqrt{x^5}$ (lower branch of graph) has a maximum at $x = \frac{\sqrt[3]{20}}{5}$. The graph has no points of inflexion and no asymptotes.

1441. Defined for $x \geqq 0$, two-valued. The function $y = x^2 + \sqrt{x^5}$ (upper branch of graph) increases monotonically. The function $y = x^2 - \sqrt{x^5}$ (lower branch of graph) has a maximum at $x = \frac{16}{25}$. The abscissa of the point of inflexion of the lower branch is $\frac{64}{225}$. No asymptotes.

1442. Defined for $x \geqq -1$, two-valued. No extrema. The graph is symmetrical with respect to the axis of abscissae, has points of inflexion $(0, 1)$ and $(0, -1)$. No asymptotes.

1443. Defined in the intervals $[-1, 0]$ and $[1, \infty]$, two-valued. The graph is symmetrical with respect to the axis of abscissae. $|y|_{\max} = \frac{\sqrt[4]{12}}{3}$ for $x = -\frac{\sqrt{3}}{3}$. The abscissa of the point of inflexion is $\sqrt{1 + \frac{\sqrt{12}}{3}}$. No asymptotes.

1444. Defined for $x \geqq 0$, two-valued. The graph is symmetrical with respect to the axis of abscissae. $|y|_{\max} = \frac{\sqrt{12}}{9}$ for $x = \frac{1}{3}$. The graph has no points of inflexion. No asymptotes.

1445. Defined for $x = 0$ and for $x \geqq 1$. The origin is an isolated point. The graph is symmetrical with respect to the axis of abscissae. No extremals. Points of inflexion at $\left(\frac{4}{3}, \pm \frac{4\sqrt{3}}{9}\right)$. No asymptotes.

1446. Defined for $x < 0$ and for $x \geqq \sqrt[3]{2}$, two-valued. The graph is symmetrical with respect to the axis of abscissae. $|y|_{\max} = 1$ for $x = -1$. No points of inflexion. Asymptotes $x = 0$ and $y = \pm \frac{x\sqrt{3}}{3}$

1447. Defined for $x \leqq -2$ and for $x > 0$, two-valued. The graph is symmetrical with respect to the straight line $y = x$. $y_{\max} = -2$ for $x = 1$. No points of inflexion. Asymptotes $x = 0$, $y = 0$ and $x + y = 0$.

1448. Defined for $-a \leqq x < a$, two-valued. The graph is symmetrical with respect to the axis of abscissae. $|y|_{\max} = a\sqrt{\frac{5\sqrt{5} - 11}{2}}$ for $x = -\frac{a}{2}(\sqrt{5} - 1)$. No points of inflexion. Asymptote $x = a$.

1449. Defined for $0 \leqq x \leqq 4$, two-valued. The graph is symmetrical with respect to the axis of abscissae. $|y|_{\max} = \sqrt{3}$ for $x = 3$. Abscissa of the point of inflexion is $3 - \sqrt{3}$. No asymptotes.

1450. Defined for $-2 \leqq x \leqq 2$, two-valued. The graph is symmetrical with respect to the coordinate axes. $|y|_{\max} = \dfrac{3\sqrt{3}}{5}$ for $x = \pm 1$. Points of inflexion are (0, 0) and $\left(\pm\sqrt{3},\ \pm\dfrac{\sqrt{3}}{5}\right)$. No asymptotes.

1451. Defined for $-1 \leqq x < 1$, two-valued. The graph is symmetrical with respect to the coordinate axes. $|y|_{\max} = \dfrac{1}{2}$ for $x = \pm\dfrac{\sqrt{2}}{2}$. Point of inflexion at (0, 0). No asymptotes.

1452. Defined for $x \geqq 1$, two-valued. The graph is symmetrical with respect to the axis of abscissae. $|y|_{\max} = 1$ for $x = 2$. Abscissa of point of inflexion is $\dfrac{6 + 2\sqrt{3}}{3}$. Asymptote $y = 0$.

1453. Defined for $0 \leqq x < 2a$, two-valued. The graph is symmetrical with respect to the axis of abscissae. No extrema. No points of inflexion. Asymptote $x = 2a$.

1454. Defined for $x < 0$, for $0 < x \leqq 1$ and for $x \geqq 2$, two-valued. The graph is symmetrical with respect to the axis of abscissae, has asymptotes $x = 0$ and $y = \pm 1$ and two points of inflexion. No extrema.

1455. Defined for $-a \leqq x < 0$ and for $0 < x < a$, two-valued. The graph is symmetrical with respect to the axis of abscissae. No extrema. Points of inflexion are $\left[a(\sqrt{3} - 1),\ \pm a\sqrt[4]{\dfrac{27}{4}}\right]$. Asymptote $x = 0$.

1456. Defined for $-1 \leqq x \leqq 1$ and for $x = \pm 2$, two-valued. The graph is symmetrical with respect to the coordinate axes and has two isolated points: $(\pm 2, 0)$. $|y|_{\max} = 1$ for $x = 0$. No points of inflexion or asymptotes.

1457. Defined for $-1 \leqq x \leqq 1$, two-valued. The graph is symmetrical with respect to the coordinate axes. $|y|_{\max} = 1$ for $x = 0$. Points of inflexion are $\left(\pm\dfrac{\sqrt{2}}{2},\ \pm\dfrac{\sqrt{2}}{4}\right)$. No asymptotes.

1458. Defined for $x \leqq -1$ and for $x \geqq 1$, two-valued. The graph is symmetrical with respect to the coordinate axes. No extrema. Points of inflexion are $\left(\pm\sqrt{2}, \pm\frac{1}{2}\right)$. Asymptotes $y = \pm x$.

1459. Defined for $x \geqq 0$, two-valued. The graph is symmetrical with respect to the axis of abscissae. $|y|_{\max} = 1$ for $x = \frac{1}{2}$. The abscissa of the point of inflexion is $\frac{1+\sqrt{2}}{2}$. Asymptote $y = 0$.

1460. Defined everywhere except for $x = 0$. No extrema. Point of inflexion is $\left(-\frac{1}{2}, e^{-2}+\frac{1}{2}\right)$. Asymptotes $x = 0$ and $x + y = 1$.

1461. Defined everywhere except at $x = \frac{\pi}{2} + k\pi$, where $k = 0$, $\pm 1, \pm 2, \ldots$ No extrema. The graph has no points of inflexion. Asymptotes $x = \frac{\pi}{2} + k\pi$.

1462. Defined everywhere. The graph is symmetrical with respect to the axis of ordinates. Extremal points satisfies the equation $x = \tan x$. Asymptote $y = 0$.

1463. Defined everywhere. No extrema. No points of inflexion. For $x \leqq 0$ the function is identically equal to the linear function $y = 1 - x$. Asymptote $x + y = 3$. (0, 1) is a node where there are two different tangents.

1464. Defined everywhere. The graph is symmetrical with respect to the axis of ordinates. $y_{\max} = 3$ for $x = 0$, $y_{\min} = -1$ for $x = \pm 2$. The graph has no points of inflexion and no asymptotes; the right-hand side of it is part of the parabola $y = x^2 - 4x + 3$, lying to the right of the axis of ordinates. (0, 3) is a node with two different tangents.

1465. Exists and is continuous for any t; $(-3, 3)$ is a maximum, $(5, -1)$ a minimum, $(1, 1)$ a point of inflexion. No asymptotes. The slope of the curve with respect to the axis of abscissae tends to 45° as $x \to \infty$.

1466. Exists and is continuous for any t. Asymptotes $y = x$ and $y = x + 6\pi$; $\left(-1 - 3\pi, -1 + \frac{3\pi}{2}\right)$ is a maximum, $\left(1 - 3\pi, 1 - \frac{3\pi}{2}\right)$ a minimum, $(-3\pi, 0)$ a point of inflexion.

1467. Exists for all values of t except $t = -1$. Asymptote is $x + y + 1 = 0$. The curve cuts itself at (0, 0), where the tangents are the coordinate axes. No points of inflexion. A closed loop in the first quadrant.

1468. Exists for all values of t. For $x < -\frac{1}{\mathrm{e}}$, y is not defined as a function of x, the function being two-valued for $-\frac{1}{\mathrm{e}} < x < 0$, and single-valued for $x > 0$. The curve is symmetrical with respect to the straight line $x + y = 0$. $\left(\mathrm{e}, \frac{1}{\mathrm{e}}\right)$ is a maximum. There are two points of inflexion. The asymptotes are the coordinate axes.

1469. Exists for all values of t. The curve is symmetrical with respect to the axis of abscissae and consists of a closed curve with a cusp at $(a, 0)$.

1470. A closed three-petal rose. The pole is a triple point of self-intersection at which the curve touches the polar axis and the straight lines $\varphi = \frac{\pi}{3}$ and $\varphi = -\frac{\pi}{3}$. Extrema at $\varphi = \frac{\pi}{6}$, $\varphi = \frac{5\pi}{6}$ and $\varphi = \frac{\pi}{2}$. Is it sufficient to investigate the curve for $0 \leqq \varphi < \pi$. It is superimposed on itself for further values.

1471. Exists for all values of φ in the interval $[0, 2\pi]$ except $\varphi = \frac{\pi}{2}$ and $\varphi = \frac{3\pi}{2}$. Symmetrical with respect to the polar axis and the straight line $\varphi = \frac{\pi}{2}$. The pole is a point of self-intersection with polar axis as double tangent. The straight lines $x = -a$ and $x = a$ are asymptotes†.

1472. Exists for all values of φ in the interval $[0, 2\pi]$ except for $\varphi = \frac{\pi}{2}$ and $\varphi = \frac{3\pi}{2}$. The pole is a point of self-osculation with the straight line $\varphi = \frac{3}{4}\pi$ as double tangent. Asymptotes $x = a$ and $x = -a$.

1473. Exists for all values of φ. A maximum equal to $2a$ at $\varphi = 0$, a minimum $= 0$ for $\varphi = \pi$. The curve is closed, symmetrical with respect to the polar axis. The pole is a cusp.

† In this and following problems the asymptotes are given in the Cartesian system of coordinates, in which the axis of abscissae is the polar axis and the axis of ordinates is the perpendicular to the polar axis through the pole.

1474. Exists for all values of φ. A maximum equal to $a(1+b)$ at $\varphi = 0$, a minimum equal to $a(1-b)$ at $\varphi = \pi$. The pole is a point of self-intersection.

1475. Exists for $\varphi > 0$. Point of inflexion $(\sqrt{2\pi}, 0{\cdot}5)$. The polar axis is an asymptote. The curve winds spirally about the pole, approaching it asymptotically.

1476. Exists for all values of φ. For $\varphi \geqq 0$ the curve is a spiral starting from the pole and approaching the circle $\varrho = 1$ asymptotically. For $\varphi < 0$, we have the same curve but mirrored in the straight line $\varphi = \frac{\pi}{2}$.

1477. Exists for $-1 \leqq t \leqq 1$, situated entirely to the right of the axis of ordinates. A closed curve. A maximum at $t = 0$ ($\varphi = 1$ radian, $\varrho = 1$). No points of inflexion. Touches the axis of ordinates at $t = \pm 1$.

1478. A four-petal rose. The origin is a double point of self-osculation.

1479. The curve lies entirely in the strip $-\frac{a\sqrt{2}}{2} \leqq x \leqq \frac{a\sqrt{2}}{2}$. Symmetrical with respect to the origin. Asymptote $x = 0$. $(0, 0)$ is a point of inflexion, with the axis of abscissae as tangent. There are two further points of inflexion.

1480. Symmetrical with respect to the four axes $x = 0$, $y = 0$, $y = x$, $y = -x$; a closed curve with four cusps at $(a, 0)$, $(0, a)$, $(-a, 0)$ and $(0, -a)$. The origin is an isolated point.

1481. The curve is symmetrical with respect to the coordinate axes and the bisectors of the quadrants. Asymptotes are $(x \pm y)^2 = = \frac{1}{2}$. The origin is a quadruple point of self-intersection, at which the branches of the curve touch the coordinate axes. The curve is shaped like a windmill.

1485. The remaining roots are simple.

1486. $0{\cdot}1 < x < 0{\cdot}2$. **1487.** $-0{\cdot}7 < x_1 < -0{\cdot}6$ and $0{\cdot}8 < x_2 < 0{\cdot}9$.

1488. $0{\cdot}32 < x < 0{\cdot}33$. **1489.** $-3{\cdot}11 < x_1 < -3{\cdot}10$, $0{\cdot}22 < x_2 < 0{\cdot}23$ and $2{\cdot}88 < x_3 < 2{\cdot}89$. **1490.** $0{\cdot}38 < x_1 < 0{\cdot}39$ and $1{\cdot}24 < x_2 < 1{\cdot}25$.

1491. $-0{\cdot}20 < x < -0{\cdot}19$. **1492.** $0{\cdot}84 < x < 0{\cdot}85$.

1493. $1{\cdot}63 < x < 1{\cdot}64$. **1494.** $1{\cdot}537 < x < 1{\cdot}538$. **1495.** $0{\cdot}826 < < x < 0{\cdot}827$. **1496.** $1{\cdot}096 < x < 1{\cdot}097$. **1497.** $0{\cdot}64 < x < 0{\cdot}65$. For $0 < a < 1$ there exists a unique real number equal to its logarithm and less than 1. For $1 < a < e^{\frac{1}{e}}$ there exist two distinct numbers equal to their logarithms: one in the interval $(1, e)$, the other in the interval $(e, +\infty)$. For $a = e^{\frac{1}{e}}$ there exists a unique number equal to its logarithm: the number e (it is the double root of the equation $\log_{e^{\frac{1}{e}}} x = x$). For $e^{\frac{1}{e}} < a < \infty$ there exist no real numbers equal to their logarithms.

1498. $(x-4)^4 + 11(x-4)^3 + 37(x-4)^2 + 21(x-4) - 56.$

1499. $(x+1)^3 - 5(x+1) + 8.$

1500. $(x-1)^{10} + 10(x-1)^9 + 45(x-1)^8 + 120(x-1)^7 +$
$+ 210(x-1)^6 + 249(x-1)^5 + 195(x-1)^4 +$
$+ 90(x-1)^3 + 15(x-1)^2 - 5(x-1) - 1.$

1501. $x^6 - 9x^5 + 30x^4 - 45x^3 + 30x^2 - 9x + 1.$

1502. $f(-1) = 143;\ f'(0) = -60;\ f''(1) = 26.$

1503. $-1 - (x+1) - (x+1)^2 - \ldots - (x+1)^n +$
$+ (-1)^{n+1} \dfrac{(x+1)^{n+1}}{[-1+\theta(x+1)]^{n+2}}$ where $0 < \theta < 1.$

1504. $x + \dfrac{x^2}{1} + \dfrac{x^3}{2!} + \ldots + \dfrac{x^n}{(n-1)!} + \dfrac{x^{n+1}}{(n+1)!}(\theta x + n + 1)\, e^{\theta x},$
where $0 < \theta < 1.$

1505. $2 + \dfrac{x-4}{4} - \dfrac{(x-4)^2}{64} + \dfrac{(x-4)^3}{512} - \ldots +$
$+ (-1)^{n-1} \dfrac{(2n-2)!\,(x-4)^n}{n!\,(n-1)!\,2^{4n-2}} + \dfrac{(2n)!\,(x-4)^{n+1}}{2^{2n+1} n!\,(n+1)!\,\sqrt{[4+\theta(x-4)]^{2n+1}}},$
where $0 < \theta < 1.$

1506. $1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots + \dfrac{x^{2n}}{(2n)!} + \dfrac{x^{2n+1}}{(2n+1)!} \cdot \dfrac{e^{\theta x} - e^{-\theta x}}{2},$
where $0 < \theta < 1.$

1507. $(x-1) + \dfrac{5}{2!}(x-1)^2 + \dfrac{11}{3!}(x-1)^3 + \dfrac{6}{4!}(x-1)^4 + \ldots +$
$+ \dfrac{(-1)^n 6(x-1)^n}{(n-3)(n-2)(n-1)n} +$
$+ \dfrac{(-1)^{n+1} 6(x-1)^{n+1}}{(n-2)(n-1)n(n+1)[1+\theta(x-1)]^{n-2}}$ where $0 < \theta < 1.$

1508. $\dfrac{2x^2}{2!} - \dfrac{2^3x^4}{4!} + \dfrac{2^5x^6}{6!} - \dfrac{2^7x^8}{8!} + \ldots + (-1)^{n-1} \dfrac{2^{2n-1}x^{2n}}{(2n)!} +$
$+ \dfrac{(-1)^n 2^{2n} x^{2n+1}}{(2n+1)!} \sin 2\theta x$, where $0 < \theta < 1.$

1509. $2 - (x-2) + (x-2)^2 - (x-2)^3 + \dfrac{(x-2)^4}{[1+\theta(x-2)]^5},$
where $0 < \theta < 1.$

1510. $x + \frac{x^3}{3} \cdot \frac{1 + 2\sin^2 \theta x}{\cos^4 \theta x}$, where $0 < \theta < 1$.

1511. $x + \frac{x^3}{6} + \frac{x^4}{4!} \frac{9\theta x + 6\theta^3 x^3}{(1 - \theta^2 x^2)^{\frac{7}{2}}}$, where $0 < \theta < 1$.

1512. $1 - \frac{1}{2}(x - 1) + \frac{1 \cdot 3}{2^2 \cdot 2}(x - 1)^2 - \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!}(x - 1)^3 +$
$+ \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4 \cdot 4!} \frac{(x - 1)^4}{\sqrt{[1 + \theta(x - 1)]^9}}$, where $0 < \theta < 1$.

1513*. $\lim\limits_{h \to 0} \theta = \frac{1}{n + 2}$. We have by virtue of the existence of the third derivative:

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a + \theta_1 h).$$

We obtain by comparing with the expression in the text:

$$\frac{h^2}{2!}[f''(a + \theta h) - f''(a)] = \frac{h^3}{3!} f'''(a + \theta_1 h),$$

i.e.

$$\frac{f''(a + \theta h) - f''(a)}{h} = \frac{1}{3} f'''(a + \theta_1 h).$$

It remains to pass to the limit as $h \to 0$.

1514. The function is decreasing. (0, 3) is a point of inflexion of the graph.

1515. The function has a minimum, equal to 1.

1516. The function has a minimum, equal to 2.

1517. The function has a maximum, equal to -11.

1518. The function is increasing. (0, 0) is a point of inflexion of the graph.

1519. The function is increasing. (0, 4) is a point of inflexion of the graph.

1520. $f(x) = 1 - 6(x - 1) + (x - 1)^2 + \ldots$; $f(1{\cdot}03) \approx 0{\cdot}82$.

1521. $f(x) = 321 + 1087(x - 2) + 1648(x - 2)^2 + \ldots$;
$f(2{\cdot}02) \approx 343{\cdot}4$; $f(1{\cdot}97) \approx 289{\cdot}9$.

1522. $f(x) = 1 + 60(x - 1) + 2570(x - 1)^2 + \ldots$;
$f(1{\cdot}005) \approx 1{\cdot}364$.

1523. $f(x) = -6 + 21(x - 2) + 50(x - 2)^2 + \ldots$;
$f(2{\cdot}1) \approx -3{\cdot}4$; $f(2{\cdot}1) = -3{\cdot}36399$; $\delta = 0{\cdot}036$; $\delta' \approx 0{\cdot}011 = 1{\cdot}1\%$.

1524. 1·65. **1525.** 0·78, $\delta < 0{\cdot}01$. **1526.** 0·342020. **1527.** 0·985.

1528. 0·40, $\delta < 0{\cdot}01$.

1529. $\frac{\sqrt{2}}{4}$. **1530.** $\frac{a}{b^2}$; $\frac{b}{a^2}$. **1531.** 36. **1532.** 0·128. **1533.** $\frac{\sqrt{2}}{4}$.

1534. 0. **1535.** 1. **1536.** $\frac{8\sqrt{2}}{3a}$. **1537.** $\frac{6|x|}{(1+9x^4)^{\frac{3}{2}}}$.

1538. $\frac{a^4b^4}{(b^4x^2+a^4y^2)^{\frac{3}{2}}}$. **1539.** $|\cos x|$. **1540.** $\frac{1}{3\sqrt[3]{a|xy|}}$.

1541. $\frac{|(m-1)(ab)^{2m}(xy)^{m-2}|}{(b^{2m}x^{2m-2}+a^{2m}y^{2m-2})^{\frac{3}{2}}}$. **1542.** $\frac{1}{a\cosh^2\frac{x}{a}}$. **1543.** $\frac{1}{6}$.

1544. $\frac{2}{3a|\sin 2t_1|}$. **1545.** $\frac{2}{\pi a}$. **1546.** $\frac{3}{8a\left|\sin\frac{t}{2}\right|}$.

1547. $\frac{1}{\sqrt{1+\ln^2 a}}$. **1548.** $\frac{2+\varphi^2}{a(1+\varphi^2)^{\frac{3}{2}}}$. **1549.** $\frac{\varphi^2+k^2+k}{a\varphi^{k-1}(\varphi^2+k^2)^{\frac{3}{2}}}$.

1550. $\frac{(a^2+b^2)^{\frac{3}{2}}}{2ab\sqrt{2}}$. **1554.** $(x+4)^2+\left(y-\frac{7}{2}\right)^2=\frac{125}{4}$.

1555. $(x-2)^2+(y-2)^2=2$. **1556.** $(x+2)^2+(y-3)^2=8$.

1557. $\left(x-\frac{\pi-10}{4}\right)^2+\left(y-\frac{9}{4}\right)^2=\frac{125}{16}$.

1558. $\left(x+\frac{7}{3}a\right)^2+\left(y-\frac{8}{3}a\right)^2=\frac{125}{9}a^2$. **1559.** $\left(\frac{a}{4},\frac{a}{4}\right)$.

1560. $\left(\frac{\sqrt{2}}{2},-\frac{1}{2}\ln 2\right)$. **1561.** $\left(-\frac{1}{2}\ln 2,\frac{\sqrt{2}}{2}\right)$. **1562.** For $t=k\pi$.

1563. $\frac{3}{4}a$. **1566.** $a=3$, $b=-3$, $c=1$.

1567. $y=-x^5-0{\cdot}6x^4+4{\cdot}5x^3+0{\cdot}1x^2$.

1568. $\xi=x-\frac{[1+n^2x^{2(n-1)}]\,x}{n-1}$, $\eta=x^n+\frac{1+n^2x^{2(n-1)}}{n(n-1)x^{n-2}}$.

1569. $\xi=\frac{(a^2+b^2)\,x^3}{a^4}$, $\eta=-\frac{(a^2+b^2)\,y^3}{b^4}$; $(a\xi)^{\frac{2}{3}}-(b\eta)^{\frac{2}{3}}=$
$=(a^2+b^2)^{\frac{2}{3}}$.

1570. $\xi = x + 3x^{\frac{1}{3}} y^{\frac{2}{3}}$, $\eta = y + 3x^{\frac{2}{3}} y^{\frac{1}{3}}$; $(\xi + \eta)^{\frac{2}{3}} +$
$+ (\xi - \eta)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

1571. $\xi = \pm \frac{4}{3}\sqrt{\frac{y}{a}}(3y + a)$, $\eta = -\frac{9y^2 + 2ay}{2a}$.

1572. $\xi = -\frac{4}{3}t^3$, $\eta = 3t^2 - \frac{3}{2}$, $\xi^2 = \frac{16}{243}\left(\eta + \frac{3}{2}\right)^3$.

1573. $\left(\frac{3\eta}{8}\right)^4 + 6a^2\left(\frac{3\eta}{8}\right)^2 + 3a^3\xi = 0$. **1574.** $\xi^{\frac{2}{3}} + \eta^{\frac{2}{3}} = (2a)^{\frac{2}{3}}$.

1576. Yes, it is possible. **1579.** $2p\left[\sqrt{\left(\frac{x+p}{3p}\right)^3} - 1\right]$.

1580. $\frac{4(a^3 - b^3)}{ab}$. **1581.** $6a$.

1582*. $16a$. Having obtained the parametric equations of the evolute, transform them to new coordinates and parameter by putting $x = -x_1$, $y = y_1$, $t = t_1 + \pi$.

1583*. Use the relationship between the length of the evolute and the increment of radius of curvature.

1584. 233. **1585.** 0·073. **1586.** (3·00; 2·46). **1587.** (−0·773; −0·841).

1588. (1·38; 4·99). **1589.** (0·57; −3·62). **1590.** 0·0387. **1591.** (2·327; 0·845).

Chapter V

1592. (1) $\int_0^3 (x^2 + 1)\,dx$; (2) $\int_a^b (e^x + 2)\,dx$; (3) $\int_0^\pi \sin x\,dx$;

(4) $\int_{-2}^{2} (8 - 2x^2)\,dx$; (5) $\int_0^1 (\sqrt{x} - x^2)\,dx$; (6) $\int_1^e (\ln x - \ln^2 x)\,dx$.

1593. $20 - \frac{4}{n}$ and $20 + \frac{4}{n}$; $\alpha = \frac{4}{n}$; $\sigma = \frac{1}{5n}$.

1594. $\alpha = \frac{149}{600} \approx 0{\cdot}248$, $\delta \approx 0{\cdot}039$. **1595.** 31·5. **1596.** $10\frac{2}{3}$.

1597. $\frac{2}{3}ah = 40$ cm². **1598.** $10\frac{2}{3}$. **1599.** 8. **1600.** $21\frac{1}{3}$.

1601. $2\frac{7}{8}$. **1602.** 140 cm. **1603.** $\approx 122{\cdot}6$ m. **1604.** $20\frac{5}{6}$ cm.

1605. 62·5 kg. **1606.** 4 cm.

1607. (a) $m_n = \sum_{i=0}^{n-1} v(\xi_i)(t_{i+1} - t_i)$, $t_0 = T_0$, $t_n = T_1$;

(b) $m = \int_{T_0}^{T_1} v(t)\,dt$.

1608. (a) $\theta_n = \sum_{i=0}^{n-1} \psi(\xi_i)(t_{i+1} - t_i)$, $t_0 = T_0$, $t_n = T_1$;

(b) $\theta = \int_{T_0}^{T_1} \psi(t)\,dt$.

1609. $Q_n = \sum_{i=0}^{n-1} I(\xi_i)(t_{i+1} - t_i)$, $t_0 = 0$, $t_n = T$; $\quad Q = \int_0^T I(t)\,dt$.

1610. (a) $A_n = \sum_{i=0}^{n-1} \varphi(\xi_i)\psi(\xi_i)(t_{i+1} - t_i)$, $t_0 = T_0$, $t_n = T_1$;

(b) $A = \int_{T_0}^{T_1} \varphi(t)\,\psi(t)\,dt$.

1611. 1500 coulombs. **1612.** $\approx$ 67,600 joules. **1613.** 2880 joules.

1614. (a) $P_n = \sum_{i=0}^{n-1} a\xi_i(x_{i+1} - x_i)$, $x_0 = 0$, $x_n = b$; (b) $P = \int_0^b ax\,dx$.

1615. (a) $\frac{ab^2}{2} = 18{\cdot}75$ kg; (b) the line must be drawn at a distance $\frac{b}{\sqrt{2}} \approx 17{\cdot}7$ cm from the surface.

1616. $e - 1$. **1617.** $\frac{b^{k+1} - a^{k+1}}{k+1}$. **1618.** (1) 50; (2) 4; (3) $\frac{7a^3}{24}$; (4) $\frac{7}{3}ab^2$; (5) $a\left(a^2 - \frac{a}{2} + 1\right)$; (6) $\frac{4}{3}m$; (7) 31·5; (8) $\frac{(a-b)^3}{6}$; (9) $\frac{a^2}{3}$; (10) $\frac{a^3 - 3ab + 3b^2}{3(a-b)^2}$; (11) 4; (12) $16\frac{2}{15}$; (13) 0.

1619*. $\frac{1}{k+1}$; $\approx 1{\cdot}67 \times 10^{11}$. Write the expression whose limit is sought as the nth integral sum of a certain function.

1620. ln 2. **1621.** ln 2. **1622*.** $\ln a$, $\ln 3 \approx 1{\cdot}1$. See problems 1620 and 1621.

1623*. (1) $ae^a - e^a + 1$; (2) $a \ln a - a + 1$; (3) $\frac{(\ln b)^2 - (\ln a)^2}{2}$.

The expression $q + 2q^2 + \ldots + nq^n$ is found with the aid of differentiation of the terms of a geometric progression.

1624. $\int\limits_0^{2\pi} |\sin x|\, dx = 2\int\limits_0^{\pi} \sin x\, dx$. **1625.** $\frac{1}{2}$. **1626.** $\frac{64}{3}$. **1627.** $\frac{8}{5}$.

1630. $8 < I < 9{\cdot}8$. **1631.** $3 < I < 5$. **1632.** $\pi < I < 2\pi$.

1633. $\frac{20}{29} < I < 1$. **1634.** $\frac{\pi}{9} < I < \frac{2\pi}{3}$.

1635. $\frac{e^2 - 1}{e^{e^2-1}} < I < \frac{e^2 - 1}{e^2}$.

1636. (a) The first; (2) the second.

1637. (1) The first; (2) the second; (3) the first; (4) the second.

1640*. $0{\cdot}85 < I < 0{\cdot}90$. Use the result of problem 1639.

1641. (a) $1 < I < \sqrt{2} \approx 1{\cdot}414$; (b) $1 < I < \frac{1 + \sqrt{2}}{2} \approx 1{\cdot}207$; (c) $1 < I < \sqrt{\frac{6}{5}} \approx 1{\cdot}095$.

1642. $y_{av} = \frac{k(x_1 + x_2)}{2} + b$; $\frac{x_1 + x_2}{2}$.

1643. $y_{av} = \frac{a}{3}(x_1^2 + x_1x_2 + x_2^2)$. If $x_1x_2 \geqq 0$, at one point; if $x_1 < 0$ and $x_2 > 0$, at two points if the inequalities are satisfied: $-\frac{x_1}{2} \leqq x_2 \leqq -2x_1$, otherwise at one point.

1644. $24{\cdot}5$ **1645.** $\frac{\pi a}{4}$. **1646.** 0. **1647.** $\frac{2}{3}h = 1$ m.

1648. $11a$. **1649.** ≈ 1558 V. **1650.** (1) $\frac{x^3}{3}$; (2) $\frac{x^6 - a^6}{6}$; (3) $\frac{x^4 - x^5}{20}$.

1651. $s = \frac{2}{3}t^3$. **1652.** $A = 100\, s + 25\, s^2$ ergs, s is the path in cm.

1653. $A = \frac{1}{R}\left(\frac{\alpha^2}{3}t^3 + \alpha\beta t^2 + \beta^2 t\right)$, where $\alpha = \frac{E_2 - E_1}{t_2 - t_1}$, $\beta = \frac{E_1t_2 - E_2t_1}{t_2 - t_1}$.

1654. $Q = C_0t + \frac{\alpha}{2}t^2 + \frac{\beta}{3}t^2$. **1655.** $dS = 10$, $\Delta S = 10, 10\,033 \ldots$

1656. $dS = 1$.

1657.

Δx	ΔS	dS	α	δ
1	92·25	64	28·25	0·442
0·1	6·644	6·4	0·244	0·0382
0·01	0·6424	0·64	0·0024	0·00376

1658. $\frac{1}{3}$. **1659.** $0; \frac{\sqrt{2}}{2}$; 1. **1660.** $\frac{d}{dx}\int_x^a f(x)\,dx = -f(x)$.

1661. $-1, -\frac{5}{4}$. **1662.** $\frac{\sin 2x}{x}$. **1663.** (1) x; (2) $-4x \ln x$.

1664*. $2 \ln^2 2x - \ln^2 x$. Write the integral $\int_x^{2x} \ln^2 x\,dx$ as the sum of integrals $\int_x^a \ln^2 x\,dx + \int_a^{2x} \ln^2 x\,dx$, where $a > 0$.

1665. $y' = -\frac{\cos x}{e^y}$. **1666.** (1) $\frac{dy}{dx} = \cot t$; (2) $\frac{dy}{dx} = -t^2$.

1667. -2. **1668.** Minimum at $x = 0$ ($I(0) = 0$). **1669.** 1.

1670. $y_{\max} = \frac{5}{6}$ for $x = 1$, $y_{\min} = \frac{2}{3}$ for $x = 2$. The graph has a point of inflexion at $\left(\frac{3}{2}, \frac{3}{4}\right)$.

1672. (1) $\frac{3}{4}$; (2) $-\frac{15}{32}$; (3) 52; (4) $4\frac{5}{6}$; (5) $45\frac{1}{6}$; (6) $\approx 0{\cdot}08$;

(7) $2 - \sqrt{2}$; (8) $6\frac{2}{3}$; (9) $3\left(\frac{1}{\sqrt[3]{a}} - \frac{1}{\sqrt[3]{b}}\right)$;

(10) $\frac{z_1^2 - z_0^2}{2} - \frac{4}{3}\left(\sqrt{z_1^3} - \sqrt{z_0^3}\right) + z_1 - z_0$.

1673. (1) 2; (2) 0; (3) $e^3 - 1$; (4) 1; (5) $\frac{\pi}{4}$; (6) $\frac{\pi}{6}$. **1674.** 0.

1675. $1 - \sqrt{3}$; -1.

Chapter VI

1676. $\frac{2}{3}\sqrt{x^3} + C$. **1677.** $\frac{mx^{\frac{n}{m}+1}}{n + m} + C$. **1678.** $C - \frac{1}{x}$.

1679. $\approx 0{\cdot}4343 \times 10^x + C$. **1680.** $\frac{(ae)^x}{1 + \ln a} + C$.

1681. $\sqrt{x} + C$. **1682.** $\sqrt{\frac{2h}{g}} + C$. **1683.** $\approx 4{\cdot}1x^{0{\cdot}83} + C$.

1684. $u - u^2 + C$. **1685.** $\frac{2}{5} x^2 \sqrt{x} + x + C$.

1686. $C - \frac{2}{3x\sqrt{x}} - e^x + \ln |x|$.

1687. $C - 10x^{-0{\cdot}2} + 15x^{0{\cdot}2} - 3{\cdot}62x^{1{\cdot}38}$.

1688. $z - 2 \ln |z| - \frac{1}{z} + C$. **1689.** $\frac{2x^2 - 12x - 6}{3\sqrt{x}} + C$.

1690. $\frac{3}{2} \sqrt[3]{x^2} + \frac{18}{7} x \sqrt[6]{x} + \frac{9}{5} x \sqrt[3]{x^2} + \frac{6}{13} x^2 \sqrt[6]{x} + C$.

1691. $\frac{6}{7} \sqrt[6]{x^7} - \frac{4}{3} \sqrt[4]{x^3} + C$. **1692.** $\frac{1}{\sqrt{3}} \arcsin x + C$.

1693. $3x - \frac{2(1{\cdot}5)^x}{\ln 1{\cdot}5} + C$. **1694.** $\frac{1}{2} (\tan x + x) + C$.

1695. $C - \cot x - \tan x$. **1696.** $\tan x - x + C$.

1697. $C - \cot x - x$. **1698.** $x - \sin x + C$.

1699. $\arctan x - \frac{1}{x} + C$. **1700.** $\ln |x| + 2 \arctan x + C$.

1701. $\tan x + C$. **1702.** $\frac{\pi}{2} x + C$. **1703.** $\frac{\sin^2 x}{2} + C$.

1704. $\frac{\tan^4 x}{4} + \mathrm{C}$. **1705.** $2\sqrt{1 + x^2} + C$. **1706.** $\frac{(x+1)^{16}}{16} + C$.

1707. $C - \frac{1}{8(2x - 3)^4}$. **1708.** $\frac{(a + bx)^{1-C}}{b(1 - C)} + C_1$.

1709. $C - \frac{5}{33} (8 - 3x)^{\frac{11}{5}}$. **1710.** $C - \frac{\sqrt{(8 - 2x)^3}}{3}$.

1711. $\frac{3m}{b} \sqrt[3]{a + bx} + C$. **1712.** $\frac{2}{3} \sqrt{(x^2 + 1)^3} + C$.

1713. $C - \frac{1}{3} \sqrt{(1 - x^2)^3}$. **1714.** $\frac{5}{18} \sqrt[5]{(x^3 + 2)^6} + C$.

1715. $\sqrt{x^2 + 1} + C$. **1716.** $\frac{2}{5} \sqrt{4 + x^5} + C$.

1717. $\frac{3}{8} \sqrt[3]{(x^4 + 1)^2} + C$. **1718.** $\sqrt{3x^2 - 5x + 6} + C$.

1719. $\frac{1}{4}\sin^4 x + C$. **1720.** $\sec x + C$. **1721.** $3\sqrt[3]{\sin x} + C$.

1722. $C - \frac{2}{5}\cos^5 x$. **1723.** $\frac{2}{3}\sqrt{(\ln x)^3} + C$.

1724. $\frac{(\arctan x)^3}{3} + C$. **1725.** $C - \frac{1}{2(\arcsin x)^2}$.

1726. $2\sqrt{1 + \tan x} + C$. **1727.** $\sin 3x + C$.

1728. $\tan(1 + \ln x) + C$. **1729.** $\frac{1}{3}\sin 3x + C$.

1730. $x\cos\alpha - \frac{1}{2}\sin 2x + C$. **1731.** $C - \frac{1}{2}\cos(2x - 3)$.

1732. $C - \frac{1}{2}\sin(1 - 2x)$.

1733. $\frac{1}{2}\tan\left(2x - \frac{\pi}{4}\right) + C$ or $\frac{1}{2}(\tan 4x - \sec 4x) + C$.

1734. $C - \cos(e^x)$. **1735.** $\ln(1 + x^2) + C$.

1736. $\ln|\arcsin x| + C$. **1737.** $\ln(x^2 - 3x + 8) + C$.

1738. $\frac{1}{2}\ln|2x - 1| + C$. **1739.** $\frac{1}{c}\ln|cx + m| + C_1$.

1740. $\frac{1}{2}\ln(x^2 + 1) + C$. **1741.** $\frac{1}{3}\ln|x^3 + 1| + C$.

1742. $\ln(e^x + 1) + C$. **1743.** $\frac{1}{2}\ln(e^{2x} + a^2) + C$.

1744. $C - \ln|\cos x|$. **1745.** $\ln|\sin x| + C$.

1746. $C - \frac{1}{3}\ln|\cos 3x|$. **1747.** $\frac{1}{2}\ln|\sin(2x + 1)| + C$.

1748. $C - \ln(1 + \cos^2 x)$. **1749.** $\ln|\ln x| + C$.

1750. $\frac{\ln^{m+1} x}{m + 1} + C$, if $m \neq -1$ and $\ln|\ln x| + C$, if $m = -1$.

1751. $e^{\sin x} + C$. **1752.** $e^{\sin x} + C$. **1753.** $\frac{a^3x}{3\ln a} + C$.

1754. $C - \frac{a^{-x}}{\ln a}$. **1755.** $C - \frac{e^{1-3x}}{3}$. **1756.** $0{\cdot}5\,e^{x^2} + C$.

1757. $C - \frac{1}{3}e^{-x^3}$. **1758.** $\arcsin\frac{x}{3} + C$. **1759.** $\frac{1}{5}\arcsin 5x + C$

1760. $\frac{1}{3}\arctan 3x + C$. **1761.** $\arcsin\frac{x}{2} + C$.

1762. $\frac{1}{3\sqrt{2}} \arctan \frac{\sqrt{2}}{3} x + C.$ **1763.** $\frac{1}{3} \arcsin \frac{3x}{2} + C.$

1764. $\frac{1}{2} \arctan x^2 + C.$ **1765.** $\frac{1}{2} \arcsin \frac{x^2}{a} + C.$

1766. $\frac{1}{6} \arctan \frac{x^3}{2} + C.$ **1767.** $\frac{1}{4} \arcsin x^4 + C.$

1768. $\frac{1}{2} \arctan \frac{e^x}{2} + C.$ **1769.** $\frac{\arcsin 2^x}{\ln 2} + C.$

1770. $\frac{1}{a} \arctan \frac{\sin \alpha}{a} + C.$ **1771.** $e^x + e^{-x} + C.$

1772. $\frac{1}{3} e^{3x} + \frac{3}{2} e^{2x} + 3e^x + x + C.$

1773. $\arcsin x - \sqrt{1 - x^2} + C.$

1774. $\frac{3}{2} \ln (x^2 + 9) - \frac{1}{3} \arctan \frac{x}{3} + C.$

1775. $\arcsin x + \sqrt{1 - x^2} + C.$

1776. $\frac{1}{2} \arctan x^4 - \frac{1}{4} \ln (x^4 + 1) + C.$

1777. $\arcsin x + \frac{1}{\sqrt{1 - x^2}} + C.$

1778. $\frac{2}{3} [x^3 - \sqrt{(x^2 - 1)^3}] - x + C.$

1779. $C - 2\sqrt{1 - x^2} - \frac{2}{3} \sqrt{(\arcsin x)^3}.$

1780. $C - \frac{1}{9} [\sqrt{1 - 9x^2} + (\arccos 3x)^3].$

1781. $x - 4 \ln |x + 4| + C.$ **1782.** $\frac{1}{2} \left[x - \frac{1}{2} \ln |2x + 1|\right] + C.$

1783. $\frac{A}{b} \left[x - \frac{a}{b} \ln |bx + a|\right] + C.$ **1784.** $C - x - 6 \ln |3 - x|.$

1785. $2x + 3 \ln |x - 2| + C.$ **1786.** $\frac{1}{2} x + \frac{5}{4} \ln |2x - 1| + C.$

1787. $x + \ln (x^2 + 1) + C.$ **1788.** $x - 2 \arctan x + C.$

1789. $C - \frac{1}{4} x^4 - \frac{1}{3} x^3 - \frac{1}{2} x^2 - x - \ln |1 - x|.$

1790. $\frac{x^3}{3} - x + \arctan x + C.$ **1791.** $\ln \left|\frac{x - 1}{x}\right| + C.$

1792. $\ln\left|\frac{x}{x+1}\right| + C$. **1793.** $\frac{1}{5}\ln\left|\frac{2x-3}{x+1}\right| + C$.

1794. $\frac{1}{b-a}\ln\left|\frac{b-x}{a-x}\right| + C$. **1795.** $x + \ln\left|\frac{x-1}{x+1}\right| + C$.

1796. $\frac{1}{3}\ln\left|\frac{x-5}{x-2}\right| + C$.

1797. $\frac{1}{7}\ln\left|\frac{x-2}{x+5}\right| + C$. **1798.** $\frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + C$.

1799. $\frac{1}{2\sqrt{6}}\ln\left|\frac{\sqrt{2}+x\sqrt{3}}{\sqrt{2}-x\sqrt{3}}\right| + C$. **1800.** $\frac{1}{2}\arctan\frac{x-1}{2} + C$.

1801. $\frac{1}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}} + C$. **1802.** $\frac{2}{3}\arctan\frac{1-2x}{3} + C$.

1803. $\frac{1}{4}\arctan\frac{2x+1}{2} + C$. **1804.** $\frac{1}{2}\arcsin(2x+3) + C$.

1805. $\arcsin(x-2) + C$. **1806.** $\frac{1}{3}\arcsin\frac{3x-1}{3} + C$.

1807. $\frac{1}{3}\arcsin\frac{3x+1}{\sqrt{3}} + C$. **1808.** $\frac{x}{2} + \frac{\sin 2x}{4} + C$.

1809. $\frac{x}{2} - \frac{\sin 2x}{4} + C$. **1810.** $C - \cot\frac{x}{2}$.

1811. $\tan\left(\frac{x}{2} - \frac{\pi}{4}\right) + C$. **1812.** $2\tan\frac{x}{2} - x + C$.

1813. $2\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) - x + C$. **1814.** $\frac{1}{3}\tan^3 x + C$.

1815. $\ln(2 + \sin 2x) + C$. **1816.** $C - \frac{1}{4}\left(\frac{\cos 4x}{2} + \cos 2x\right)$.

1817. $\frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C$. **1818.** $\frac{1}{6}\sin 3x - \frac{1}{14}\sin 7x + C$.

1819. $\frac{1}{8}\left(2x + \sin 2x + \frac{1}{2}\sin 4x + \frac{1}{3}\sin 6x\right) + C$.

1820. $\ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$. **1821.** $\ln(1 + \sin x) + C$.

1822. $\frac{\cos^2 x}{2} - \ln|\cos x| + C$. **1823.** $\frac{1}{\sin x} - \frac{1}{3\sin^3 x} + C$.

1824. $2\sqrt{\cos\alpha}\left(\frac{\cos^2\alpha}{5} - 1\right) + C$. **1825.** $\tan x + \frac{1}{3}\tan^3 x + C$.

1826. $\sin x - \frac{\sin^3 x}{3} + C.$ **1827.** $\frac{1}{3}\tan^3 x - \tan x + x + C.$

1828. $C - \cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x.$

1829. $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$

1830. $\frac{1}{2}\tan^2 x + \ln|\cos x| + C.$

1831. $C - \cot x - \frac{2}{3}\cot^3 x - \frac{1}{5}\cot^5 x.$

1832. $\frac{1}{4}\sin 2x - \frac{1}{2}x\cos 2x + C.$ **1833.** $x\sin x + \cos x + C.$

1834. $C - e^{-x}(x + 1).$ **1835.** $\frac{3^x}{\ln^2 3}(x\ln 3 - 1) + C.$

1836. $\frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C.$

1837. $\frac{x^2+1}{2}\operatorname{arc\,tan} x - \frac{x}{2} + C.$

1838. $x \operatorname{arc\,cos} x - \sqrt{1 - x^2} + C.$

1839. $x \operatorname{arc\,tan}\sqrt{x} - \sqrt{x} + \operatorname{arc\,tan}\sqrt{x} + C.$

1840. $2\sqrt{x+1}\operatorname{arc\,sin} x + 4\sqrt{1-x} + C.$

1841. $x\tan x - \frac{x^2}{2} + \ln|\cos x| + C.$

1842. $\frac{x^2}{4} + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + C.$

1843. $C - \frac{1}{2x^2}\log\left(x\sqrt{e}\right).$

1844. $\sqrt{1+x^2}\operatorname{arc\,tan} x - \ln\left(x + \sqrt{1+x^2}\right) + C.$

1845. $2\left(\sqrt{x} - \sqrt{1-x}\operatorname{arc\,sin}\sqrt{x}\right) + C.$

1846. $x\ln(x^2+1) - 2x + 2\operatorname{arc\,tan} x + C.$

1847. $C - \frac{x}{2(1+x^2)} + \frac{1}{2}\operatorname{arc\,tan} x.$

1848. $x^2\sqrt{1+x^2} - \frac{2}{3}\sqrt{(1+x^2)^3} + C.$

1849. $\frac{(x^3+1)\ln(1+x)}{3} - \frac{x^3}{9} + \frac{x^2}{6} - \frac{x}{3} + C.$

1850. $C - e^{-x}(2 + 2x + x^2).$ **1851.** $e^x(x^3 - 3x^2 + 6x - 6) + C.$

1852. $a^x\left(\frac{x^2}{\ln a} - \frac{2x}{\ln^2 a} + \frac{2}{\ln^3 a}\right) + C.$

1853. $C - x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x.$

1854. $\frac{1}{6} x^3 + \frac{1}{4} x^2 \sin 2x + \frac{1}{4} x \cos 2x - \frac{1}{8} \sin 2x + C.$

1855. $x(\ln^2 x - 2 \ln x + 2) + C.$

1856. $C - \frac{1}{x}(\ln^3 x + 3 \ln^2 x + 6 \ln x + 6).$

1857. $C - \frac{8}{27 \sqrt{x^3}}\left(\frac{9}{4} \ln^2 x + 3 \ln x + 2\right).$

1858. $x(\arcsin x)^2 + (2 \arcsin x)\sqrt{1 - x^2} - 2x + C.$

1859. $\frac{x^2 + 1}{2}(\arctan x)^2 - x \arctan x + \frac{1}{2} \ln(1 + x^2) + C.$

1860. $\frac{e^x(\sin x - \cos x)}{2} + C.$ **1861.** $\frac{e^{3x}}{13}(\sin 2x - 5 \cos 2x) + C.$

1862. $\frac{e^{ax}}{a^2 + n^2}(n \sin nx + a \cos nx) + C.$

1863. $\frac{x}{2}(\sin \ln x - \cos \ln x) + C.$

1864. $\frac{x}{2}(\cos \ln x + \sin \ln x) + C.$

1865*. $C - \frac{x}{2}\sqrt{1 - x^2} + \frac{1}{2} \arcsin x.$ (Put $dv = \frac{x\,dx}{\sqrt{1 - x^2}}$ and further, transform $\int \sqrt{1 - x^2}\,dx$ to the form $\int \frac{1 - x^2}{\sqrt{1 - x^2}}\,dx.$)

1866*. $\frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) + C.$ (Put $u = \sqrt{a^2 + x^2}$).

1867. $\frac{x - 2}{x + 2} e^x + C.$

1868. $\frac{1}{2}[(x^2 - 1) \sin x - (x - 1)^2 \cos x] e^x + C.$

1869. $2[\sqrt{x + 1} - \ln(1 + \sqrt{x + 1})] + C.$

1870. $\frac{2\sqrt{x - 1}}{35}(5x^3 + 6x^2 + 8x + 16) + C.$

1871. $C - \frac{11}{2(x - 2)^2} - \frac{4}{x - 2}.$ **1872.** $\ln\left|\frac{\sqrt{x + 1} - 1}{\sqrt{x + 1} + 1}\right| + C.$

1873. $2\sqrt{x-2}+\sqrt{2}\arctan\sqrt{\dfrac{x-2}{2}}+C.$

1874. $2[\sqrt{x}-\ln(1+\sqrt{x})]+C.$ **1875.** $2\arctan\sqrt{x}+C.$

1876. $2(\sqrt{x}-\arctan\sqrt{x})+C.$

1877. $\dfrac{3}{2}(x+1)^{\frac{2}{3}}-3(x+1)^{\frac{1}{3}}+3\ln\left|1+\sqrt[3]{x+1}\right|+C.$

1878. $\dfrac{2}{a}[\sqrt{ax+b}-m\ln|\sqrt{ax+b}+m|]+C.$

1879. $x+\dfrac{6\sqrt[6]{x^5}}{5}+\dfrac{3\sqrt[3]{x^2}}{2}+2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+$

$+6\ln|\sqrt[6]{x}-1|+C.$

1880. $3\sqrt[3]{x}+3\ln|\sqrt[3]{x}-1|+C.$

1881. $2\sqrt{x}-4\sqrt[4]{x}+4\ln(1+\sqrt[4]{x})+C.$

1882. $\dfrac{6}{5}[\sqrt[6]{x^5}+2\sqrt[12]{x^5}+2\ln|\sqrt[12]{x^5}-1|]+C.$

1883. $\dfrac{4}{21}(3e^x-4)\sqrt[4]{(e^x+1)^3}+C.$ **1884.** $\ln\left|\dfrac{\sqrt{1-e^x}-1}{\sqrt{1+e^x}+1}\right|+C.$

1885. $2\sqrt{1+\ln x}-\ln|\ln x|+2\ln|\sqrt{1+\ln x}-1|+C.$

1886. $0{\cdot}4\sqrt{(1+\cos^2 x)^3}(3-2\cos^2 x)+C.$

1887. $\dfrac{1}{2}\ln^2\tan x+C.$ **1888.** $C-\dfrac{2}{9}\sqrt{a^3-x^3}(2a^3+x^3).$

1889. $\dfrac{x^2-4}{2}-\dfrac{8}{x^2-4}+4\ln|x^2-4|+C.$

1890. $C-\dfrac{\sqrt{x^2+a^2}}{a^2x}.$ **1891.** $\dfrac{a^2}{2}\arcsin\dfrac{x}{a}-\dfrac{x}{2}\sqrt{a^2-x^2}+C.$

1892. $C-\dfrac{1}{a}\arcsin\dfrac{a}{x}.$ **1893.** $C-\dfrac{\sqrt{(1+x^2)^3}}{3x^3}.$

1894. $C-\dfrac{\sqrt{1-x^2}}{x}-\arcsin x.$ **1895.** $\dfrac{x}{a^2\sqrt{x^2+a^2}}+C.$

1896. $C-\dfrac{\sqrt{(9-x^2)^5}}{45x^5}.$ **1897.** $\dfrac{\sqrt{x^2-9}}{9x}+C.$

1898. $\ln\dfrac{|x|}{1+\sqrt{x^2+1}}+C.$ **1899.** $C-\dfrac{x}{a^2\sqrt{x^2-a^2}}.$

1900. $\frac{x}{4}(x^2-2)\sqrt{4-x^2}+2\arcsin\frac{x}{2}+C.$

1901. $\frac{1}{4\sqrt{15}}\ln\left|\frac{x\sqrt{15}+2\sqrt{4x^2+1}}{x\sqrt{15}-2\sqrt{4x^2+1}}\right|+C.$

1902*. $\arccos\frac{1}{x}-\frac{\sqrt{x^2-1}}{x}+C$. (The substitution $x=\frac{1}{z}$ can be used.)

1903*. $2\arcsin\sqrt{x}+C$. (The substitution $x=\sin^2 z$ can be used.)

1904*. $\ln\left|\frac{xe^x}{1+xe^x}\right|+C$. (Multiply numerator and denominator by e^x and put $xe^x=z$.)

1905. $2e^{\sqrt{x}}(\sqrt{x}-1)+C.$

1906. $3\left[(2-\sqrt[3]{x^2})\cos\sqrt[3]{x}+2\sqrt[3]{x}\sin\sqrt[3]{x}\right]+C.$

1907. $\frac{x\arcsin x}{\sqrt{1-x^2}}+\frac{1}{2}\ln|1-x^2|+C.$

1908. $x\arctan x-\frac{1}{2}\ln(1+x^2)-\frac{1}{2}(\arctan x)^2+C.$

1909. $\ln\frac{|x|}{\sqrt{1+x^2}}-\frac{1}{x}\arctan x-\frac{1}{2}(\arctan x)^2+C.$

1910. $\frac{1}{3}\sqrt{(x^2+2x)^3}+C.$ **1911.** $\frac{1}{9}(1+e^{3x})^3+C.$ **1912.** $2e^{\sqrt{x}}+C.$

1913. $e^{-\cos x}+C.$ **1914.** $C-\frac{2}{3}(1-e^x)^{\frac{3}{2}}.$ **1915.** $\frac{1}{2}\sin x^2+C.$

1916. $C-\frac{5}{24}\left(2-3x^{\frac{4}{5}}\right)^{\frac{6}{5}}.$ **1917.** $C-\frac{1}{3}\ln|1+3x^3-x^6|.$

1918. $\frac{2}{3}\ln\left(1+x^{\frac{3}{2}}\right)+C.$ **1919.** $C-\ln(3+e^{-x}).$

1920. $C-\arcsin e^{-x}.$ **1921.** $2\sqrt{1+x^2}+3\ln\left(x+\sqrt{1+x^2}\right)+C.$

1922. $\frac{1}{9}\left[2\sqrt{9x^2-4}-3\ln\left|3x+\sqrt{9x^2-4}\right|\right]+C.$

1923. $2\sin\sqrt{x}+C.$ **1924.** $\arcsin\frac{\ln x}{\sqrt{3}}+C.$

1925. $C-\frac{1}{2}\ln|1-\ln^2 x|.$

1926. $\frac{1}{\sqrt{x^2+1}}+\ln\left(x+\sqrt{x^2+1}\right)+C.$

1927. $\dfrac{(\operatorname{arc\,tan} x)^{n+1}}{n+1} + C$, if $n \neq -1$, and $\ln|\operatorname{arc\,tan} x|$, if $n = -1$.

1928. $C - 2\cot 2\varphi$. **1929.** $2x - \tan x + C$. **1930.** $\dfrac{1}{5}\tan^5 x + C$.

1931. $\dfrac{2}{45}\sqrt{\tan^5 x}(5\tan^2 x + 9) + C$.

1932. $\dfrac{1}{3}(\tan 3x + \ln\cos^2 3x) + C$.

1933. $\dfrac{x^3}{3} - \dfrac{x^2}{2} + x - \ln|x+1| + C$.

1934. $C - \dfrac{1}{x-1} - \dfrac{1}{2(x-1)^2}$.

1935. $\dfrac{\sqrt{2+4x}(x-1)}{6} + C$. **1936.** $x\sqrt{1+2x} - \dfrac{1}{3}\sqrt{(1+2x)^3} + C$.

1937. $\dfrac{2}{15}(3x - 2a)\sqrt{(a+x)^3} + C$.

1938. $\dfrac{x}{2} + \dfrac{1}{4}\sin 2x + \dfrac{4}{3}\sqrt{\sin^3 x} - \cos x + C$.

1939. $\dfrac{a^{mx}b^{nx}}{m\ln a + n\ln b} + C$.

1940. $C - \ln\left[1 - x + \sqrt{5 - 2x + x^2}\right]$.

1941. $\dfrac{1}{3}\ln\left(3x - 1 + \sqrt{9x^2 - 6x + 2}\right) + C$.

1942. $\dfrac{1}{3}\operatorname{arc\,sin}\dfrac{3x-2}{\sqrt{2}} + C$.

1943. $C - 8\sqrt{5 + 2x - x^2} - 3\operatorname{arc\,sin}\dfrac{x-1}{\sqrt{6}}$.

1944. $\dfrac{1}{2}\ln(x^2 + 2x + 2) + \operatorname{arc\,tan}(x+1) + C$.

1945. $C - \sqrt{3 - 2x - x^2} - 4\operatorname{arc\,sin}\dfrac{x-1}{2}$.

1946. $\dfrac{3}{8}\left[\ln(4x^2 - 4x + 17) + \dfrac{1}{6}\operatorname{arc\,tan}\dfrac{2x-1}{4}\right] + C$.

1947. $3\sqrt{x^2 + 2x + 2} - 4\ln\left(x + 1 + \sqrt{x^2 + 2x + 2}\right) + C$.

1948. $\ln\dfrac{(x-4)^2}{|x-3|} + C$.

1949. $\frac{2}{9}\sqrt{9x^2+6x+2}+\frac{13}{9}\ln\left(3x+1+\sqrt{9x^2+6x+2}\right)+C.$

1950. $C-\ln|2x^2-3x+1|.$

1951. $\frac{29}{45}\operatorname{arc\,tan}\frac{5x+3}{9}-\frac{3}{10}\ln(5x^2+6x+18)+C.$

1952. $\frac{61}{16}\ln\left|8x+9+4\sqrt{4x^2+9x+1}\right|-\frac{5}{4}\sqrt{4x^2+9x+1}+C,$

1953. $\frac{1}{3}\sqrt{3x^2-11x+2}+\frac{11}{6\sqrt{3}}\ln\left|x-\frac{11}{6}+\sqrt{x^2-\frac{11}{3}x+\frac{2}{3}}\right|+C.$

1954. $\frac{1}{2}\sqrt{2x^2+3x}-\frac{3}{4\sqrt{2}}\ln\left(x+\frac{3}{4}+\sqrt{x^2+\frac{3x}{2}}\right)+C.$

1955. $\sqrt{(a-x)(x-b)}-(a-b)\operatorname{arc\,tan}\sqrt{\frac{a-x}{x-b}}+C.$

1956. $x\operatorname{arc\,tan}x-\frac{1}{2}\ln(1+x^2)+C.$

1957. $\frac{1}{8}\sin 2x-\frac{1}{4}x\cos 2x+C.$

1958. $\frac{1}{\omega^3}\left[(\omega^2x^2-2)\sin\omega x+2\omega x\cos\omega x\right]+C.$

1959. $e^{2x}\left(\frac{1}{2}x^3-\frac{3}{4}x^2+\frac{3}{4}x-\frac{3}{8}\right)+C.$

1960. $(\tan x)\ln(\cos x)+\tan x-x+C.$ **1961.** $\ln|\ln\sin x|+C.$

1962. $\frac{1}{4}\left[\ln(1+x^4)+\frac{1}{1+x^4}\right]+C.$

1963. $\frac{1}{3}\left(\ln\left|\tan\frac{3x}{2}\right|+\cos 3x\right)+C.$ **1964.** $\frac{1}{3}\tan\left(\frac{\pi}{4}+\frac{3x}{2}\right)+C.$

1965. $C-\frac{1}{8}\ln\frac{2+\cos 2x}{2-\cos 2x}.$ **1966.** $\ln\frac{e^x}{e^x+1}+C.$

1967. $2\ln\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)+C.$ **1968.** $e^{e^x}+C.$ **1969.** $\frac{1}{4}e^{2x^2}+C.$

1970. $\frac{1}{\sqrt{2}}\left[3\ln\left(x+\sqrt{1+x^2}\right)+\frac{1}{3}(x^2-2)\sqrt{1+x^2}\right]+C.$

1971. $x-\sqrt{1-x^2}\operatorname{arc\,sin}x+C.$ **1972.** $C-\frac{1}{2}\left(\frac{x}{\sin^2 x}+\cot x\right).$

1973. $\frac{e^x}{2}\left(1-\frac{2\sin 2x+\cos 2x}{5}\right)+C.$

1974. $\frac{1}{2}(\tan x + \ln|\tan x|) + C.$

1975. $\ln|\sin x + \cos x| + C.$ **1976.** $\frac{1}{2}\ln\left|\tan\left(\frac{\varphi}{2} + \frac{\pi}{6}\right)\right| + C.$

1977. $\sec x - \tan x + x + C.$ **1978.** $\sin x - \operatorname{arc\,tan}\sin x + C.$

1979. $\sqrt{2}\ln\left|\tan\frac{x}{4}\right| + C.$ **1980.** $(\ln x)\ln\ln x - \ln x + C.$

1981. $\frac{e^{x^2}(x^2-1)}{2} + C.$ **1982.** $C - \frac{1}{2}e^{-x^2}(x^4 + 2x^2 + 2).$

1983. $\frac{1}{6}(x^2 - 1)\sqrt{1 + 2x^2} + C.$

1984. $C - \frac{x(x^2-3)}{2\sqrt{1-x^2}} - \frac{3}{2}\arcsin x.$

1985. $\frac{1}{5}\sqrt{(x^2-a^2)^5} - \frac{a^2}{3}\sqrt{(x^2-a^2)^3} + a^4\sqrt{x^2-a^2} + a^5\arcsin\frac{a}{x} + C.$

1986. $\frac{\sqrt{4+x^2}(x^2-2)}{24x^3} + C.$ **1987.** $\frac{\sqrt{(x^2-8)^3}}{24x^3} + C.$

1988. $\frac{\sqrt{(4+x^2)^3}(x^2-6)}{120x^5} + C.$ **1989.** $\frac{\sqrt{x^2-3}(2x^2+3)}{27x^3} + C.$

1990. $\frac{4}{3}\left[\sqrt[4]{x^3} - \ln\left(\sqrt[4]{x^3} + 1\right)\right] + C.$

1991. $x + 4\sqrt{x+1} + 4\ln(\sqrt{1+x} - 1) + C.$

1992. $2\operatorname{arc\,tan}\sqrt{1+x} + C.$ **1993.** $\ln\frac{x}{\left(\sqrt[6]{x} + 1\right)^6} + C.$

1994. $\sqrt{x^2+2x} + \ln|x + 1 + \sqrt{x^2+2x}| + C.$

1995*. $\frac{x^8}{8(1-x^2)^4} + C.$ ($x = \sin u$ is a convenient substitution.)

1996. $\frac{2}{\sqrt{ab}}\operatorname{arc\,tan}\sqrt{\frac{ax}{b}} + C.$ **1997.** $C - \frac{(1+x^8)^{\frac{3}{2}}}{12x^{12}}.$

1998. $\frac{x^2}{2\sqrt{1-x^4}} + C.$ **1999.** $\frac{1}{4}x^2\sqrt{x^4+4} - \ln(x^2 + \sqrt{x^4+4}) + C.$

2000. $\ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C.$ **2001.** $C - \frac{2}{3}\sqrt{\frac{1-x^3}{x^3}} - \frac{2}{3}\arcsin\sqrt{x^3}.$

2002. $C - \frac{x^3}{4(1+x^2)^2} - \frac{3x}{8(1+x^2)} + \frac{3\operatorname{arc\,tan} x}{8}.$

2003. $\dfrac{(x^2+1)\arctan x}{\sqrt{x}} - 2\sqrt{x} + C.$

2004. $\arcsin e^x - \sqrt{1-e^{2x}} + C.$

2005. $2\sqrt{e^x-1} - 2\arctan\sqrt{e^x-1} + C.$

2006*. $C - \dfrac{1}{2}\ln^2\left(1+\dfrac{1}{x}\right)$; $\left(\text{substitute } u = 1+\dfrac{1}{x}\right)$.

2007. $\arctan x + \dfrac{1}{x} - \dfrac{1}{3x^3} + C.$

2008. $x \arccos\sqrt{\dfrac{x}{x+1}} + \sqrt{x} - \arctan\sqrt{x} + C.$

2009. $x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C.$

2010. $\dfrac{3}{55}\sqrt[3]{\tan^5 x}(5\tan^2 x + 11) + C.$

2011. $\dfrac{\sqrt{2}}{5}(\tan^2 x + 5)\sqrt{\tan x} + C.$ 2012. $\ln\dfrac{|x+1|}{\sqrt{2x+1}} + C.$

2013. $\dfrac{1}{5}\ln\left[(x-2)^2\sqrt{2x+1}\right] + C.$ 2014. $\ln\left|\dfrac{(x-1)^4(x-4)^5}{(x+3)^7}\right| + C.$

2015. $\dfrac{3}{11}\ln|3x+1| + \dfrac{2}{33}\ln|2x-3| - \dfrac{1}{3}\ln|x| + C.$

2016. $\dfrac{x^3}{3} + \dfrac{x^2}{2} + 4x + \ln\left|\dfrac{x^2(x-2)^5}{(x+2)^3}\right| + C.$

2017. $\dfrac{1}{4}x + \ln|x| - \dfrac{7}{16}\ln|2x-1| - \dfrac{9}{16}\ln|2x+1| + C.$

2018. $\ln|2x-1| - 6\ln|2x-3| + 5\ln|2x-5| + C.$

2019. $\ln\sqrt{\dfrac{x^2-2}{x^2-1}} + C.$

2020. $\dfrac{1}{2\sqrt{2}}\ln\left|\dfrac{x-\sqrt{2}}{x+\sqrt{2}}\right| + \dfrac{1}{2\sqrt{3}}\ln\left|\dfrac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C.$

2021. $\dfrac{x^2}{2} + \ln\left|\dfrac{x(x-2)\sqrt{(x-1)(x+1)^3}}{x+2}\right| + C.$

2022. $\ln\left|\dfrac{x^2}{x+1}\right| + \dfrac{6}{x+1} + C.$

2023. $4\ln|x| - 3\ln|x-1| - \dfrac{9}{x-1} + C.$

2024. $\frac{4}{x+2}+\ln|x+1|+C$. **2025.** $x+\frac{1}{x}+\ln\frac{(x-1)^2}{|x|}+C$.

2026. $C-\frac{1}{3(x-2)^3}+\frac{1}{2(x-2)^2}+\ln|x-2|$.

2027. $\frac{1}{x}+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$. **2028.** $2\ln\left|\frac{x+4}{x+2}\right|-\frac{5x+12}{x^2+6x+8}+C$.

2029. $\frac{3}{2(x-2)^2}+\ln|x-5|+C$.

2030. $\frac{x}{8}-\ln|x+1|-\frac{9x^2+12x+5}{3(x+1)^3}+C$.

2031. $\frac{(x+2)^2}{2}-\frac{1}{4(x-1)^2}-\frac{9}{4(x-1)}+\frac{31}{8}\ln|x-1|+$
$+\frac{1}{8}\ln|x+1|+C$.

2032. $\frac{1}{x-1}+\ln\frac{\sqrt{(x-1)(x-3)}}{|x|}+C$.

2033. $\frac{3}{2x}-\frac{5}{4}\ln|x|+20\ln|x-3|-\frac{47}{4}\ln|x-2|+C$.

2034. $\frac{1}{4}\ln\left|\frac{x}{x-2}\right|-\frac{1}{x}\left(1+\frac{1}{2x}\right)-\frac{1}{2(x-2)}+C$.

2035. $C-\frac{x}{(x^2-1)^2}$. **2036.** $\ln\frac{|x|}{\sqrt{x^2+1}}+C$.

2037. $\frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1}+\frac{1}{\sqrt{3}}\operatorname{arc\,tan}\frac{2x-1}{\sqrt{3}}+C$.

2038. $\frac{1}{3}\ln\frac{|x-1|}{\sqrt{x^2+x+1}}+\frac{1}{\sqrt{3}}\operatorname{arc\,tan}\frac{2x+1}{\sqrt{3}}+C$.

2039. $\ln\frac{\sqrt{(x^2-2x+5)^3}}{|x-1|}+\frac{1}{2}\operatorname{arc\,tan}\frac{x-1}{2}+C$.

2040. $\frac{(x+1)^2}{2}+\ln\frac{|x-1|}{\sqrt{x^2+1}}-\operatorname{arc\,tan}x+C$.

2041. $\frac{1}{4}\ln\left|\frac{1+x}{1-x}\right|-\frac{1}{2}\operatorname{arc\,tan}x+C$.

2042. $\frac{1}{4}\ln\frac{x^4}{(x+1)^2(x^2+1)}-\frac{1}{2}\operatorname{arc\,tan}x+C$.

2043. $\frac{1}{2}\ln|x+1|-\frac{1}{4}\ln(x^2+1)-\frac{1}{2(x+1)}+C$.

2044. $\frac{1}{4}\left[\ln\frac{\sqrt{x^2+1}}{|x-1|}+\arctan x-\frac{7}{(x-1)^2}\right]+C.$

2045. $\frac{x^2}{2}-2x-\frac{2}{x}+2\ln(x^2+2x+2)-2\arctan(x+1)+C.$

2046. $\ln\frac{x^2+4}{\sqrt{x^2+2}}+\frac{3}{2}\arctan\frac{x}{2}-\frac{3\sqrt{2}}{2}\arctan\frac{x\sqrt{2}}{2}+C.$

2047*. $\frac{1}{4\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{\sqrt{2}}{4}\arctan\frac{x\sqrt{2}}{1-x^2}+C.$ (Add and subtract $2x^2$ in the denominator of the integrand.)

2048. $\frac{2-x}{4(x^2+2)}+\frac{\ln(x^2+2)}{2}-\frac{1}{4\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+C.$

2049. $\frac{1}{16}\ln|x|-\frac{1}{18}\ln(x^2+1)+\frac{7}{288}\ln(x^2+4)-$
$-\frac{1}{24(x^2+4)}+C.$

2050. $\frac{13x-159}{8(x^2-6x+13)}+\frac{53}{16}\arctan\frac{x-3}{2}+C.$

2051. $\frac{3}{8}\arctan(x+1)-\frac{5x^3+15x^2+18x+8}{8(x^2+2x+2)^2}+C.$

2052. $\frac{x}{216(x^2+9)}+\frac{x}{36(x^2+9)^2}+\frac{1}{648}\arctan\frac{x}{3}+C.$

2053. $\frac{x-1}{2(x^2+1)}-\frac{1}{2}\ln|x+1|+\frac{1}{4}\ln(1+x^2)+C.$

2054. $\frac{15x^5+40x^3+33x}{48(1+x^2)^3}+\frac{15}{48}\arctan x+C.$

2055. $\frac{1}{4}\left(\frac{2x^6-3x^2}{x^4-1}+\frac{3}{2}\ln\left|\frac{x^2-1}{x^2+1}\right|\right)+C.$

2056. $\frac{x}{x^2+x+1}+\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}-2\ln(x^2+x+1)+$
$+\frac{x^4}{4}-\frac{2x^3}{3}+\frac{x^2}{2}+2x+C.$

2057. $\frac{3x^2-x}{(x-1)(x^2+1)}+\ln\frac{(x-1)^2}{x^2+1}+\arctan x+C.$

2058. $C-6\ln\left|\frac{x-1}{x}\right|-\frac{12x^2-5x-1}{2(x^3-x^2)}.$

2059. $\frac{1}{x^2(x^2+1)}+\ln\sqrt{x^2+1}+C.$

2060. $\frac{1}{2}\cdot\frac{1+x}{(1+x^2)^2}+\frac{1}{4}\cdot\frac{x-2}{x^2+1}+\frac{1}{4}\operatorname{arc\,tan} x+C.$

2061. $\frac{2}{3}\ln\left|\frac{x^3+1}{x^3}\right|-\frac{1}{3x^3}-\frac{1}{3(x^3+1)}+C.$

2062. $\frac{1}{648}\left[\operatorname{arc\,tan}\frac{x+1}{3}+\frac{3(x+1)}{x^2+2x+10}+\frac{18(x+1)}{(x^2+2x+10)^2}\right]+C.$

2063. $\frac{3}{8}\operatorname{arc\,tan}(x+1)+\frac{3}{8}\cdot\frac{x+1}{x^2+2x+2}+\frac{x}{4(x^2+2x+2)^2}+C.$

2064. $C-\frac{x}{8(x^2+4)}-\frac{2x+5}{2(x^2+4x+5)}-\frac{1}{16}\operatorname{arc\,tan}\frac{x}{2}-$
$-\operatorname{arc\,tan}(x+2).$

2065. $C-\frac{57x^4-103x^2+32}{8x(x^2+1)^2}-\frac{57}{8}\operatorname{arc\,tan} x.$

2066. $\frac{3-7x-2x^2}{2(x^3-x^2-x+1)}+\ln\frac{|x-1|}{(x+1)^2}+C.$

2067. $\left(-\frac{1}{2}x^4+\frac{5}{4}x^2-\frac{3}{5}\right)\frac{1}{x(3-2x^2)^2}+\frac{1}{8\sqrt{6}}\ln\left|\frac{\sqrt{3}+x\sqrt{2}}{\sqrt{3}-x\sqrt{2}}\right|+C.$

2068. $\ln\frac{x}{\left(1+\sqrt[10]{x}\right)^{10}}+\frac{10}{\sqrt[10]{x}}-\frac{5}{\sqrt[5]{x}}+\frac{10}{3\sqrt[10]{x^3}}-\frac{5}{2\sqrt[5]{x^2}}+C.$

2069. $2\sqrt{x}-3\sqrt[3]{x}-8\sqrt[4]{x}+6\sqrt[6]{x}+48\sqrt[12]{x}+3\ln(1+\sqrt[12]{x})+$

$+\frac{33}{2}\ln(\sqrt[6]{x}-\sqrt[12]{x}+2)-\frac{171}{\sqrt{7}}\operatorname{arc\,tan}\frac{2\sqrt[12]{x}-1}{\sqrt{7}}+C.$

2070. $6\left[\frac{1}{9}(x+1)^{\frac{3}{2}}-\frac{1}{8}(x+1)^{\frac{4}{3}}+\frac{1}{7}(x+1)^{\frac{7}{6}}-\frac{1}{6}(x+1)+\right.$

$\left.+\frac{1}{5}(x+1)^{\frac{5}{6}}-\frac{1}{4}(x+1)^{\frac{2}{3}}\right]+C.$

2071. $\ln\left|\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right|+2\operatorname{arc\,tan}\sqrt{\frac{1-x}{1+x}}+C.$

2072. $(\sqrt{x}-2)\sqrt{1-x}-\operatorname{arc\,sin}\sqrt{x}+C.$

2073. $6\sqrt[3]{(1+x)^2}\left[\frac{(1+x)^2}{16}-\frac{1+x}{5}+\frac{\sqrt{1+x}}{7}+\frac{1}{4}\right]+C.$

2074. $\ln\frac{|u^2-1|}{\sqrt{u^4+u^2+1}}+\sqrt{3}\operatorname{arc\,tan}\frac{1+2u^2}{\sqrt{3}}+C,$

where $u=\sqrt[3]{\frac{1-x}{1+x}}$.

2075*. $\frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}}+C$. Multiply numerator and denominator of the fraction by $\sqrt[4]{x-1}$ and take out the factors from behind the radical.

2076. $\frac{2}{3}x\sqrt{x}+\frac{24}{11}x\sqrt[6]{x^5}+\frac{36}{13}x^2\sqrt[6]{x}+\frac{8}{5}x^2\sqrt{x}+\frac{6}{17}x^2\sqrt[6]{x^5}+C$.

2077. $3\left[\ln\left|\frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}\right|+\frac{2\sqrt[3]{x}+3}{2\left(1+\sqrt[3]{x}\right)^2}\right]+C$.

2078. $\frac{1}{2}\ln\left(\sqrt[3]{x^2+1}-1\right)-\frac{1}{4}\ln\left[\sqrt[3]{(x^2+1)^2}+\sqrt[3]{x^2+1}+1\right]+$
$+\frac{\sqrt{3}}{2}\arctan\frac{2\sqrt[3]{x^2+1}+1}{\sqrt{3}}+C$.

2079. $\frac{1}{8}\sqrt[3]{(1+x^3)^8}-\frac{1}{5}\sqrt[3]{(1+x^3)^5}+C$.

2080. $\frac{1}{6}\ln\frac{u^2+u+1}{(u-1)^2}-\frac{1}{\sqrt{3}}\arctan\frac{2u+1}{\sqrt{3}}+C$, where $u=\frac{\sqrt[3]{x^3+1}}{x}$.

2081. $\frac{1}{4}\ln\frac{\sqrt[4]{1+x^4}+x}{\sqrt[4]{1+x^4}-x}-\frac{1}{2}\arctan\frac{\sqrt[4]{1+x^4}}{x}+C$.

2082. $\frac{1}{4}\ln\frac{\sqrt{1-x^4}+1}{x^2}-\frac{1}{4}\frac{\sqrt{1-x^4}}{x^4}+C$.

2083. $\frac{3}{7}\left(4\sqrt{x}+\sqrt[4]{x}-3\right)\sqrt[3]{1+\sqrt[4]{x}}+C$.

2084. $6u+2\ln\frac{u-1}{\sqrt{u^2+u+1}}-2\sqrt{3}\arctan\frac{2u+1}{\sqrt{3}}+C$,
where $u=\sqrt[3]{1+\sqrt{x}}$.

2085. $\frac{1}{5}\ln\frac{|u-1|}{\sqrt{u^2+u+1}}+\frac{\sqrt{3}}{5}\arctan\frac{1+2u}{\sqrt{3}}+C$, where $u=\sqrt[3]{1+x^5}$.

2086. $C - \frac{\sqrt[3]{1+x^3}}{x} + \frac{1}{\sqrt{3}} \arctan \frac{2\sqrt[3]{1+x^3}+x}{x\sqrt{3}} -$

$- \frac{1}{3} \ln \left| \frac{\sqrt[3]{1+x^3} - x}{\sqrt{\sqrt[3]{(1+x^3)^2} + x\sqrt[3]{1+x^3} + x^2}} \right|.$

2087. $C - \frac{1}{10}\sqrt{\left(\frac{1+x^4}{x^4}\right)^5} + \frac{1}{3}\sqrt{\left(\frac{1+x^4}{x^4}\right)^3} - \frac{1}{2}\sqrt{\frac{1+x^4}{x^4}}.$

2088. $\frac{u}{2(u^3+1)} - \frac{1}{6}\ln\frac{u+1}{\sqrt{u^2-u+1}} - \frac{1}{2\sqrt{3}}\arctan\frac{2u-1}{\sqrt{3}} + C,$

where $u = \sqrt[3]{\frac{1-x^2}{x^2}}.$

2089. $12\left[\frac{\sqrt[3]{u^{13}}}{13} - \frac{3\sqrt[3]{u^{10}}}{10} + \frac{3\sqrt[3]{u^7}}{7} - \frac{\sqrt[3]{u^4}}{4}\right] + C,$ where $u = 1 + \sqrt[4]{x}$.

2090. $\frac{1}{15}\cos^3 x(3\cos^2 x - 5) + C.$ **2091.** $\frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C.$

2092. $\ln|\tan x| - \frac{1}{2\sin^2 x} + C.$

2093. $\tan x + \frac{1}{4}\sin 2x - \frac{3}{2}x + C.$

2094. $\frac{1}{2}(\tan^2 x - \cot^2 x) + 2\ln|\tan x| + C.$

2095. $\frac{(\tan^2 x - 1)(\tan^4 x + 10\tan^2 x + 1)}{3\tan^3 x} + C.$

2096. $\frac{1}{\cos x - 1} + C.$ **2097.** $\frac{1}{2}\cot\frac{x}{2} - \frac{1}{6}\cot^3\frac{x}{2} + C.$

2098. $\frac{5}{16}x + \frac{1}{12}\sin 2x\left(\cos^4 x + \frac{5}{4}\cos^2 x + \frac{15}{8}\right) + C.$

2099. $x - \frac{1}{3}\cot^3 x + \cot x + C.$

2100. $\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C.$

2101. $x - \frac{1}{7}\cot^7 x + \frac{1}{5}\cot^5 x - \frac{1}{3}\cot^3 x + \cot x + C.$

2102. $C - \frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right|$.

2103. $\frac{1}{4}\ln\left|\frac{1+\tan x}{1-\tan x}\right| + \frac{1}{2}\sin x\cos x + C$.

2104. $C - \frac{1}{1+\tan x}$. **2105.** $\frac{\sqrt{2}}{2}\ln\left|\tan\left(\frac{\pi}{8}+\frac{x}{2}\right)\right| + C$.

2106. $\frac{1}{\sqrt{a^2+b^2}}\ln\left|\tan\frac{x+\arctan\frac{a}{b}}{2}\right| + C$.

2107. $\ln\frac{|C\sin x|}{\sqrt{\cos 2x}}$. **2108.** $\ln\frac{|C\sin x|}{\sqrt{1-4\sin^2 x}}$.

2109. $\frac{1}{2}[x + \ln|\sin x + \cos x|] + C$.

2110. $\frac{1}{2}\arctan\left(2\tan\frac{x}{2}\right) + C$. **2111.** $\frac{2}{3}\arctan\frac{5\tan\frac{x}{2}+4}{3} + C$.

2112. $\ln(2+\cos x) + \frac{4}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) + C$.

2113. $\frac{\cos x(\cos x - \sin x)}{4} - \frac{1}{4}\ln|\cos x - \sin x| + C$.

2114. $\frac{4}{25}x - \frac{3}{25}\ln|\tan x + 2| + \frac{2}{5(\tan x + 2)} - \frac{3}{25}\ln|\cos x| + C$.

2115. $\frac{\cos 2x - 15}{15(4+\sin 2x)} + \frac{4}{15\sqrt{15}}\arcsin\frac{4\sin 2x + 1}{4+\sin 2x} + C$.

2116. $\frac{1}{2-\tan\frac{x}{2}} + C$. **2117.** $\frac{1}{3}\arctan(3\tan x) + C$.

2118. $\frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan x) + C$.

2119. $\frac{1}{2}\tan x + \frac{1}{2\sqrt{2}}\arctan(\sqrt{2}\tan x) + C$.

2120. $\frac{1}{ab}\arctan\frac{a\tan x}{b} + C$.

2121. $C - \frac{1}{2}\left[\cot x + \frac{1}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right)\right]$.

2122. $\ln \dfrac{\left|\sqrt[3]{\tan x - 1}\right|}{\sqrt[6]{\tan^2 x + \tan x + 1}} - \dfrac{\sqrt{3}}{3} \arctan \dfrac{2 \tan x + 1}{\sqrt{3}} + C.$

2123. $2\left(\sin \dfrac{x}{2} - \cos \dfrac{x}{2}\right) + C$ for values of x satisfying the inequality $\sin \dfrac{x}{2} + \cos \dfrac{x}{2} \geqq 0$, and $-2\left(\sin \dfrac{x}{2} - \cos \dfrac{x}{2}\right) + C$ for values of x satisfying the inequality $\sin \dfrac{x}{2} + \cos \dfrac{x}{2} \leqq 0$.

2124. $2\sqrt{\tan x} + C.$

2125*. $C - \dfrac{4\sqrt{2}}{5}\sqrt{\cot^5 x}$. (Substitute $u = \cot x$.)

2126. $4\sqrt[4]{\tan x} + C.$ **2127.** $\dfrac{1}{\sqrt{2}} \ln\left(\sqrt{2} \tan x + \sqrt{1 + 2\tan^2 x}\right) + C.$

2128. $2 \arcsin \sqrt{\sin x} + C.$

2129. $C - \dfrac{1}{3} \tan x\, (2 + \tan^2 x) \sqrt{4 - \cot^2 x}.$

2130. $\dfrac{4}{\sqrt{\cos \frac{x}{2}}} + 2 \arctan \sqrt{\cos \dfrac{x}{2}} - \ln \dfrac{1 + \sqrt{\cos \frac{x}{2}}}{1 - \sqrt{\cos \frac{x}{2}}} + C.$

2131. $\dfrac{1}{\sqrt{2}}\left[\ln\left(\sin x + \cos x - \sqrt{\sin 2x}\right) + \arcsin(\sin x - \cos x)\right] + C.$

2132. $\sinh x + C.$ **2133.** $\cosh x + C.$ **2134.** $\tanh x + C.$

2135. $x + C.$ **2136.** $\dfrac{1}{2a} \sinh 2ax + C.$

2137. $\dfrac{\sinh x \cosh x - x}{2} + C.$ **2138.** $x - \tanh x + C.$

2139. $x - \coth x + C.$ **2140.** $\dfrac{1}{3}\cosh^3 x - \cosh x + C.$

2141. $\sinh x + \dfrac{1}{3}\sinh^3 x + C.$ **2142.** $x - \tanh x - \dfrac{1}{3}\tanh^3 x + C.$

2143. $\dfrac{1}{3}\sinh^3 x + \dfrac{1}{5}\sinh^5 x + C.$

2144. $\ln|\sinh x| - \dfrac{1}{2}\coth^2 x - \dfrac{1}{4}\coth^4 x + C.$

2145. $\ln|\tanh x| + C$. **2146.** $\ln\left|\tanh\frac{x}{2}\right| + C$.

2147. $\frac{1}{2}\tanh\frac{x}{2} - \frac{1}{6}\tanh^3\frac{x}{2} + C$.

2148. $\frac{1}{2}\ln\frac{1+\sqrt{\tanh x}}{|1-\sqrt{\tanh x}|} - \arctan\sqrt{\tanh x} + C$.

2149. $x\tanh x - \ln\cosh x + C$. **2150.** $C - \frac{e^{3x}}{3\sinh^3 x}$.

2151*. $\ln\frac{|\cos x|}{2 + x + 2\sqrt{x^2+x+1}}$. (e.g. the substitution $x = \frac{1}{z}$ can be used.)

2152. $\frac{1}{2}\arccos\frac{2-x}{x\sqrt{2}} + C$. **2153.** $\arcsin\frac{x-1}{x\sqrt{2}} + C$.

2154. $C - \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2+x-x^2}+\sqrt{2}}{x} + \frac{1}{2\sqrt{2}}\right|$.

2155. $\ln|x+1+\sqrt{2x+x^2}| - \frac{4}{x+\sqrt{2x+x^2}} + C$.

2156. $C - \frac{1}{\sqrt{3}}\ln\left|\frac{3+3x+2\sqrt{3(x^2+x+1)}}{x-1}\right|$.

2157. $C - \frac{1}{\sqrt{15}}\ln\left|\frac{x+6+\sqrt{60x-15x^2}}{2x-3}\right|$.

2158. $\frac{1}{2}(x-1)\sqrt{x^2-2x-1} - \ln|x-1+\sqrt{x^2-2x-1}| + C$.

2159. $\frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{3x^2-3x+1} +$

$+ \frac{1}{8\sqrt{3}}\ln\left|\sqrt{3x^2-3x-1} + \frac{\sqrt{3}}{2}(2x-1)\right| + C$.

2160. $\frac{1}{2}\left[(x+1)\sqrt{1-4x-x^2} + 5\arcsin\frac{x+2}{\sqrt{5}}\right] + C$.

2161. $C - \frac{3}{2(2x-1-2\sqrt{x^2-x+1})} -$

$-\frac{3}{2}\ln|2x-1-2\sqrt{x^2-x+1}| + 2\ln|x-\sqrt{x^2-x+1}|$.

2162. $\ln\left|\frac{x+\sqrt{x^2+1}}{x}\right| - \frac{\sqrt{1+x^2}}{x} + C$.

2163. $\dfrac{1-\sqrt{x^2+2x+2}}{x+1}+\ln(x+1+\sqrt{x^2+2x+2})+C.$

2164. $\dfrac{1}{2}(3-x)\sqrt{1-2x-x^2}+2\arcsin\dfrac{x+1}{\sqrt{2}}+C.$

2165. $x\sqrt{x^2-2x+5}-5\ln(x-1+\sqrt{x^2-2x+5})+C.$

2166. $C-\dfrac{1}{2}(3x-19)\sqrt{3-2x-x^2}+14\arcsin\dfrac{x+1}{2}.$

2167. $(x^2-5x+20)\sqrt{x^2+4x+5}-15\ln(x+2+$
$+\sqrt{x^2+4x+5})+C.$

2168. $\left(\dfrac{1}{3}x^2-\dfrac{5}{6}x+\dfrac{1}{6}\right)\sqrt{x^2+2x+2}+\dfrac{5}{2}\ln(x+1+$
$+\sqrt{x^2+2x+2})+C.$

2169. $(x^2+5x+36)\sqrt{x^2-4x-7}+112\ln|x-2+$
$+\sqrt{x^2-4x-7}|+C.$

2170. $\left(\dfrac{1}{4}x^3-\dfrac{7}{6}x^2+\dfrac{95}{24}x-\dfrac{145}{12}\right)\sqrt{x^2+4x+5}+$
$+\dfrac{35}{8}\ln(x+2+\sqrt{x^2+4x+5})+C.$

2171. $\dfrac{\sqrt{x^2+2x-3}}{8(x+1)^2}+\dfrac{1}{16}\arccos\dfrac{2}{x+1}+C.$

2172. $\dfrac{1}{2\sqrt{2}}\ln\dfrac{\sqrt{2+2x^2}-x}{\sqrt{2+2x^2}+x}+\ln(x+\sqrt{x^2+1})+C.$

2173. $\dfrac{\sqrt{2x^2-2x+1}}{x}+C.$

2174. $\ln\dfrac{\sqrt{x^2+2x+4}-1}{\sqrt{x^2+2x+4}+1}-\dfrac{1}{\sqrt{2}}\arctan\dfrac{\sqrt{2(x^2+2x+4)}}{x+1}+C.$

2175. $C-\dfrac{1}{8(x-1)^8}-\dfrac{1}{3(x-1)^9}-\dfrac{3}{10(x-1)^{10}}-\dfrac{1}{11(x-1)^{11}}.$

2176. $\dfrac{1}{3}[x^3+\sqrt{(x^2-1)^3}]+C.$ **2177.** $\dfrac{3(4x-3a)\sqrt[3]{(a+x)^4}}{28}+C.$

2178. $\dfrac{1}{m\sqrt{ab}}\arctan\left(e^{mx}\sqrt{\dfrac{a}{b}}\right)+C.$

2179. $\dfrac{1}{2}\arcsin x-\dfrac{x+2}{2}\sqrt{1-x^2}+C.$

2180. $\frac{x^2}{2} - 2x + \frac{1}{6}\ln\frac{|x-1|\,(x+2)^{32}}{|x+1|^3} + C.$

2181. $\frac{1}{4}\ln\left|\frac{1+x}{1-x}\right| + \frac{1}{2}\arctan x + C.$

2182. $\frac{3}{8}\arctan x - \frac{x}{4(x^4-1)} - \frac{3}{16}\ln\left|\frac{x-1}{x+1}\right| + C.$

2183. $2\sqrt{x+1}\,[\ln|x+1| - 2] + C.$

2184. $\left(\frac{1}{2}x + \frac{3}{4}\right)\cos 2x + \left(\frac{1}{2}x^2 + \frac{3}{2}x + \frac{9}{4}\right)\sin 2x + C.$

2185. $x^2\cosh x - 2x\sinh x + 2\cosh x + C.$

2186. $x\arctan(1+\sqrt{x}) - \sqrt{x} + \ln|x + 2\sqrt{x} + 2| + C.$

2187. $\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| - \frac{\arcsin x}{x} + C.$

2188. $3e^{\sqrt[3]{x}}(\sqrt[3]{x^2} - 2\sqrt[3]{x} + 2) + C.$

2189. $3e^{\sqrt[3]{x}}(\sqrt[3]{x^5} - 5\sqrt[3]{x^4} + 20x - 60\sqrt[3]{x^2} + 120\sqrt[3]{x} - 120) + C.$

2190. $e^{3x}\left(\frac{1}{3}x^3 - x^2 + \frac{2}{3}x + \frac{13}{9}\right) + C.$

2191. $2(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}) + C.$

2192. $\frac{\sqrt{x-1}\,(3x+2)}{4x^2} + \frac{3}{4}\arctan\sqrt{x-1} + C.$

2193. $\frac{x^2}{2} + \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln|x + \sqrt{x^2-1}| + C.$

2194. $\ln(x + \sqrt{1+x^2}) - \frac{\sqrt{(1+x^2)^5}}{5x^5} - \frac{\sqrt{(1+x^2)^3}}{3x^3} - \frac{\sqrt{1+x^2}}{x} + C.$

2195. $\left(\frac{1}{4}x^3 - \frac{3}{8}x\right)\sqrt{x^2+1} + \frac{3}{8}\ln(x + \sqrt{x^2+1}) + C.$

2196. $3[\ln|u| - \ln(1 + \sqrt{1-u^2}) - \arcsin u] + C,$ where $u = \sqrt[3]{x}$

2197. $\frac{15x^2 + 5x - 2}{4x^2\sqrt{1+x}} + \frac{15}{8}\ln\left|\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right| + C.$

2198. $C - \frac{\sqrt{2x+1}}{x} + \ln\left|\frac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1}\right|.$

2199. $\frac{1}{15}\left[\frac{1}{2}\ln\frac{(z-1)^2}{z^2+z+1} - \sqrt{3}\arctan\frac{2z+1}{\sqrt{3}}\right] + C,$ where $z = x^5.$

2200. $C - \frac{1}{4}\ln\left|\tan\frac{x}{2}\right| + \frac{1}{8\sin^2\frac{x}{2}}$. **2201.** $\frac{1}{\sqrt{2}}\arctan\frac{\tan x}{\sqrt{2}} + C$.

2202. $\frac{2}{b^2\sin 2\alpha}\ln\left|\frac{\sin(\alpha - x)}{\sin(\alpha + x)}\right| + C$, where $\alpha = \arccos\frac{a}{b}$, if $a^2 < b^2$: $\frac{1}{a^2\sin\alpha}\arctan\frac{\tan x}{\sin\alpha} + C$, where $\alpha = \arccos\frac{b}{a}$, if $a^2 > b^2$.

2203. $\frac{1}{2}x^2\ln(1 + x^3) - \frac{3}{4}x^2 + \frac{1}{4}\ln(x^2 - x + 1) - \frac{1}{2}\ln(x + 1) + \frac{\sqrt{3}}{2}\arctan\frac{2x - 1}{\sqrt{3}} + C$.

2204. $\frac{x}{\ln x} + C$. **2205.** $\arctan\sqrt{x^2 - 1} - \frac{\ln x}{\sqrt{x^2 - 1}} + C$.

2206. $\frac{1}{2}e^x[(x^2 - 1)\cos x + (x - 1)^2\sin x] + C$.

2207. $\frac{x^2 e^{x^2}}{2} + C$. **2208.** $\frac{2}{3}\,\frac{\tan^2 x - 3}{\sqrt{\tan x}} + C$.

2209. $\frac{1}{4}(\tan^4 x - \cot^4 x) + 2(\tan^2 x - \cot^2 x) + 6\ln|\tan x| + C$.

2210. $\arctan(\tan^2 x) + C$. **2211.** $\ln\left|1 + \tan\frac{x}{2}\right| + C$.

2212. $\frac{\sqrt{2}}{2}\left[\arctan\frac{\tan x}{\sqrt{2 + \tan^2 x}} + \ln\left(\sqrt{2 + \tan^2 x} + \tan x\right)\right] + C$.

2213. $\ln\frac{x^2 + 1 + \sqrt{x^4 + 3x^2 + 1}}{x} + C$.

2214. $C - \frac{1}{\sqrt{15}}\ln\left|\frac{x + 6 + \sqrt{60x - 15x^2}}{2x - 3}\right|$. **2215.** $\frac{e^x}{1 + x} + C$.

2216. $2x\sqrt{1 + e^x} - 4\sqrt{1 + e^x} - 2\ln\frac{\sqrt{1 + e^x} - 1}{\sqrt{1 + e^x} + 1} + C$.

2217. $\frac{1}{6}\ln\frac{1 + x^2}{x^2} - \frac{\arctan x}{3x^3} - \frac{1}{6x^2} + C$.

2218. $C - \frac{\arctan x}{2(1 + x^2)} + \frac{\arctan x}{4} + \frac{x}{4(1 + x^2)}$.

2219. $\frac{1}{4}\ln\frac{|x + 1|}{\sqrt{x^2 + 1}} - \frac{\arctan x}{2(1 + x)^2} - \frac{1}{4(x + 1)} + C$.

2220. $x - \log_2|1 - 2^x| + \frac{1}{\ln 2}\left[\frac{1}{1 - 2^x} + \frac{1}{2(1 - 2^x)^2} + \right.$
$\left. + \frac{1}{3(1 - 2^x)^3}\right] + C.$

2221. $\arctan(e^x - e^{-x}) + C.$

2222. $\ln\frac{\sqrt{1 + e^x + e^{2x}} - e^x - 1}{\sqrt{1 + e^x + e^{2x}} - e^x + 1} + C.$

2223. $x - \frac{2}{\sqrt{3}}\arctan\frac{1 + 2\tan x}{\sqrt{3}} + C.$

2224. $\frac{35}{128}x - \frac{1}{4}\sin 2x + \frac{7}{128}\sin 4x + \frac{1}{24}\sin^3 2x +$
$+ \frac{1}{1024}\sin 8x + C.$

2225. $\frac{1}{2}x^2 + \frac{3}{2}\ln(1 + x^2) + \frac{1}{(1 + x^2)^2} + C.$

2226. $\frac{8}{49(x - 5)} - \frac{27}{49(x + 2)} + \frac{30}{343}\ln\left|\frac{x - 5}{x + 2}\right| + C.$

2227. $C - \frac{\sqrt{2}}{2}\arctan(\sqrt{2}\cot 2x).$ **2228.** $x\tan\frac{x}{2} + C.$

2229. $\frac{1}{\sqrt{2}}\arccos\frac{x\sqrt{2}}{x^2 + 1} + C.$ (Divide numerator and denominator by x^2 and use the substitution $x + \frac{1}{x} = z$.)

2230. $e^{\sin x}(x - \sec x) + C.$

Chapter VII

2231. $\frac{2}{3}(\sqrt{8} - 1).$ **2232.** $\frac{7}{72}.$ **2233.** $-5(\sqrt[5]{16} - 1).$ **2234.** $7\frac{2}{3}.$

2235. $\frac{T}{\pi}\cos\varphi_0.$ **2236.** 12. **2237.** $0{\cdot}2(e - 1)^5.$ **2238.** $3\ln\frac{b}{b - a}.$

2239. $\frac{1}{4}.$ **2240.** $\frac{\pi}{2}.$ **2241.** $1 + \frac{1}{2}\log e.$ **2242.** $e - \sqrt{e}.$ **2243.** $\frac{\pi}{6n}$

2244. 2. **2245.** $\frac{4}{3}.$ **2246.** $\ln\frac{3}{2}.$ **2247.** $0{\cdot}2\ln\frac{4}{3}.$ **2248.** $\arctan\frac{1}{7}.$

2249. $\frac{1}{2}\ln\frac{8}{5}.$ **2250.** $\frac{\pi}{6}.$ **2251.** 2. **2252.** $\frac{2}{7}.$ **2253.** $\frac{4}{3}.$

2254. $\frac{\pi}{2\omega}$. **2255.** $-0{\cdot}083\ldots$ **2256.** $\frac{2}{3}+\frac{\pi}{4}-\alpha+\frac{\cot^3 3}{3}\alpha-\cot\alpha$

2257. 1. **2258.** $-\frac{\sqrt{2}}{3}$. **2259.** $1-\frac{2}{e}$. **2260.** $\frac{\pi}{2}-1$.

2261. $\frac{\pi(9-4\sqrt{3}}{36}+\frac{1}{2}\ln\frac{3}{2}$. **2262.** $\pi^3-6\pi$. **2263.** $2-\frac{3}{4\ln 2}$.

2264. 1. **2265.** $\frac{141\,a^3\sqrt[3]{a}}{20}$. **2266.** $\frac{\pi a^2}{4}$. **2267.** $\frac{e^{\pi}-2}{5}$.

2268. $6-2e$. **2269.** (a) $\frac{8}{15}$; (b) $\frac{7\,.\,5\,.\,3\,.\,1}{8\,.\,6\,.\,4\,.\,2}\cdot\frac{\pi}{2}\approx 0{\cdot}429$;
(c) $\frac{10\,.\,8\,.\,6\,.\,4\,.\,2}{11\,.\,9\,.\,7\,.\,5\,.\,3}=\frac{256}{693}$.

2270. $J_{m,n}=\frac{n-1}{m+n}J_{m,n-2}=\frac{m-1}{m+n}J_{m-2,n}$.

If n is odd, then

$$J_{m,n}=\frac{(n-1)(n-3)\ldots 4\,.\,2}{(m+n)(m+n-2)\ldots(m+3)(m+1)};$$

if m is odd, then

$$J_{m,n}=\frac{(m-1)(m-3)\ldots 4\,.\,2}{(m+n)(m+n-2)\ldots(n+3)(n+1)};$$

if m is even, n even, then

$$J_{m,n}=\frac{(n-1)(n-3)\ldots 3\,.\,1\times(m-1)(m-3)\ldots 3\,.\,1}{(m+n)(m+n-2)(m+n-4)\ldots 4\,.\,2}\cdot\frac{\pi}{2}.$$

2271. $(-1)^n\,n!\left[1-\frac{1}{e}\left(\frac{1}{n!}+\frac{1}{(n-1)!}+\ldots+\frac{1}{1!}+1\right)\right]$.

2272. $\frac{11}{48}+\frac{5\pi}{64}$.

2274*. $\frac{p!q!}{(p+q+1)!}$. Put $x=\sin^2 z$ and use the result of problem 2270.

2275. $7+2\ln 2$. **2276.** $2-\frac{\pi}{2}$. **2277.** $\frac{32}{3}$.

2278. $\frac{5}{3}-2\ln 2$. **2279.** $\ln\frac{e+\sqrt{1+e^2}}{1+\sqrt{2}}$. **2280.** $8+\frac{3\sqrt{3}}{2}\pi$.

2281*. $\frac{5}{16}\pi$. Putting $x=2z$, the given integral is transformed to $2\int\limits_0^{\frac{\pi}{2}}\sin^6 z\,dz$. (See *Course*, sec. 106.)

2282*. $\frac{8}{35}$. Put $x = \frac{z}{2}$. **2283.** $\frac{\pi}{32}$.

2284. $\sqrt{2} - \frac{2}{\sqrt{3}} + \ln \frac{2 + \sqrt{3}}{1 + \sqrt{2}}$. **2285.** $\frac{8}{15}$. **2286.** $\sqrt{3} - \frac{\pi}{3}$.

2287. $-\frac{1}{8}\left(\pi + \frac{7\sqrt{3}}{8}\right)$. **2288.** $\frac{3}{16}\pi$. **2289.** $\frac{\pi}{16}$.

2290. $\frac{\sqrt{3}}{2} + \ln(2 - \sqrt{3})$. **2291.** $\frac{\pi}{4}$. **2292.** $\frac{\sqrt{12}}{48}$. **2293.** $-\frac{\pi}{3}$.

2294. $\arctan\frac{1}{2}$. **2295.** $\frac{\sqrt{6}}{27} + \frac{\pi\sqrt{2}}{48}$. **2296.** $\frac{20}{9}$.

2297. $2\ln\frac{6}{5} \approx 0{\cdot}365$. **2298.** $\frac{2}{\pi}$; $\frac{1}{2}$. **2299.** $2 + \ln\frac{2}{e^2 + 1}$.

2300. For $a = e$. **2301.** $\frac{1}{2}\ln\frac{8}{5}$. **2302.** $\frac{2}{45}$.

2303. $8\ln 3 - 15\ln 2 + \frac{13}{8}$. **2304.** $\frac{5}{192}(5 + 7\sqrt[5]{5^3})$. **2305.** $\frac{\pi}{6}$.

2306. $a^2[\sqrt{2} - \ln(\sqrt{2} + 1)]$. **2307.** $\sqrt{3} - \frac{1}{2}\ln(2 + \sqrt{3})$.

2308. $\frac{848}{105}$. **2309.** $4 - \pi$. **2310.** $\ln\frac{7 + 2\sqrt{7}}{9}$. **2311.** $\frac{\pi}{4} - \frac{1}{2}$.

2312. $\frac{2}{\sqrt{5}}\arctan\frac{1}{\sqrt{5}}$. **2313.** $\frac{\pi}{2}\sqrt{\frac{6}{7}}$. **2314.** $\frac{\pi^4}{16} - 3\pi^2 + 24$.

2315. $\frac{16\pi}{3} - 2\sqrt{3}$. **2316.** $\frac{19}{27} - \frac{5}{6\sqrt{6}}$. **2317.** $\frac{1}{a^2 - b^2}\ln\left|\frac{a}{b}\right|$.

2319. $x = 2$. **2320.** $x = \ln 8$. **2322*.** Use the relationship $4 - x^2 \geqq 4 - x^2 - x^3 \geqq 4 - 2x^2$, which holds for $0 \leqq x \leqq 1$.

2323*. Use the inequalities

$$\sqrt{1 - x^2} \leqq \sqrt{1 - x^{2n}} \leqq 1, \text{ where } -1 \leqq x \leqq 1 \text{ and } n \geqq 1.$$

2324. $1{\cdot}098 < I < 1{\cdot}110$.

2325*. Use the inequality $1 + x^4 < (1 + x^2)^2$ for the lower estimate, and Bunyakovskii's inequality for the upper estimate.

2326. $I(1) \approx 1{\cdot}66$ is the maximum value, $I\left(-\frac{1}{2}\right) \approx -0{\cdot}11$ is the minimum value.

2327. Minimum at $x = 1$ $\left(y = -\frac{17}{12}\right)$; points of inflexion are $\left(2, -\frac{4}{3}\right)$ and $\left(\frac{4}{3}, -\frac{112}{81}\right)$.

2332*. (a) Substitute for the variable of integration in accordance with the formula $t = -x$, divide the interval $[-a, -x]$ into two intervals: $[-a, a]$ and $[a, -x]$ and use the fact that the integral of an odd function over the interval $[-a, a]$ is zero. (b) No, if $a \neq 0$; yes, if $a = 0$.

2333*. Put $t = \frac{1}{z}$.

2338. Each of the integrals is equal to $\frac{\pi}{4}$.

2339*. Put $x = \pi - z$. The integral is equal to $\frac{\pi^2}{4}$.

2340*. Divide the interval of integration $[a, a + T]$ into $[a, 0]$, $[0, T]$ and $[T, a + T]$, then show by using the property $f(x) = f(x + T)$ that

$$\int_0^a f(x)\,dx = \int_T^{a+T} f(x)\,dx.$$

2341*. The equality required for the proof is equivalent to

$$\int_x^{x+T} f(z)\,dz = 0.$$

Show that the integral on the left of this equality is independent of x, then put $x = -\frac{T}{2}$. **2342.** $\frac{2 \cdot 4 \cdot 6 \ldots 2n}{1 \cdot 3 \cdot 5 \ldots (2n + 1)}$.

2343. The substitution $z = \tan\frac{x}{2}$ is not permissible because the function $\tan\frac{x}{2}$ is discontinuous for $x = \pi$.

2344*. To estimate I_n, use the fact that I_n decreases as n increases.

2345*. Replace the variable of integration in accordance with the formula $z = \frac{x + t}{2}$ and use the property of the integral of an even function (see *Course*, sec. 107.)

2346*. Replace the variable of integration in accordance with the formula $z = k\omega^2 x^2$ and then use l'Hôpital's rule.

2347. By the rectangle rule $\pi \approx 2{\cdot}904$ (under-value) and $\pi \approx 3{\cdot}305$ (over-value). By the trapezium formula $\pi \approx 3{\cdot}104$. By Simpson's formula $\pi \approx 3{\cdot}127$.

2348. By the rectangle rule $\pi \approx 3{\cdot}04$ (under-value) and $\pi \approx 3{\cdot}24$ (over-value). By the trapezium formula $\pi \approx 3{\cdot}140$. By Simpson's formula $\pi \approx 3{\cdot}1416$ (correct to all places).

2349. $\ln 10 = 2{\cdot}31$, $M = \frac{1}{\ln 10} \approx 0{\cdot}433$. **2350.** $\approx 0{\cdot}837$.

2351. $\approx 1{\cdot}09$. **2352.** $\approx 2{\cdot}59$. **2353.** $\approx 0{\cdot}950$. **2354.** $\approx 1{\cdot}53$.
2355. $\approx 0{\cdot}985$. **2356.** $\approx 0{\cdot}957$.
2357. ≈ 239 m^2 (by Simpson's formula).
2358. $\approx 11{\cdot}7$ m^2 (by Simpson's formula).
2359. ≈ 1950 mm^2. **2360.** $\approx 10{\cdot}9$. **2361.** $\approx 36{\cdot}2$. **2362.** $\approx 98{\cdot}2$.
2363. $\approx 9{\cdot}2$. **2364.** ≈ 569 mm^2. **2365.** ≈ 138 mm^2.

2366. $\frac{1}{3}$. **2367.** Divergent. **2368.** $\frac{1}{a}$. **2369.** Divergent.

2370. π. **2371.** Divergent. **2372.** $1 - \ln 2$. **2373.** $\frac{1}{2}$.

2374. $\frac{\pi}{2}$. **2375.** $\ln \frac{\sqrt{a^4+1}+1}{a^2}$. **2376.** $\frac{1}{2}$. **2377.** $\frac{1}{2}$.

2378. Divergent. **2379.** 2. **2380.** $\frac{1}{2}$.

2381. $\frac{a}{a^2+b^2}$, if $a > 0$, divergent if $a \leqq 0$.

2382. $\frac{\pi}{4} + \frac{1}{2}\ln 2$. **2383.** $\frac{2\pi}{3\sqrt{3}}$. **2384.** $\frac{\pi}{2}$. **2385.** $\frac{1}{2} + \frac{\pi}{4}$.

2386. Convergent. **2387.** Divergent. **2388.** Convergent.
2389. Divergent. **2390.** Convergent. **2391.** Divergent.
2392. Divergent.

2393. Convergent. **2394.** $\frac{\pi}{2}$. **2395.** Divergent. **2396.** $\frac{8}{3}$.

2397. $-\frac{1}{4}$. **2398.** 1. **2399.** Divergent. **2400.** 2. **2401.** π.

2402. $\frac{1}{2}\pi(a+b)$. **2403.** $\frac{33\pi}{2}$. **2404.** $\frac{\pi}{3\sqrt{3}}$. **2405.** $\frac{\pi}{\sqrt{3}}$.

2406. $14\frac{4}{7}$. **2407.** $\frac{10}{7}$. **2408.** Divergent. **2409.** $6 - \frac{9}{2}\ln 3$.

2410. $-\frac{2}{e}$. **2411.** Divergent. **2412.** Convergent.

2413. Divergent. **2414.** Convergent. **2415.** Convergent.
2416. Divergent. **2417.** Convergent. **2418.** No.
2419. Convergent for $k < -1$, divergent for $k \geqq -1$.
2420. (1) Convergent for $k > 1$, divergent fr $k \leqq 1$;

(2) $I = \frac{1}{(k-1)(\ln 2)^{k-1}}$ if $k > 1$; divergen i $k \leqq 1$.

2421. Convergent for $k < 1$, divergent for $k \geqq 1$.

2422. Divergent for any k.

2423. Convergent when the inequalities $k > -1$ and $t > k + 1$ are fulfilled simultaneously.

2424. Convergent for $m < 3$, divergent for $m \geqq 3$.

2425. Convergent for $k < 1$, divergent for $k \geqq 1$.

2426. π. **2427*.** $\frac{5\pi}{3}$. Put $x = \cos\varphi$ and integrate by parts.

2428. $\frac{3 + 2\sqrt{3}}{4}\pi - \frac{3}{2}\ln 2$.

2429. $\frac{1 \cdot 3 \cdot 5 \ldots (2n-3)}{2 \cdot 4 \cdot 6 \ldots (2n-2)} \frac{\pi}{2a^{2n-1}}$. **2430.** $n!$. **2431.** $\frac{n!}{2}$.

2432. $(-1)^n n!$.

2433*. (a) $\frac{(m-1)(m-3)\ldots 3 \cdot 1}{m(m-2)\ldots 4 \cdot 2} \frac{\pi}{2}$;

(b) $\frac{(m-1)(m-3)\ldots 4 \cdot 2}{m(m-2)\ldots 3 \cdot 1}$. Put $x = \sin\varphi$.

2434*. $2\frac{2n(2n-2)\ldots 4 \cdot 2}{(2n+1)(2n-1)\ldots 3 \cdot 1}$. Put $x = \sin^2\varphi$.

2435. $\frac{\pi - \alpha}{\sin\alpha}$ ($I = 1$ with $\alpha = \pi$).

2436*. To prove that the integrals are equal, put $x = \frac{1}{z}$ in one of them. Then evaluate their sum, using the identity

$$\frac{1+x^2}{1+x^4} = \frac{1}{2}\left(\frac{1}{1+x^2+x\sqrt{2}} + \frac{1}{1+x^2-x\sqrt{2}}\right).$$

2437*. Write the integral as the sum of two integrals: $\int_0^\infty = \int_0^1 + \int_1^\infty$; put $x = \frac{1}{y}$ in the second integral.

2438. 0. **2439.** $\frac{1}{2}\sqrt{\frac{\pi}{a}}$. **2440.** $\sqrt{\pi}$. **2441*.** $\frac{\sqrt{\pi}}{4}$. Integrate by parts.

2442. $\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n} \frac{\sqrt{\pi}}{2}$. **2443.** $\frac{\pi}{2}$.

2444. $\frac{\pi}{2}$ if $a > 0$; 0 if $a = 0$; $-\frac{\pi}{2}$ if $a < 0$.

2445. $\frac{\pi}{2}$ if $a > b$; $\frac{\pi}{4}$ if $a = b$; 0 if $a < b$.

2446*. $\frac{\pi}{2}$. Integrate by parts.

2447*. $\frac{\pi}{4}$. Write the numerator as a difference between sines of multiple angles.

2448*. $\frac{\pi}{4}$. Use the methods of solution of problems 2446 and 2447.

2449*. Put $y = \frac{\pi}{2} - z$ and reduce $\varphi(x)$ to the form

$$\varphi(x) = \int\limits_{\frac{\pi}{2}}^{\frac{\pi}{2}-x} \ln \sin z \, dz.$$

Split the integral into three by using the formula $\sin z = 2 \sin \frac{z}{2} \cos \frac{z}{2}$; one integral is then obtained directly. The other two reduce to integrals of the original type by substituting for the variable; $\varphi\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \ln 2$.

2450. $-\frac{\pi}{2} \ln 2$. **2451.** $-\frac{\pi^2}{2} \ln 2$.

2452*. $\frac{\pi}{2} \ln 2$. Integrate by parts.

2453*. $\frac{\pi}{2} \ln 2$. Reduces to the previous problem by substituting for the variable.

2454. $-\frac{\pi}{2} \ln 2$.

Chapter VIII

2455. $\frac{16}{3}$. **2456.** $\frac{9}{4}$. **2457.** $\frac{16}{3} p^2$. **2458.** $\frac{1}{3}$. **2459.** $\frac{32}{3} \sqrt{6}$.

2460. $2\frac{1}{4}$. **2461.** $2\pi + \frac{4}{3}$ and $6\pi - \frac{4}{3}$.

2462. $\frac{4}{3}(4\pi + \sqrt{3})$ and $\frac{4}{3}(8\pi - \sqrt{3})$.

2464. $\frac{b^2 c}{a} - ab \ln \frac{c + b}{a} = b\left[\varepsilon b - a \ln\left(\varepsilon + \sqrt{\varepsilon^2 - 1}\right)\right]$, where ε is the eccentricity.

2465. $a^2\left[\frac{\pi}{6}-\frac{\sqrt{2}}{8}\ln(\sqrt{3}+\sqrt{2})\right]$; $a^2\left[\frac{\pi}{6}-\frac{\sqrt{2}}{8}\ln(\sqrt{3}+\sqrt{2})\right]$ and $a^2\left[\frac{2\pi}{3}+\frac{\sqrt{2}}{4}\ln(\sqrt{3}+\sqrt{2})\right]$.

2466. $S_1 = S_3 = \pi - \frac{\sqrt{2}}{2}\ln 3 - 2\arcsin\sqrt{\frac{2}{3}} \approx 0{\cdot}46$.
$S_2 = 2(\pi - S_1)$

2467. $\frac{\pi}{2}-\frac{1}{3}$. **2468.** $\frac{1}{12}$. **2469.** $\frac{1}{12}$.

2470. $\left|\frac{m-n}{m+n}\right|$; $4\left|\frac{m-n}{m+n}\right|$, if m and n are both even: $2\left|\frac{m-n}{m+n}\right|$ if m and n are both odd; $\left|\frac{m-n}{m+n}\right|$ if m is odd and n even or vice versa.

2471. (a) $\frac{3}{14}$; (b) $73\frac{1}{7}$.

2472. 1 (the figure consists of two parts whose areas are equal).

2473. $\frac{8}{15}$. **2474.** $\frac{3}{4}\pi$. **2475.** $\frac{4}{3}$.

2476. $\frac{\pi a^2}{8}$. **2477.** $8\left(\sqrt{1+\frac{2}{3}\sqrt{3}} - \arctan\sqrt{1+\frac{2}{3}\sqrt{3}}\right)$.

2478. $e+\frac{1}{e}-2$. **2479.** 4. **2480.** $\frac{3}{e}(e^3-4)$. **2481.** $\frac{18}{e^2}-2$.

2482. (a) $b(\ln b-1)-a(\ln a-1)$; (b) $b-a$. **2483.** $3-e$.

2484. $\frac{3-2\ln 2-2\ln^2 2}{16}$. **2485.** $2-\sqrt{2}$. **2486.** $\frac{1}{3}+\ln\frac{\sqrt{3}}{2}$.

2487. $\frac{5}{3}\sqrt{2}$. **2488.** $\sqrt{2}-1$. **2489.** $\frac{\pi}{4}$. **2490.** $3\pi a^2$.

2491. $\frac{3}{8}\pi a^2$. **2492.** $6\pi a^2$.

2493. (1) $\frac{\pi R^2}{n^2}(n+1)(n+2)$; (2) $\frac{\pi R^2}{n^2}(n-1)(n-2)$.

2494. (1) $\frac{72}{5}\sqrt{3}$; (2) $\frac{8}{15}$. **2495.** (1) $\frac{4}{3}\pi^3 a^2$; (2) $\frac{76a^2\pi^3}{3}$.

2496. $\frac{\pi a^2}{2}$ (four-petal rose). **2497.** $\frac{\pi a^2}{4}$. **2498.** $18\pi a^2$.

2499. $\frac{a^2}{8}(4-\pi)$. **2500.** $\frac{37\pi}{6} - 5\sqrt{3}$. **2501.** $\frac{51\sqrt{3}}{16}$. **2502.** a^2.

2505*. $a^2\frac{5\pi + 18\sqrt{3}}{32}$. To construct the curve, the variation of φ from 0 to 3π must be taken into account.

2506. $\frac{\pi}{4}$. **2507.** a^2 **2508.** $a^2\left(1 + \frac{\pi}{6} - \frac{\sqrt{3}}{2}\right)$.

2509. $\frac{\pi}{2}(a^2 + b^2)$. **2510.** a^2. **2511.** $\pi\sqrt{2}$. **2512.** π. **2513.** 2.

2514. $3\pi a^2$. **2515.** 4π. **2516*.** (1) $\frac{\sqrt{\pi}}{2}$; (2) $\sqrt{\pi}$. Use the fact that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \text{ (Poisson's integral).}$$

2517. $\frac{\pi a^2}{2}$. **2518.** $2 - \frac{\pi}{2}$ and $2 + \frac{\pi}{2}$. **2519.** $\frac{a}{2}\left(e^{\frac{b}{a}} - e^{-\frac{b}{a}}\right)$.

2520. $\frac{y}{2p}\sqrt{y^2 + p^2} + \frac{p}{2}\ln\frac{y + \sqrt{y^2 + p^2}}{p}$.

2521. $1 + \frac{1}{2}\ln\frac{3}{2}$. **2522.** $\ln 3 - \frac{1}{2}$.

2523. $\ln\frac{e^b - e^{-b}}{e^a - e^{-a}}$. **2524.** $\frac{8}{9}\left(\frac{5}{2}\sqrt{\frac{5}{2}} - 1\right)$.

2525. $4\frac{26}{27}$. **2526.** $4a\sqrt{3}$.

2527. $\frac{\pi}{2} + 2\ln\tan\frac{3\pi}{8} = \frac{\pi}{2} + 2\ln(\sqrt{2} + 1)$.

2528. $\frac{1}{6} + \frac{1}{4}\ln 3$. **2529.** 2. **2530.** 8.

2531. At $t = \frac{2\pi}{3}$. $\left[x = a\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right),\ y = \frac{3a}{2}\right]$.

2532. At $t = \frac{\pi}{6}$, $\left(x = \frac{3\sqrt{3}}{8}R,\ y = \frac{R}{8}\right)$.

2533*. $4\frac{a^2 + ab + b^2}{a + b}$. Put $x = a\cos^3 t$, $y = b\sin^3 t$.

2534. $5a\left[1 + \frac{1}{2\sqrt{3}}\ln(2 + \sqrt{3})\right]$. **2535.** $a\ln\frac{a}{y}$. **2536.** $\frac{\pi^2}{2}R$.

2537. $\frac{\pi^3}{3}$. **2538.** $4\sqrt{3}$. **2541.** $2(e^t - 1)$.

2543. $\pi a\sqrt{1 + 4\pi^2} + \frac{a}{2}\ln(2\pi + \sqrt{1 + 4\pi^2})$.

2545. $\ln\frac{3}{2} + \frac{5}{12}$. **2546.** $8a$. **2547.** $\frac{3}{2}\pi a$.

2549. k must have the form $\frac{2N + 1}{2N}$ or $\frac{2N}{2N - 1}$, where N is an integer.

2550. 4. **2551.** $\ln\frac{\pi}{2}$.

2554*. Prove that the length of the ellipse can be written in the form $L = 4\int_0^{\frac{\pi}{4}}\left(\sqrt{a^2\cos^2 t + b^2\sin^2 t} + \sqrt{a^2\sin^2 t + b^2\cos^2 t}\right)dt$, and use the theorem on estimating an integral.

2555. 2π. **2556.** (1) $\frac{4}{3}\pi ab^2$; (2) $\frac{4}{3}\pi a^2 b$.

2557. $\frac{8}{15}\pi h^2 a$. **2558.** $\frac{\pi h^2}{3}(3a + h)$. **2559.** $\frac{\pi}{4}(e^2 - 1)$.

2560. $\frac{\pi}{4}\left[\frac{e^{2b} - e^{-2b}}{2} - \frac{e^{2a} - e^{-2a}}{2} + 2(b - a)\right]$. **2561.** $\frac{3\pi}{10}$.

2562. $\frac{\pi}{2}(15 - 16\ln 2)$. **2563.** $\pi\left(\frac{\pi^2}{4} - 2\right)$. **2564.** $\frac{8\pi}{3}$.

2565. $2\pi^2$. **2566.** $\frac{\pi a^3}{4}\left[\sqrt{2}\ln(1 + \sqrt{2}) - \frac{2}{3}\right]$.

2567. (1) $\frac{2}{3}\pi a^3$; (2) $\frac{\pi^2}{16}$. **2568.** $5\pi^2 a^3$. **2569.** $\pi a^3\left(\frac{3\pi^2}{2} - \frac{8}{3}\right)$.

2570. $\frac{32}{105}\pi a^3$. **2571.** $\frac{16\pi c^6}{105ab^2}$. **2572.** $\frac{\pi^2}{2}$.

2573. $\frac{\pi e}{2}$. **2574*.** (1) π; (2) $\pi\sqrt{\frac{\pi}{2}}$. See hint on problem 2516.

2575*. $\frac{3\sqrt{2\pi}}{32}$. See hint on problem 2516.

2576*. π^2. Use the fact that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ (Dirichlet's integral).

2577*. $2\pi^2 a^3$. It is advisable to pass to the parametric form by putting $x = 2a\sin^2 t$, $y = \frac{2a\sin^3 t}{\cos t}$.

2578. $\frac{2}{3}\pi a^3$.

2579*. $\frac{4}{3}\pi abc$. Use the formula $v = \int\limits_{x_1}^{x_2} S(x)\,dx$, where $S(x)$ is the cross-sectional area.

2580. (1) $\pi\sqrt{2}$; (2) 36π.

2581. $v_1 = \pi\sqrt{2}\left(2\sqrt{6} - \frac{11}{3}\right)$, $v_2 = \pi\sqrt{2}\left(2\sqrt{6} + \frac{11}{3}\right)$.

2582. $v_1 = v_3 = 4\pi(\sqrt{6} + \sqrt{3} - 4)$, $v_2 = 8\pi(4 - \sqrt{3})$.

2583. $\frac{8\pi\sqrt{6}}{3}$. **2584.** 8π.

2585*. $\frac{2}{3}R^2H = 400\ \text{cm}^3$. Take the axis of symmetry of the base as axis of abscissae.

2586. $\frac{4}{15}ahH = 128\ \text{cm}^3$. **2587.** $\frac{2}{3}abH = 133\frac{1}{3}\ \text{cm}^3$.

2588*. $\frac{2}{3}\pi R^2H$. The area of a symmetrical parabolic segment is equal to $\frac{2}{3}ah$, where a is the base of the segment, h the "height" (see *Course*, sec. 84).

2589*. $\frac{R^2H}{6}\left(\pi + \frac{4}{3}\right)$ and $\frac{R^2H}{6}\left(\pi - \frac{4}{3}\right)$. (See hint on problem 2588).

2590. $\frac{8}{3}a^3$. **2591.** $\frac{8}{3}\pi r^3$. **2592.** $\frac{16}{3}R^3$. **2593.** $\frac{4}{3}R^2H$.

2594. $\frac{56}{3}\pi a^2$. **2595.** $\frac{\pi}{9}\left(\sqrt{(1 + a^4)^3} - 1\right)$. **2596.** $\frac{\pi a^2}{4}(e^2 - e^{-2} + 4)$.

2597. $2\pi b^2 + \frac{2\pi ab}{\varepsilon}\arcsin\varepsilon$ and $2\pi a^2 + \frac{\pi b^2}{\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon}$, where ε is the eccentricity of the ellipse.

2598. $2\pi[\sqrt{2} + \ln(1 + \sqrt{2})]$.

2599. $\pi\left[\sqrt{5} - \sqrt{2} + \ln\frac{2\sqrt{2} + 2}{\sqrt{5} + 1}\right]$. **2600.** $3\pi a^2$.

2601. $\pi a^2\sqrt{2}\left(2 - \frac{\pi}{2}\right)$. **2602.** $\frac{2\pi\sqrt{2}}{5}(e^\pi - 2)$. **2603.** $\frac{12}{5}\pi a^2$.

2604. $8\pi a^2\left(\pi - \frac{4}{3}\right)$. **2605.** $\frac{32}{5}\pi a^2$. **2606.** $4\pi^2r^2$.

2607. $2\pi a^2 (2 - \sqrt{2})$. **2608.** $\pi[\sqrt{2} + \ln(1 + \sqrt{2})]$. **2609.** $4\pi a^2$

2610. $\frac{ah^2}{2}$. **2611.** $\frac{a^3}{6}, \frac{a^3}{6}, \frac{a^3\sqrt{2}}{12}$.

2613. The centre of gravity lies on the axis of symmetry of the segment at a distance $\frac{2}{5}h$ from the base.

2614. For S_1: $\xi = \frac{3}{5}a$, $\eta = \frac{3}{8}b$; for S_2: $\xi = \frac{3}{10}a$, $\eta = \frac{3}{4}b$.

2615. $\xi = 0$, $\eta = \frac{2r}{\pi}$. **2616.** $\xi = 0$, $\eta = \frac{4r}{3\pi}$.

2617. The centre of gravity lies on the bisector of the central angle subtended by the arc, at a distance $2r\frac{\sin\frac{\alpha}{2}}{\alpha}$ from the centre.

2618. $\xi = \frac{a}{5}$, $\eta = \frac{a}{5}$. **2619.** $\xi = \frac{4a}{3\pi}$, $\eta = \frac{4b}{3\pi}$.

2620. $\frac{b^2}{2} + \frac{ab}{2\varepsilon}\arcsin\varepsilon$, where ε is the eccentricity of the ellipse.

2621. $\xi = \frac{\pi}{2}$, $\eta = \frac{\pi}{8}$. **2622.** $\frac{\pi}{2} + \frac{4}{5}$. **2623.** $\frac{\pi}{12} + \frac{\sqrt{3}}{8}$.

2624. $\frac{3}{20}$. **2625.** $\xi = \frac{5}{8}a$, $\eta = 0$.

2626. $\xi = 0$, $\eta = a\frac{e^4 + 4e^2 - 1}{4e(e^2 - 1)}$.

2628. $\xi = \pi a$, $\pi = \frac{4}{3}a$. **2629.** $\xi = \pi a$, $\eta = \frac{5}{6}a$.

2630. $\xi = \frac{2}{5}a$, $\eta = \frac{2}{5}a$. **2631.** $\xi = \frac{256a}{315\pi}$, $\eta = \frac{256a}{315\pi}$.

2633. $\xi = \frac{6a(4 - \pi^2)}{\pi^3}$, $\eta = \frac{2a(\pi^2 - 6)}{\pi^3}$.

2634. The centre of gravity lies on the axis of symmetry of the sector at a distance $\frac{2}{3}\frac{r\sin\alpha}{\alpha}$ from the centre of the circle.

2635. $\xi = \frac{5}{6}a$, $\eta = 0$. **2636.** $\xi = \frac{\sqrt{2}}{8}\pi a$, $\eta = 0$.

2638. $\xi = -\frac{a}{5}\,\frac{2e^{2\pi} + e^{\pi}}{e^{\pi} - e^{\frac{\pi}{2}}}$, $\eta = \frac{a}{5}\,\frac{e^{2\pi} - 2e^{\pi}}{e^{\pi} - e^{\frac{\pi}{2}}}$.

2639. $\xi = \frac{4}{5}a$, $\eta = \frac{4}{5}a$. **2640.** $\frac{3}{8}R$.

2641. The centre of gravity is on the axis of symmetry at a distance $\frac{R}{2}$ from the centre.

2642. $\frac{H}{3}$, $\frac{H\sqrt{R^2 + H^2}}{3(R + \sqrt{R^2 + H^2})}$, $\frac{H}{4}$. **2643.** $\frac{h}{3}$.

2644. $\frac{l}{3}(a^2 + ab + b^2)$.

2645. $\frac{\pi R^2}{4} = M\frac{R^2}{2}$ (M is the mass of the semi-circular disc).

2646. $\frac{\sqrt{(1 + e)^3} - 2\sqrt{2}}{3}$.

2647. $I_x = \frac{256}{15}a^3$; $I_y = 16a^3\left(\pi^2 - \frac{128}{45}\right)$.

2648. $\frac{ab^3}{3}$. **2649.** (1) $\frac{bh^3}{12}$; (2) $\frac{bh^3}{4}$; (3) $\frac{bh^3}{36}$. **2650.** $\frac{\pi R^4}{8}$.

2651. $\frac{\pi R^4}{2}$. **2652.** $\frac{\pi}{4}ab^3$. and $\frac{\pi}{4}ba^3$. **2653.** $\frac{1}{2}\pi R^4 H$.

2654. $\frac{1}{10}\pi R^4 H$. **2655.** $\frac{8}{15}\pi R^5$.

2656. $\frac{8}{15}\pi ab^4$, where $2a$ is the length of the axis about which the rotation takes place.

2657. $\frac{1}{6}\pi R^4 H$. **2658.** $\frac{56\pi}{15}$.

2659. (1) $I_x = \frac{\pi(e^4 - 1)}{8}$; (2) $I_y = 4\pi(3 - e)$.

2660. MR^2, where M is the mass of the lateral surface of the cylinder.

2661. $\frac{1}{2}MR^2$. **2662.** $\frac{2}{3}MR^2$. **2663.** $\frac{9}{2}\pi a^3$. **2664.** $6\pi^2 ab^2$.

2665. The volume is $\frac{3\sqrt{2}}{8}\pi^2 a^3$, the surface area is $6\sqrt{2}\pi a^2$.

2666. The volume is $12\pi^3 a^3$, the surface area $32\pi^2 a^2$.

2667. The axis of rotation must be perpendicular to the diagonal of the square; the axis of rotation must be perpendicular to the median.

2668. $\approx 23{\cdot}7$ m. 2669. $x_2 = x_1 + \sin\left(\frac{2\pi t_2}{T} + \varphi_0\right) - \sin\left(\frac{2\pi t_1}{T} + \varphi_0\right)$.

2670. $\frac{kmM}{a(a+l)}$, $\frac{a+l}{a}M$, $\frac{kmM}{l}\ln\frac{r_1(r_2+l)}{r_2(r_1+l)}$.

2671. $\frac{2kmM}{\pi r^2}$. 2672. $\frac{kmMa}{\sqrt{(R^2+a^2)^3}} = \frac{kmM\cos^3\varphi}{a^2}$, where φ is the angle between the straight lines joining point C to the centre of the ring and to any point of the ring; $\frac{kmM}{R}$.

2673. $\frac{2kmM}{R^2}\left(1 - \frac{a}{\sqrt{a^2+R^2}}\right)$. 2674. $2\pi km\sigma$.

2675*. $2\pi km\gamma h\left(1 - \frac{h}{\sqrt{h^2+(R-r)^2}}\right) = 2\pi km\gamma h\,(1-\cos\alpha)$, where α is the angle between the generator of the cone and its axis. Use the solution of problem 2673.

2676. $2km\gamma$.

2678*. $\frac{kM^2}{l^2}\ln\frac{4}{3}$. First calculate the force of interaction of element ds of the first rod with the second rod (use the result of problem 2670), then find the total force of interaction.

2679. $\frac{g^2M^3}{6m^2}$ ergs (M and m in grams, g in cm/sec^2).

2680. ≈ 754 kg. 2681. $\approx 1{\cdot}63 \cdot 10^{11}$ kg.

2682. 353,250 kg.

2683. $\frac{\pi\, dR^2H^2}{12}$, $\frac{\pi\, dR^2H^2}{4}$. The work is obtained in kg in the answers to problems 2683–2686 if the distance is in metres, and the specific weight in kg/m^3.

2684. $\frac{\pi dR^4}{4} \approx 101{\cdot}8$ kg. 2685. $\frac{\pi\, dR^2H^2}{6} \approx 26{,}800$ kg.

2686. $\frac{4}{15}dabH^2 = 240$ kg. 2687. $\frac{Sl^3\omega^2\gamma}{6} \approx 4{\cdot}1\times10^7\,\text{erg} \approx 0{\cdot}418$ kg.

2688. $\frac{ab^3\, d\gamma\omega^2}{6} \approx 1{\cdot}16$ kg. 2689. $\frac{ah^3\, d\omega^2\gamma}{24} \approx 0{\cdot}05$ kg.

2690. $\frac{ha^3\, d\omega^2\gamma}{60} \approx 0{\cdot}015$ kg. 2691. $\frac{\pi R^4 H\omega^2\gamma}{4}$ erg (R and H in cm, γ in g/cm^3, ω in rad/sec).

2692. $\frac{MR^2\pi^2n^2}{3600}$ erg, $\frac{MR^2(3\pi - 8)\,\pi n^2}{3600}$ erg (R in cm, M in g).

2693. (a) $\frac{ah^2}{6}$; (b) The pressure is doubled.

2694. $\frac{a^3\sqrt{2}}{2}$. **2695.** 22·2 m. **2696.** $\frac{2}{3}da^2b$.

2697. $ab\,d\left(h + \frac{b}{2}\sin\alpha\right)$.

2699. (a) $\frac{d^2H^2S}{2} = 32$ kg; (b) $\frac{1}{2}SH^2(1 - d)^2 = 2$ kg.

2700. $\frac{4}{3}\pi R^4$. **2701.** $\approx 0{\cdot}206$ cm^2.

2702. (a) $\approx 33{\cdot}2$ sec; (b) $\approx 64{\cdot}6$ sec.

2703. ≈ 1 hour 6 min 53 sec.

2704. $\frac{2bL\sqrt{2a}}{3S\sqrt{g}}(2\sqrt{2} - 1)$. **2705.** $\frac{2b\sqrt{2g}}{3}[(H + h)^{\frac{3}{2}} - H^{\frac{3}{2}}]$; for $H = 0$: $\frac{2b\sqrt{2g}}{3}h^{\frac{3}{2}} = \frac{2\sqrt{2g}}{3}S\sqrt{h}$, where S is the area of the slit.

2706. (a) $\approx 2{\cdot}4$ sec; (b) $\approx 6{\cdot}3$ sec; (c) ≈ 53 sec; (d) as $t \to \infty$.

2707. $\approx 3{\cdot}4$ kg.

2708. 1(a) $\approx 7{\cdot}16$ kg; (b) $\approx 16{\cdot}6$ kg; (c) $\approx 23{\cdot}8$ kg; (2) the work increases indefinitely on indefinite expansion of the gas.

2709. ≈ 1600 kg. **2710.** ≈ 82 min.

2711. Slightly more than 5°. **2712.** $\frac{2\sigma}{a}$.

2713. (a) 40 erg. (b) 60 erg. **2714.** 5 cm. **2715.** ≈ 946 coulombs.

2716. ≈ 1092 coulombs. **2717.** ≈ 5110 coulombs.

2718. $\frac{E_0^2}{2}$. The effective voltage of alternating current is equal to $\frac{E_0}{\sqrt{2}}$.

2719. $\frac{E_0I_0}{2}T\cos\varphi_0$. **2720.** ≈ 7 min 2 sec. **2721.** $\approx 2{\cdot}915$ l.

2722. (a) $H_1 = H\frac{\ln a - \ln c}{\ln a - \ln b} \approx 15$ cm. (b) $\approx 0{\cdot}125\%$.

2723. $\frac{1}{1024}$ of the original quantity. **2724.** $\approx 2{\cdot}49$ g. **2725.** $\frac{8}{9}$ g.

2726. $\approx 37{\cdot}3$ min.

Chapter IX

2727*. $S_n = 1 - \dfrac{1}{n+1}$, $S = 1$. Write each term of the series as a sum of two terms.

2728. $S_n = \dfrac{1}{2}\left(1 - \dfrac{1}{2n+1}\right)$, $S = \dfrac{1}{2}$.

2729. $S_n = \dfrac{1}{3}\left(1 - \dfrac{1}{3n+1}\right)$, $S = \dfrac{1}{3}$.

2730. $S_n = \dfrac{1}{3}\left(1 + \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{n+1} - \dfrac{1}{n+2} - \dfrac{1}{n+3}\right)$, $S = \dfrac{11}{18}$.

2731. $S_n = \dfrac{1}{6}\left(1 + \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{2n+1} - \dfrac{1}{2n+3} - \dfrac{1}{2n+5}\right.$, $S = \dfrac{23}{90}$.

2732. $S_n = \dfrac{1}{2}\left[\dfrac{1}{2} - \dfrac{1}{(n+1)(n+2)}\right]$, $S = \dfrac{1}{4}$.

2733. $S_n = 1 + \dfrac{1}{2} - \dfrac{1}{2^n} - \dfrac{1}{2 \cdot 3^n}$, $S = \dfrac{3}{2}$.

2734. $S_n = 1 - \dfrac{1}{(n+1)^2}$, $S = 1$.

2735. $S_n = \dfrac{1}{8}\left[1 - \dfrac{1}{(2n+1)^2}\right]$, $S = \dfrac{1}{8}$.

2736. $S_n = \arctan \dfrac{n}{n+1}$, $S = \dfrac{\pi}{4}$.

2737. Convergent. **2738.** Convergent. **2739.** Divergent.
2740. Convergent. **2741.** Divergent. **2742.** Divergent.
2743. Convergent. **2744.** Divergent. **2745.** Divergent.
2746. Convergent. **2747.** Convergent. **2748.** Divergent.
2749. Convergent. **2750.** Divergent. **2751.** Convergent.
2752. Convergent. **2753.** Divergent. **2767.** Convergent.
2768. Divergent. **2769.** Convergent. **2770.** Convergent.
2771. Convergent. **2772.** Divergent. **2773.** Divergent.
2774. Convergent. **2775.** Divergent. **2776.** Divergent.
2777. Divergent. **2778.** Convergent. **2779.** Convergent.
2780. Divergent. **2781.** Convergent. **2782.** Divergent.

2783. Convergent. **2784*.** Divergent. Use the formula

$$\sin\alpha + \sin 2\alpha + \ldots + \sin k\alpha = \frac{\sin\frac{k+1}{2}\alpha \sin\frac{k}{2}\alpha}{\sin\frac{\alpha}{2}}$$ or the inequality $\sin x > \frac{2}{\pi}x$ if $0 < x < \frac{\pi}{2}$.

2790. Convergent, but not absolutely.

2791. Absolutely convergent. **2792.** Non-absolutely convergent.

2793. Absolutely convergent. **2794.** Absolutely convergent.

2795. Divergent. **2796.** Non-absolutely convergent.

2797. Absolutely covergent. **2798.** Non-absolutely convergent.

2799. Divergent. **2802.** $-1 < x < 1$. **2803.** $\frac{1}{e} < x < e$.

2804. $-1 < x < 1$. **2805.** $-1 \leqq x \leqq 1$. **2806.** $-1 \leqq x < 1$.

2807. $x < -1$ and $x > 1$. **2808.** $-1 < x < 1$. **2809.** $-1 \leqq x < 1$.

2810. $x \neq \pm 1$. **2811.** For any x. **2812.** $-2 < x < 2$.

2813. For any x. **2814.** $x > 0$. **2815.** $x > 0$. **2816.** $x \geqq 0$.

2822. 11 terms. **2823*.** Use the inequality $\ln(1+\alpha) \leqq \alpha$.

2825. $f(0) = \frac{1}{9}$; $f\left(\frac{\pi}{2}\right) = -\frac{1}{101}$; $f\left(\frac{\pi}{3}\right) = \frac{44}{1001}$; $f(1) = 0{\cdot}049$; $f(-0{\cdot}2) = 0{\cdot}108$.

2827. $\frac{1}{4}\ln\frac{1+x}{1-x} - \frac{1}{2}\arctan x$.

2828. $\frac{1}{2}\arctan x + \frac{1}{4}\ln\frac{1+x}{1-x}$.

2829. $(x+1)\ln(x+1) - x$. **2830.** $\frac{1}{2}$. **2831.** $0{\cdot}2$.

2832*. $\ln\frac{3}{2}$. Use the relationship $\cos\frac{x}{2}\cos\frac{x}{4}\ldots\cos\frac{x}{2^n}\ldots = \frac{\sin x}{x}$.

2833*. $\frac{\pi^3}{12}$. Use the formula $\sum_{n=1}^{n=\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

2834. (1) $\frac{1}{3}\left(\ln 2 + \frac{\pi}{\sqrt{3}}\right)$; (2) $\frac{1}{2\sqrt{2}}\left[\ln(1+\sqrt{2}) + \frac{\pi}{2}\right]$.

2835. $\ln 2$. **2836.** $\frac{2-\sqrt{2}}{2}$.

2837. The given series cannot be differentiated term by term in any interval. For, the general term in the series of derivatives has the form $\pi \cos(2^n \pi x)$. No matter how small the interval (α, β), and no matter where it lies on the real axis, a number of the form $\frac{k}{2N}$ can always be found inside it, where k is an integer and N a sufficiently large positive integer. But the series of derivatives is divergent with $x = \frac{k}{2N}$, since its terms become equal to π for all $n > N$.

2838. $\frac{1}{(1-x)^2}$ and $\frac{1}{(1-x)^3}$.

2841. $(x-1) - \frac{(x-1)^2}{2} + \ldots + (-1)^{n+1}\frac{(x-1)^n}{n} + \ldots$

2842. $1 + \frac{3}{2}\left[(x-1) + \frac{1}{2}\,\frac{(x-1)^2}{2!} - \frac{1}{2^2}\,\frac{(x-1)^3}{3!} + \ldots\right.$
$\left.\ldots + (-1)^n \frac{1\,.\,3\ldots(2n-5)}{2^{n-1}}\,\frac{(x-1)^n}{n!} + \ldots\right]$.

2843. $\frac{1}{3} - \frac{x-3}{9} + \frac{(x-3)^2}{27} - \ldots + (-1)^{n+1}\frac{(x-3)^{n-1}}{3^n} + \ldots$

2844. $1 - \left(\frac{\pi}{4}\right)^2 \frac{(x-2)^2}{2!} + \ldots + (-1)^{n+1}\left(\frac{\pi}{4}\right)^{2n-2}\frac{(x-2)^{2n-2}}{(2n-2)!} + \ldots$

2845. $1 + \frac{x^2}{2!} + \ldots + \frac{x^{2n-2}}{(2n-2)!} + \ldots$

2846. $x^2 + \frac{x^3}{1!} + \frac{x^4}{2!} + \ldots + \frac{x^{n+1}}{(n-1)!} + \ldots$

2847. $\cos\alpha\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^{n+1}\frac{x^{2n-2}}{(2n-2)!} + \ldots\right] -$
$- \sin\alpha\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!} + \ldots\right]$.

2848. $x + x^2 + \frac{2x^3}{3!} - \frac{4x^5}{5!} + \ldots + \sqrt{2^n}\sin\frac{\pi n}{4}\cdot\frac{x^n}{n!} + \ldots$

2849. $1 - \frac{4x^4}{4!} + \frac{4^2x^8}{8!} + \ldots + (-1)^{n+1}\frac{4^{n-1}x^{4(n-1)}}{(4n-4)!} + \ldots$

2850. $\ln 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \ldots$

2851. $e\left(1 - \frac{x^2}{2} + \frac{x^4}{6} - \ldots\right)$.

2852. $1 - \frac{nx^2}{2} + \frac{3n^2 - 2n}{24} x^4 + \dots$

2853. $\frac{x^2}{2} + \frac{x^4}{12} + \dots$ **2854.** $1 + x^2 - \frac{x^3}{2} + \frac{5x^4}{6} + \dots$

2855. $1 + 2x + \frac{(2x)^2}{2!} + \dots + \frac{(2x)^{n-1}}{(n-1)!} + \dots$

2856. $1 - x^2 + \frac{x^4}{2!} - \dots + (-1)^{n+1} \frac{x^{2(n-1)}}{(n-1)!} + \dots$

2857. $1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots + \frac{x^{n-1}}{n!} + \dots$

2858. $1 + \frac{x^6}{3!} + \frac{x^{12}}{5!} + \dots + \frac{x^{6(n-1)}}{(2n-1)!} + \dots$

2859. $\frac{x}{2} - \frac{x^3}{2^3 \cdot 3!} + \dots + (-1)^{n+1} \frac{x^{2n-1}}{2^{2n-1}(2n-1)!} + \dots$

2860. $1 - \left[x^2 - \frac{(2x)^4}{2 \cdot 4!} + \dots + (-1)^{n+1} \frac{2^{2n-1} x^{2n}}{(2n)!} + \dots\right].$

2861. $1 - \frac{x^2}{3!} + \dots + (-1)^{n+1} \frac{x^{2(n-1)}}{(2n-1)!} + \dots$

2862. $-\frac{2x^3}{3!} + \frac{4x^5}{5!} - \dots + (-1)^n \frac{2n x^{2n+1}}{(2n+1)!} + \dots$

2863. $\ln 10 + \left[\frac{x}{10} - \frac{x^2}{2 \cdot 10^2} + \dots + (-1)^{n+1} \frac{x^n}{n \cdot 10^n} + \dots\right].$

2864. $x^2 - \frac{x^3}{2} + \dots + (-1)^n \frac{x^{n+1}}{n} + \dots$

2865. $1 + \left[\frac{x^2}{2} - \frac{1}{2} \cdot \frac{x^4}{4} + \dots + \right.$

$$\left. + (-1)^{n+1} \frac{1 \cdot 3 \dots (2n-3)\, x^{2n}}{2 \cdot 4 \dots (2n-2)\, 2n} + \dots\right].$$

2866. $2 - 2\left[\frac{1}{3}\left(\frac{x}{2}\right)^3 - \frac{2}{3 \cdot 6}\left(\frac{x}{2}\right)^6 + \dots \right.$

$$\left. \dots + (-1)^{n+1} \frac{2 \cdot 5 \dots (3n-4)}{3^n \cdot n!} \left(\frac{x}{2}\right)^{3n} + \dots\right].$$

2867. $1 - \left[\frac{1}{3} x^3 - \frac{1 \cdot 4}{3^2\, 2!} x^6 + \dots + \right.$

$$\left. + (-1)^{n+1} \frac{1 \cdot 4 \dots (3n-2)}{3^n \cdot n!} x^{3n} + \dots\right].$$

2868. $x^2 + \left[\frac{1}{2} x^4 + \frac{1 \cdot 3}{2 \cdot 4} x^6 + \dots + \frac{1 \cdot 3 \dots (2n-1)}{2^n \cdot n!} x^{2n+2} + \dots\right].$

2869. $1 + 2^2x + \ldots + n^2x^{n-1} + \ldots$, $S = 12$.

2870. (1) $-7!$, (2) $\frac{105}{16}$, (3) $\frac{10!}{4!}$ and (4) $\frac{8}{3}$.

2871. $\frac{1}{6}$. **2872.** $\frac{1}{4}$. **2873.** 1. **2874.** $\frac{1}{2}$. **2875.** $\frac{2}{3}$. **2876.** $\frac{1}{3}$.

2877. $\frac{1}{60}$. **2878.** $-\frac{1}{10} < x < \frac{1}{10}$. **2879.** $-1 < x \leqq 1$.

2880. $-10 \leqq x < 10$. **2881.** $x = 0$. **2882.** $-\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}$.

2883. $-\infty < x < \infty$. **2884.** $-\frac{1}{3} < x < \frac{1}{3}$. **2885.** $-1 \leqq x \leqq 1$.

2886. $-\frac{1}{e} \leqq x < \frac{1}{e}$. **2887.** $x = 0$. **2888.** $-1 \leqq x < 1$.

2889. $-\frac{1}{e} < x < \frac{1}{e}$. **2890.** $x - \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} - \ldots$

$$\ldots + (-1)^{n+1}\frac{1 \cdot 3 \ldots (2n-1)}{2^n \cdot n!}\frac{x^{2n+1}}{2n+1} + \ldots (-1 \leqq x \leqq 1).$$

2891. $x + \frac{x^3}{3} + \ldots + \frac{x^{2n+1}}{2n+1} + \ldots (-1 < x < 1)$.

2892. $x^2 + \frac{x^4}{2 \cdot 3} + \ldots + \frac{x^{2n}}{n(2n-1)} + \ldots (-1 < x < 1)$.

2893. $4\left(\frac{x^3}{3!} + \frac{2x^5}{5!} + \ldots + \frac{nx^{2n+1}}{(2n+1)!} + \ldots\right)$

$(-\infty < x < \infty)$, $\frac{1}{2e}$.

2894. 1·39, error 0·01. **2895.** 0·3090, error 0·0001.

2896. 2·154, error 0·001. **2897.** 7·389. **2898.** 1·649. **2899.** 0·3679.

2900. 0·7788. **2901.** 0·0175. **2902.** 1·000. **2903.** 0·17365. **2904.** 0·9848.

2905. 3·107. **2906.** 4·121. **2907.** 7·937. **2908.** 1·005. **2909.** 3·017.

2910. 5·053. **2911.** 2·001. **2912.** 1·0986. **2913.** 0·434294. **2914.** 0·6990.

2915. $1 + 2x + \frac{5}{2}x^2 + \ldots +$

$$+ \left[2 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{(n-1)!}\right]x^{n-1} + \ldots$$

2916. $x - \frac{3}{2}x^2 + \frac{11}{6}x^3 - \ldots +$

$$+ (-1)^{n+1}\left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right]x^n + \ldots$$

2917. $1 - \frac{x}{3} + \frac{x^3}{81} + \dots$ **2918.** $-\frac{x}{2} + \frac{5x^3}{32} + \dots$

2919. $x - x^2 + 2x^3 + \dots$

2920. $C + x - \frac{x^3}{3 \cdot 3!} + \dots + (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)\,(2n+1)!} + \dots$ $(-\infty < x < \infty)$.

2921. $C + \ln|x| - \frac{x^2}{2\,.\,2!} + \frac{x^4}{4\,.\,4!} - \dots$
$\dots + (-1)^n \frac{x^{2n}}{2n\,.\,(2n)!} + \dots (-\infty < x < 0$ and $0 < x < \infty)$.

2922. $C + \ln|x| + x + \frac{x^2}{2\,.\,2!} + \dots + \frac{x^n}{n\,.\,n!} + \dots$
$(-\infty < x < 0$ and $0 < x < \infty)$.

2923. $C - \frac{1}{x} + \ln|x| + \frac{x}{2} + \frac{x^2}{2\,.\,3!} + \dots + \frac{x^n}{n(n+1)!} + \dots$
$(-\infty < x < 0$ and $0 < x < \infty)$.

2924. $x - \frac{x^3}{3} + \frac{x^5}{5\,.\,2!} - \dots + (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)\,(n-1)!} + \dots$
$(-\infty < x < \infty)$.

2925. $x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \dots + (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)^2} + \dots$
$(-1 \leqq x \leqq 1)$.

2926. $x + \frac{1}{2} \cdot \frac{x^5}{5} + \frac{1\,.\,3}{2\,.\,4} \frac{x^9}{9} + \dots + \frac{1\,.\,3 \dots (2n-3)}{2^{n-1}(n-1)!} \frac{x^{4n-3}}{4n-3} + \dots$
$(-1 \leqq x \leqq 1)$.

2927. $x + \frac{1}{2} \frac{x^4}{4} - \frac{1}{2\,.\,4} \frac{x^7}{7} + \dots +$
$+ (-1)^n \frac{1\,.\,3 \dots (2n-5)}{2^{n-1}(n-1)!} \frac{x^{3n-2}}{3n-2} + \dots$ $(-1 \leqq x \leqq 1)$.

2928. $x + \frac{x^{10}}{10} + \frac{x^{19}}{19} + \dots + \frac{x^{9n-8}}{9n-8} + \dots$
$(-1 \leqq x < 1)$.

2929. $\frac{1}{4} \cdot \frac{x^3}{3} - \frac{3}{4.8} \frac{x^7}{7} + \dots +$
$+ (-1)^{n+1} \frac{3 \cdot 7 \dots (4n-5)}{4^n\,.\,n!} \frac{x^{4n-1}}{4n-1} + \dots$ $(-1 \leqq x \leqq 1)$.

2930. 0·3230, error 0·0001. **2931.** 0·24488, error 0·00001.

2932. 0·4971, error 0·0001. **2933.** 3·518, error 0·001.

2934. 0·012, error 0·001. **2935.** 32·831. **2936.** 0·487. **2937.** 0·006.

2938. 0·494. **2940.** 3·141592654.

2941. $x + \frac{2}{1 \cdot 3} x^3 + \frac{2^2}{1 \cdot 3 \cdot 5} x^5 + \ldots + \frac{2^{n-1}}{1 \cdot 3 \cdot 5 \ldots (2n-1)} x^{2n-1} + \ldots$

2942*. $1 - \frac{1}{2^2} + \frac{1}{3^3} - \ldots + (-1)^{n+1} \frac{1}{(n+1)^{n+1}} + \ldots$ Write x^x in the form $e^{x \ln x}$, expand as a power series in $x \ln x$, and integrate the expression of the form $x^n \ln^n x$.

2943. 0·6449. **2944.** 0·511. **2945.** 1·015.

2946*. 3·71. Evaluation of the area with the aid of the formula $S = 4 \int_0^1 \sqrt[4]{1 - x^4}\, dx$ is inconvenient because the corresponding series is slowly convergent for $x = 1$. One should evaluate the area of the sector bounded by the curve, the axis of ordinates and the bisector of the first quadrant. This yields a rapidly convergent series.

2947. 0·2505. **2948.** 3·821. **2949.** 0·119. **2950.** 1·225. **2951.** (0·347; 2·996). **2952.** (1·71; 0·94).

Chapter X

2953. $z = \frac{\pi}{3} (x^2 y - y^3)$.

2954. $S = \frac{1}{4} \sqrt{(x + y + z)(x + y - z)(x - y + z)(y + z - x)}$.

2955.

y \ x	0	1	2	3	4	5
0	1	3	5	7	9	11
1	−2	0	2	4	6	8
2	−5	−3	−1	1	3	5
3	−8	−6	−4	−2	0	2
4	−11	−9	−7	−5	−3	−1
5	−14	−12	−10	−8	−6	−4

2956.

$y \backslash x$	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1
0	0·00	0·10	0·20	0·30	0·40	0·50	0·60	0·70	0·80	0·90	1·00
0·1	0·10	0·14	0·22	0·32	0·41	0·51	0·61	0·71	0·81	0·90	1·00
0·2	0·20	0·22	0·28	0·36	0·45	0·54	0·63	0·73	0·82	0·92	1·01
0·3	0·30	0·32	0·36	0·42	0·50	0·58	0·67	0·76	0·85	0·95	1·04
0·4	0·40	0·41	0·45	0·50	0·57	0·64	0·72	0·81	0·89	0·98	1·08
0·5	0·50	0·51	0·54	0·58	0·64	0·71	0·78	0·86	0·94	1·03	1·12
0·6	0·60	0·61	0·63	0·67	0·72	0·78	0·85	0·92	1·00	1·08	1·16
0·7	0·70	0·71	0·73	0·76	0·81	0·86	0·92	0·99	1·06	1·14	1·22
0·8	0·80	0·81	0·82	0·85	0·89	0·94	1·00	1·06	1·13	1·20	1·28
0·9	0·90	0·91	0·92	0·95	0·98	1·03	1·08	1·14	1·20	1·27	1·34
1	1·00	1·00	1·02	1·04	1·08	1·12	1·16	1·22	1·28	1·34	1·41

2957. (1) $\frac{9}{16}$; (2) 1; (3) 16; 2; 2.

2958. $\dfrac{\varphi(a)\psi\left(\frac{1}{a}\right) - \psi(a)\varphi\left(\frac{1}{a}\right)}{\varphi(1)\psi(1)}$; $a - \frac{1}{a}$.

2959. The second function varies more rapidly.

2960. A second-order parabola; (1) no, (2) no.

2961. Put $m = \frac{1}{x}$.

2965. The function is not single-valued.

2966. (1) 1; (2) 1; (3) $\frac{1}{5}$; (4) not defined; (5) 1.

2967. $z = (x + y)^{x-y} + (x + y)^{y-x}$, $(x + y > 0)$; z is a rational function of u and v, but not of w, t, x and y.

2968. $z = (x + y)^{xy} + (xy)^{2x}$.

2969. $u = (x^2 + y^2 + z^2)^2 - \dfrac{x^2 + y^2 + z^2}{4}[(x^2 + y^2 + z^2)^2 + 3(x + y + z)^4]$; u is an integral rational function of ξ and η, x, y and z, but not of ω and φ.

2970. $z = \left(\dfrac{u + v}{u - v}\right)^v + u$; $u = x^2 + y^2$; $v = xy$.

2971. $x = \text{const}$ gives a parabola, $y = \text{const}$ a parabola, $z = \text{const} \neq 0$ a hyperbola, $z \neq 0$ a pair of straight lines.

2972. $x = \text{const}$, $y = \text{const}$ are straight lines, $z = \text{const} \neq 0$ is a hyperbola, $z = 0$ is a pair of straight lines.

2973. $x = \text{const}$ is a parabola, $y = \text{const}$ is a cubical parabola, $z = \text{const} \neq 0$ is a curve of the third order, $z = 0$ is a semicubical parabola.

2974. $z = \text{const} > 0$ is an ellipse, $x = \text{const}$ and $y = \text{const}$ are curves of the third order (semicubical parabolas with $x = 0$ and $y = 0$).

2975. $0 < y < 2$; $-1 < y - \frac{1}{2}x < 0$. **2976.** $x^2 \leqq y \leqq \sqrt{x}$.

2977. $0 < y < x\sqrt{3}$; $y < (a - x)\sqrt{3}$.

2978. $(x - a)^2 + (y - b)^2 < R^2$; $-\infty < z < \infty$.

2979. $(x - a)^2 + (y - b)^2 + (z - c)^2 \leqq R^2$.

2980. (a) $x^2 + y^2 < 4R^2$; (b) $-\infty < x < \infty$; $-\infty < y < \infty$.

2981. $v = \frac{1}{6}\, xy(2R \pm \sqrt{4R^2 - x^2 - y^2})$; the function is not single-valued. The domain of definition of the function is $x^2 + y^2 \leqq 4R^2$; $x > 0$, $y > 0$. The domain of definiteness of the analytic expression is $x^2 + y^2 \leqq 4R^2$.

2982. With $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ $S = xy$;
with $0 \leqq x \leqq 1$, $1 \leqq y$ $S = x$;
with $1 \leqq x$ $0 \leqq y \leqq 1$ $S = y$;
with $1 \leqq x \leqq 2$; $1 \leqq y \leqq 2$ $S = xy - x - y + 2$;
with $1 \leqq x \leqq 2$; $2 \leqq y$ $S = x$;
with $2 \leqq x$, $1 \leqq y \leqq 2$ $S = y$;
with $2 \leqq x$, $2 \leqq y$ $S = 2$.
The function is not defined for $x < 0$ and $y < 0$.

2983. $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leqq 1$. **2984.** $y^2 > 4x - 8$.

2985. The whole of the plane except for points of the circle $x^2 + y^2 = R^2$.

2986. The interior of the right-hand vertical angle between the bisectors of the first and fourth quadrants, including the bisectors themselves:

$$x + y \geqq 0, \quad x - y \geqq 0.$$

2987. The same as in problem 2986, but without the boundaries.

2988. The interior of the right and left-hand angles formed by the straight lines $y = 1 + x$ and $y = 1 - x$, including these straight lines, but with their points of intersection:

$$1 - x \leqq y \leqq 1 + x \ (x > 0),$$
$$1 + x \leqq y \leqq 1 - x \ (x < 0)$$

(the function is not defined for $x = 0$).

2989. The part of the plane lying inside the first and third quadrants (without the boundaries).

2990. The closed domain lying between the positive semi-axis of abscissae and the parabola $y = x^2$ (including the boundary):

$$x \geqq 0, \quad y \geqq 0; \quad x^2 \geqq y.$$

2991. The ring, including its circumferences, between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

2992. The part of the plane lying inside the parabola $y^2 = 4x$, between the parabola and the circle $x^2 + y^2 = 1$, including the arc of the parabola except for its vertex and excluding the arc of the circle.

2993. The part of the plane lying outside the circles of unit radii with centres at the points $(-1, 0)$ and $(1, 0)$. Points of the circumference of the first circle belong to the domain, points of the second do not.

2994. Only points of the circle $x^2 + y^2 = R^2$.

2995. The whole of the plane except for the straight lines

$$x + y = n$$

(n is any integer, positive, negative or zero).

2996. The interior of the circle $x^2 + y^2 = 1$ and of the ring

$$2n \leqq x^2 + y^2 \leqq 2n + 1$$

(n is an integer), including the boundary.

2997. $\left.\begin{array}{l} \text{If } x \geqq 0, \text{ then} \\ 2n\pi \leqq y \leqq (2n+1)\pi; \\ \text{if } x < 0, \text{ then} \\ (2n+1)\pi \leqq y \leqq (2n+2)\pi; \end{array}\right\} n \text{ is an integer.}$

2998. $x > 0$; $2n\pi < y < 2(n+1)\pi$ (n is an integer).

2999. The open domain shown in Fig. 83.

$\left\{\begin{array}{l} \text{For } x > 0,\ y > x + 1, \\ \text{for } x < 0,\ x < y < x + 1. \end{array}\right.$

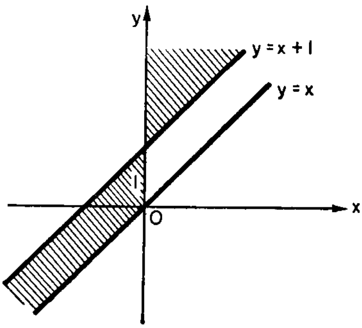

FIG. 83.

3000. The part of the plane lying between the curve $y = \dfrac{1}{1+x^2}$ and its asymptote, including the boundary.

3001. $x > 0$, $y > 0$, $z > 0$.

3002. The part of space lying between the spheres $x^2 + y^2 + z^2 = r^2$ and $x^2 + y^2 + z^2 = R^2$, including the surface of the exterior and excluding the surface of the interior sphere.

3003. 2. **3004.** 0. **3005.** 0.

3006. The function has no limit as $x \to 0$, $y \to 0$.

3007. 0. **3008.** 1.

3009. (a) $y = 0$ or $y = x^\alpha$ $(\alpha > 1)$, $x \to 0$ in accordance with any law; (b) $y = \dfrac{x}{3}$, $x \to 0$ in accordance with any law.

3010. The point $(0, 0)$; near this point the function can take positive values as large as desired.

3011. All points with integral coordinates.

3012. On the straight line $y = x$.

3013. On the straight lines $x = m$, $y = n$ (m and n are integers).

3014. On the parabola $y^2 = 2x$.

3015. (1) continuous; (2) discontinuous; continuous with respect to x and y separately; (3) continuous; (4) discontinuous; (5) discontinuous; (6) discontinuous. Pass to polar coordinates.

3016. Circles with centres at the origin and radii respectively $1, \dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{3}}{3}, \dfrac{1}{2}$.

3017. Circles through points A and B.

3025. The straight lines $y = ax + b$, where $a = \ln b$.

3026. Concentric spheres with centre at the point A and radii equal to 1, 2, 3, 4.

3027. Ellipsoids of revolution with foci at points A and B:

$$\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2} + \sqrt{(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2} = \text{const.}$$

3028. The spheres $x^2 + y^2 + z^2 = \left(\dfrac{c-1}{c+1}\right)^2$, where $c > 1$.

3029. The paraboloids of revolution $x^2 + y^2 = cz$.

3030. (1) The planes $2x + 3y - z = C$; (2) the hyperboloids of revolution or cone $x^2 + y^2 - 2z^2 = C$.

3032. $\dfrac{1}{v}\dfrac{\partial v}{\partial T}$ for $T = T_0$.

3033. $\frac{\partial \theta}{\partial t}$ is the rate of change of temperature at the given point; $\frac{\partial \theta}{\partial x}$ is the rate of change of temperature at the given instant with respect to distance (along the rod).

3034. $\frac{\partial S}{\partial h} = b$ is the rate of change of the area as a function of the height of the rectangle; $\frac{\partial S}{\partial b} = h$ is the rate of change of the area as a function of the base of the rectangle.

3036. $\frac{\partial z}{\partial x} = 1,\ \frac{\partial z}{\partial y} = -1.$

3037. $\frac{\partial z}{\partial x} = 3x^2y - y^3;\ \frac{\partial z}{\partial y} = x^3 - 3y^2x.$

3038. $\frac{\partial \theta}{\partial x} = ae^{-t};\ \frac{\partial \theta}{\partial t} = -axe^{-t} + b.$

3039. $\frac{\partial z}{\partial u} = \frac{1}{v} - \frac{v}{u^2};\ \frac{\partial z}{\partial v} = -\frac{u}{v^2} + \frac{1}{u}.$

3040. $\frac{\partial z}{\partial x} = \frac{x^4 + 3x^2y^2 - 2xy^3}{(x^2 + y^2)^2};$

$$\frac{\partial z}{\partial y} = \frac{y^4 + 3x^2y^2 - 2x^3y}{(x^2 + y^2)^2}.$$

3041. $\frac{\partial z}{\partial x} = 30xy(5x^2y - y^3 + 7)^2;$

$$\frac{\partial z}{\partial y} = 3(5x^2y - y^3 + 7)^2\,(5x^2 - 3y^2).$$

3042. $\frac{\partial z}{\partial x} = \sqrt{y} - \frac{y}{3\sqrt[3]{x^4}};\ \frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}} + \frac{1}{\sqrt[3]{x}}.$

3043. $\frac{\partial z}{\partial x} = \frac{1}{\sqrt{x^2 + y^2}};\ \frac{\partial z}{\partial y} = \frac{y}{x^2 + y^2 + x\sqrt{x^2 + y^2}}.$

3044. $\frac{\partial z}{\partial x} = \frac{y}{x^2 + y^2};\ \frac{\partial z}{\partial y} = -\frac{x}{x^2 + y^2}.$

3045. $\frac{\partial z}{\partial x} = \frac{y}{(x^2 + y^2)\left(\arctan\frac{y}{x}\right)^2};$

$$\frac{\partial z}{\partial y} = -\frac{x}{(x^2 + y^2)\left(\arctan\frac{y}{x}\right)^2}.$$

3046. $\dfrac{\partial z}{\partial x} = yx^{y-1}$; $\dfrac{\partial z}{\partial y} = x^y \ln x$.

3047. $\dfrac{\partial z}{\partial x} = \dfrac{2x}{x^2 + y^2}$; $\dfrac{\partial z}{\partial y} = \dfrac{2y}{x^2 + y^2}$.

3048. $\dfrac{\partial z}{\partial x} = -\dfrac{2}{\sqrt{x^2 + y^2}}$; $\dfrac{\partial z}{\partial y} = \dfrac{2x}{y\sqrt{x^2 + y^2}}$.

3049. $\dfrac{\partial z}{\partial x} = \dfrac{xy\sqrt{2}}{(x^2 + y^2)\sqrt{x^2 - y^2}}$; $\dfrac{\partial z}{\partial y} = -\dfrac{x^2\sqrt{2}}{(x^2 + y^2)\sqrt{x^2 - y^2}}$.

3050. $\dfrac{\partial z}{\partial x} = \dfrac{2}{y \sin \dfrac{2x}{y}}$; $\dfrac{\partial z}{\partial y} = -\dfrac{2x}{y^2 \sin \dfrac{2x}{y}}$.

3051. $\dfrac{\partial z}{\partial x} = -\dfrac{1}{y} e^{-\frac{x}{y}}$; $\dfrac{\partial z}{\partial y} = \dfrac{x}{y^2} e^{-\frac{x}{y}}$.

3052. $\dfrac{\partial z}{\partial x} = \dfrac{1}{x + \ln y}$; $\dfrac{\partial z}{\partial y} = \dfrac{1}{y(x + \ln y)}$.

3053. $\dfrac{\partial u}{\partial v} = -\dfrac{w}{v^2 + w^2}$; $\dfrac{\partial u}{\partial w} = \dfrac{v}{v^2 + w^2}$.

3054. $\dfrac{\partial z}{\partial x} = \dfrac{1}{y} \cos \dfrac{x}{y} \cos \dfrac{y}{x} + \dfrac{y}{x^2} \sin \dfrac{x}{y} \sin \dfrac{y}{x}$;

$\dfrac{\partial z}{\partial y} = -\dfrac{x}{y^2} \cos \dfrac{x}{y} \cos \dfrac{y}{x} - \dfrac{1}{x} \sin \dfrac{x}{y} \sin \dfrac{y}{x}$.

3055. $\dfrac{\partial z}{\partial x} = \dfrac{y}{x^2} 3^{-\frac{y}{x}} \ln 3$; $\dfrac{\partial z}{\partial y} = -\dfrac{1}{x} 3^{-\frac{y}{x}} \ln 3$.

3056. $\dfrac{\partial z}{\partial x} = y^2(1 + xy)^{y-1}$;

$\dfrac{\partial z}{\partial y} = xy(1 + xy)^{y-1} + (1 + xy)^y \ln (1 + xy)$.

3057. $\dfrac{\partial z}{\partial x} = y \ln (x + y) + \dfrac{xy}{x + y}$; $\dfrac{\partial z}{\partial y} = x \ln (x + y) + \dfrac{xy}{x + y}$.

3058. $\dfrac{\partial z}{\partial x} = x^{x^y} x^{y-1}(y \ln x + 1)$; $\dfrac{\partial z}{\partial y} = x^y x^{x^y} \ln^2 x$.

3059. $\dfrac{\partial u}{\partial x} = yz$; $\dfrac{\partial u}{\partial y} = xz$; $\dfrac{\partial u}{\partial z} = xy$.

3060. $\dfrac{\partial u}{\partial x} = y + z$; $\dfrac{\partial u}{\partial y} = x + z$; $\dfrac{\partial u}{\partial z} = x + y$.

3061. $\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$; $\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$;

$\frac{\partial u}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$.

3062. $\frac{\partial u}{\partial x} = 3x^2 + 3y - 1$; $\frac{\partial u}{\partial y} = z^2 + 3x$; $\frac{\partial u}{\partial z} = 2yz + 1$.

3063. $\frac{\partial w}{\partial x} = yz + vz + vy$; $\frac{\partial w}{\partial y} = xz + zv + vx$;

$\frac{\partial w}{\partial z} = xy + yv + vx$; $\frac{\partial w}{\partial v} = yz + xz + xy$.

3064. $\frac{\partial u}{\partial x} = (3x^2 + y^2 + z^2)\, e^{x(x^2+y^2+z^2)}$;

$\frac{\partial u}{\partial y} = 2xye^{x(x^2+y^2+z^2)}$; $\frac{\partial u}{\partial z} = 2xze^{x(x^2+y^2+z^2)}$.

3065. $\frac{\partial u}{\partial x} = 2x \cos (x^2 + y^2 + z^2)$; $\frac{\partial u}{\partial y} = 2y \cos (x^2 + y^2 + z^2)$

$\frac{\partial u}{\partial z} = 2z \cos (x^2 + y^2 + z^2)$.

3066. $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = \frac{1}{x + y + z}$.

3067. $\frac{\partial u}{\partial x} = \frac{y}{z} x^{\frac{y}{z} - 1}$; $\frac{\partial u}{\partial y} = \frac{1}{z} x^{\frac{y}{z}} \ln x$; $\frac{\partial u}{\partial z} = -\frac{y}{z^2} x^{\frac{y}{z}} \ln x$.

3068. $\frac{\partial u}{\partial x} = y^z x^{y^z - 1}$; $\frac{\partial u}{\partial y} = zy^{z-1} x^{y^z} \ln x$; $\frac{\partial u}{\partial z} = y^z x^{y^z} \ln x \ln y$.

3069. $\frac{2}{5}$. **3070.** $\frac{1}{2}$.

3071. $\frac{\partial z}{\partial x} = 2(2x + y)^{2x+y}[1 + \ln (2x + y)]$;

$\frac{\partial z}{\partial y} = (2x + y)^{2x+y}[1 + \ln (2x + y)]$.

3072. $\frac{\partial z}{\partial x} = \frac{3}{x \ln y}\left(1 + \frac{\ln x}{\ln y}\right)^2$; $\frac{\partial z}{\partial y} = -\frac{3 \ln x}{y \ln^2 y}\left(1 + \frac{\ln x}{\ln y}\right)^2$.

3073. $\frac{\partial z}{\partial x} = y\, e^{\sin \pi xy}(1 + \pi xy \cos \pi xy)$;

$\frac{\partial z}{\partial y} = x\, e^{\sin \pi xy}(1 + \pi xy \cos \pi xy)$.

3074. $$\frac{\partial z}{\partial x}=\frac{1-x^2-y^2-\sqrt{x^2+y^2}}{(1+\sqrt{x^2+y^2})^2}\,2x;$$
$$\frac{\partial z}{\partial y}=\frac{1-x^2-y^2-\sqrt{x^2+y^2}}{(1+\sqrt{x^2+y^2})^2}\,2y.$$

3075. $$\frac{\partial z}{\partial x}=\frac{y\sqrt{x^y}}{2x(1+x^y)};\quad \frac{\partial z}{\partial y}=\frac{\sqrt{x^y}\ln x}{2(1+x^y)}.$$

3076. $$\frac{\partial z}{\partial x}=-\frac{y}{(1+\sqrt{xy})\sqrt{xy-x^2y^2}};$$
$$\frac{\partial z}{\partial y}=-\frac{x}{(1+\sqrt{xy})\sqrt{xy-x^2y^2}}.$$

3077. $$\frac{\partial z}{\partial x}=\frac{y^2+2xy}{\sqrt{1+(xy^2+yx^2)^2}};\quad \frac{\partial z}{\partial y}=\frac{x^2+2xy}{\sqrt{1+(xy^2+yx^2)^2}}.$$

3078. $$\frac{\partial z}{\partial x}=-\frac{1}{x^2}\sqrt{\frac{xy-x-y}{xy+x+y}};\quad \frac{\partial z}{\partial y}=-\frac{1}{y^2}\sqrt{\frac{xy-x-y}{xy+x+y}}.$$

3079. $$\frac{\partial z}{\partial x}=\frac{y\left[\left(1+\operatorname{arc\,tan}^2\frac{y}{x}\right)^2+2\operatorname{arc\,tan}^3\frac{y}{x}\right]}{(x^2+y^2)\left(1+\operatorname{arc\,tan}^2\frac{y}{x}\right)\left(1+\operatorname{arc\,tan}\frac{y}{x}\right)^2};$$
$$\frac{\partial z}{\partial y}=-\frac{x\left[\left(1+\operatorname{arc\,tan}^2\frac{y}{x}\right)^2+2\operatorname{arc\,tan}^3\frac{y}{x}\right]}{(x^2+y^2)\left(1+\operatorname{arc\,tan}^2\frac{y}{x}\right)\left(1+\operatorname{arc\,tan}\frac{y}{x}\right)^2}.$$

3080. $$\frac{\partial u}{\partial x}=-\frac{4kx}{(x^2+y^2+z^2)^3};\quad \frac{\partial u}{\partial y}=-\frac{4ky}{(x^2+y^2+z^2)^3};$$
$$\frac{\partial u}{\partial z}=-\frac{4kz}{(x^2+y^2+z^2)^3}.$$

3081. $$\frac{\partial u}{\partial x}=\frac{z(x-y)^{z-1}}{1+(x-y)^{2z}};\quad \frac{\partial u}{\partial y}=-\frac{z(x-y)^{z-1}}{1+(x-y)^{2z}};$$
$$\frac{\partial u}{\partial z}=\frac{(x-y)^z\ln(x-y)}{1+(x-y)^{2z}}.$$

3082. $$\frac{\partial u}{\partial x}=\frac{y}{z}x^{\frac{y-z}{z}};\quad \frac{\partial u}{\partial y}=x^{\frac{y}{z}}\frac{\ln x}{z};\quad \frac{\partial u}{\partial z}=-\frac{y\ln x}{z^2}x^{\frac{y}{z}}.$$

3083. $$\frac{\frac{\partial u}{\partial x}}{x}=\frac{\frac{\partial u}{\partial y}}{y}=\frac{\frac{\partial u}{\partial z}}{z}=\frac{2}{r(r^2-1)},\ \text{where } r=\sqrt{x^2+y^2+z^2}.$$

3084. $\dfrac{\partial w}{\partial x} = (2xy^2 - yzv)\tan^3\alpha$; $\dfrac{\partial w}{\partial y} = (2x^2y - xzv)\tan^3\alpha$;
$\dfrac{\partial w}{\partial z} = (2zv^2 - xyv)\tan^3\alpha$; $\dfrac{\partial w}{\partial v} = (2z^2v - xyz)\tan^3\alpha$.
where $\alpha = x^2y^2 + z^2v^2 - xyzv$.

3085. 4. **3086.** $\left(\dfrac{\partial u}{\partial z}\right)_{\substack{z=b\\t=a}} = \dfrac{3b}{2}\sqrt{\dfrac{ab}{b^2 - a^2}}$;
$\left(\dfrac{\partial u}{\partial t}\right)_{\substack{z=b\\t=a}} = -\dfrac{3a}{2}\sqrt{\dfrac{ab}{b^2 - a^2}}$.

3087. 1 and -1. **3088.** $\dfrac{\sqrt{2}}{2}$. **3089.** $\dfrac{3}{2}$. **3090.** $-\dfrac{13}{22}$.

3091. 45°. **3092.** 30°. **3093.** $\arctan\dfrac{4}{7}$.

3094. $d_xz = (y^3 - 6xy^2)\,dx$; $d_yz = (3xy^2 - 6x^2y + 8y^3)\,dy$.

3095. $d_xz = \dfrac{x\,dx}{\sqrt{x^2 + y^2}}$; $d_yz = \dfrac{y\,dy}{\sqrt{x^2 + y^2}}$.

3096. $d_xz = \dfrac{y(y^2 - x^2)\,dx}{(x^2 + y^2)^2}$; $d_yz = \dfrac{x(x^2 - y^2)\,dy}{(x^2 + y^2)^2}$.

3097. $d_xu = \dfrac{3x^2\,dx}{x^3 + 2y^3 - z^3}$; $d_yu = \dfrac{6y^2\,dy}{x^3 + 2y^3 - z^3}$;
$d_zu = \dfrac{-3z^2\,dz}{x^3 + 2y^3 - z^3}$.

3098. $\dfrac{1}{270}$. **3099.** $\approx 0{\cdot}0187$. **3100.** $\dfrac{97}{600}$.

3101. $xy[(2y^3 - 3xy^2 + 4x^2y)\,dx + (4y^2x - 3yx^2 + 2x^3)\,dy]$.

3102. $\dfrac{x\,dx + y\,dy}{x^2 + y^2}$. **3103.** $\dfrac{2(s\,dt - t\,ds)}{(s - t)^2}$.

3104. $\dfrac{y\,dx - x\,dy}{y\sqrt{y^2 - x^2}}$. **3105.** $(x\,dy + y\,dx)\cos(xy)$.

3106. $\dfrac{dx}{1 + x^2} + \dfrac{dy}{1 + y^2}$. **3107.** $\dfrac{4xy(x\,dy - y\,dx)}{(x^2 - y^2)^2}$.

3108. $\dfrac{x\,dy + y\,dx}{1 + x^2y^2}$.

3109. $x^{zy-1}(yz\,dx + zx\ln x\,dy + xy\ln x\,dz)$.

3110. 0·08. **3111.** 0·25e. **3112.** $\dfrac{1}{36}$. **3113.** $\approx 7{\cdot}5$. **3114.** $\approx 0{\cdot}005$.

3115. $\approx 1{\cdot}08$. **3116.** 5. **3117.** $1{\cdot}8 \pm 0{\cdot}2$. **3118.** 4730 ± 100.

3119. $2\delta' + \dfrac{\delta'_B B \sin C}{\sin B \sin \overline{B+C}} + \dfrac{\delta'_C C \sin B}{\sin C \sin \overline{B+C}}$.

3120. Increases at a rate of 444 cm^2/sec.

3121. By $\approx 2575\ \text{cm}^3$.

3123. $dr = \dfrac{s}{p}\,ds + \left(\dfrac{1}{2} - \dfrac{s^2}{2p^2}\right) dp = 0{\cdot}16$ cm, i.e. about 1%.

3124. $e^{\sin t - 2t^2}(\cos t - 6t^2)$. **3125.** $\sin 2t + 2e^{2t} + e^t(\sin t + \cos t)$.

3126. $\dfrac{3 - 12t^2}{\sqrt{1 - (3t - 4t^3)^2}}$.

3127. $\dfrac{\partial z}{\partial u} = 3u^2 \sin v \cos v(\cos v - \sin v)$;

$\dfrac{\partial z}{\partial v} = u^3(\sin v + \cos v)\,(1 - 3 \sin v \cos v)$.

3128. $\dfrac{\partial z}{\partial u} = 2\dfrac{u}{v^2} \ln (3u - 2v) + \dfrac{3u^2}{v^2(3u - 2v)}$;

$\dfrac{\partial z}{\partial v} = -\dfrac{2u^2}{v^3} \ln (3u - 2v) - \dfrac{2u^2}{v^2(3u - 2v)}$.

3129. $\dfrac{\partial u}{\partial x} = \dfrac{e^x}{e^x + e^y}$; $\dfrac{du}{dx} = \dfrac{e^x + 3e^{x^3}x^2}{e^x + e^{x^3}}$.

3130. $\dfrac{dz}{dx} = \dfrac{e^x(x + 1)}{1 + x^2e^{2x}}$. **3131.** $\dfrac{du}{dx} = \dfrac{1}{1 + x^2}$.

3132. $\dfrac{dz}{dt} = \left(3 - \dfrac{4}{t^3} - \dfrac{1}{2\sqrt{t}}\right) \sec^2 \left(3t + \dfrac{2}{t^2} - \sqrt{t}\right)$.

3133. $\dfrac{du}{dx} = e^{ax} \sin x$.

3134. $dz = \dfrac{y^2\,dx + x^2\,dy}{(x + y)^2} \operatorname{arc\,tan} (xy + x + y) +$

$+ \dfrac{xy[(y + 1)\,dx + (x + 1)\,dy]}{(x + y)^2\,[1 + (xy + x + y)^2]}$.

3135. $\dfrac{e^{\frac{x^2 + y^2}{xy}}}{x^2y^2}\,[(y^4 - x^4 + 2xy^3)\,x\,dy + (x^4 - y^4 + 2x^3y)\,y\,dx]$.

3136. $\dfrac{\partial z}{\partial x} = 2x\dfrac{\partial f}{\partial u} + y\,e^{xy}\dfrac{\partial f}{\partial v}$, $\dfrac{\partial z}{\partial y} = -2y\dfrac{\partial f}{\partial u} + x\,e^{xy}\dfrac{\partial f}{\partial v}$; $\quad \left\{ u = x^2 - y^2;\ v = e^{xy}. \right.$

3145. $\dfrac{3x^2y - y^3}{3xy^2 - x^3}$. **3146.** $\dfrac{x(y^2 - 2x^2)}{y(2y^2 - x^2)}$. **3147.** $\dfrac{y\,e^{xy} - y\,e^x - e^y}{x\,e^y - e^x - x\,e^{xy}}$.

3148. $-\frac{x}{y}\cdot\frac{2(x^2+y^2)-a^2}{2(x^2+y^2)+a^2}$. **3149.** $\frac{y}{x}\cdot\frac{2x+e^{xy}-\cos xy}{\cos xy-e^{xy}-x}$.

3150. $-\sqrt[3]{\frac{y}{x}}$. **3151.** $\frac{y^2}{1-xy}$. **3152.** $\frac{a^2}{(x+y)^2}$.

3153. $\frac{2y}{x(y-1)}$. **3154.** $\frac{y}{y-1}$. **3155.** $\frac{y^2}{x^2}\,\frac{\ln x-1}{\ln y-1}$.

3157. $\left(\frac{dy}{dx}\right)_{\substack{x=6\\y=2}}=\frac{4}{3}$; $\left(\frac{dy}{dx}\right)_{\substack{x=6\\y=8}}=-\frac{4}{3}$; the tangents to the circles at these points have the same angle of inclination to Ox.

3158. -1. **3161.** $\frac{\partial z}{\partial x}=-\frac{c^2x}{a^2z}$; $\frac{\partial z}{\partial y}=-\frac{c^2y}{b^2z}$.

3162. $\frac{\partial z}{\partial x}=\frac{2-x}{z+1}$; $\frac{\partial z}{\partial y}=\frac{2y}{z+1}$.

3163. $\frac{\partial z}{\partial x}=-\frac{yz}{xy+z^2}$; $\frac{\partial z}{\partial y}=-\frac{xz}{xy+z^2}$.

3164. $\frac{\partial z}{\partial x}=\frac{z}{x(z-1)}$; $\frac{\partial z}{\partial y}=\frac{z}{y(z-1)}$.

3167. $dz=-\frac{\sin 2x\,dx+\sin 2y\,dy}{\sin 2z}$. **3168.** $z=\frac{x^2-y^2}{4}$.

3169. $z=\frac{3xy-x^3}{2}$. **3170.** $z=k\arctan\frac{y}{x}$.

3171. $dz=\frac{x\,dx}{z}-\frac{y\,dy}{z}$. **3172.** $dz=\frac{x\,dx}{a}+\frac{y\,dy}{a}$.

3173. $dz=\sqrt{z}(x\,dx-y\,dy)$.

3174. $dz=e^{-u}[(v\cos v-u\sin v)\,dx+(u\cos v+v\sin v)\,dy]$.

3176. $2(x\,dx+y\,dy)$. **3177.** $2(x\,dx+y\,dy)$.

3185. $\frac{\partial^2 z}{\partial x^2}=\frac{2x^2+y^2}{\sqrt{x^2+y^2}}$; $\frac{\partial^2 z}{\partial y^2}=\frac{x^2+2y^2}{\sqrt{x^2+y^2}}$; $\frac{\partial^2 z}{\partial x\,\partial y}=\frac{xy}{\sqrt{x^2+y^2}}$.

3186. $\frac{\partial^2 z}{\partial x^2}=-\frac{x}{(x^2+y^2)^{\frac{3}{2}}}$; $\frac{\partial^2 z}{\partial y^2}=\frac{x^3+(x^2-y^2)\sqrt{x^2+y^2}}{(x^2+y^2)^{\frac{3}{2}}(x+\sqrt{x^2+y^2})^2}$.

3187. $\frac{\partial^2 z}{\partial x^2}=-\frac{2x}{(1+x^2)^2}$; $\frac{\partial^2 z}{\partial y^2}=-\frac{2y}{(1+y^2)^2}$; $\frac{\partial^2 z}{\partial x\,\partial y}=0$.

3188. $\frac{\partial^2 z}{\partial x^2}=2a^2\cos 2(ax+by)$; $\frac{\partial^2 z}{\partial y^2}=2b^2\cos 2(ax+by)$;

$\frac{\partial^2 z}{\partial x\,\partial y}=2\,ab\cos 2(ax+by)$.

3189. $\dfrac{\partial^2 z}{\partial x^2} = e^{x e^y + 2y}$; $\dfrac{\partial^2 z}{\partial y^2} = x(1 + x e^y)\, e^{x e^y + y}$;

$\dfrac{\partial^2 z}{\partial x\, \partial y} = (1 + x e^y)\, e^{x e^y + y}$.

3190. $\dfrac{\partial^2 z}{\partial x^2} = -\dfrac{4y}{(x+y)^3}$; $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{4x}{(x+y)^3}$; $\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{2(x-y)}{(x+y)^3}$.

3191. $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\ln y(\ln y + 1)}{x^2}\, e^{\ln x \ln y}$; $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\ln x(\ln x - 1)}{y^2}\, e^{\ln x \ln y}$;

$\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{\ln x \ln y + 1}{xy}\, e^{\ln x \ln y}$.

3192. $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{xy^3}{\sqrt{(1 - x^2y^2)^3}}$; $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{x^3 y}{\sqrt{(1 - x^2y^2)^3}}$;

$\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{1}{\sqrt{(1 - x^2y^2)^3}}$.

3193. $\dfrac{(x-z)\, y}{\sqrt{(x^2 + y^2 + z^2 - 2xz)^3}}$. **3194.** $2y^3(2 + xy^2)\, e^{xy^2}$.

3195. $\dfrac{4x(3y^2 - x^2)}{(x^2 + y^2)^3}$. **3196.** $-x(2 \sin xy + xy \cos xy)$.

3197. $(x^2y^2z^2 + 3xyz + 1)\, e^{xyz}$.

3198. $mn(n-1)(n-2)\, p(p-1)\, x^{m-1} y^{n-3} z^{p-2}$. **3204.** $a = -3$.

3209.
$$\frac{d^2 y}{dx^2} = -\frac{\dfrac{\partial^2 f}{\partial x^2}\cdot\left(\dfrac{\partial f}{\partial y}\right)^2 - 2\,\dfrac{\partial^2 f}{\partial x\,\partial y}\cdot\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial f}{\partial y} + \dfrac{\partial^2 f}{\partial y^2}\cdot\left(\dfrac{\partial f}{\partial x}\right)^2}{\left(\dfrac{\partial f}{\partial y}\right)^3} =$$
$$= \frac{1}{\left(\dfrac{\partial f}{\partial y}\right)^3}\begin{vmatrix} 0 & \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\,\partial y} \\ \dfrac{\partial f}{\partial y} & \dfrac{\partial^2 f}{\partial x\,\partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix}.$$

3219. $-2y\, dx^2 + 4(y - x)\, dx\, dy + 2x\, dy^2$.

3220. $-\dfrac{(dx - dy)^2}{(x - y)^2}$.

3221. $\dfrac{(3x^2 - y^2)\, dx^2 + 8xy\, dx\, dy + (3y^2 - x^2)\, dy^2}{(x^2 + y^2)^3}$.

3222. $2 \sin 2y\, dx\, dy + 2x \cos 2y\, dy^2$.

3223. $e^{xy}\,[(y\, dx + x\, dy)^2 + 2\, dx\, dy]$.

3224. $2(z\,dx\,dy + y\,dx\,dz + x\,dy\,dz)$.

3225. $-\cos(2x+y)(2dx+dy)^3$; $(2dx+dy)^3$; 0.

3226. $-\sin(x+y+z)(dx+dy+dz)^2$.

3227. $-\dfrac{c^4}{z^3}\left[\left(\dfrac{x^2}{a^2}+\dfrac{z^2}{c^2}\right)\dfrac{dx^2}{a^2}+\dfrac{2xy}{a^2b^2}\,dx\,dy+\left(\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right)\dfrac{dy^2}{b^2}\right]$.

3228. $-\dfrac{2z\,[xy^3\,dx^2+(x^2y^2+2xyz^2-z^4)\,dx\,dy+x^3y\,dy^2]}{(z^2-xy)^3}$.

3229. $-31{\cdot}5\,dx^2+354\,dx\,dy+275\,dy^2$. **3230.** $\dfrac{d^2y}{dt^2}+y$.

3231. $y''-5y'+y$. **3232.** $y-x$. **3233.** $\dfrac{d^2y}{dt^2}+ay$. **3234.** $-\dfrac{x'''}{x'^5}$.

3235. $-\dfrac{v''+2v}{v^3}$. **3236.** $\dfrac{d\varrho}{d\varphi}=\varrho$. **3237.** $\dfrac{2\varrho'-\varrho\varrho''+\varrho^2}{(\varrho'^2+\varrho^2)^{\frac{3}{2}}}$.

3238. $-\dfrac{\partial z}{\partial v}$. **3239.** $\dfrac{\partial^2u}{\partial\varrho^2}+\dfrac{1}{\varrho^2}\dfrac{\partial^2u}{\partial\varphi^2}+\dfrac{1}{\varrho}\dfrac{\partial u}{\partial\varrho}$.

3240. $\omega''(r)+\dfrac{1}{r}\omega'(r)+k\omega(r)$. **3241.** $-4\dfrac{\partial^2\omega}{du^2}+2$.

Chapter XI

3242. $x^3+2y^3-xy+h(3x^2-y)+k(6y^2-x)+3xh^2-hk+$ $+6yk^2+h^3+2k^3$.

3243. $\Delta z=15h^2-6hk+k^2+h^3$.

3244. $\Delta z=-2h+7k-4h^2+4hk+2k^2-2h^3-h^2k+$ $+\dfrac{5}{2}hk^2+\dfrac{1}{4}k^3-h^3k+\dfrac{1}{2}h^2k^2+\dfrac{1}{4}hk^3$; $f(1{\cdot}02;\ 2{\cdot}03)\approx 2{\cdot}1726$.

3245. $Ax^2+By^2+Cz^2+Dxy+Eyz+Fzx+$ $+(2Ax+Dy+Fz)h+(2By+Dx+Ez)k+$ $+(2Cz+Ey+Fx)l+Ah^2+Bk^2+Cl^2+Dhk+Ekl+Fhl$.

3246.
$$z=\frac{1}{2}+\frac{1}{2}\left(x-\frac{\pi}{4}\right)+\frac{1}{2}\left(y-\frac{\pi}{4}\right)-$$
$$-\frac{1}{4}\left[\left(x-\frac{\pi}{4}\right)^2-2\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right)+\left(y-\frac{\pi}{4}\right)^2\right]-$$
$$-\frac{1}{6}\left[\cos\xi\cos\eta\left(x-\frac{\pi}{4}\right)^3+3\sin\xi\cos\eta\left(x-\frac{\pi}{4}\right)^2\left(y-\frac{\pi}{4}\right)+\right.$$
$$\left.+3\cos\xi\sin\eta\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right)^2+\sin\xi\cos\eta\left(y-\frac{\pi}{4}\right)^3\right].$$

3247. $z = 1 + (x-1) + (x-1)(y-1) + $
$+\frac{1}{2}(x-1)^2(y-1) + \ldots;\ z_1 \approx 1{\cdot}1021.$

3248. $e^x\Big[\sin y + h\sin y + k\cos y + \frac{1}{2}(h^2\sin y + 2hk\cos y -$
$- k^2\sin y) + \frac{1}{6}(h^3\sin y + 3h^2k\cos y - 3hk^2\sin y - k^3\cos y)\Big] + \ldots;$
$z_1 \approx 1{\cdot}1051.$

3249. $y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + \ldots$

3250. $y + \frac{1}{2!}(2xy - y^2) + \frac{1}{3!}(3x^2y - 3xy^2 + 2y^3) + \ldots$

3251. $1 + (x+y) + \ldots + \frac{x^{n+1} - y^{n+1}}{x-y} + \ldots$

3252*. $x - y - \frac{1}{3}(x^3 - y^3) + \frac{1}{5}(x^5 - y^5) - \ldots +$
$+\frac{(-1)^n}{2n+1}(x^{2n+1} - y^{2n+1}) + \ldots$ Note that $\operatorname{arc\,tan}\frac{x-y}{2+xy} =$
$= \operatorname{arc\,tan} x - \operatorname{arc\,tan} y.$

3253. $\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)\left(\sum_{n=1}^{\infty}\frac{y^n}{n}\right) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^ny^m}{nm}.$

3254. $\sum_{n=2}^{\infty}\frac{(x+y)^n - x^n - y^n}{n}.$ **3255.** $\sum_{n=0}^{\infty}(-1)^n\frac{(x^2+y^2)^{2n+1}}{(2n+1)!}.$

3256. $\sum_{n=0}^{\infty}\frac{x^m}{m!}\sum_{n=0}^{\infty}(-1)^n\frac{y^{2n}}{(2n)!} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\frac{x^my^{2n}}{m!\,(2n)!}.$

3257. $z = 1 + (x-1) + \frac{1}{4}(y-1) + \frac{1}{4}(x-1)(y-1) +$
$+\frac{9}{64}(y-1)^2 + \ldots$

3259. $(0, 0),\ \left(-\frac{5}{3}, 0\right),\ (-1, 2),\ (-1, -2).$

3260. $\left(\frac{1}{2}, -1\right).$ **3261.** $(0, 0),\ (0, a),\ (a, 0),\ \left(\frac{a}{3}, \frac{a}{3}\right).$

3262. $(0, 0),\ (0, 2b),\ (a, b),\ (2a, 0),\ (2a, 2b).$ **3263.** $\left(\frac{\pi}{6}, \frac{\pi}{6}\right).$

3264. $\left(\frac{b}{a}, \frac{c}{a}\right).$ **3265.** $\left(-\frac{2}{3}, -\frac{2}{3}\right).$ **3266.** $(2, 1, 7).$

3267. $(6, 4, 10).$

3268. A and C are maxima, B a minimum; in the neighbourhood of D the surface is saddle-shaped, the function has a constant value along EF.

3269. $(-2, 0)$, $\left(\frac{16}{7}, 0\right)$. **3270.** $(1, 1)$, $(-1, -1)$.

3271*. $(0, 0)$. To show that there is a maximum at the point obtained, the function only needs to be written in the form $z = 10 - (x-y)^2 - 2x^2 - y^2$.

3272. $(2, -2)$. **3273.** $(-1, 1)$. **3277.** A maximum at the point $(6, 4)$.

3278. There is no extremum at $(0, 0)$. A minimum at $(1, 1)$.

3279. The greatest and least values lie on the boundary of the domain; the greatest is $z = 4$ at the points $(2, 0)$ and $(-2, 0)$; the least is $z = -4$ at the points $(0, 2)$ and $(0, -2)$. The stationary point $(0, 0)$ does not give an extremum.

3280. The greatest value $z = 17$ at the point $(1, 2)$; the least value $z = -3$ at the point $(1, 0)$; the stationary point $(-4, 6)$ lies outside the specified domain.

3281. The greatest value $z = 4$ at the stationary point $(2, 1)$ (there is therefore a maximum at the point). The least value is $z = -64$ at the point $(4, 2)$ on the boundary.

3282. The greatest value is $z = \frac{3}{e}$ at the points $(0, \pm 1)$ (maximum). The least value is $z = \frac{12}{e^4}$ at the point $(0, 4)$ (on the boundary).

3283. $z_{\max} = \frac{3}{2}\sqrt{3}$ at the point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ (maximum), $z_{\min} = 0$ at the point $(0, 0)$ (on the boundary).

3284. All the terms of the sum are equal.

3285. All the factors are equal.

3286. $\left(\frac{8}{5}, \frac{16}{5}\right)$. **3287.** $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$.

3288. $x = \frac{\sum_{i=1}^{n} x_i}{n}$, $y = \frac{\sum_{i=1}^{n} y_i}{n}$. **3289.** $(3, \sqrt{39}, 0)$; $(3, -\sqrt{39}, 0)$.

3290. A cube. **3291.** A minimum $z = 2$ at the point $(1, 1)$.

3292. (a, a) or $(-a, -a)$, $z = a^2$ (maximum), $(a, -a)$ or $(-a, a)$, $z = -a^2$ (minimum).

3293. $(-a\sqrt{2}, -a\sqrt{2})$, $z = -\frac{\sqrt{2}}{a}$ (minimum), $(a\sqrt{2}, a\sqrt{2})$, $z = \frac{\sqrt{2}}{a}$ (maximum).

3294. Stationary points $x = -\frac{1}{2}\ \text{Arctan}\,\frac{b}{a}$, $y = \frac{1}{2}\ \text{Arctan}\,\frac{a}{b}$.

3295. (3, 3, 3), $u = 9$ (minimum).

3296. Two of the variables are equal to 2, the third equal to 1 (minimum, equal to 4); two of the variables are equal to $\frac{4}{3}$, the third equal to $\frac{7}{3}$ $\left(\text{maximum, equal to } \frac{112}{7}\right)$.

3297*. Investigate the minimum of the function $\dfrac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}$ for $x_1 + x_2 + \ldots + x_n = A$. The relationship $\dfrac{\sum x_i^k}{n} \geqq \left(\dfrac{\sum x_i}{n}\right)^k$, is satisfied in general if $k \geqq 1$.

3299. $u_{\min} = abc/(bc + ca + ab)$ for $x = bc/(bc + ca + ab)$; $y = ac/(bc + ac + ab)$; $z = ab/(bc + ac + ab)$.

3300. $x = \pm\frac{1}{2}$, $y = \pm\frac{1}{3}$, $z = \pm\frac{1}{6}$. **3301.** $\left(\frac{21}{13}, 2, \frac{63}{26}\right)$.

3302. $(3, -1, 1)$. **3303.** (a) $(-2, 0, 0)$; (b) $(2, 0, 0)$. **3304.** A cube.

3305. A cube. **3306.** $\dfrac{8abc}{3\sqrt{3}}$.

3307. If R is the radius of the base of the marquee, H the height of the cylindrical part, h the height of the conical top, the following relationships must hold:

$$R = \frac{h\sqrt{5}}{2}, \quad H = \frac{h}{2}.$$

3308. If l is the lateral side of the trapezoid, b the base and α the angle of inclination of the lateral side, the following relationship must hold: $l = b = \dfrac{2\sqrt{A}}{\sqrt[4]{3}}$, $\alpha = \dfrac{\pi}{3}$, where A is the given cross-sectional area. The perimeter of the section is now

$$u = 2\sqrt[4]{3}\sqrt{A} \approx 2{\cdot}632\sqrt{A}.$$

3309. A cube. **3310.** The sides of the base are each equal to $2\alpha + \sqrt[3]{2v}$, the height is half as great: $\left(\alpha + \frac{1}{2}\sqrt[3]{2v}\right)$.

3311. a^3 (a cube). **3312.** The least area is equal to $3\sqrt{3}ab$.

3313. $x = \pm\dfrac{4}{\sqrt{5}}$, $y = \pm\dfrac{3}{\sqrt{5}}$. **3314.** $\left(-\frac{5}{9}, -\frac{1}{9}\right)$. **3315.** (3, 5)

3316. $z_{\max} = +\sqrt{\dfrac{ab - h^2}{abc + 2fgh - af^2 - bg^2 - ch^2}}$.

3317. The sides of the triangle are $\sqrt{2S}$, $\sqrt{2S}$ and $2\sqrt{S}$.

3318. The height is $\dfrac{H}{3}$, the sides of the base are $\dfrac{2a\sqrt{2}}{3}$, and $\dfrac{2b\sqrt{2}}{3}$, the volume $V = \dfrac{8}{27}abH$.

3319. A tetrahedron.

3320. The normal to the ellipse at the required point must be perpendicular to the line joining the given points.

3321. The normal is drawn at the point with coordinates

$$\left(\pm a\sqrt{\frac{a}{a+b}},\ \pm b\sqrt{\frac{b}{a+b}}\right).$$

3322. $\left(9, \dfrac{1}{8}, \dfrac{3}{8}\right)$; $\left(-9, -\dfrac{1}{8}, -\dfrac{3}{8}\right)$. **3323.** $2\sqrt{2}$.

3324. $x + y = 2$; $y = x$. **3325.** $x - y + a = 0$; $x + y - 3a = 0$.
3326. $x + 2y - 1 = 0$; $2x - y - 2 = 0$.
3327. $x - y + 2 = 0$; $x + y - 2 = 0$. **3328.** $(0, 0)$. **3329.** $(0, 0)$.
3330. $(0, 0)$. **3331.** $(a, 0)$. **3332.** $(0, a)$, $(0, -a)$, $(a, 0)$, $(-a, 0)$.
3333. $(2, 0)$, $(-2, 0)$. **3334.** $(0, 3)$, $(-3, 0)$, $(-6, 3)$.
3335. $(0, 0)$ is a double point. **3336.** $(0, 0)$ is an isolated point.
3337. $(0, 0)$ is a break-point. **3338.** $k\pi$; $k = 0, 1, 2, \ldots$ are cusps.
3339. $(a, 0)$ is a cusp. **3340.** $(0, 0)$.
3341. $x = -f'(a)$, $y = f(a) - af'(a)$; $y = x \arcsin x + \sqrt{1 - x^2}$.

3342. $16y^3 + 27x^4 = 0$. **3343.** $y^2 = 4ax$. **3344.** $y = \dfrac{x}{2}$ and $y = -\dfrac{x}{2}$.

3345. $y = -\dfrac{x^4}{4}$. **3346.** $y = 0$ and $16y = x^4$.

3347. $y = x$ and $y = x - \dfrac{4}{27}$. The first is the locus of singular points, the second the envelope.

3348. $x^2 + \dfrac{2}{3\sqrt{3}}y^2 = 0$ and $x^2 - \dfrac{2}{3\sqrt{3}}y^3 = 0$.

3349. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = d^{\frac{2}{3}}$.

3350. 4 straight lines $x \pm y = \pm R$. **3351.** $2by(x^2 + y^2) + x^2 = 0$.

3352. The parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

3353. The cycloid $x = \frac{R}{2}(t - \sin t)$, $y = \frac{R}{2}(1 - \cos t)$.

3354. The ellipse $x^2 + \frac{y^2}{2} = R^2$. **3355.** The hyperbola $xy = \frac{a}{4}$.

3357. The evolute of the parabola $y^2 = \frac{8}{27p}(x - p)^3$.

3359. The hyperbolas $xy = \frac{1}{2}$ and $xy = -\frac{1}{2}$.

3361. (a) $2\boldsymbol{r}\cdot\frac{d\boldsymbol{r}}{dt} = 2|\boldsymbol{r}|\frac{d|\boldsymbol{r}|}{dt}$, (b) $\left(\frac{d\boldsymbol{r}}{dt}\right)^2 + \boldsymbol{r}\frac{d^2\boldsymbol{r}}{dt^2}$;
(c) $\boldsymbol{r}\times\frac{d^2\boldsymbol{r}}{dt^2}$; (d) $\left(\boldsymbol{r}\frac{d\boldsymbol{r}}{dt}\frac{d^3\boldsymbol{r}}{dt^3}\right)$.

3362. It follows from the equation $\frac{d\boldsymbol{r}}{dt} = \alpha(t)\boldsymbol{r}$ that:

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d\alpha}{dt}\boldsymbol{r} + \alpha\frac{d\boldsymbol{r}}{dt} = \left(\frac{d\alpha}{dt} + \alpha^2\right)\boldsymbol{r} = \beta(t)\,\boldsymbol{r} \text{ and so on.}$$

3363. Differentiation of the equation $\boldsymbol{r}^2 = \text{const}$ (see problem 3361) gives us: $\boldsymbol{r}\cdot\frac{d\boldsymbol{r}}{dt} = 0$. The tangent to a spherical curve (i.e. to a curve drawn on a sphere) is perpendicular to the radius of the sphere through the point of contact. The converse also holds.

3368. $\frac{d\boldsymbol{r}}{dx} = \frac{d\boldsymbol{r}}{du}\varphi'$; $\frac{d^2\boldsymbol{r}}{dx^2} = \frac{d^2\boldsymbol{r}}{du^2}\varphi'^2 + \frac{d\boldsymbol{r}}{du}\varphi''$;

$$\frac{d^3\boldsymbol{r}}{dx^3} = \frac{d^3\boldsymbol{r}}{du^3}\varphi'^3 + 3\frac{d^2\boldsymbol{r}}{du^2}\varphi'\varphi'' + \frac{d\boldsymbol{r}}{du}\varphi'''.$$

3370. It follows from the equation $\boldsymbol{a}\frac{d\boldsymbol{r}(\tau)}{dt} = 0$, where $t_1 < \tau < t_2$, that a point can be found on a closed curve (closed by virtue of the equation $\boldsymbol{r}(t_1) = \boldsymbol{r}(t_2)$) at which the tangent is perpendicular to any previously assigned direction.

3371. The hodograph of the velocity $\boldsymbol{v}\{a\cos t,\ a\sin t,\ 2bt\}$ is a helix; the hodograph of the acceleration $\boldsymbol{w}\{-a\sin t,\ a\cos t,\ 2b\}$ is a circle.

3372. Scalar multiplication by $\boldsymbol{a}$ and by $\boldsymbol{r}$ gives: $\boldsymbol{a}\frac{d\boldsymbol{r}}{dt} = 0$, $\boldsymbol{r}\frac{d\boldsymbol{r}}{dt} = 0$. Hence $\boldsymbol{ar} = \text{const}$ is the equation of a plane and $\boldsymbol{r}^2 = \text{const}$ is the equation of a sphere. The required trajectory is a circle, the plane of which is perpendicular to vector $\boldsymbol{a}$.

3374. An ellipse. The velocity will be a maximum at the instant when the material particle is at an end of the minor semi-axis, and a minimum when it is at an end of the major semi-axis. The acceleration is a maximum (minimum) at the instant when the velocity is a minimum (maximum).

3375. The velocity components are $\dfrac{d\varrho}{dt}$; $\varrho\dfrac{d\varphi}{dt}$; $\varrho \sin\varphi \dfrac{d\theta}{dt}$.

Hint. Find the scalar products $\dfrac{d\boldsymbol{r}}{dt}\boldsymbol{e}_\varrho$; $\dfrac{d\boldsymbol{r}}{dt}\boldsymbol{e}_\varphi$; $\dfrac{d\boldsymbol{r}}{dt}\boldsymbol{e}_\theta$.

3376. $\dfrac{x-\dfrac{t^4}{4}}{t^2} = \dfrac{y-\dfrac{t^3}{3}}{t} = \dfrac{z-\dfrac{t^2}{2}}{1}$; $t^2x + ty + z = \dfrac{t^6}{4} + \dfrac{t^4}{3} + \dfrac{t^2}{2}$.

3377. $\dfrac{x-\dfrac{a\sqrt{2}}{2}}{-a\sqrt{2}} = \dfrac{y-\dfrac{a\sqrt{2}}{2}}{a\sqrt{2}} = \dfrac{z-\dfrac{k}{8}}{\dfrac{k}{\pi}}$;

$$-x + y + \frac{k}{\pi a\sqrt{2}}z = \frac{k^2}{8\pi a\sqrt{2}}.$$

3378. $x - 6a = \dfrac{y-18a}{6} = \dfrac{z-72a}{36}$; $x + 6y + 36z = 2706a$.

3379. $\dfrac{x-\dfrac{\pi}{2}+1}{1} = \dfrac{y-1}{1} = \dfrac{z-2\sqrt{2}}{\sqrt{2}}$;

$$x + y + \sqrt{2}z = \frac{\pi}{2} + 4.$$

3380. $\dfrac{x-1}{12} = \dfrac{y-3}{-4} = \dfrac{z-4}{3}$; $12x - 4y + 3z - 12 = 0$

3381. $\dfrac{x+2}{27} = \dfrac{y-1}{28} = \dfrac{z-6}{4}$; $27x + 28y + 4z + 2 = 0$.

3382. $\dfrac{x-x_0}{z_0} = \dfrac{y-y_0}{z_0} = \dfrac{z-z_0}{y_0+x_0}$; $\dfrac{x+y}{x_0+y_0} + \dfrac{z}{z_0} = 2$.

3383. $\dfrac{x-x_0}{y_0^2z_0^2} = \dfrac{y-y_0}{x_0^2z_0^2} = \dfrac{z-z_0}{-x_0^2y_0^2}$;

$$\frac{x-x_0}{x_0^2} + \frac{y-y_0}{y_0^2} - \frac{z-z_0}{z_0^2} = 0.$$

3384. $\boldsymbol{r}_0\left\{\dfrac{\sqrt{3}}{2},\ \dfrac{1}{2},\ e^{\frac{\pi}{6}}\right\}$.

3385. $6x - 8y - z + 3 = 0$; $\frac{x-1}{6} = \frac{y-1}{-8} = \frac{z-1}{-1}$;

$\frac{x-1}{31} = \frac{y-1}{26} = \frac{z-1}{-22}$.

3386. $\sqrt{b}(x - x_0) - \sqrt{a}(y - y_0) = 0$; $\frac{x-x_0}{\sqrt{b}} = \frac{y-y_0}{-\sqrt{a}} = \frac{z-z_0}{0}$;

$\frac{x-x_0}{\sqrt{2az_0}} = \frac{y-y_0}{\sqrt{2bz_0}} = \frac{z-z_0}{-(a+b)}$.

3387. $\frac{1}{e}x - ey - \sqrt{2}z + 2 = 0$; $\frac{x-e}{-\frac{1}{e}} = \frac{y-\frac{1}{e}}{e} = \frac{z-\sqrt{2}}{\sqrt{2}}$;

$\frac{x-e}{1} = \frac{y-\frac{1}{e}}{1} = \frac{z-\sqrt{2}}{-\sqrt{2}\sinh 1}$.

3389. $\frac{x-1}{2} + \frac{y}{-1} = \frac{z-1}{3}$; $2x - y + 3z - 5 = 0$;

$\frac{x-1}{3} = \frac{y}{3} = \frac{z-1}{-1}$; $3x + 3y - z - 2 = 0$;

$\frac{x-1}{8} = \frac{y}{-11} = \frac{z-1}{-9}$; $8x - 11y - 9z + 1 = 0$.

3390. $\frac{x-1}{1} = \frac{y-1}{-1} = \frac{z-1}{0}$; $x - y = 0$;

$\frac{x-1}{0} = \frac{y-1}{0} = \frac{z-1}{1}$; $z = 1$; $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{0}$;

$x + y - 2 = 0$.

3391. $\frac{x-\frac{\sqrt{2}}{2}}{\sqrt{2}} = \frac{y-\frac{\sqrt{2}}{2}}{-\sqrt{2}} = \frac{z-1}{4}$; $\sqrt{2}x - \sqrt{2}y + 4z = 4$;

$\frac{x-\frac{\sqrt{2}}{2}}{\sqrt{2}} = \frac{y-\frac{\sqrt{2}}{2}}{3\sqrt{2}} = \frac{z-1}{1}$; $\sqrt{2}x + 3\sqrt{2}y + z - 5 = 0$;

$\frac{x-\frac{\sqrt{2}}{2}}{13} = \frac{y-\frac{\sqrt{2}}{2}}{-3} = \frac{z-1}{-4\sqrt{2}}$; $-13 + 3y + 4\sqrt{2}z + \sqrt{2} = 0$.

3392. $\dfrac{x+1}{2} = \dfrac{y-13}{3} = \dfrac{z}{6}$; $2x + 3y + 6z = 37$;

$\dfrac{x+1}{6} = \dfrac{y-13}{2} = \dfrac{z}{-3}$; $6x + 2y - 3z = 20$;

$\dfrac{x+1}{3} = \dfrac{y-13}{-6} = \dfrac{z}{2}$; $3x - 6y + 2z = -81$.

3393. The equation of the osculating plane is $3x - 2y - 11 = 0$ for any point of the curve, i.e. the curve lies entirely in this plane.

3394. The osculating plane is the same for all points of the curve. Its equation is

$$\begin{vmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

3395. $\dfrac{\cosh^2 t}{\sinh t}$. **3396.** $R = \sqrt{2} \cdot \operatorname{cosec} 2\theta$.

3398. $k = \sqrt{\dfrac{(y'z'' - z'y'')^2 + y''^2 + z''^2}{(1 + y'^2 + z'^2)^3}}$.

3399. $\boldsymbol{\tau}_1 = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}'|}$, $\boldsymbol{\beta}_1 = \dfrac{\boldsymbol{r}' \times \boldsymbol{r}''}{|\boldsymbol{r}' \times \boldsymbol{r}''|}$, $\boldsymbol{\nu}_1 = \dfrac{(\boldsymbol{r}' \times \boldsymbol{r}'') \times \boldsymbol{r}'}{|\boldsymbol{r}'| \cdot |\boldsymbol{r} \times \boldsymbol{r}'|}$.

3400. $\boldsymbol{\tau}_1 = \boldsymbol{\nu}_1 \times \boldsymbol{\beta}_1$; $\boldsymbol{\nu}_1 = \boldsymbol{\beta}_1 \times \boldsymbol{\tau}_1$; $\boldsymbol{\beta}_1 = \boldsymbol{\tau}_1 \times \boldsymbol{\nu}_1$.

3401. The required vector $\boldsymbol{\omega}$ (if it exists) can be written as

$$\boldsymbol{\omega} = (\boldsymbol{\omega}\boldsymbol{\tau}_1)\,\boldsymbol{\tau}_1 + (\boldsymbol{\omega}\boldsymbol{\nu}_1)\,\boldsymbol{\nu}_1 + (\boldsymbol{\omega}\boldsymbol{\beta}_1)\,\boldsymbol{\beta}_1. \tag{1}$$

It follows from the condition of the problem (taking into account Frenet's formulae) that

$$\boldsymbol{\omega} \times \boldsymbol{\tau}_1 = k\boldsymbol{\gamma}_1;\quad \boldsymbol{\omega} \times \boldsymbol{\nu}_1 = -k\boldsymbol{\tau}_1 + T\boldsymbol{\beta}_1;\quad \boldsymbol{\omega} \times \boldsymbol{\beta}_1 = -T\boldsymbol{\nu}_1. \tag{2}$$

On forming the scalar product of each side of these equations with $\boldsymbol{\nu}_1, \boldsymbol{\beta}_1, \boldsymbol{\tau}_1$ respectively, we find that $\boldsymbol{\omega}\boldsymbol{\tau}_1 = T$, $\boldsymbol{\omega}\boldsymbol{\nu}_1 = 0$, $\boldsymbol{\omega}\boldsymbol{\beta}_1 = k$, so that $\boldsymbol{\omega} = T\boldsymbol{\tau}_1 + k\boldsymbol{\beta}_1$. Substitution in (2) shows that this vector satisfies the condition of the problem.

3402. $99 + \ln 10 \approx 101{\cdot}43$. **3403.** $a \ln\left(1 + \sqrt{2}\right) = a \ln \tan \dfrac{3\pi}{8}$.

3404. $\sqrt{3}\,(e^t - 1)$. **3405.** 5. **3406.** $4a$. **3407.** $z\sqrt{2}$.

3408. $a \ln \dfrac{\sqrt{2a} + \sqrt{x}}{\sqrt{2a} - \sqrt{x}}$. **3409.** $\dfrac{a}{2}\left(1 + \dfrac{1}{2} \ln 3\right)$.

3410. $8x - 8y - z = 4$; $\dfrac{x-2}{8} = \dfrac{y-1}{-8} = \dfrac{z-4}{-1}$.

3411. $x + y - z - 1 = 0$; $\dfrac{x-1}{1} = \dfrac{y-1}{1} = \dfrac{z-1}{-1}$.

3412. $z + a = 0$, $x = a$, $y = a$.

3413. $17x + 11y + 5z = 60$; $\dfrac{x-3}{17} = \dfrac{y-4}{11} = \dfrac{z+7}{5}$.

3414. $x - y + 2z - \dfrac{\pi}{2} = 0$; $\dfrac{x-1}{1} = \dfrac{y-1}{-1} = \dfrac{z-\frac{\pi}{2}}{2}$.

3415. $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = \sqrt{3}$;

$$a\left(x - \frac{a\sqrt{3}}{3}\right) = b\left(y - \frac{b\sqrt{3}}{3}\right) = c\left(z - \frac{c\sqrt{3}}{3}\right).$$

3416. $x + 11y + 5z - 18 = 0$; $\dfrac{x-1}{1} = \dfrac{y-2}{11} = \dfrac{z+1}{5}$.

3417. $3x - 2y - 2z + 1 = 0$; $\dfrac{x-1}{3} = \dfrac{y-1}{-2} = \dfrac{z-1}{-2}$.

3418. $2x + y + 11z - 25 = 0$; $\dfrac{x-1}{2} = \dfrac{y-1}{1} = \dfrac{z-2}{11}$.

3419. $5x + 4y + z - 28 = 0$; $\dfrac{x-2}{5} = \dfrac{y-3}{4} = \dfrac{z-6}{1}$.

3421. $x - y + 2z = \sqrt{\dfrac{11}{2}}$ and $x - y + 2z = -\sqrt{\dfrac{11}{2}}$.

3422. $x + y + z = \sqrt{a^2 + b^2 + c^2}$.

3424. All the planes pass through the origin.

3425. $x_0x + y_0y + z_0z = a^2$; $\dfrac{x}{x_0} = \dfrac{y}{y_0} = \dfrac{z}{z_0}$.

3426. $\dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} = 2(z + z_0)$; $\dfrac{a(x - x_0)}{bx_0} = -\dfrac{b(y - y_0)}{ay_0} = \dfrac{z - z_0}{-2ab}$.

3428. $\dfrac{9}{2}a^3$. **3430.** $2x + y - z = 2$. **3434.** $4x - 2y - 3z = 3$.

3435. $(0, 3, 3)$ and $(0, 3, -7)$.

3436. (a) $6u_0v_0x - 3(u_0+v_0)\,y + 2z + (u_0+v_0)\,(u_0^2 - 4u_0v_0 + v_0^2) = 0$; (b) $3(x_0^2 - y_0^2)\,x - 6x_0y_0y + 2z + 4z_0 = 0$.

3437. $2z(x^2 + y^2 + z^2) + p(x^2 + y^2) = 0$.

3438. $(x^2 + y^2 + z^2)^3 = 27a^3xyz$.

3439. (1) $\{-2, 1\}$; (2) $\{10xy - 3y^3, 5x^2 - 9xy^2 + 4y^3\}$.

3440. (1) $6\boldsymbol{i} + 4\boldsymbol{j}$; (2) $\dfrac{1}{3}(2\boldsymbol{i} + \boldsymbol{j})$; (3) $\dfrac{-y_0\boldsymbol{i} + x_0\boldsymbol{j}}{x_0^2 + y_0^2}$.

3441. (1) $\tan\varphi \approx 0{\cdot}342$, $\varphi \approx 18°52'$; (2) $\tan\varphi \approx 4{\cdot}87$, $\varphi \approx 78°24'$.

3442. The negative z semi-axis. **3443.** (1) $\cos\alpha \approx 0{\cdot}99$, $\alpha \approx 8°$; (2) $\cos\alpha \approx -0{\cdot}199$, $\alpha \approx 101°30'$.

3444. (1) $\left(-\frac{1}{3}, \frac{3}{4}\right)$; $\left(\frac{7}{3}, -\frac{3}{4}\right)$;

(2) Points lying on the circle $x^2 + y^2 = \frac{2}{3}$.

3447. (1) $\{3x_0^2y_0^2z_0,\ 2x_0^3y_0z_0,\ x_0^3y_0^2\}$;

(2) $\dfrac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|}$, where $\boldsymbol{r}$ is the radius vector.

3450. (1) $2\boldsymbol{r}$; (2) $2\dfrac{\boldsymbol{r}}{|\boldsymbol{r}|}$; (3) $2F'(\boldsymbol{r}^2)\,\boldsymbol{r}$; (4) $\boldsymbol{a}(\boldsymbol{br})+\boldsymbol{b}(\boldsymbol{ar})$; (5) $\boldsymbol{a}\times\boldsymbol{b}$.

3451. (1) 0; (2) $\dfrac{\sqrt{2}}{2}$; (3) $\sqrt{5}$; (4) $\dfrac{\cos\alpha + \sin\alpha}{2}$.

3452. $\dfrac{\sqrt{2}}{3}$. **3453.** $\dfrac{1}{2}$. **3455.** (1) 5; (2) $\dfrac{98}{13}$.

3456. 22. **3459.** $\dfrac{1}{r^2}$.

Chapter XII

3460. $M = \iint\limits_D \gamma(x, y)\,\mathrm{d}\sigma$. **3461.** $E = \iint\limits_D \sigma(x, y)\,\mathrm{d}\sigma$.

3462. $T = \dfrac{1}{2}\omega^2 \iint\limits_D y^2\gamma(x, y)\,\mathrm{d}\sigma$.

3463. $Q = (t_2 - t_1)\iint\limits_D c(x, y)\,\gamma(x, y)\,\mathrm{d}\sigma$.

3464. $M = \iiint\limits_\Omega \gamma(x, y, z)\,\mathrm{d}v$. **3465.** $E = \iiint\limits_\Omega \delta(x, y, z)\,\mathrm{d}v$.

3466. $8\pi(5 - \sqrt{2}) < I < 8\pi(5 + \sqrt{2})$. **3467.** $36\pi < I < 100\pi$.

3468. $2 < I < 8$. **3469.** $-8 < I < \frac{2}{3}$. **3470.** $0 < I < 64$.

3471. $4 < I < 36$. **3472.** $4 < I < 8(5 - 2\sqrt{2})$. **3473.** $4\pi < I < 22\pi$.

3474. $0 < I < \frac{4}{3}\pi R^5$. **3475.** $24 < I < 72$.

3476. $28\pi\sqrt{3} < I < 52\pi\sqrt{3}$. **3477.** 1. **3478.** $(e - 1)^2$. **3479.** $\dfrac{\pi}{12}$.

3480. $\ln\frac{4}{3}$. **3481.** $\ln\frac{2+\sqrt{2}}{1+\sqrt{3}}$. **3482.** $\pi-2$. **3483.** 2. **3484.** $-\frac{\pi}{16}$.

3485. $\int\limits_{3}^{5} dx \int\limits_{\frac{3x+1}{2}}^{\frac{3x+4}{2}} f(x, y)\, dy$. **3486.** $\int\limits_{0}^{2} dx \int\limits_{0}^{2-x} f(x, y)\, dy$.

3487. $\int\limits_{0}^{1} dx \int\limits_{0}^{\sqrt{1-x^2}} f(x, y)\, dy$. **3488.** $\int\limits_{0}^{1} dx \int\limits_{x-1}^{1-x} f(x, y)\, dy$.

3489. $\int\limits_{-\sqrt{2}}^{\sqrt{2}} dx \int\limits_{x^2}^{4-x^2} f(x, y)\, dy$. **3490.** $\int\limits_{-2}^{2} dx \int\limits_{-\frac{3}{2}\sqrt{4-x^2}}^{\frac{3}{2}\sqrt{4-x^2}} f(x, y)\, dy$.

3491. $\int\limits_{0}^{4} dx \int\limits_{3-\sqrt{4x-x^2}}^{3+\sqrt{4x-x^2}} f(x, y)\, dy$. **3492.** $\int\limits_{0}^{1} dx \int\limits_{x^2}^{\sqrt{x}} f(x, y)\, dy$.

3493. $\int\limits_{x}^{2} dx \int\limits_{x}^{2x} f(x, y)\, dy + \int\limits_{2}^{3} dx \int\limits_{x}^{6-x} f(x, y)\, dy$.

3494. $\int\limits_{-\frac{2}{3}}^{\frac{1}{3}} dx \int\limits_{1-2x}^{x+3} f(x, y)\, dy + \int\limits_{\frac{1}{3}}^{\frac{2}{3}} dx \int\limits_{x}^{x+3} f(x, y)\, dy + \int\limits_{\frac{2}{3}}^{\frac{5}{3}} dx \int\limits_{x}^{5-2x} f(x, y)\, dy$.

3495. $\int\limits_{0}^{1} dx \int\limits_{\frac{x}{2}}^{2x} f(x, y)\, dy + \int\limits_{1}^{2} dx \int\limits_{\frac{x}{2}}^{\frac{2}{x}} f(x, y)\, dy$.

3496. $\int\limits_{0}^{2} dx \int\limits_{-2\sqrt{2x}}^{2x} f(x, y)\, dy + \int\limits_{2}^{\frac{9}{2}} dx \int\limits_{-2\sqrt{2x}}^{2\sqrt{2x}} f(x, y)\, dy +$

$+\int\limits_{\frac{9}{2}}^{8} dx \int\limits_{-2\sqrt{2x}}^{24-4x} f(x, y)\, dy$.

3497. $\int\limits_{-3}^{-2} dx \int\limits_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\, dx + \int\limits_{-2}^{2} dx \int\limits_{-\sqrt{1+x^2}}^{\sqrt{1+x^2}} f(x, y)\, dy +$

$+\int\limits_{2}^{3} dx \int\limits_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\, dy$.

3498. $\int_0^1 dx \int_{x^2}^{x} f(x, y)\, dy$. **3499.** $\int_0^1 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y)\, dx$.

3500. $\int_0^r dy \int_{-r\sqrt{r^2-y^2}}^{y} f(x, y)\, dx$. **3501.** $\int_{-\sqrt{2}}^{\sqrt{2}} dy \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} f(x, y)\, dx$.

3502. $\int_1^2 dy \int_1^y f(x, y)\, dx + \int_2^4 dy \int_{\frac{y}{2}}^{2} f(x, y)\, dx$.

3503. $\int_0^4 dy \int_0^{\frac{y}{2}} f(x, y)\, dx + \int_4^6 dy \int_0^{6-y} f(x, y)\, dx$.

3504. (1) $\int_0^1 dy \int_y^{2-y} f(x, y)\, dx$; (2) $\int_0^1 dy \int_{\sqrt{y}}^{3-2y} f(x, y)\, dx$;

(3) $\int_0^1 dy \int_{y^{\frac{3}{2}}}^{2-\sqrt{2y-y^2}} f(x, y)\, dx$.

3505. (1) $\int_0^2 dy \int_{\frac{y}{2}}^{2y} f(x, y)\, dx + \int_2^4 dy \int_{2y-3}^{\frac{y+6}{2}} f(x, y)\, dx$;

(2) $\int_1^3 dy \int_{\frac{y+1}{2}}^{\frac{9-y}{2}} f(x, y)\, dx$; (3) $\int_{-1}^3 dx \int_0^{1+\sqrt{3+2x-x^2}} f(x, y)\, dy$;

(4) $\int_0^1 dy \int_{1-\sqrt{1-y^2}}^{3+\sqrt{1-y^2}} f(x, y)\, dx + \int_1^2 dy \int_{2-\sqrt{2y-y^2}}^{2+\sqrt{2y-y^2}} f(x, y)\, dx$.

3506. (1) $\frac{2}{3} a^{\frac{3}{2}}$; (2) 9; (3) $\frac{1}{2}$. **3507.** 0. **3508.** $\frac{33}{140}$. **3509.** $\frac{9}{4}$.

3510. -2. **3511.** $\frac{\pi}{6}$. **3512.** $\frac{4}{135}$. **3513.** 4. **3514.** 3. **3515.** $12\frac{2}{3}$.

3516. $\frac{2}{3}R$. **3517.** 6. **3518.** $\frac{abc(a+b+c)}{2}$. **3519.** $\frac{a^6}{48}$. **3520.** $\frac{a^{11}}{110}$.

3521. $2e - 5$. **3522.** $\frac{1}{2}\left(\ln 2 - \frac{5}{8}\right)$. **3523.** $\frac{1}{180}$. **3524.** $\frac{\pi^2}{16} - \frac{1}{2}$.

3525. (1) $\int\limits_0^{2\pi} d\varphi \int\limits_0^R f(\varrho\cos\varphi, \varrho\sin\varphi)\, \varrho\, d\varrho$;

(2) $\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int\limits_0^{a\cos\varphi} f(\varrho\cos\varphi, \varrho\sin\varphi)\, \varrho\, d\varrho$;

(3) $\int\limits_0^{\pi} d\varphi \int\limits_0^{b\sin\varphi} f(\varrho\cos\varphi, \varrho\sin\varphi)\, \varrho\, d\varrho$.

3526. $\int\limits_{\frac{\pi}{4}}^{\arctan 2} d\varphi \int\limits_{4\cos\varphi}^{8\cos\varphi} f(\varrho\cos\varphi, \varrho\sin\varphi)\, \varrho\, d\varrho$.

3527. $\int\limits_0^{\arctan\frac{a}{b}} d\varphi \int\limits_0^{b\sin\varphi} f(\varrho\cos\varphi, \varrho\sin\varphi)\, \varrho\, d\varphi +$

$+ \int\limits_{\arctan\frac{a}{b}}^{\frac{\pi}{2}} d\varphi \int\limits_0^{a\cos\varphi} f(\varrho\cos\varphi, \varrho\sin\varphi)\, \varrho\, d\varrho$.

3528. $\int\limits_0^{\frac{\pi}{4}} d\varphi \int\limits_0^{\sec\varphi} f(\varrho\cos\varphi, \varrho\sin\varphi)\, \varrho\, d\varphi$.

3529. $\int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_{\sqrt{2}\sec\left(\varphi-\frac{\pi}{4}\right)}^{2} f(\varrho\cos\varphi, \varrho\sin\varphi)\, \varrho\, d\varphi$.

3530. $\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int\limits_0^{a\sqrt{\cos 2\varphi}} f(\varrho\cos\varphi, \varrho\sin\varphi)\, \varrho\, d\varrho$.

3531. $\int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^{a\sin 2\varphi} f(\varrho\cos\varphi, \varrho\sin\varphi)\, \varrho\, d\varrho$.

3532. $\int\limits_0^{\frac{\pi}{2}} d\varphi \int\limits_0^{R} f(\varrho\cos\varphi, \varrho\sin\varphi)\, \varrho\, d\varrho$.

3533. $\int\limits_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\varphi \int\limits_{\frac{R}{2\sin\varphi}}^{2R\sin\varphi} f(\varrho\cos\varphi, \varrho\sin\varphi)\, \varrho\, d\varrho$.

3534. $\frac{\pi}{2}\int_0^R f(\varrho^2)\,\varrho\,d\varrho$. **3535.** $\frac{R^2}{2}\int_0^{\arctan R} f(\tan\varphi)\,d\varphi$.

3536. $\frac{\pi}{4}[(1+R^2)\ln(1+R^2)-R^2]$. **3537.** $\frac{\pi(\pi-2)}{8}$.

3538. $\pi R^2 h$. **3539.** $\frac{R^3}{3}\left(\pi-\frac{4}{3}\right)$. **3540.** $\frac{\pi^2}{6}$.

3542. $x = 2\varrho\cos\varphi,\quad y = 3\varrho\sin\varphi;$

$$I = 6\int_0^{2\pi} d\varphi\int_0^1 f(2\varrho\cos\varphi, 3\varrho\sin\varphi)\,\varrho\,d\varrho.$$

3543. $x = \varrho\cos\varphi,\quad y = \sqrt{3}\varrho\sin\varphi;$

$$I = \sqrt{3}\int_0^{\pi} d\varphi\int_0^{\sqrt{3}\cos^2\varphi\sin\varphi} f(\varrho\cos\varphi,\ \sqrt{3}\varrho\sin\varphi)\,\varrho\,d\varrho.$$

3544. $x = a\varrho\cos\varphi,\ y = b\varrho\sin\varphi;\ I = ab\int_0^{\frac{\pi}{2}} d\varphi\int_1^2 f\sqrt{(4-\varrho^2)}\,\varrho\,d\varrho.$

3545. $\frac{a^2b^2}{8}$. **3546.** $\frac{1}{\sqrt[4]{6}}$.

3547. $\int_0^1 dz\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi\int_0^R f(\varrho\cos\varphi, \varrho\sin\varphi, z)\,\varrho\,d\varrho.$

3548. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\int_0^{2\cos\varphi}\varrho\,d\varrho\int_0^{\varrho^2} f(\varrho\cos\varphi,\ \varrho\sin\varphi, z)\,dz.$

3549. $\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\int_0^{\frac{\pi}{2}} d\varphi\int_0^R f(\varrho\cos\varphi\sin\theta,\ \varrho\sin\varphi\sin\theta,\ \varrho\cos\theta)\,\varrho^2\,d\varrho.$

3550. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi\int_0^{R\sqrt{\cos 2\varphi}}\varrho\,d\varrho\int_{-\sqrt{R^2-\varrho^2}}^{\sqrt{R^2-\varrho^2}} f(\varrho\cos\varphi, \varrho\sin\varphi, z)\,dz.$

3551. $\int_0^{2\pi} d\varphi\int_0^{\frac{R\sqrt{3}}{2}}\varrho\,d\varrho\int_{R-\sqrt{R^2-\varrho^2}}^{\sqrt{R^2-\varrho^2}} f(\varrho\cos\varphi, \varrho\sin\varphi, z)\,dz$ or

$$\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{3}} \sin\theta\, d\theta \int_0^{R} f(\varrho\cos\varphi\sin\theta,\ \varrho\sin\varphi\sin\theta,\ \varrho\cos\theta)\,\varrho^2\, d\varrho +$$

$$+\int_0^{2\pi} d\varphi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{2R\cos\theta} f(\varrho\cos\varphi\sin\theta,\ \varrho\sin\varphi\sin\theta,\ \varrho\cos\theta)\,\varrho^2\, d\varrho.$$

3552. $\frac{\pi a}{2}$. **3553.** $\frac{8}{9}a^2$. **3554.** $\frac{4}{15}\pi R^5$. **3555.** $\frac{\pi}{8}$.

3556. $\frac{4}{15}\pi(R^5 - r^5)$. **3557.** $\frac{2\pi}{3}$.

3558. $\pi\left[3\sqrt{10} + \ln\frac{\sqrt{2}-1}{\sqrt{10}-3} - \sqrt{2} - 8\right]$.

3559. $186\frac{2}{3}$. **3560.** $\frac{ab}{6}\left(\frac{a^2}{p} + \frac{b^2}{q}\right)$. **3561.** $\frac{abc}{6}$. **3562.** 12.

3563. $\frac{1}{6}$. **3564.** $78\frac{15}{32}$. **3565.** $\frac{48}{5}\sqrt{6}$. **3566.** 16. **3567.** 45.

3568. $13\frac{1}{3}$. **3569.** $16\frac{1}{5}$. **3570.** $ar^2\left(\frac{\pi}{4} - \frac{1}{3}\right)$. **3571.** 22π.

3572. $\frac{16}{3}R^3$. **3573.** $12\frac{4}{21}$. **3574.** $\frac{4R^5}{15a^2}$. **3575.** 27. **3576.** $\frac{3}{8}$.

3577. $\frac{88}{105}$. **3578.** $\frac{1}{3}abc$. **3579.** $\frac{\pi a^3}{4}$. **3580.** $2\left(e^2 - \frac{2e^3+1}{9}\right)$.

3581. $3e - 8$. **3582*.** $4e - e^2 - 1$. The solid is symmetrical with respect to the plane $y = x$.

3583. $2\left(\pi^2 - \frac{35}{9}\right)$. **3584.** $\frac{1}{45}$. **3585.** $\frac{16}{9}$. **3586.** $\frac{\pi}{4}$.

3587. 40π. **3588.** 2π. **3589.** $\frac{5}{2}\pi R^3$. **3590.** $\frac{3}{2}\pi a^3$.

3591. $\frac{4}{3}a^3\left(\frac{\pi}{2} - \frac{2}{3}\right)$. **3592.** $\frac{a^3}{24}$. **3593.** $\frac{15}{8}\left(\frac{3\pi}{8} + 1\right)$.

3594. $\frac{3}{2}\left(\frac{\pi}{2} - 1\right)$. **3595.** $\frac{\pi\sqrt{2}}{24}$. **3596.** $\frac{\pi^2R^2h}{16}$. **3597.** $\frac{1}{2}$.

3598. 2. **3599.** πab. **3600.** $\frac{ab}{6}$. **3601.** $\frac{16}{3}$. **3602*.** $\frac{5}{8}\pi a^2$. Transform to polar coordinates. **3603.** $\frac{3}{4}\pi$. **3604.** $2a^2$. **3605.** $\frac{2}{3}$. **3606.** $\frac{1}{60}$.

3607. $\frac{1}{1260}$. **3608*.** (1) $\frac{a^2b^2}{2c^2}$, (2) $\frac{39}{25}\pi$. Use the result of problem

3541. **3609.** 8. **3610.** $\frac{7}{12}$. **3611.** $\frac{3}{35}$. **3612.** $4(4 - 3\ln 3)$.

3613*. $\frac{\pi}{2}$. The projection of the solid on xOy is a circle.

3614. $\frac{\pi}{8}$. Transfer the origin to the point $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

3615*. $\frac{19}{6}\pi$ and $\frac{15}{2}\pi$. Transform to cylindrical coordinates.

3616. $\frac{5}{12}\pi R^3$. **3617.** $\frac{\pi}{96}$. **3618.** $\frac{92}{75}\pi R^3$.

3619*. $\frac{1}{3}\pi a^3$. Transform to spherical coordinates.

3620. $\frac{a^3}{360}$. **3621.** $\frac{4}{21}\pi a^3$. **3622.** $\frac{4}{3}\pi a^3$. **3623.** $\frac{64}{105}\pi a^3$.

3624. $\frac{\pi^2 a^3}{6}$. **3625.** $\frac{21(2-\sqrt{2})}{4}\pi$. **3626.** 14. **3627.** 36. **3628.** 8π.

3629. $2\sqrt{2}\,\pi p^2$. **3630*.** $2\pi R^2$. Project the surface on to yOx.

3631. $8\sqrt{2}ab$. **3632.** $\frac{16}{3}(\sqrt{8}-1)$. **3633.** $\frac{2\pi}{3}\left\{(1+R^2)^{\frac{3}{2}}-1\right\}$.

3634. $\frac{2\pi}{3}(\sqrt{8}-1)$. **3635.** $4\pi a(a-\sqrt{a^2-R^2})$.

3636. $2R^2(\pi-2)$. **3637.** $2R^2(\pi+4-4\sqrt{2})$.

3638. $\frac{\pi}{4}\left\{3\sqrt{2}-\sqrt{3}-\frac{\sqrt{2}}{2}\ln 2+\sqrt{2}\ln(\sqrt{3}+\sqrt{2})\right\}$. **3639.** $\frac{2a^2}{\sin 2\alpha}$.

3640*. $\frac{\pi R^2}{12}(\sqrt{3}-\sqrt{2}) \approx 3{\cdot}42\times 10^8$ km^2. Transform to spherical coordinates.

3641. $\frac{16}{3}\pi a^2$. **3642.** $8R^2$. **3643.** $\frac{ab^2}{2}$. **3644.** $\frac{2}{3}R^3$. **3645.** πR^3.

3646. $\frac{9}{4}a^3$. **3647.** The statical moment is equal to $\frac{ah^2}{6}$.

3648. The centre of gravity lies on the minor axis at a distance $\frac{4b}{3\pi}$ from the major axis (b is the minor semi-axis).

3649. $\xi = \left(1-\frac{\pi}{4}\right)(\sqrt{2}+1)$, $\eta = \frac{1}{8}\left(\frac{\pi}{2}-1\right)(\sqrt{2}+2)$.

3650. The centre of gravity lies on the bisector of the angle α at a distance $\frac{4}{3}R\frac{\sin\frac{\alpha}{2}}{\alpha}$ from the centre of the circle.

3651. The centre of gravity lies on the bisector of the angle α at a distance $\frac{4}{3}R\frac{\sin^3\frac{\alpha}{2}}{\alpha-\sin\alpha}$ from the centre of the circle.

3652. $\xi=\frac{3\pi}{16}$, $\eta=0$. **3653.** $\frac{5}{4}\pi R^4$. **3654.** $\frac{2}{3}a^4$.

3655. $\frac{\pi ab}{4}(a^2+b^2)$. **3656.** $\frac{ab(a^2+b^2)}{12}$. **3657.** $\frac{ah}{48}(a^2+12h^2)$.

3658. $\frac{3\pi R^4}{2}$. **3659.** $ah\left(\frac{2h^2}{7}+\frac{a^2}{30}\right)$.

3662*. Choose the system of coordinates so that the origin coincides with the centre of gravity of the figure and one of the coordinate axes is parallel to the axis with respect to which the moment of inertia is being sought.

3663. $\frac{a^2bc}{2}$, $\frac{ab^2c}{2}$ and $\frac{abc^2}{2}$. **3664.** $\frac{\pi R^2H^2}{4}$. **3665.** $\frac{\pi abc^2}{4}$.

3666. $\xi=\frac{14}{15}$, $\eta=\frac{26}{15}$, $\zeta=\frac{8}{3}$. **3667.** $\xi=\frac{3}{8}a$, $\eta=\frac{3}{8}b$, $\zeta=\frac{3}{8}c$.

3668. $\xi=\frac{6}{5}$, $\eta=\frac{12}{5}$, $\zeta=\frac{8}{5}$. **3669.** $\xi=\frac{18}{7}$, $\eta=\frac{15}{16}\sqrt{6}$, $\zeta=\frac{12}{7}$.

3670. $\xi=0$, $\eta=0$, $\zeta=\frac{5a}{83}(6\sqrt{3}+5)$.

3671. $\xi=\frac{3R}{8}(1+\cos\alpha)$, $\eta=0$, $\zeta=0$.

3672. $\xi=0$, $\eta=0$, $\zeta=\frac{9a}{20}$. **3673.** $\xi=\frac{R}{2}$, $\eta=\frac{R}{2}$, $\zeta=\frac{R}{2}$.

3674. $\xi=0$, $\eta=0$, $\zeta=\frac{55+9\sqrt{3}}{130}$.

3675. $\frac{1}{3}M(b^2+c^2)$, $\frac{1}{3}M(c^2+a^2)$, $\frac{1}{3}M(a^2+b^2)$ and $\frac{1}{12}M(a^2+b^2+c^2)$.

3676. $\frac{7}{5}MR^2$. **3677.** $\frac{1}{5}M(b^2+c^2)$, $\frac{1}{5}M(c^2+a^2)$, $\frac{1}{5}M(a^2+b^2)$.

3678. $M\left(\frac{R^2}{4}+\frac{H^2}{3}\right)$ and $\frac{M}{12}(H^2+3R^2)$. **3679.** $\frac{2}{5}M\frac{R^5-r^5}{R^3-r^3}$.

3680. $\frac{1}{36}\pi R^2H(3R^2+H^2)$. **3681.** $\frac{1}{2}M\left(R^2+\frac{1}{6}H^2\right)$.

3682. $\frac{55+9\sqrt{3}}{65}Mc^2$.

3683. $\frac{1}{2}\pi l\frac{R^4-r^4}{R-r}$, where l is the generator of the cone.

3684. $\frac{4}{3}a^2$. **3685.** $2\pi r(R-r)$. **3686.** $\frac{4}{3}\gamma ab^2$.

3687. $2\pi\gamma(R^2-r^2)$. **3688.** $\frac{\pi R^2H}{6}(3R^2+2H^2)$.

3689*. $\frac{\pi\gamma h^{n+2}\tan^2\alpha}{n+3}$. If the axis of the cone is taken as Oz, and its vertex as the origin, the equation of the cone becomes $x^2+y^2-z^2\tan^2\alpha=0$.

3690. $\frac{2}{3}\pi\gamma R^6$. **3691.** $\frac{\pi a^5}{5}\left(18\sqrt{3}-\frac{97}{6}\right)$.

3692. $\xi=0$, $\eta=0$, $\zeta=\frac{5}{4}R$. Transform to cylindrical coordinates.

3693*. $\frac{59}{480}\pi R^5$. See hint on previous problem.

3694*. Choose the system of coordinates so that the origin coincides with the centre of gravity of the body and one of the coordinate axes is parallel to the axis with respect to which the moment of inertia is being sought.

3695. $\frac{kMm}{a^2}$, where M is the mass of the sphere, and k is the gravitational constant.

3696*. Use the result of the previous problem.

3697. $\frac{17}{56}\frac{kM}{R^2}$, k is the gravitational constant.

3699. The centre of pressure lies on the axis of symmetry of the rectangle perpendicular to side a, at a distance $\frac{2}{3}b$ from the side lying on the surface. In the second case (side a situated at a depth h) the distance of the centre of pressure from the upper side will be

$\frac{2b}{3}\frac{b+\frac{3}{2}l}{b+2l}$, where $l = \frac{h}{\sin\alpha}$. (With $l \gg b$ the centre of pressure almost coincides with the centre of the rectangle).

3700. (a) $\frac{h}{2}\sin\alpha$; (b) $\frac{3}{4}h\sin\alpha$.

3701. The centre of pressure lies on the major axis of the ellipse, at a distance $a + \frac{a^2}{4(a+h)}$ from its upper end.

3702*. Choose the system of coordinates so that one of the coordinate planes coincides with the plane of the lamina and one of the axes coincides with the line of intersection of the fluid surface and the plane of the lamina.

3703. Divergent. **3704.** 2π. **3705.** $\frac{\pi}{4a^2}$. **3706.** 4. **3707.** 2. **3708.** $\frac{1}{4}$.

3709*. $\frac{\alpha}{2\sin\alpha}$. Transform to polar coordinates.

3710*. $\frac{1}{2}$. Change the order of integration.

3711. $\frac{1}{16}$. See the hint on the previous problem.

3712. Convergent. **3713.** Divergent. **3714.** Convergent.

3715. Divergent. **3716.** No. **3717.** $\frac{8}{15}$. **3718.** $\frac{\pi}{16}$.

3719*. $\pi\sqrt{\pi}$; use Poisson's integral $\int\limits_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

3720. Divergent. **3721.** Convergent. **3722.** Divergent.

3723. $\frac{8}{3}\pi R^3\left(\ln R - \frac{1}{3}\right)$. **3724*.** π. (See hint on problem 3719).

3725. $\frac{\pi}{4}$. **3726.** $\frac{\sqrt{\pi}}{2}$. **3727.** $2\pi km\,\gamma(R + H - \sqrt{R^2+H^2})$. The force is directed along the axis of the cylinder.

3728. $\frac{2\pi km\,\gamma H}{l}(l - H)$, where l is the generator of the cone. The force is directed along the axis of the cone.

3729* (a) $a = 4\sigma_c - 3\sigma_0$, $b = \frac{4}{R}(\sigma_c - \sigma_0)$; (b) $\frac{4}{3}\pi k R\sigma_c = \frac{kM}{R^2}$.

3730. Defined everywhere except at $x = 0$. **3731.** 3π.

3733. $\dfrac{b}{8a^4}\left\{\dfrac{5a^2 + 3b^2}{(a^2 + b^2)^2} + \dfrac{3}{ab}\arctan\dfrac{b}{a}\right\}$.

3734. $\dfrac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}\,\dfrac{\pi}{2a^{2n-1}}$ $(n > 1)$.

3735. $\dfrac{(n-1)!}{a^n}$. **3736*.** $\dfrac{\pi(a^2 + b^2)}{4\,|ab|^3}$. Differentiate with respect to a and b and add the results. **3737.** $\ln(1 + a)$. **3738.** $\dfrac{1}{2}\ln(1 + a)$.

3739. $\dfrac{\pi}{2}\ln(a + \sqrt{1 + a^2})$. **3740.** $\pi(\sqrt{1 - a^2} - 1)$.

3741. $\dfrac{\pi}{2}\ln(1 + a)$, if $a \geqq 0$; $-\dfrac{\pi}{2}\ln(1 - a)$ if $a \leqq 0$.

3742. $\pi\ln\dfrac{1 + \sqrt{1 - a^2}}{2}$. **3743.** $\pi\arcsin a$. **3744.** $\pi\arcsin a$.

3745. $\sqrt{\pi a}$. **3746*.** $\sqrt{\pi}\,(\sqrt{b} - \sqrt{a})$. Differentiate with respect to a or with respect to b.

3747*. $\arctan\dfrac{b}{a} - \arctan\dfrac{c}{a} = \arctan\dfrac{a(b - c)}{a^2 + bc}$. Differentiate with respect to b or with respect to c.

3748. $\dfrac{1}{2}\ln\dfrac{a^2 + b^2}{a^2 + c^2}$. **3749*.** $\pi\ln\dfrac{a + b}{2}$. Differentiate with respect to a or with respect to b.

3750. $\dfrac{\pi}{2}\ln(1 + a)$ if $a > 0$; $-\dfrac{\pi}{2}\ln(1 - a)$ if $a < 0$;

$\displaystyle\int_0^{\frac{\pi}{2}}\frac{x}{\tan x}\,dx = \frac{\pi}{2}\ln 2$. **3751*.** $\ln\dfrac{1 + \beta}{1 + \alpha}$. Integrate with respect to parameter n between the limits α and β. **3752.** $\sqrt{\pi}(b - a)$.

3753. $\displaystyle\int_0^\infty\frac{\cos x\,dx}{\sqrt{x}} = \int_0^\infty\frac{\sin x\,dx}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}$. **3755.** $\dfrac{\pi}{2}\ln\dfrac{a}{b}$.

3756. $\dfrac{1}{n}\ln\dfrac{b}{a}$. **3757*.** $\displaystyle I = \lim_{\varepsilon\to 0}\left[\int_\varepsilon^\infty\frac{f(ax) - f(bx)}{x}\,dx\right] =$

$$= \lim_{\varepsilon\to 0}\left[\int_\varepsilon^\infty\frac{f(ax)}{x}\,dx - \int_\varepsilon^\infty\frac{f(bx)}{x}\,dx\right] = \lim_{\varepsilon\to 0}\int_{a\varepsilon}^{b\varepsilon}\frac{f(x)}{x}\,dx.$$

We write inequalities for the last integral by replacing $f(x)$ by its greatest and least values in the interval $(a\varepsilon, b\varepsilon)$, and pass to the limit.

3758. $\ln\frac{b}{a}$. **3759.** $\ln\frac{b}{a}$. **3760.** $\frac{1}{2}\ln\left|\frac{a+b}{a-b}\right|$. **3761.** $ab\ln\frac{b}{a}$.

3762*. $\frac{3}{4}\ln 3$. We write $\sin^3 x$ as a difference of sines of multiple angles and reduce the problem to the previous one (with suitable choice of a and b).

3763*. Two methods can be used for the proof: (1) integration by parts; (2) change of the order of integration in the double integral, obtained after substituting an integral for $\Phi(az)$.

3764*. See hint on problem 3763.

3765*. Use the second method of solution of problem 3763. When proving the second relationship it is necessary to investigate the integral

$$\int_0^\infty \frac{\sin ax \cos (x \sin\theta)}{x}\,dx$$

with $|a| > 1$ and $|a| \leqq 1$. This is done by transforming the expression in the numerator, and recalling that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ (Dirichlet's integral).

3767*. Substitute in the left-hand side of the required equation the expressions for y' and y'' obtained by differentiation of the integral y with respect to the parameter. One of the terms obtained must be integrated by parts.

3768*. See hint on problem 3767. **3769*.** See hint on problem 3767.

Chapter XIII

3770. $\sqrt{5}\ln 2$. **3771.** 24. **3772.** $\frac{p^2}{3}(5\sqrt{5}-1)$. **3773.** $2\pi a^{2n+1}$.

3774. $\frac{ab(a^2+ab+b^2)}{3(a+b)}$. **3775.** $4\pi a\sqrt{a}$.

3776. $\int_{\varphi_1}^{\varphi_2} F(\varrho\cos\varphi, \varrho\sin\varphi)\sqrt{\varrho^2+\varrho'^2}\,d\varphi$.

3777*. $\frac{\pi a^2}{2}$. Transform to polar coordinates.

3778. $\dfrac{2a^3\sqrt{2}}{3}$. **3779.** $\dfrac{1}{12}\left[(R^2+4)^{\frac{3}{2}}-8\right]$. **3780.** $\dfrac{8a\pi^3\sqrt{2}}{3}$.

3781. $\dfrac{R^4\sqrt{3}}{32}$. **3782.** $\dfrac{2\sqrt{2}}{3}\left[(1+2\pi^2)^{\frac{3}{2}}-1\right]$. **3783.** $R^2\sqrt{2}$.

3784. $\dfrac{1}{3}\left\{(x_2^2+1)^{\frac{3}{2}}-(x_1^2+1)^{\frac{3}{2}}\right\}$. **3785.** δa.

3786. $\dfrac{b^2}{2}+\dfrac{ab}{2\varepsilon}\arcsin\varepsilon$, where ε is the eccentricity of the ellipse.

3787. $\left(2\pi a^2+\dfrac{8\pi^3b^2}{3}\right)\sqrt{a^2+b^2}$. **3788.** $(1-e^{-t})\sqrt{3}$.

3789. $\left(0, \dfrac{2a}{\pi}, \dfrac{b\pi}{2}\right)$. **3790.** $\dfrac{8k\sqrt{2}}{15}\left[(3\pi^2-1)(2\pi^2+1)^{\frac{3}{2}}+1\right]$.

3791. $I_x=I_y=\left(\dfrac{a^2}{2}+\dfrac{h^2}{3}\right)\sqrt{4\pi^2a^2+h^2}$, $\quad I_z=a^2\sqrt{4\pi^2a^2+h^2}$.

3792. $3\pi R^2$. **3793.** $\dfrac{\pi p^2}{4}$. **3794.** $\dfrac{11}{3}$. **3795.** R^2.

3796. $ka\left(a+\dfrac{b^2}{2c}\ln\dfrac{a+c}{a-c}\right)$, where $c=\sqrt{a^2-b^2}$. When $a=b$, $S=2ka^2$.

3797. $\dfrac{98}{81}p^2$. **3798.** $8R^2$. **3799.** $4R^2$. **3800.** $\dfrac{2Im}{a}$. **3801.** $\dfrac{8mI\sqrt{2}}{a}$.

3803. $\dfrac{2\pi mIa}{b^2}$, where a and b are the semi-axes of the ellipse.

3804. $\dfrac{2\pi mI}{p}$. **3805.** $\dfrac{2\pi mIR^2}{(h^2+R^2)^{\frac{3}{2}}}$. For $R=h\sqrt{2}$. **3806.** 3.

3807. $\dfrac{ab}{2}$. **3808.** $-\dfrac{56}{15}$. **3809.** $37\dfrac{1}{3}$. **3810.** 4π.

3811. (1) $\dfrac{1}{3}$; (2) $\dfrac{1}{12}$; (3) $\dfrac{17}{30}$; (4) $-\dfrac{1}{20}$.

3812. The integral is equal to 1 in all four cases.

3813. 0. **3814.** $-2\pi ab$. **3815.** $-\dfrac{4}{3}a$. **3816.** πa^2.

3817. $\dfrac{3}{16}\pi R\sqrt[3]{R}$. **3818.** 13. **3819.** 0. **3820.** $3\sqrt{3}$. **3821.** $-\dfrac{\pi R^3}{4}$.

3822. $\iint\limits_D (x^2+y^2)\,dx\,dy$. **3823.** $\iint\limits_D (y-x)\,e^{xy}\,dx\,dy$.

3824. $\frac{\pi R^4}{2}$. **3825.** (1) 0; (2) $-\frac{\pi a^3}{8}$. **3827.** $\frac{1}{3}$.

3836*. Apply Green's formula to the doubly-connected domain bounded by contour L and any circle with centre at the origin and not intersecting contour L.

3837. π. **3838.** 8. **3839.** 4. **3840.** $\ln\frac{13}{5}$. **3841.** $R_2 - R_1$.

3842. $\frac{10}{3}$. **3843.** 0. **3844.** $-\frac{9}{2}$. **3845.** $u = \frac{x^3 + y^3}{3} + C$.

3846. $u = (x^2 - y^2)^2 + C$. **3847.** $u = \ln|x + y| - \frac{y}{x + y} + C$.

3848. $u = \frac{\sqrt{x^2 + y^2} + 1}{y} + C$.

3849. $u = \ln|x - y| + \frac{y}{x - y} + \frac{x^2}{2} - \frac{y^3}{3} + C$.

3850. $u = x^2 \cos y + y^2 \cos x + C$. **3851.** $u = \frac{e^y - 1}{1 + x^2} + y + C$.

3852. $u = \frac{x - y}{(x + y)^2} + C$.

3853. $n = 1$, $u = \frac{1}{2}\ln(x^2 + y^2) + \arctan\frac{y}{x} + C$.

3854. $a = b = -1$, $u = \frac{x - y}{x^2 + y^2} + C$.

3855. $u = \ln|x + y + z| + C$. **3856.** $u = \sqrt{x^2 + y^2 + z^2} + C$.

3857. $u = \arctan xyz + C$. **3858.** $u = \frac{2x}{x - yz} + C$.

3859. $u = \frac{x - 3y}{z} + \frac{z^2}{2} + C$. **3860.** $u = e^{\frac{y}{z}}(x + 1) + e^{yz} - e^{-z}$.

3861. πab. **3862.** $\frac{3}{8}\pi a^2$. **3863.** $6\pi a^2$.

3864*. $\frac{3}{2}a^2$. Transform to the parametric form by putting $y = tx$.

3865. $\frac{1}{60}$. **3866.** $\frac{1}{210}$. **3867*.** $2a^2$. Put $y = x\tan t$.

3868*. $\frac{1}{30}$. Put $y = xt^2$. **3869.** mFR.

3870. (1) $\frac{4}{3}$; (2) $\frac{17}{12}$; (3) $\frac{3}{2}$ and 1. **3871.** (a) $\frac{a^2 - b^2}{2}$; (b) 0.

3872. 0.

3873. $\dfrac{k\sqrt{a^2+b^2+c^2}}{c}\ln 2$, where k is a coefficient of proportionality.

3874. $0{\cdot}5k\ln 2$, where k is a coefficient of proportionality.

3876. $4\sqrt{61}$. **3877.** $\dfrac{\sqrt{3}}{120}$. **3878.** $\dfrac{\pi R^3}{4}$. **3879.** 0. **3880.** πR^3.

3881. $\dfrac{2\pi R^6}{15}$. **3882.** $2\pi \arctan\dfrac{H}{R}$.

3883. $\dfrac{2\pi R}{c\,(n-2)}\left[\dfrac{1}{(c-R)^{n-2}}-\dfrac{1}{(c+R)^{n-2}}\right]$ for $n\neq 2$; $\dfrac{2\pi R}{c}\ln\dfrac{c+R}{c-R}$ for $n=2$.

3884. $\pi\left[R\sqrt{R^2+1}+\ln\left(R+\sqrt{R^2+1}\right)\right]$.

3885*. π^2R^3. Use spherical coordinates.

3886. $\dfrac{8}{3}\pi R^4$. **3887.** 3. **3888.** $\dfrac{2\pi R^7}{105}$. **3889.** $\dfrac{4}{3}\pi abc$. **3890.** 0.

3891. $\dfrac{1}{8}$. **3892.** $R^2H\left(\dfrac{2R}{3}+\dfrac{\pi H}{8}\right)$. **3893.** $\dfrac{\pi}{8}$.

3894. $2\displaystyle\iint_S (x-y)\,dx\,dy+(y-z)\,dy\,dz+(z-x)\,dx\,dz$.

3895. $-\dfrac{\pi R^6}{8}$. **3896.** $2\displaystyle\iiint_\Omega (x+y+z)\,dx\,dy\,dz$.

3897. $\displaystyle\iiint_\Omega \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$. **3898.** 0. **3899.** $\dfrac{12}{5}\pi R^5$.

Chapter XIV

3901. $1+y^2=C(1-x^2)$. **3902.** $x^2+y^2=\ln Cx^2$.

3903. $y=\sqrt[3]{C+3x-3x^2}$. **3904.** $y=C\sin x-a$.

3905. $Cx=\dfrac{y-1}{y}$. **3906.** $x\sqrt{1-y^2}+y\sqrt{1-x^2}=C$.

3907. $\sqrt{1-y^2}=\arcsin x+C$. **3908.** $e^t=C(1-e^{-s})$.

3909. $10^x+10^{-y}=C$. **3910.** $\ln\left|\tan\dfrac{y}{4}\right|=C-2\sin\dfrac{x}{2}$.

3911. $t = \frac{1}{a}\left(l + \frac{bl^{1-n}}{1-n}\right).$

3912. $t = \frac{v^2}{2\sqrt{k_1k_2}} \ln \frac{\sqrt{k_1}\,(1-x) + x\sqrt{k_2}}{\sqrt{k_1}\,(1-x) - x\sqrt{k_2}}.$

3913. $y = e^{\tan\frac{x}{2}}$. **3914.** $y = \frac{1+x}{1-x}$. **3915.** $\cos x = \sqrt{2}\cos y.$

3916. $y = \frac{b+x}{1+bx}$. **3917.** The hyperbola $xy = 6.$

3918. The tractrix $y = \sqrt{4-x^2} + 2\ln\left|\frac{2-\sqrt{4-x^2}}{x}\right|.$

3919. The parabolas $y^2 = Cx$. **3920.** $y^k = Cx$. **3921.** $y = e^{\frac{x-a}{a}}$.

3922. $(x - C)^2 + y^2 = a^2$. **3923.** $y = \frac{1}{k}\ln|C(k^2x^2 - 1)|.$

3924. $x = y^n$. **3925.** $\approx 269{\cdot}3$ cm/sec. **3927.** $0{\cdot}467$ km/hr; $85{\cdot}2$ m.

3928. $H = \left[\sqrt{h} - \frac{\sqrt{2g}}{4S}\,qT\right]^2$. **3929.** $\ln\left|\frac{\theta_0-\theta_1}{\theta-\theta_1}\right| = \frac{k_0}{2}(2t + \alpha t^2).$

3930*. If t is time, measured from midnight and expressed in hours, the differential equation of the problem becomes

$$\frac{dS}{S\sqrt{S}} = k\cos\frac{\pi(t-12)}{12}\,dt; \quad \text{whence } S = \frac{160{,}000}{\left[9 - \sin\frac{\pi(t-12)}{12}\right]^2}.$$

The function $S(t)$ is defined for $6 \leqq t \leqq 18.$

3931. $x + \cot\frac{x-y}{2} = C$. **3932.** $4y - 6x - 7 = Ce^{-2x}.$

3933. $x + C = 2u + \frac{2}{3}\ln|u-1| - \frac{8}{3}\ln(u+2)$, where $u = \sqrt{1 + x + y}.$

3934. $y - 2x = Cx^3(y + x)$. **3935.** $\arctan\frac{y}{x} = \ln C\sqrt{x^2+y^2}.$

3936. $\ln|y| + \frac{x}{y} = C$. **3937.** $x^2 + y^2 = Cy.$

3938. $y = \pm x\sqrt{2\ln|Cx|}$. **3939.** $x^2 = C^2 + 2Cy.$

3940. $e^{\frac{y}{x}} = Cy$. **3941.** $\ln|Cx| = -e^{-\frac{y}{x}}$. **3942.** $y = xe^{1+Cx}.$

3943. $(x+y)^2 = Cx^3 e^{-\frac{x}{x+y}}$. **3944.** $Cx = \varphi\left(\frac{y}{x}\right)$.

3945. $\sqrt{x^2+y^2} = e^{\frac{y}{x}\arctan\frac{y}{x}}$. **3946.** $y^3 = y^2 - x^2$.

3947. $y = -x$. **3948.** $y^2 = 5 \pm 2\sqrt{5}\cdot x$.

3949. If $\frac{y}{x} = u$, then $\ln|x| = \int \frac{du}{\varphi\left(\frac{1}{u}\right)}$; $\varphi\left(\frac{1}{u}\right) = -u^2$ or $\varphi\left(\frac{x}{y}\right) = -\frac{y^2}{x^2}$.

3950. $x = Ce^{\pm 2\sqrt{\frac{y}{x}}}$. **3951.** $x = y\ln|Cy|$. **3952.** $x^2 = 2Cy + C^2$.

3953*. A paraboloid of revolution. Let xOy be the meridian plane of the mirror surface; the required curve lies in this plane; the differential equation is obtained by equating the tangents of the angles of incidence and reflexion, expressed in terms of x, y, y'.

3954. $y = Ce^{-2x} + 2x - 1$. **3955.** $y = e^{-x^2}\left(C + \frac{x^2}{2}\right)$.

3956. $y = Cx^2 e^{\frac{1}{x}} + x^2$. **3957.** $y = (x + C)(1 + x^2)$.

3958. $y = Ce^{-x} + \frac{1}{2}(\cos x + \sin x)$.

3959. If $m \neq -a$, then $y = Ce^{-ax} + \frac{e^{mx}}{m+a}$; if $m = -a$, then $y = (C + x)e^{mx}$.

3960. $y^2 - 2x = Cy^3$. **3961.** $x = Ce^{2y} + \frac{1}{2}y^2 + \frac{1}{2}y + \frac{1}{4}$.

3962. $x = y\ln y + \frac{C}{y}$. **3963.** $y = e^x\left(\ln|x| + \frac{x^2}{2}\right) + Ce^x$.

3964. $y = Ce^{-\Phi(x)} + \Phi(x) - 1$. **3965.** $y = \frac{x}{\cos x}$.

3966. $y = \frac{e^x + ab - e^a}{x}$. **3967.** $y = \frac{x}{x+1}(x - 1 + \ln|x|)$.

3968. $x = -t\arctan t$. **3969.** (b) $\alpha + \beta = 1$.

3971. $y = Cx - x\ln|x| - 2$.

3972*. $y = Cx \pm \frac{a^2}{2x}$. The differential equation of the problem is $|xy - x^2y'| = a^2$.

3973*. $x = Cy \pm \frac{a^2}{y}$. The differential equation of the problem is $xy - y^2 \frac{dx}{dy}\Big| = 2a^2$.

3974. $v = \frac{k_1}{k}\left(t - \frac{m}{k} + \frac{m}{k}e^{-\frac{kt}{m}}\right)$.

3975. $v = (v_0 + b)\,e^{-at^2} + b(at^2 - 1)$, where $a = \frac{k_1}{2m}$, $b = \frac{2km}{k_1^2}$.

3976. $\theta - \theta_0 = e^{-kt}\int_0^t \varphi(t)\,e^{kt}\,dt$. **3977.** $9{\cdot}03a$.

3978. $I = \frac{E_0}{R^2 + \omega^2L^2}\cdot\left[\omega L e^{-\frac{Rt}{L}} + R\sin\omega t - \omega L\cos\omega t\right]$.

3979. $x = Ce^{\arctan\frac{y}{x}}$. **3980.** $y = Cx^2 + \frac{1}{x}$.

3981. $y = \frac{C}{x}\sqrt{x^2 + 1} + \frac{(1 + x^2)^2}{3x}$. **3982.** $y = Cx - 1$.

3983. $(1 + x^2)(1 + y^2) = Cx^2$. **3984.** $(x + y)^2(2x + y)^3 = C$.

3985. $x = Ce^{-\frac{x^3}{2y^3}}$. **3986.** $\sin\frac{y}{x} = Cx$.

3987. $\sin\frac{y}{x} + \ln|x| = C$. **3988.** $y = Ce^{-e^x} + e^x - 1$.

3989. $y(y - 2x)^3 = C(y - x)^2$. **3990.** $x = Ce^{\sin y} - 2(1 + \sin y)$.

3991. $x = y^2\left(1 + Ce^{\frac{1}{y}}\right)$. **3992.** $y = Ce^{-\sin x} + \sin x - 1$.

3993. $y = (C + e^x)(1 + x)^n$. **3994.** $y^4 = 4xy + C$.

3995. $y = Ce^x$ and $y = C + \frac{x^2}{2}$. **3996*.** $y^2 = \frac{2}{3}\sin x + \frac{C}{\sin^2 x}$. Reduce to an equation linear in $z = y^2$.

3997. $\arctan(x + y) = x + C$.

3999. $\arctan\frac{y}{x} + \ln(x^2 + y^2) = \frac{\pi}{4} + \ln 2$.

4000. $y = \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\left[2 + x\sqrt{1 - x^2} + \arcsin x\right]$.

4001. $(1 + y)\,e^{-y} = \ln\frac{1 + e^x}{2} + 1 - x$.

4002. $y = \frac{5}{3}\,\mathrm{e}^{x^3} - \frac{1}{3}(2 + x^3)$.

4004. $y = \frac{1}{2k}[\mathrm{e}^{kx+C} + \mathrm{e}^{-(kx+C)}]$. **4005.** $x^2 + y^2 = Cx$.

4006. $(y - x)^2(x + 2y) = 1$. **4007.** The parabolas $y = x + Cx^2$.

4008. $(2y^2 - x^2)^3 = Cx^2$. **4009.** The catenary. **4010.** $y = Cx^2$.

4011*. The pencil of straight lines $y - y_0 = C(x - x_0)$. The differential equation is $y - y_0 = y'(x - x_0)$.

4012. The circle with centre at the point (x_0, y_0): $x^2 + y^2 = 2(xx_0 + yy_0)$.

4013. Any circle with centre on Oy and touching Ox.

4014. If S is the path and t is time, we have $S = S_0 + C\mathrm{e}^{-k_2 t} - \frac{k_1}{k_2^2}\,t + \frac{k_1}{2k_2}\,t^2$, where S_0 is the initial path and k_1, k_2 are coefficients of proportionality.

4016. (1) $\frac{8}{9}$ rev/sec; (2) after 6 min 18 sec.

4017. 0·00082 sec.

4018*. $v = v_0\left(1 - \frac{m}{M_0}\,t\right)^{-1} \mathrm{e}^{-\frac{3f_0}{mv_0}\left(1 - \sqrt[3]{1 - \frac{m}{M_0}\,t}\right)}$.

The force acting is equal to $\frac{\mathrm{d}(mv)}{\mathrm{d}t}$. When solving this problem and the next two, it must be borne in mind that mass m is a variable depending on time t; the velocity v is the required function.

4019*. $v = \frac{g}{2m - k}(M_0 - mt)\left[\left(1 - \frac{m}{M_0}\,t\right)^{\frac{k}{m} - 2} - 1\right]$. See hint on the solution of problem 4018.

4020*. $v = \frac{g}{\mu}\,\mathrm{e}^{k_1\mu^{\frac{2}{3}}} \int\limits_0^t \mu\mathrm{e}^{-k_1\mu^{\frac{2}{3}}}\,\mathrm{d}t$, where $\mu = M - mt$, $k_1 = \frac{3k}{m}\sqrt[3]{\frac{9\pi}{2\gamma^2}}$.

See hint on the solution of problem 4018.

4021*. $y = m_0 + \frac{m_0}{k_1 - k_2}(k_2\mathrm{e}^{-k_1 t} - k_1\mathrm{e}^{-k_2 t})$, where t = time, y = quantity of second product. If x is the quantity of the second product formed in t units of time, then $\frac{dx}{\mathrm{d}t} = k_1(m_0 - x)$. Hence we find $x = x(t)$. The speed $\frac{\mathrm{d}y}{\mathrm{d}t}$ of formation of the second product is proportional to $x - y$.

4022. 2·97 kg salt. The maximum is attained at $t = 33\frac{1}{3}$ min. and is equal to 3·68 kg.

4023. $I = 1 + (I_0 - 1)\,\mathrm{e}^{-t^2}$.

4024*. $p = \dfrac{p_0 l \mathrm{e}^{k\omega^2 x^2}}{\int\limits_0^l \mathrm{e}^{k\omega^2 x^2}\,\mathrm{d}x}$, where $k = \dfrac{M}{2p_0 lS}$.

An important practical case is that when ω is very large (centrifuge). Instead of working out the integral in the denominator for a given ω (it is not expressed in elementary functions), $\lim\limits_{\omega\to\infty} p$ is evaluated (see problem 2439). By using the same argument as when deducing the barometric formula (see *Course*, sec. 122), we obtain the differential equation $S\,\mathrm{d}p = \omega^2 x\,\mathrm{d}m$, where $\mathrm{d}m$ is the mass of element CD. Further, $\gamma = 2kp$ (one of the forms of the Boyle–Marriotte law; the coefficient of proportionality is written as $2k$ in future for simplicity); $\mathrm{d}m = \gamma S\,\mathrm{d}x = 2kpS\,\mathrm{d}x$. As a result, the equation with separated variables $\dfrac{\mathrm{d}p}{p} = 2k\omega^2 x\,\mathrm{d}x$ is obtained. Integration of this gives: $p = C\mathrm{e}^{k\omega^2 x^2}\,\mathrm{d}x$, where C is the constant of integration. Further, $M = \int\limits_0^M \mathrm{d}m = C2kS\int\limits_0^l \mathrm{e}^{k\omega^2 x^2}\,\mathrm{d}x$, whence C is found. We have:

$$p = \frac{M\mathrm{e}^{k\omega^2 x^2}}{2kS\int\limits_0^l \mathrm{e}^{k\omega^2 x^2}\,\mathrm{d}x}, \text{ but } \gamma_0 = 2kp_0 = \frac{M}{lS}, \quad k = \frac{M}{2p_0 lS}$$

and finally

$$p = \frac{p_0 l\mathrm{e}^{k\omega^2 x^2}}{\int\limits_0^l \mathrm{e}^{k\omega^2 x^2}\,\mathrm{d}x.}.$$

4025. $(x + y - 1)^3 = C(x - y + 3)$.

4026. $x^2 - xy + y^2 + x - y = C$.

4027. $x - 2y + \ln|x + y| = C$. **4028.** $\mathrm{e}^{-2\arctan\frac{y+2}{x-3}} = C(y + 2)$.

4029. $y^2 = x + (x + 1)\ln\dfrac{C}{x + 1}$. **4030.** $y^2\mathrm{e}^{-\frac{y^2}{x}} = C$.

4031. $y = \tan\ln|Cx|$. **4032.** $x^2y^2 + 1 = Cy$.

4033. $Cx = 1 - \dfrac{x}{x^2 + y^2}$. **4034.** $(1 + Cx)\,\mathrm{e}^y = 1$.

4035. $y^4 + 2x^2y^2 + 2y^2 = C$. **4036.** $x^2 + y^2 = C(y - 1)^2$.

4037. $y = x \tan(x + C)$. **4038.** $\frac{1}{y^2} = Ce^{2x^2} + x^2 + \frac{1}{2}$.

4039. $y = \frac{1}{(1 + x)[C + \ln|1 + x|]}$.

4040. $ny^n = Ce^{\frac{-nx}{a}} + nx - a$. **4041.** $x^2 = y^2(C - y^2)$.

4042. $y(1 + \ln x + Cx) = 1$. **4043.** $y(x + C) = \sec x$.

4044. $y = \left(\frac{C + \ln|\cos x|}{x} + \tan x\right)^2$. **4045.** $y = \frac{x^4}{4}\ln^2|Cx|$.

4046. $y^2 = Ce^{\frac{-2a}{x}} - \frac{b}{a}$. **4047.** $y = \frac{\varphi(x)}{x + C}$.

4048. (1) $\frac{a}{x} + \frac{b}{y} = 1$; (2) $\frac{a}{x^2} + \frac{b}{y^2} = 1$.

4049. $\frac{\varrho - k}{\varrho} = \frac{(\varrho_0 - k)\varphi}{\varrho_0\varphi_0}$. **4050.** $x^4 - x^2y^2 + y^4 = C$.

4051. $x + \arctan\frac{y}{x} = C$. **4052.** $xe^y - y^2 = C$. **4053.** $x^y = C$.

4054. $\sqrt{x^2 + y^2} + \frac{y}{x} = C$. **4055.** $\tan(xy) - \cos x - \cos y = C$.

4056. $\frac{1}{3}\sqrt{(x^2 + y^2)^3} + x - \frac{1}{2}y^2 = C$.

4057. $\sin\frac{y}{x} - \cos\frac{x}{y} + x - \frac{1}{y} = C$.

4058. $x - \frac{y}{x} = C$. The integrating factor is $\mu(x) = \frac{1}{x^2}$.

4059*. $x^2 + \frac{2x}{y} = C$. Seek the integrating factor in the form of the function $\mu(y)$.

4060. $(x^2 + y^2)e^x = C$. **4061.** $\frac{y^2}{2} + \frac{\ln x}{y} = C$.

4062. $(x \sin y + y \cos y - \sin y)e^x = C$.

4064. $\mu = y^{-n} e^{-(n-1)\int P(x)\,dx}$.

4065. The expression $\frac{Y'_x - X'_y}{X - Y}$ must be a function of $(x + y)$.

4066. The expression $\frac{Y'_x - X'_y}{xX - yY}$ must be a function of xy.

4067. $abx + b^2y + a + bc = Ce^{bx}$. **4068.** $y = \left[Ce^{\frac{(m-1)bx}{a}} - \frac{c}{b}\right]^{\frac{1}{1-m}}$.

4069. $x^2 + 2xy - y^2 - 4x + 8y = C$.

4070. $\dfrac{2x}{x - y} + \ln|x + y| + 3\ln|y - x| = C$.

4071. $x + y = a\tan\left(C + \dfrac{y}{a}\right)$. **4072.** $y^3 - 3xy = C$.

4073. $x^2 - y^2 = Cy^3$. **4074.** $3x^2y + x^3y^3 = C$.

4075. $y\left(x^2 + \dfrac{1}{3}y^2\right) = Ce^{-x}$. **4076.** $\ln|1 + y| - \dfrac{1 + y}{x} = C$.

4077. $y^2 - 1 + Cxy = 0$. **4078.** $\dfrac{xy}{x - y} + \ln\left|\dfrac{x}{y}\right| = C$.

4079. $3\sqrt{y} = C\sqrt[4]{x^2 - 1} + x^2 - 1$. **4080.** $y = \sin x + C\cos x$.

4081. $y = \dfrac{2e^x}{C + e^x(\cos x - \sin x)}$. **4082.** $\tan x - \dfrac{\sin y}{\sin x} = C$.

4083. $xe^{\sin\frac{y}{x}} = C$. **4084.** $xy\cos\dfrac{y}{x} = C$.

4085. $\sin y = x - 1 + Ce^{-x}$. **4086.** $y = \dfrac{\tan x + \sec x}{C + \sin x}$.

4087. $\ln|Cx| = -e^{-\frac{x^2+y^2}{2}}$. **4088.** $x + ye^{\frac{x}{y}} = C$.

4089. $y = x\ln|Cx|$. **4090.** $y^2 - by - axy = C$.

4091. The circle $x^2 + y^2 - \dfrac{2k}{k + 1}(ax + by) = C$ $(k \neq -1)$ or the circle $x^2 + y^2 - \dfrac{2k}{k - 1}(ax + by) = C$ $(k \neq 1)$; if $k = -1$ or $k = 1$, the straight line $ax + by = C$.

4092. The logarithmic spiral:

$$\sqrt{x^2 + y^2} = Ce^{\pm\arctan\frac{y}{x}}.$$

4093*. $y^2 = \dfrac{x^4 + C^4}{2x^2}$. The differential equation of the problem is $y^2 = x(x - yy')$.

4094. $I = \dfrac{t}{2}$.

4095. The field vector at any point is perpendicular to the radius vector of the point. The integral curves are a family of concentric circles with centre at the origin. The equation of the family is $x^2 + y^2 = C$. The isoclines are a family of straight lines through the origin.

4096. (1) $y' = f(xy)$; (2) $y' = f\left(\frac{y^2}{x}\right)$; (3) $y' = f(x^2 + y^2)$.

4097. The straight lines $y = Cx$. The result can be stated as the following geometric theorem: if a family of parabolas with a common axis and a common vertex is cut by a straight line through the vertex, the tangents to the parabolas at their intersections with the straight line are parallel.

4099. $y' = \frac{ay + b}{x} + C$; $y' = ay + bx + C$.

4103. If $\Delta x = 0{\cdot}05$, $y \approx 0{\cdot}31$.

4104. If $\Delta x = 0{\cdot}05$, $y \approx 1{\cdot}68$.

4105. The exact solution is $y = e^{\frac{x^2}{4}} = f(x)$; $f(0{\cdot}9) = 1{\cdot}2244$. The approximate solution is $f(0{\cdot}9) = 1{\cdot}1942$. The relative error is $\approx 2{\cdot}5\%$.

4106. With the exact solution, $x = \sqrt[3]{3(e - 1)} \approx 1{\cdot}727$; numerical integration with the interval divided into 4 parts gives $x \approx 1{\cdot}72$.

4107. $y_2 = 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \frac{13}{24}x^4 + \frac{1}{4}x^5 + \frac{1}{18}x^6 + \frac{1}{63}x^7$.

4108. $-1{\cdot}28$. **4109.** $y = 1 + x + x^2 + 2x^3 + \frac{13}{4}x^4 + \ldots$

4110. $y = 1 - x + \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} + \ldots$

4111. $y = \frac{1}{3}x^3 - \frac{1}{7 \,.\, 9}x^7 - \frac{2}{7 \,.\, 11 \,.\, 27}x^{11} - \ldots$

4112. $y = 1 + 2x - x^2 + \frac{4}{3}x^3 - \frac{3}{2}x^4 + \ldots$

4113. $y = 0$. **4114.** $y = x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{11x^4}{2 \,.\, 3 \,.\, 4} + \ldots$

4115. $y = -\frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^5}{5!} - \ldots$

4116. $y = 1 + (x - 1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} + \frac{4(x-1)^4}{4!} + {}$ $+ \frac{60(x-1)^5}{5!} + \ldots$

4117. $y = Cx + C^2$; the singular integral is $x^2 + 4y = 0$.

4118. $y = Cx - 3C^3$; the singular integral is $9y \pm 2x\sqrt{x} = 0$.

4119. $y = Cx + \frac{1}{C}$; the singular integral is $y^2 = 4x$.

4120. $y = Cx + \sqrt{1 + C^2}$; the singular integral is $x^2 + y^2 = 1$.

4121. $y = Cx + \sin C$; the singular solution is

$$y = x(\pi - \arccos x) + \sqrt{1 - x^2}.$$

4122. $y^2 = 2Cx + C^3$; the singular integral is $27y^4 + 32x^3 = 0$.

4123. $y = (\sqrt{x+1} + C)^2$; singular solution $y = 0$.

4124. $y = Cx^2 + \frac{1}{C}$; singular integral $y^2 - 4x^2 = 0$.

4125. $2Cx = C^2 - y^2$; no singular integral.

4126. $x = Ce^{-p} + 2(1 - p)$, $y = x(1 + p) + p^2$; no singular integral.

4127. $e^x(y - C) = C$, singular integral $y^2 + 4e^x = 0$.

4128. $y = Cx + C + C^2$; singular solution $y = -\frac{1}{4}(x + 1)^2$.

4129. $y^{\frac{3}{2}} - x^{\frac{3}{2}} = a^{\frac{3}{2}}$. **4130.** $y - 4x = 0$.

4131. $y^2 - 4e^x = 0$. **4132.** $xy = 1$. **4133.** $2y - x^2 = 0$.

4135. The equilateral hyperbola $4xy = \pm a^2$, where a^2 is the area of the triangle; the trivial solution is any straight line of the family $y = Cx \pm a\sqrt{C}$.

4136. $(y - x - 2a)^2 = 8ax$.

4137. Ellipses and hyperbolas.

4138. $x = \dfrac{Ce^{-\frac{1}{2p^2}}(1 + p^2)}{p^2}$, $y = \dfrac{Ce^{-\frac{1}{2p^2}}}{p}$, or

$$x = \frac{(p^2 + 1)\,C}{\sqrt{p}\,\sqrt[4]{(p^2 + 2)^3}}, \quad y = \frac{-C\sqrt{p}}{\sqrt[4]{(p^2 + 2)^3}}.$$

4139. $y^2 = Cx^{-\frac{1}{k}} + \dfrac{k^2x^2}{2k + 1}$.

4140*. $$\begin{cases} y = \cos\alpha\left(C + \frac{a}{2}\sin^2\alpha\right), \\ x = \sin\alpha\left(a - C - \frac{a}{2}\sin^2\alpha\right). \end{cases}$$

We put $\frac{dy}{dx} = \tan\alpha$ in the differential equation obtained, then express x in terms of y and parameter α, find dx, replace dx by $\dfrac{dy}{\tan\alpha}$

and solve the resulting differential equation, y being taken as a function of α.

4141. $S = at^2$, where a is some definite constant.

4142. $x^2 + y^2 = 2a^2 \ln |Cx|$. **4143.** $y = Ce^{-\frac{x}{2}}$.

4144. $y = C(x^2 + y^2)$. **4145.** $(x^2 + y^2)^2 = C(y^2 + 2x^2)$.

4146. If the parameter of the parabola is equal to $2p$ and the straight line is taken as the axis of ordinates, the equation of the trajectory becomes:

$$y = C + \frac{2}{3}\sqrt{\frac{2x^3}{p}}.$$

4147. Tractrices.

4148. On measuring angle α in one of the two possible directions, we obtain the equation of the family as $xy - \frac{\sqrt{3}}{2}(x^2 + y^2) = C$.

4149. On measuring angle α in one of the two possible directions, we obtain the equation of the family as

$$\ln (2x^2 + xy + y^2) + \frac{6}{\sqrt{7}} \operatorname{arc\,tan} \frac{x + 2y}{x\sqrt{7}} = C.$$

4150*. We can assume say that the wind blows along Ox. The sound propagation curves in the xOy plane will be the orthogonal trajectories of the family of circles $(x - at)^2 + y^2 = (v_0 t)^2$, where t is the time that has passed since the departure of the sound wave from the source, and v_0 is the velocity of sound in stationary air.

For any fixed t, the differential equation of the required orthogonal trajectories is $y' = \frac{y}{(x - at)}$ together with the equation of the family of circles.

On eliminating t, we obtain a Lagrange equation. Its general solution is

$$x = C(\cos \varphi + b)\left(\tan \frac{\varphi}{2}\right)^{\frac{1}{b}}, \quad y = C \sin \varphi \left(\tan \frac{\varphi}{2}\right)^{\frac{1}{b}},$$

where $b = \pm \frac{a}{v_0}$, φ is a parameter.

4151. $x = C \sin t + R(\cos t + t \sin t)$,

$y = -C \cos t + R(\sin t - t \cos t)$.

4152. $x = \frac{C}{\cosh t} + a\,(t - \tanh t)$, $y = C \tanh t + \frac{a}{\cosh t}$.

4153. $x = a(\cos t + t\sin t) - \cos t\left(\frac{at^2}{2} + C\right)$,

$y = a(\sin t + t\cos t) - \sin t\left(\frac{at^2}{2} + C\right)$.

4154. $x = C\sin t + 2\tan t,\ y = \tan^2 t - C\cos t - 2$.

4155. $y = \frac{x^3}{6} - \sin x + C_1x + C_2$.

4156. $y = \frac{\operatorname{arc\,tan} x}{2}(x^2 - 1) - \frac{x}{2}\ln(1 + x^2) + C_1x + C_2$.

4157. $y = \frac{x^2}{2}\left[\ln x - \frac{2}{3}\right] + C_1x + C_2$. **4158.** $y = C_1x^2 + C_2$.

4159. $y = C_1e^x + C_2 - x - \frac{x^2}{2}$. **4160.** $y = \frac{1}{3}x^3 + C_1x^2 + C_2$.

4161. $y = (1 + C_1^2)\ln|x + C_1| - C_1x + C_2$.

4162. $y = (C_1x - C_1^2)\,e^{\frac{x}{C_1}+1} + C_2$. **4163.** $y = \frac{1}{12}(x + C_1)^3 + C_2$.

4164. $y = \frac{2}{3C_1}\sqrt{(C_1x - 1)^3} + C_2$.

4165. $y = -\frac{1}{3}\sin^3 x + C_1\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) + C_2$.

4166. $(x + C_2)^2 = 4C_1(y - C_1)$. **4167.** $y = C_1(x + C_2)^{\frac{2}{3}}$.

4168. $y = C_1e^{\frac{x}{a}} + C_2e^{-\frac{x}{a}}$.

4169. $x = \frac{4}{3}\left(y^{\frac{1}{2}} - 2C_1\right)\sqrt{y^{\frac{1}{2}} + C_1} + C_2$. **4170.** $y = \frac{x + C_1}{x + C_2}$.

4171. $(x + C_2)^2 - y^2 = C_1$. **4172.** $y = C_1e^{C_2x}$.

4173. $y\cos^2(C_1 \pm x) = C_2$. **4174.** $(x + C_2)\ln y = x + C_1$.

4175. If the arbitrary constant introduced by the first integration is positive $(+C_1^2)$, then $y = C_1\tan(C_1x + C_2)$; if it is negative $(-C_1^2)$, then $y = C_1\frac{1 + e^{2(C_1x+C_2)}}{1 - e^{2(C_1x+C_2)}} = -C_1\coth(C_1x + C_2)$; if $C_1 = 0$, then $y = -\frac{1}{x + C_2}$.

4176. $x = C_1 + \cos C_2 \ln\left|\tan\frac{y + C_2}{2}\right|$.

4177. $C_1x + C_2 = \ln\left|\frac{y}{y+C_1}\right|.$

4178. $\frac{x+C_2}{2} = C_1 \operatorname{arc\,tan}(C_1 \ln y).$

4179. $\ln|C_1 y| = 2\tan(2x + C_2).$

4180. $y = \ln|x^2 + C_1| + \frac{a}{\sqrt{-C_1}} \ln\left|\frac{x - \sqrt{-C_1}}{x + \sqrt{-C_1}}\right| + C_2$, if $C_1 < 0$,
and $y = \ln|x^2 + C_1| + \frac{2a}{\sqrt{C_1}} \operatorname{arc\,tan} \frac{x}{\sqrt{C_1}} + C_2$ if $C_1 > 0$.

4181*. After substituting $y' = p$ the equation splits into two, one of which is of Clairaut's type. Its general solution is

$y = C_1 + C_2e^{C_1x}$, whilst the singular solution is $y = \frac{4}{C - x}$.
The second equation is $y' = 0$.

4182. $y = C_1x(x - C_1) + C_2$ and the singular solutions are $y = \frac{x^3}{3} + C$.

4183. $y^2 = C_1x^4 + C_2$. **4184.** $y = \ln\left|\frac{C_1x^{C_1}}{C_2 - x^{C_1}}\right|.$

4185. $y = \sqrt{\frac{1}{3}x^3 + C_1x + C_2}$. **4186.** $y = C_1x + \frac{C_2}{x}$.

4187. $y = C_1xe^{\frac{C_2}{x}}$. **4188.** $\ln|y + C_1| + \frac{C_1}{y + C_1} = x + C_2$.

4189. $y = x^3 + 3x + 1$. **4190.** $y = 2 + \ln\frac{x^2}{4}$.

4191. $y = \frac{2}{5}x^2\sqrt{2x} - \frac{16}{5}$. **4192.** $y = \frac{4}{(x+4)^2}$.

4193. $y - x = 2\ln|y|$. **4194.** $y = \sqrt{2x - x^2}$.

4195. $y = \sqrt{1 + e^{2x}}$. **4196.** $y = -\ln|1 - x|$. **4197.** $y = \frac{x+1}{x}$.

4198*. $y = x$. Make the substitution $y = ux$. **4199.** $y = 2e^{\frac{1}{2}x^2} - 1$.

4200*. The differential equation of the curve is $dx = \frac{dy}{\sqrt{(C_1y)^{\frac{2}{k}} - 1}}$,

where k is a coefficient of proportionality.

If $k = 1$, then $y = \frac{1}{2C_1}\left[e^{C_1(x-C_2)} + e^{-C_1(x-C_2)}\right]$; this is a catenary. If $k = -1$, then $(x + C_2)^2 + y^2 = C_1^2$; this is a circle. If $k = 2$, then $(x + C_2)^2 = 4C_1(y - C_1)$; this is a parabola. If $k = -2$, $dx = \frac{y\,dy}{\sqrt{y - C_1y^2}}$; this is the differential equation of a cycloid.

4201. $e^{\frac{y}{a}} = C_2 \sec\left(\frac{x}{a} + C_1\right)$. **4202.** $Cx = y^{2k-1}$.

4203. A catenary. **4204.** $v = \sqrt{\frac{mgv_0^2}{mg + kv_0^2}}$. **4205.** A parabola.

4206. $S = \frac{m}{3k}\left[\sqrt{\left(\frac{2k}{m}t + C\right)^3} - \sqrt{C^3}\right]$.

4207*. Let the axis of abscissae be directed vertically downwards, let the origin be on the fluid surface, and let the equation of the ray be $y = f(x)$. At a depth x we have $\frac{\sin\alpha}{\sin(\alpha + d\alpha)} = \frac{(m + dm)}{m}$, where m is the refractive index at depth x, and α is the angle between the vertical and the tangent to the light ray. Obviously, $\tan\alpha$ is equal to y'. We obtain from the equation $m\sin\alpha = (m + dm)(\sin\alpha\cos d\alpha + \cos\alpha\sin d\alpha)$, on removing the brackets and neglecting higher order infinitesimals: $m\,d\alpha = -dm\tan\alpha$, whence $\frac{dm}{m} = -\frac{dy'}{y'(1 + y'^2)}$. On integrating this equation, we find y' as a function of m. On replacing m by its expression in terms of x and integrating again, we obtain the answer:

$$y = \frac{m_0h\sin\alpha_0}{m_2 - m_1}\ln\left|m + \sqrt{m^2 - m_0^2\sin^2\alpha_0}\right| + C,$$

where $m = \frac{(m_2 - m_1)x + m_1h}{h}$.

4208. $y = x^2\ln\sqrt{x} + C_1x^2 + C_2x + C_3$.

4209. $y = -\frac{1}{8}\sin 2x + C_1x^2 + C_2x + C_3$.

4210. $y = \frac{e^{ax}}{a^{10}} + P_9$ (P_9 is a polynomial of the ninth degree in x with arbitrary coefficients).

4211. $y = C_1\frac{x^2}{2} + C_2x + C_3 - C_1^2(x + C_1)\ln|x + C_1|$.

4212. $y = C_1x^5 + C_2x^3 + C_3x^2 + C_4x + C_5$.

4213. $y = \frac{1}{3}(C_1 - 2x)^{\frac{3}{2}} + C_2x + C_3$. **4214.** $x = C_1y^2 + C_2y + C_3$.

4215. The solution can be written in three forms:

$y = C_1 \sin (C_2x + C_3)$, or $y = C_1 \sinh (C_2x + C_3)$, or

$$y = C_1 \cosh (C_2x + C_3).$$

4216. $(x + C_2)^2 + (y + C_3)^2 = C_1^2$.

4217. $y = C_2\left(xe^{C_1x} - \frac{1}{C_1}e^{C_1x}\right) + C_3$.

4219. (2) $y = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{3x^4}{4!} + \frac{14x^5}{5!} + \ldots$

4220. $y = 1 - \frac{(x-1)^2}{2!} - \frac{2(x-1)^4}{4!} + \frac{3(x-1)^5}{5!} + \ldots$

4221. $y = \frac{\pi}{2}(x - 1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} - \frac{(x-1)^4}{4!} - \frac{4(x-1)^5}{5!} + \ldots$

4222. $y = 1 + x + \frac{x^3}{3!} + \frac{2x^4}{4!} + \frac{3x^5}{5!} + \ldots$ If $f(x) \approx 1 + x + \frac{x^3}{3!} + \frac{2x^4}{4!}$, an alternating numerical series is obtained with $x = -0{\cdot}5$, and the value of the first of the neglected terms is less than 0·001.

4223. $y = 1 - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{4x^5}{5!} - \frac{14x^6}{6!} + \ldots$; the fifth.

4224. $y = x^2 - \frac{1}{10}x^5 + \frac{1}{80}x^8 - \frac{7}{4400}x^{11} + \ldots$; 0·318; 0·96951.

4225*. The differential equation of the problem is $E = L\frac{d^2Q}{dt^2} + \frac{dQ}{dt}\frac{V_0 - kQ}{k_1}$, where Q is the quantity of electricity that has flowed through the circuit from the start of the experiment till the instant t. Having expressed Q in terms of V (V is the quantity of water in the vessel at the instant t) and found the coefficients from the conditions of the problem, we arrive at the equation $V'' + aVV' + b = 0$, where $a = \frac{1}{k_1L} = 0{\cdot}005$, $b = \frac{kE}{L} = 0{\cdot}00935$. On integrating this with the initial conditions: $V_0 = 1000\ \text{cm}^3$, $V_0' = -kI_0 = -0{\cdot}00187\ \text{cm}^3/\text{sec}$, we obtain the series $V = 1000 - 0{\cdot}00187t -$

$-10^{-9}\cdot[2{\cdot}91t^3 - 3{\cdot}64t^4 + 3{\cdot}64t^5 - 3{\cdot}04t^6 + 2{\cdot}17t^7 - \ldots]$. The series is alternating and the coefficients, as from the sixth, are decreasing and tend to zero, which is convenient for computations.

4226*. The differential equation of the problem is

$$L\frac{d^2Q}{dt^2} + \frac{dQ}{dt}\,\frac{k_1}{M_0 - kQ} = E.$$

On taking the quantity y of hydrochloric acid not decomposed at the instant t as the required function, we reduce the equation to the form $yy'' + ay' + by = 0$, where $a = \frac{k_1}{L} = 50$, $b = \frac{kE}{L} = 0{\cdot}0191$. Integration of this equation with the initial conditions $y_0 = M_0 = 10$, $y_0' = -kI_0 = -0{\cdot}00381$ gives the series

$$y = 10 - 0{\cdot}00381t + 10^{-10}\,t^3\,(1{\cdot}21 - 1{\cdot}52t + \ldots).$$

4227. $x^2y'' - 6xy' + 12y = 0$.

4228. $xy'' - (2x + 1)\,y' + (x + 1)\,y = 0$.

4229. $(x^3 - 3x^2 + 3x)\,y''' - (x^3 - 3x + 3)\,y'' - 3x\,(1 - x)\,y' + 3(1 - x)\,y = 0$.

4230. $y = 3x^2 - 2x^3$.

4231. (a) $\frac{\sin^2 x}{\cos^2 x} \neq \text{const}$; (b) $y''\sin 2x - 2y'\cos 2x = 0$.

4232*. (3) By Ostrogradskii's formula:

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = Ce^{-\int P(x)\,dx},$$

or, on expanding the determinant (Wronskian): $y_1y_2' - y_1'y_2 = Ce^{-\int P(x)\,dx}$. We divide both sides of the equation by y_1^2; then $\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{C}{y_1^2}e^{-\int P(x)\,dx}$, whence the required relationship follows.

4233. $y = C_1x\ln\left|\frac{1 + x}{1 - x}\right| - 2C_1 + C_2x$.

4234. $y = C_1\frac{\sin x}{x} + C_2\frac{\cos x}{x}$. **4235.** $y = x^2 - e^{x-1}$.

4236*. Functions P and Q must be connected by the relationship $Q' + 2PQ = 0$. Substitute $y_1 = \frac{1}{y_2}$ in the formula (following from Ostrogradskii's formula) of problem 4232, differentiate twice the relationship obtained, and substitute y_2', y_2'' in the given equation.

4237*. $y = C_1(4x^3 - 3x) + C_2\sqrt{1 - x^2}(4x^2 - 1)$. We put by hypothesis: $y_1 = Ax^3 + Bx^2 + Cx + D$. Having substituted y_1 in the given equation, we get $B = 0$, $D = 0$, $A : C = \frac{4}{-3}$, or $A = 4k$, $C = -3k$. Hence the particular solution will be $y_1 = k(4x^3 - 3x)$. In accordance with the property of linear equations, we can take $k = 1$, so that $y_1 = 4x^3 - 3x$. Knowing one particular solution, a second can be found in the usual way and the general solution obtained.

4238. $y = C_1 \sin x + C_2\left[1 - \sin x \ln \left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|\right]$.

4239. $y = C_1 x + C_2 x \int \frac{e^x\,dx}{x^2}$. **4240.** $y = C_1 x + C_2(x^2 - 1)$.

4241. $y = C_1 x + C_2 x^2 + C_3 x^3$. **4242.** $y = x^3 + x(C_1 + C_2 \ln |x|)$.

4243. $y = C_1 e^x + C_2 x - x^2 - 1$.

4244. $y = C_1 x^3 + C_2(x + 1) - x$.

4245. $y = 2 + 3x + x\left(\frac{\pi}{2} + 2 \arctan x\right) + x^2$.

4246. $y = -2 + 2x - x^2 + \frac{x^3}{3} - \frac{x^4}{4} + \frac{7x^5}{60} - \ldots$

4247. $y = 1 + \frac{2x^4}{4!} - \frac{2x^5}{5!} + \frac{2x^6}{6!} - \frac{2x^7}{7!} + \frac{62x^8}{8!} - \ldots$

4248. $y = \frac{x^2}{2} + \left[\frac{x^4}{4!} + \frac{3x^6}{6!} + \frac{5x^8}{8!} + \ldots + \frac{(2n-1)\,x^{2n+2}}{(2n+2)!} + \ldots\right]$.

4249. $y = C_1\left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{24} + \ldots\right) +$
$+ C_2\left(x + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{30} + \ldots\right)$.

4250. $y = C_1\left(1 + \frac{x^4}{12} + \ldots\right) + C_2\left(x - \frac{x^3}{6} + \frac{3x^5}{40}\right) + \ldots\Big)$.

4251. $y = C_1 e^x + C_2 e^{-2x}$. **4252.** $y = C_1 e^{3x} + C_2 e^{-3x}$.

4253. $y = C_1 e^{4x} + C_2$. **4254.** $y = C_1 e^{(1+\sqrt{2})x} + C_2 e^{(1-\sqrt{2})x}$.

4255. $y = C_1 e^{2x} + C_2 e^{-\frac{4}{3}x}$. **4256.** $y = C_1 \cos x + C_2 \sin x$.

4257. $y = e^{-3x}(C_1 \cos 2x + C_2 \sin 2x)$.

4258. $y = e^x\left(C_1 \cos\frac{x}{2} + C_2 \sin\frac{x}{2}\right)$. **4259.** $y = e^x (C_1 + C_2 x)$.

4260. $x = (C_1 + C_2 t)\, e^{2.5t}$. **4261.** $y = (C_1 + C_2 x)\, e^{-\frac{1}{4}x}$.

4262. $y = 4e^x + 2e^{3x}$. **4263.** $y = 3e^{-2x} \sin 5x$.

4264. $y = e^{-\frac{x}{2}}(2 + x)$. **4265.** $y = [1 + (1 - m)x]e^{mx}$.

4266. $y = \cos 3x - \frac{1}{3}\sin 3x$.

4267. If $k > 0$, then $y = \frac{a}{\sqrt{k}}\sin[\sqrt{k}(x - x_0)] + y_0\cos[\sqrt{k}(x - x_0)]$; if $k < 0$, then

$$y = \frac{1}{2\sqrt{k_1}}[(y_0\sqrt{k_1} + a)e^{\sqrt{k_1}(x-x_0)} + (y_0\sqrt{k_1} - a)e^{-\sqrt{k_1}(x-x_0)}],$$

where $k_1 = -k$.

4268. $y = C_1e^{-x} + C_2e^{\frac{x}{2}} + e^x$.

4269. $y = C_1\cos ax + C_2\sin ax + \frac{e^x}{a^2 + 1}$.

4270. $y = C_1e^{6x} + C_2e^x + \frac{5\sin x + 7\cos x}{74}$.

4271. $y = e^{-x}(C_1\cos 2x + C_2\sin 2x) - \frac{1}{2}\cos 2x - 2\sin 2x$.

4272. $y = (C_1 + C_2x)e^{3x} + \frac{2}{9}x^2 + \frac{5}{27}x + \frac{11}{27}$.

4273. $y = e^x(C_1\cos x + C_2\sin x) + x + 1$.

4274. $y = C_1e^x + C_2e^{-5x} - 0{\cdot}2$.

4275. $y = C_1e^x + C_2e^{2x} + \bar{y}$, where $\bar{y}$ is equal to: (1) $\frac{5}{3}e^{-x}$;

(2) $3xe^{2x}$; (3) $\frac{3}{5}\cos x + \frac{1}{5}\sin x$; (4) $x^3 + \frac{9}{2}x^2 + \frac{21}{2}x - \frac{15}{4}$;

(5) $-\frac{8}{5}e^x\left[\cos\frac{x}{2} + 2\sin\frac{x}{2}\right]$; (6) $\frac{1}{2}x + \frac{5}{4} - \frac{1}{12}e^{-2x}$;

(7) $e^x(2x^2 + x)$; (8) $\frac{3}{2}x + \frac{1}{4}(9 + 3\cos 2x - \sin 2x)$;

(9) $-2xe^x - \frac{1}{12}e^{-2x}$;

(10) $\frac{1}{20}\cos x - \frac{3}{20}\sin x + \frac{7}{260}\cos 3x + \frac{9}{260}\sin 3x$;

(11) $-\frac{1}{12}e^{-x} - \frac{1}{2}xe^x$.

4276. $y = C_1 + C_2 e^{-\frac{5}{2}x} + \bar{y}$, where $\bar{y}$ is equal to:

(1) $\frac{1}{3}x^3 - \frac{3}{5}x^2 + \frac{7}{25}x$; (2) $\frac{1}{7}e^x$; (3) $5\sin x - 2\cos x$;

(4) $\frac{1}{10}x + \frac{5}{164}\sin 2x - \frac{1}{41}\cos 2x$;

(5) $\cos 2{\cdot}5x + \sin 2{\cdot}5x - 0{\cdot}02xe^{-2{\cdot}5x}$;

(6) $\left(-5x - \frac{16}{29}\right)\cos x - \left(2x - \frac{185}{29}\right)\sin x$;

(7) $e^{-x}[(10x + 18)\sin x - (20x + 1)\cos x]$;

(8) $\frac{3}{10}\left(\frac{1}{5}e^{\frac{5}{2}x} - xe^{-\frac{5}{2}x}\right)$;

4277. $y = e^{2x}(C_1 + C_2 x) + \bar{y}$, where $\bar{y}$ is equal to: (1) $\frac{1}{4}$;

(2) $\frac{1}{9}e^{-x}$; (3) $\frac{3}{2}x^2e^{2x}$; (4) $\frac{1}{4}\cos 2x + \frac{1}{2}x + \frac{1}{2}$;

(5) $\frac{1}{169}\left(\frac{-5}{2}\sin 3x + 6\cos 3x\right) - \frac{1}{50}(3\sin x + 4\cos x)$;

(6) $\frac{3}{100}(3\sin x + 4\cos x) + \frac{1}{676}(5\sin 3x - 12\cos 3x)$;

(7) $2x^2 + 4x + 3 + 4x^2e^{2x} + \cos 2x$; (8) $\frac{1}{4}\left(x^2e^{2x} - \frac{1}{8}e^{-2x}\right)$;

(9) $\frac{1}{2}\left(e^x - \frac{1}{9}e^{-x}\right) + \frac{1}{25}(3\sin x + 4\cos x)$;

(10) $e^x - \frac{1}{2}e^{x-1} + \frac{1}{18}e^{1-x}$.

4278. $y = C_1\cos x + C_3\sin x + \bar{y}$, where $\bar{y}$ is equal to:

(1) $2x^3 - 13x + 2$; (2) $\cos 3x$; (3) $\frac{1}{2}x\sin x$;

(4) $-\frac{1}{2}x\cos x - e^{-x}$; (5) $\frac{1}{4}\left(x\sin x - \frac{1}{4}\cos 3x\right)$;

(6) $9 + 4\cos 2x - 0{\cdot}2\cos 4x$; (7) $0{\cdot}5\cosh x$;

(8) $0{\cdot}5 + 0{\cdot}1\cosh 2x$.

4279. $y = e^{\frac{3}{5}x}\left(C_1\cos\frac{4}{5}x + C_2\sin\frac{4}{5}x\right) + \bar{y}$, where $\bar{y}$ is equal to:

(1) $\frac{25}{16}e^{\frac{3}{5}x}$ (2) $\frac{15}{219}\sin\frac{4}{5}x + \frac{40}{219}\cos\frac{4}{5}x$;

(3) $\frac{1}{13}e^{2x} + \frac{1}{5}\left(2x^3 + \frac{36}{5}x^2 + \frac{107}{25}x - \frac{908}{125}\right)$;

(4) $\left(-\frac{5}{9}\cos x\right)e^{\frac{3}{5}x}$; (5) $-\frac{1}{8}xe^{\frac{3}{5}x}\cos\frac{4}{5}x$; (6) $0{\cdot}5e^{2x} + 1{\cdot}3$.

4280. $y = 2 + C_1\cos x + C_2\sin x + \cos x\ln\left|\tan\frac{x}{2}\right|$.

4281. $y = e^x(C_1 + C_2x - \ln\sqrt{x^2+1} + x\arctan x)$.

4282*. (1) $y = e^x(x + C_1) - (e^x + 1)\ln(e^x + 1) + C_2$;

(2) $y = \frac{1}{2}e^x[\arcsin e^x + e^x\sqrt{1 - e^{2x}} + C_1] + \frac{1}{3}\sqrt{(1 - e^{2x})^3} + C_2$;

(3) $y = C_1e^x - \cos e^x + C_2$.

All three results are easily obtained with the aid of the general formulae (see *Course*, sec. 206).

4283. $y = (1 + x)e^{-\frac{3}{2}x} + 2e^{-\frac{5}{2}x}$.

4284. $y = e^x(0{\cdot}16\cos 3x + 0{\cdot}28\sin 3x) + x^2 + 2{\cdot}2x + 0{\cdot}84$.

4285. $y = e^x + x^2$. **4286.** $y = e^x(e^x - x^2 - x + 1)$.

4287. $y = \frac{1}{3}\sin 2x - \frac{1}{3}\sin x - \cos x$.

4288*. Differentiate twice the expressions quoted for y; on substituting for $\bar{y}$, $\bar{y}'$ and $\bar{y}''$ in the equation an identity is obtained.

4289. $y = x^3(C_1 + C_2x^4)$.

4290. $y = \frac{x}{2} + C_1\cos\ln|x| + C_2\sin\ln|x|$.

4291. $y = x[C_1 + C_2\ln|x| + \ln^2|x|]$.

4292. $y = x\ln|x| + C_1x + C_2x^2 + x^3$.

4293. If $\frac{1}{m\alpha} > \omega^2$, then $y = C_1\cos kt + C_2\sin kt +$ $+ \frac{g}{k^2 - \omega^2}\cos\omega t + \frac{e\omega^2}{k^2}$, where $k^2 = \frac{1}{m\alpha} - \omega^2$. If $\frac{1}{m\alpha} < \omega^2$, then $y = C_1e^{kt} + C_2e^{-kt} - \frac{g}{k^2 + \omega^2}\cos\omega t - \frac{e\omega^2}{k^2}$, where $k^2 = \omega^2 - \frac{1}{m\alpha}$.

4294. $s = \frac{1}{5}(4e^t + e^{-4t})$.

4295. $s = e^{-0{\cdot}2t}[10\cos(0{\cdot}245t) + 8{\cdot}16\sin(0{\cdot}245t)]$; $s|_{t=3} \approx 7{\cdot}07$ cm.

4296. $t = \sqrt{\frac{am}{f}}\ln\frac{F + \sqrt{f(2F - f)}}{F - f}$.

4297. $s = e^{0\cdot245t}\,[2\cos(156\cdot6t) + 0\cdot00313\sin(156\cdot6t)]$.

4298*. $k = 33\frac{1}{3}\frac{\mathrm{g}}{\mathrm{cm}} = 33\frac{1}{3}g\frac{\mathrm{dyn}}{\mathrm{cm}}$; $t = 0\cdot38$ sec; the height of the submerged part of the block is $x = 5[3 + \cos(8\cdot16t)]$. Take $g = 1000$ cm/sec² when forming the equation.

4299*. $r = \frac{a_0}{2}(e^{\omega t} + e^{-\omega t})$. The entire situation is as though the tube were stationary, except that a force acts on the sphere equal to $m\omega^2 r$ (r is the distance of the sphere from the axis of rotation).

4300. If $k > m\omega^2$, then $r = \frac{a_0}{k - m\omega^2}\left[k - m\omega^2\cos\left(t\sqrt{\frac{k}{m} - \omega^2}\right)\right]$;

if $k = m\omega^2$; then $r = a_0\left(1 + \frac{k}{2m}t^2\right)$;

if $k < m\omega^2$, then $r = \frac{a_0}{m\omega^2 - k}\left[m\omega^2\cosh\left(t\sqrt{\omega^2 - \frac{k}{m}}\right) - k\right]$.

4301. $y = C_1\cos 3x + C_2\sin 3x + C_3$.

4302. $y = C_1e^{2x} + C_2e^{-2x} + C_3e^{3x} + C_4e^{-3x}$.

4303. $y = (C_1 + C_2x)\,e^{2x} + (C_3 + C_4x)\,e^{-2x}$.

4304. $y = C_1e^{2x} + C_2e^{-2x} + C_3\cos 2x + C_4\sin 2x$.

4305. $y = C_1e^{-x} + C_2e^{-3x} + C_3e^{4x}$.

4306. $y = C_1e^{x} + C_2xe^{x} + C_3x^2e^{x}$.

4307. $y = C_1 + C_2x + C_3e^{-x} + C_4xe^{-x}$.

4308. $y = C_1e^{x} + C_2e^{-x} + C_3x^{n-3} + C_4x^{n-4} + \ldots + C_{n-1}x + C_n$.

4309. $$y = e^{\frac{x}{\sqrt{2}}}\left(C_1\cos\frac{x}{\sqrt{2}} + C_2\sin\frac{x}{\sqrt{2}}\right) + e^{-\frac{x}{\sqrt{2}}}\left(C_3\cos\frac{x}{\sqrt{2}} + C_4\sin\frac{x}{\sqrt{2}}\right).$$

4310. $y = (C_1 + C_2x + C_3x^2)\cos\frac{x}{2} + (C_4 + C_5x + C_6x^2)\sin\frac{x}{2} + C_7x + C_8$.

4311. $y = e^{-x}(C_1 + C_2x + C_3x^2 + \ldots + C_nx^{n-1})$.

4312. $y = 1 + \cos x$. **4313.** $y = e^{x} + \cos x - 2$.

4314. $y = (C_1 + C_2x)\,e^{x} + C_3e^{2x} - x - 4$.

4315. $y = (C_1 + C_2x)\,e^{x} + C_3e^{-2x} + (x^2 + x - 1)e^{-x}$.

4316. $y = (C_1 + C_2x)\cos 2x + (C_3 + C_4x)\sin 2x + \frac{1}{9}\cos x$.

4317. $y = (C_1 + C_2x)\cos ax + (C_3 + C_4x)\sin ax - \dfrac{x^2 \cos ax}{8a^2}.$

4318. $y = \dfrac{1}{60}x^5 - \dfrac{1}{2}x^3 + C_1x^2 + C_2x + C_3 + C_4\cos x + C_5 \sin x.$

4319. $y = C_1e^x + C_2e^{-x} + C_3\sin x + C_4\cos x + \dfrac{x^2 - 3x}{8}e^x -$
$- \dfrac{1}{4}x\sin x.$

4320. $y = (C_1 + C_2x + x^2)\,e^x + (C_3 + C_4x + x^2)\,e^{-x} + \sin x +$
$+ \cos x.$

4321. $y = 4 - 3e^{-x} + e^{-2x}.$ **4322.** $y = e^x + x^3.$

4323. $y = x(C_1 + C_2 \ln|x| + C_3 \ln^2|x|).$

4324. $\begin{cases} x = e^{-6t}(C_1\cos t + C_2 \sin t), \\ y = e^{-6t}[(C_2 + C_1)\cos t + (C_2 - C_1)\sin t]. \end{cases}$

4325. $\begin{cases} x = C_1e^t - C_2e^{-t} - \dfrac{1}{2}(e^t + e^{-t}) + \dfrac{1}{2}t(e^t - e^{-t}), \\ y = C_1e^t + C_2e^{-t} + \dfrac{1}{2}t(e^t + e^{-t}). \end{cases}$

4326. $\begin{cases} x = C_1e^{-4t} + C_2e^{-7t} + \dfrac{7}{40}e^t + \dfrac{1}{5}e^{-2t}, \\ y = \dfrac{1}{2}C_1e^{-4t} + C_2e^{-7t} + \dfrac{1}{40}e^t + \dfrac{3}{10}e^{-2t}. \end{cases}$

4327. $\begin{cases} z = C_1y; \\ zy^2 - \dfrac{3}{2}x^2 = C_2. \end{cases}$

4328. $\begin{cases} y = \dfrac{\sqrt{C_1 + x^2}}{\ln\left|\dfrac{C_2}{x + \sqrt{x^2 + C_1}}\right|}, \\ z = \sqrt{C_1 + x^2}\ln\left|\dfrac{C_2}{x + \sqrt{x^2 + C_1}}\right|. \end{cases}$

4329. $\begin{cases} \dfrac{y}{x} = C_1, \\ x^2 + y^2 + z^2 = C_2. \end{cases}$ **4330.** $\begin{cases} x^2 + y^2 + z^2 = C_1y, \\ z = C_2y. \end{cases}$

4331. $\begin{cases} y^2 - z^2 = C_1, \\ yz - y^2 - x = C_2. \end{cases}$ **4332.** $\begin{cases} x = C_1e^{-t} + C_2e^{-3t}, \\ y = C_1e^{-t} + 3C_2e^{-3t} + \cos t. \end{cases}$

4333. $\begin{cases} x = C_1e^t + C_2e^{-t} + C_3\cos t + C_4\sin t, \\ y = C_1e^t + C_2e^{-t} - C_3\cos t - C_4\sin t. \end{cases}$

4334. $\begin{cases} x = C_1 + C_2t + C_3t^2 - \dfrac{1}{6}t^3 + e^t, \\ y = C_4 - (C_1 + 2C_3)t - \dfrac{1}{2}(C_2 - 1)t^2 - \dfrac{1}{3}C_3t^3 + \dfrac{t^4}{24} - e^t. \end{cases}$

4335. $\begin{cases} x + y + z = C_1, \\ x^2 + y^2 + z^2 = C_2. \end{cases}$ **4336.** $\begin{cases} z = x - y, \\ y(y - 2x)^3 = (x - y)^2. \end{cases}$

4337. $\begin{cases} x = \dfrac{t}{3}. \\ y = -\dfrac{t}{3}. \end{cases}$

4338. $\begin{cases} x = \dfrac{1}{3}e^{-t} + \dfrac{1}{6}e^{2t} + \dfrac{1}{2}e^{-2t}, \\ y = \dfrac{1}{3}e^{-t} + \dfrac{1}{6}e^{2t} - \dfrac{1}{2}e^{-2t}, \\ z = -\dfrac{1}{3}e^{-t} + \dfrac{1}{3}e^{2t}. \end{cases}$ **4339.** $\begin{cases} x = -e^{-t}, \\ y = e^{-t}, \\ z = 0. \end{cases}$

4340. The curves $y_1 = \dfrac{C_1x^2 - C_2}{2x}$ and $y_2 = -\dfrac{C_1x^2 + C_2}{2x}$. Given the initial conditions, we obtain the hyperbolas

$$y_1 = \frac{3 - x^2}{2x}, \quad y_2 = \frac{3 + x^2}{2x}.$$

4341. $y = e^{2x}$. **4342.** The plane curve $\begin{cases} x - y + z = 0; \\ x = \pm\dfrac{z\ln|z|}{\sqrt{2}}. \end{cases}$

4343. $\begin{cases} x = \dfrac{1}{2}\left[gt^2 + (l_1 - l_0)\left(1 - \cos\dfrac{\pi t}{2T}\right)\right], \\ y = \dfrac{1}{2}\left[gt^2 + l_0 + l_1 + (l_1 - l_0)\cos\dfrac{\pi t}{2T}\right]. \end{cases}$

4344. $\begin{cases} x = 10\cosh 2t - \dfrac{4}{49}\cos 14t + \dfrac{200}{49}, \\ y = 10\cosh 2t + \dfrac{6}{49}\cos 14t - \dfrac{300}{49}. \end{cases}$

Here x is the path of the heavier sphere, and y that of the lighter.

4345. $A = \dfrac{k\alpha^2}{2k_1}\left[1 - \left(\dfrac{1-\beta e^{\alpha kt}}{1+\beta e^{\alpha kt}}\right)^2\right]$, $B = \alpha\,\dfrac{1-\beta e^{\alpha kt}}{1+\beta e^{\alpha kt}}$, where

$$\alpha = \sqrt{B_0^2 + \frac{2k_1}{k}A_0}, \quad \beta = \frac{\alpha - B_0}{\alpha + B_0}.$$

4346*. If T is the quantity of poison, then $\dfrac{dN}{dt} = aN - bNT$, $\dfrac{dT}{dt} = cN$ and $\dfrac{dN}{dt} = 0$ at the instant when $N = M$.

4347. $h_1 = \dfrac{S_1H_1 + S_2H_2}{S_1 + S_2} + \dfrac{S_2}{S_1 + S_2}(H_1 - H_2)\, e^{-\alpha\frac{S_1+S_2}{S_1S_2}}$,

$$h_2 = \frac{S_1H_1 + S_2H_2}{S_1 + S_2} - \frac{S_1}{S_1 + S_2}(H_1 - H_2)\, e^{-\alpha\frac{S_1+S_2}{S_1S_2}}.$$

4348. (1) $\theta - \theta_0 + 0{\cdot}002\,(\theta^2 - \theta_0^2) = 0{\cdot}00008\,\dfrac{E_1^2 t^3}{R_0 T^2}$; by 53°;

(2) $\theta - \theta_0 + 0{\cdot}002(\theta^2 - \theta_0^2) = \dfrac{6E_0^2}{\pi R_0 \times 10^7}(200\pi t - \sin 200\pi t)$; by 76°.

4349. (1) 44·5°; (2) 46·2°.

4350.

x	1·00	1·05	1·10	1·15	1·20	1·25
y	1·000	1·000	0·997	0·992	0·984	0·973

x	1·30	1·35	1·40	1·45	1·50
y	0·959	0·942	0·923	0·901	0·876

4351. $y|_{x=1} = 3{\cdot}43656\ldots$

y_1	y_2	y_3	y_4	y_5
2·5	3·16667	3·37500	3·42500	3·43472

y_5 gives a relative error of the order of 0·1%.

4352. 0·46128; Simpson's formula with $2n = 10$ gives the same. All the places are correct.

4353. $y_4 = 1 + x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{7x^4}{12} + \frac{5x^5}{12} + \frac{16x^6}{75} + \ldots;$

$y_4\,(0{\cdot}3) \approx 1{\cdot}543;$

$f(x) = 1 + x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{7x^4}{12} + \frac{11x^5}{20} + \frac{22x^6}{45} + \ldots \quad f(0{\cdot}3) \approx 1{\cdot}545.$

The error is less than 0·2%.

4354. 0·808.

4355*. 1·001624. The result is obtained most rapidly if the required function is sought directly in the form of a power series.

4356*. 1·0244. See the hint on the previous problem.

4357. $y = z + \frac{2}{4!}x^4 + \frac{2\,.\,5}{7!}x^7 + \ldots +$

$$+ \frac{2\,.\,5\ldots(3n-1)}{(3n+1)!}x^{3n+1} + \ldots; \quad k = 0{\cdot}2297.$$

Chapter XV

4358. $\sin^{2k} x = \frac{C_{2k}^{k}}{2^{2k}} + \frac{(-1)^k}{2^{2k-1}}[\cos 2kx - C_{2k}^{1}\cos(2k-2)x +$

$+ C_{2k}^{2}\cos(2k-4)x - \ldots + (-1)^{k-1}(C_{2k}^{k-1}\cos 2x];$

$\sin^{2k+1} x = \frac{(-1)^k}{2^{2k}}[\sin(2k+1)x - C_{2k+1}^{1}\sin(2k-1)x +$

$+ C_{2k+1}^{2}\sin(2k-3)x - \ldots + (-1)^k C_{2k+1}^{k}\sin x];$

$\cos^{2k} x = \frac{C_{2k}^{k}}{2^{2k}} + \frac{1}{2^{2k-1}}[\cos 2kx + C_{2k}^{1}\cos(2k-2)x +$

$+ C_{2k}^{2}\cos(2k-4)x + \ldots + C_{2k}^{k-1}\cos 2x];$

$\cos^{2k+1} x = \frac{1}{2^{2k}}[\cos(2k+1)x + C_{2k+1}^{1}\cos(2k-1)x +$

$+ C_{2k+1}^{2}\cos(2k-3)x + \ldots + C_{2k+1}^{k}\cos x].$

4360. $\cos nx = \cos^n x - C_n^2\cos^{n-2}x\sin^2 x + C_n^4\cos^{n-4}x\sin^4 x \ldots$

Since there are only even powers of $\sin x$, $\cos nx$ can be expressed rationally in terms of $\cos x$.

4363. (1) $\varphi = \nu\frac{2\pi}{n}$ and $\varphi = \nu\frac{2\pi}{n+1}$, where $\nu = 0, 1, 2, \ldots, n;$

(2) $\varphi = \nu\frac{2\pi}{n}$, where $\nu = 1, 2, \ldots, n-1$ for n odd and $\nu = 1, 2, \ldots, n$

for n even, and $\varphi = (2\nu - 1)\dfrac{\pi}{n+1}$, where $\nu = 1, 2, \ldots, n + 1$.

4365*. Notice that $\int\limits_0^{2\pi} \Phi_n(\varphi)\,d\varphi = 0$.

4366. Yes, since the function satisfies the conditions of the first fundamental theorem (see *Course*, sec. 212).

4371. (a) $b_1 = b_2 = b_3 = \ldots = 0$ and $a_1 = a_3 = a_5 = \ldots = 0$;
(b) $a_0 = a_1 = a_2 = \ldots = 0$ and $b_1 = b_3 = b_5 = \ldots = 0$.

4372. $\dfrac{4}{\pi}\sum\limits_{n=0}^{\infty}\dfrac{\sin(2n+1)x}{2n+1}$. **4373.** $\sum\limits_{n=1}^{\infty}\dfrac{\sin 2nx}{2n}$.

4374. $x = 2\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{\sin nx}{n}\quad(-\pi, \pi)$;

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty}\frac{\sin nx}{n}\quad(0, 2\pi).$$

4375. $\dfrac{2}{\pi}\sum\limits_{n=0}^{\infty}\dfrac{\cos(2n+1)x}{(2n+1)^2}$.

4376. (1) $\dfrac{\pi^2}{3} + 4\sum\limits_{n=1}^{\infty}(-1)^n\dfrac{\cos nx}{n^2}$;

(2) $\dfrac{4\pi^2}{3} + 4\sum\limits_{n=1}^{\infty}\dfrac{\cos nx}{n^2} - 4\pi\sum\limits_{n=1}^{\infty}\dfrac{\sin nx}{n}$;

$$S_1 = \frac{\pi^2}{6},\quad S_2 = \frac{\pi^2}{12},\quad S_3 = \frac{\pi^2}{8}.$$

4377. $\dfrac{2}{\pi}\sum\limits_{n=1}^{\infty}(-1)^{n+1}\left\{\dfrac{\pi^2}{n} + \dfrac{2}{n^3}[(-1)^n - 1]\right\}\sin nx$.

4378. $\sum\limits_{n=1}^{\infty}(-1)^n\left(\dfrac{12}{n^3} - \dfrac{2\pi^2}{n}\right)\sin nx$.

4379. $2 + \dfrac{4}{\pi}\sum\limits_{n=0}^{\infty}\dfrac{\sin(2n+1)x}{2n+1}$.

4380. $\dfrac{2h}{\pi}\left[\dfrac{1}{2} + \sum\limits_{n=1}^{\infty}\dfrac{\sin nh}{nh}\cos nx\right]$.

4381. $\dfrac{2h}{\pi}\left[\dfrac{1}{2} + \sum\limits_{n=1}^{\infty}\left(\dfrac{\sin nh}{nh}\right)^2\cos nx\right]$.

4382. $\dfrac{l}{2} - \dfrac{4l}{\pi^2} \sum_{n=0}^{\infty} \dfrac{\cos\left[\dfrac{(2n+1)\,\pi x}{l}\right]}{(2n+1)^2}$.

4383. $\dfrac{e^{2\pi} - 1}{\pi} \left[\dfrac{1}{2} + \sum_{n=1}^{\infty} \left(\dfrac{\cos nx}{1+n^2} - \dfrac{n \sin nx}{1+n^2}\right)\right] - 1$.

4384. $$\frac{e^l - e^{-l}}{2l} + l(e^l - e^{-l}) \sum_{n=1}^{\infty} \frac{(-1)^n \cos \dfrac{n\pi x}{l}}{l^2 + n^2\pi^2} +$$

$$+ \pi(e^l - e^{-l}) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n \sin \dfrac{n\pi x}{l}}{l^2 + n^2\pi^2} =$$

$$= \sinh l \left[\frac{1}{l} + 2 \sum_{n=1}^{\infty} (-1)^n \frac{l \cos \dfrac{n\pi x}{l} - \pi n \sin \dfrac{n\pi x}{l}}{l^2 + n^2\pi^2}\right].$$

4385. $\dfrac{2 \sin \pi a}{\pi} \left(\dfrac{1}{2a} + \dfrac{a \cos x}{1 - a^2} - \dfrac{a \cos 2x}{2^2 - a^2} + \ldots\right)$.

4386. $\dfrac{2 \sin \pi a}{\pi} \left(\dfrac{\sin x}{1 - a^2} - \dfrac{2 \sin 2x}{2^2 - a^2} + \dfrac{3 \sin 3x}{3^2 - a^2} - \ldots\right)$.

4387. $$\sin ax = \begin{cases} \dfrac{4a}{\pi}\left[\dfrac{\cos x}{a^2 - 1} + \dfrac{\cos 3x}{a^2 - 3^2} + \dfrac{\cos 5x}{a^2 - 5^2} + \ldots\right] \\ (a \text{ even}); \\ \dfrac{4a}{\pi}\left[\dfrac{1}{2a^2} + \dfrac{\cos 2x}{a^2 - 2^2} + \dfrac{\cos 4x}{a^2 - 4^2} + \ldots\right] \\ (a \text{ odd}). \end{cases}$$

4388. $$\cos ax = \begin{cases} -\dfrac{4}{\pi}\left[\dfrac{\sin x}{a^2 - 1^2} + \dfrac{3 \sin 3x}{a^2 - 3^2} + \dfrac{5 \sin 5x}{a^2 - 5^2} + \ldots\right] \\ (a \text{ even}) \\ -\dfrac{4}{\pi}\left[\dfrac{2 \sin 2x}{a^2 - 2^2} + \dfrac{4 \sin 4x}{a^2 - 4^2} + \dfrac{6 \sin 6x}{a^2 - 6^2} + \ldots\right] \\ (a \text{ odd}). \end{cases}$$

4389. $\dfrac{2 \sinh a\pi}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{n}{a^2 + n^2} \sin nx$.

4390. $\dfrac{\sinh \pi}{\pi}\left[1 + 2\sum_{n=1}^{\infty}(-1)^n \dfrac{\cos nx}{1+n^2}\right]$;

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n \cosh \pi}{1+n^2} n \sin nx.$$

4391. $f(x) = \dfrac{1}{3} + \dfrac{2}{\pi}\sum_{n=1}^{\infty}\left[\dfrac{\sin \dfrac{2\pi n}{3}}{n} - \dfrac{3\left(1-\cos\dfrac{2\pi n}{3}\right)}{2\pi n^2}\right]\cos\dfrac{2\pi n x}{3}$

$$= \frac{1}{3} + \frac{\sqrt{3}}{\pi}\left(\frac{\cos\frac{2\pi x}{3}}{1} - \frac{\cos\frac{4\pi x}{3}}{2} + \frac{\cos\frac{8\pi x}{3}}{4} - \dots\right) -$$

$$-\frac{9}{2\pi^2}\left(\frac{\cos\frac{2\pi x}{3}}{1^2} + \frac{\cos\frac{4\pi x}{3}}{2^2} + \frac{\cos\frac{8\pi x}{3}}{4^2} + \dots\right).$$

4392*. $f(x) = \dfrac{\pi}{6} + \dfrac{3}{2\pi}\sum_{n=1}^{\infty}\dfrac{\sin\dfrac{n\pi}{3}}{n^2}\left(\cos\dfrac{n\pi}{3}\sin 2nx - \sin\dfrac{n\pi}{3}\cos 2nx\right)$

$$= \frac{\pi}{6} + \frac{3\sqrt{3}}{8\pi}\left(\frac{\sin 2x}{1^2} - \frac{\sin 4x}{2^2} + \frac{\sin 8x}{4^2} - \frac{\sin 10x}{5^2} + \dots\right) -$$

$$-\frac{9}{8\pi}\left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 8x}{4^2} + \frac{\cos 10x}{5^2} + \dots\right).$$

Use the result of problem 4368.

4393*. (1) $f(x) = \dfrac{4}{\pi}\left(\dfrac{\sin\alpha \sin x}{1^2} + \dfrac{\sin 3\alpha \sin 3x}{3^2} + \dots\right)$;

(2) $f(x) = \dfrac{\alpha(\pi-\alpha)}{\pi} - \dfrac{1}{\pi}\sum_{n=1}^{\infty}\dfrac{1-\cos 2n\alpha}{n^2}\cos 2nx$

$$= \frac{\alpha(\pi-\alpha)}{\pi} - \frac{2}{\pi}\left(\frac{\sin^2\alpha\cos 2x}{1^2} + \frac{\sin^2 2\alpha \cos 4x}{2^2} + \dots\right).$$

Use the result of problem 4371.

4394 (4094). $\dfrac{8}{\pi}\sum_{n=1}^{\infty}\dfrac{\sin(2n-1)x}{(2n-1)^3}$; $\dfrac{\pi^3}{32}$.

4395 (4095). (b) $\dfrac{8}{15}\pi^4 - 48\sum_{n=1}^{\infty}(-1)^n\dfrac{\cos nx}{n^4}$; (c) $\dfrac{7}{720}\pi^4$.

4396*. $\dfrac{\pi - x}{2} - \sum_{n=1}^{\infty}\dfrac{\sin nx}{n(n^2+1)}$ (see problem 4374).

4397*. $\frac{x}{2}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n-1}{n(n^2+1)}\sin nx$ (see problem 4374).

4398*. $\frac{(\pi-x)^2}{4}-\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{n^2-1}{n^2(n^4+1)}\cos nx.$

Differentiate the series and use the solution of problem 4374 and the fact that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ (see problem 4376).

4399. $\frac{\pi^3}{32}+\frac{\pi}{4}-\frac{\pi x^2}{8}-2\cos x+\sum_{n=2}^{\infty}\frac{\sin n\frac{\pi}{2}}{n^3(n^2-1)}\cos nx$ $\left(-\frac{\pi}{2}<x<\frac{\pi}{2}\right)$;

Use the series $\frac{\pi}{4}=\sum_{n=1}^{\infty}\frac{\sin\frac{\pi n}{2}}{n}\cos nx$ $\left(\text{see problem 4380 with } h=\frac{\pi}{2}\right)$ and the fact that $\sum\frac{(-1)^{n-1}}{n^3}=\frac{\pi^3}{32}$ (see problem 4394).

4400. $f_1(x)\approx 27{\cdot}8+6{\cdot}5\cos x-0{\cdot}1\sin x-3{\cdot}2\cos 2x+0{\cdot}1\sin 2x$;
$f_2(x)\approx 0{\cdot}24+0{\cdot}55\cos x+0{\cdot}25\sin x-0{\cdot}08\cos 2x-0{\cdot}13\sin 2x$;
$f_3(x)\approx 0{\cdot}12+1{\cdot}32\cos x+0{\cdot}28\sin x-0{\cdot}07\cos 2x+0{\cdot}46\sin 2x$.

Chapter XVI

4401. Straight lines parallel to the vector $A\{a, b, c\}$: $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$.

4402. Circles with centres at the origin: $x^2+y^2=R^2$.

4403. Helices with pitch $\frac{2\pi h}{\omega}$, lying on cylinders whose axes coincide with the z axis: $x=R\cos(\omega t+\alpha)$, $y=R\sin(\omega t+\alpha)$, $z=ht+z_0$, where R, α and z_0 are arbitrary constants.

4404. The circles formed by cutting the sphere with centre at the origin by planes parallel to the bisector plane $y-z=0$: $x^2+y^2+z^2=R^2$, $y-z+C=0$, where R and C are arbitrary con stants.

(2) Circles formed by cutting a sphere with centre at the origin by planes which cut off from the coordinate axes segments equal in direction and magnitude: $x^2+y^2+z^2=R^2$, $x+y+z=C$.

(3) The curves of intersection of the sphere $x^2 + y^2 + z^2 = R^2$ and the hyperbolic paraboloids $zy = Cx$.

4405. $\operatorname{div} \boldsymbol{A} = 3$, $\operatorname{curl} \boldsymbol{A} = 0$.

4406. $\operatorname{div} \boldsymbol{A} = 0$, $\operatorname{curl} \boldsymbol{A} = 2[(y - z)\,\boldsymbol{i} + (z - x)\,\boldsymbol{j} + (x - y)\,\boldsymbol{k}]$.

4407. $\operatorname{div} \boldsymbol{A} = 6xyz$, $\operatorname{curl} \boldsymbol{A} = x(z^2 - y^2)\,\boldsymbol{i} + y(x^2 - z^2)\,\boldsymbol{j} + z(y^2 - x^2)\,\boldsymbol{k}$.

4408. $\operatorname{div} \boldsymbol{A} = 6$, $\operatorname{curl} \boldsymbol{A} = 0$.

4409. $\operatorname{div} \boldsymbol{A} = 0$, $\operatorname{curl} \boldsymbol{A} = 0$.

4410. $\operatorname{div} \boldsymbol{A} = \frac{k}{r^3}$, where k is a coefficient of proportionality, $\boldsymbol{r}$ the distance from the point of application of the force to the origin, $\operatorname{curl} \boldsymbol{A} = 0$.

4411. $\operatorname{div} \boldsymbol{A} = 0$, $\operatorname{curl} \boldsymbol{A} = 0$.

4412. $\operatorname{div} \boldsymbol{A} = 0$, $\operatorname{curl} \boldsymbol{A} = 0$. The field is not defined at points of Oz.

4413. $\operatorname{div} \boldsymbol{A} = -\frac{k}{z\sqrt{x^2 + y^2 + z^2}}$, where k is a coefficient of proportionality. The field is not defined at points of the Oxy plane.

4414. $3\boldsymbol{a}$. **4416.** $\operatorname{div} \boldsymbol{b}(\boldsymbol{ra}) = (\boldsymbol{ab})$, $\operatorname{div} \boldsymbol{r}(\boldsymbol{ra}) = 4(\boldsymbol{ra})$.

4417. 0. **4418.** (1) 0, (2) 0, (3) 0.

4419. $\operatorname{div} \boldsymbol{A} = \frac{2f(r)}{r} + f'(r)$ if the field is spatial; $\operatorname{div} \boldsymbol{A} = \frac{f(r)}{r} + f'(r)$ if the field is plane.

4421. $\varphi \operatorname{curl} \boldsymbol{A} + (\operatorname{grad} \varphi \times \boldsymbol{A})$. **4422.** $\frac{\boldsymbol{r} \times \boldsymbol{A}}{|\boldsymbol{r}|}$.

4423. $2\boldsymbol{a}$. **4424.** $2\omega \boldsymbol{n}^0$, where $\boldsymbol{n}^0$ is the unit vector parallel to the axis of rotation.

4431. $u = -\frac{1}{2} k(x^2 + y^2 + z^2) + C$.

4432. No. **4433.** No. **4434.** $u = -\frac{1}{2} \ln (x^2 + y^2) + C$.

4435. No. **4437.** $\frac{2}{3}, \frac{1}{3}, \frac{1}{2}$. **4438.** $k\delta \ln \frac{\sqrt{(l - x)^2 + y^2} + l - x}{\sqrt{(l + x)^2 + y^2} - l - x}$.

4439. $4k(\sqrt{2} - 1)$.

4440. $\frac{k\delta\sqrt{a^2 + b^2}}{b} \ln \frac{2\pi b + \sqrt{a^2 + 4\pi^2 b^2}}{a}$.

4441. $2k\delta a(1+\sqrt{2})$. **4442.** $\dfrac{2\pi k}{\sqrt{1-h^2}}\arccos h$, if $h<1$, $2\pi k$ if $h=1$, $\dfrac{2\pi k}{\sqrt{h^2-1}}\ln\left(h+\sqrt{h^2-1}\right)$, if $h>1$.

4443*. (1) $2k\pi R\delta\ln\dfrac{H+\sqrt{H^2+R^2}}{R}$, (2) $4k\pi R\delta\ln\dfrac{H+\sqrt{H^2+4R^2}}{2R}$

Divide the cylinder into two by cutting it parallel to the base, and work out the potential of the lateral surface of the cylinder as the sum of the potentials of the lateral surfaces of the two halves, by using the result of (1).

4444. $2k\pi R\delta$.

4445*. (1) $k\pi\delta\left[H\sqrt{R^2+H^2}-H^2+R^2\ln\dfrac{H+\sqrt{R^2+H^2}}{R}\right]$,

(2) $\dfrac{k\pi\delta}{2}\left[H\sqrt{4R^2+H^2}-H^2+4R^2\ln\dfrac{H+\sqrt{4R^2+H^2}}{2R}\right]$;

see the hint on problem 4443.

4446. $\pi k\delta H(l-H)$, where l is the generator of the cone.

4447. $u=\dfrac{2}{3}k\dfrac{\pi R^3\delta}{a}\left[\left(1+\dfrac{a^2}{R^2}\right)^{\frac{3}{2}}-\left(\dfrac{a}{R}\right)^{\frac{3}{2}}-\dfrac{3a}{2R}+1\right]$ for $a\geqq R$;

$$u=\frac{2}{3}k\pi a^2\delta\left[\left(1+\frac{R^2}{a^2}\right)^{\frac{3}{2}}-\left(\frac{R}{a}\right)^{\frac{3}{2}}+\frac{3}{2}\left(\frac{R}{a}\right)^2-2\right]\text{ for } a\leqq R;$$

$$u=\frac{k\pi R^2\delta}{3}\left(4\sqrt{2}-3\right)\text{ for } a=R.$$

4448*. $u=\dfrac{4k\pi\delta}{3a}(R^3-r^3)=\dfrac{kM}{a}$ (M is the mass of the body) for $a\geqq R$;

$u=2k\pi\delta(R^2-r^2)$ for $a\leqq r$:

$u=\dfrac{4k\pi\delta}{3a}(a^3-r^3)+2\pi\delta(R^2-a^2)$ for $r\leqq a\leqq R$.

Draw a concentric sphere of radius a and apply the results of the two previous problems.

4449. $\dfrac{kM}{R}\left[\dfrac{R}{a}+\dfrac{2}{7}\left(\dfrac{R}{a}\right)^3\right]$, where M is the mass of the sphere.

4450. Both the flux and circulation are equal to 0.

4451. The flux is equal to $2aS$, where S is the area of the domain bounded by the contour L. The circulation is 0.

4452. Both the flux and circulation are equal to 0.

4453. The flux is $\frac{3}{2}\pi R^4$, the circulation $2\pi R^2$.

4454. When the origin lies inside the contour, the flux is equal to 2π, otherwise the flux is equal to 0. The circulation is 0 in both cases.

4455. The circulation is equal to 2π if the origin lies inside the contour, and is equal to 0 if outside the contour. The flux is 0 in both cases.

4456. 2. **4458.** $2\pi R^2 H$. **4459.** $\pi R^2 H$.

4460. 4π. Work out the flux through the base of the cone and use the result of problem 4457.

4461. $\frac{3\pi}{16}$. **4462*.** $\frac{1}{6}$. Use Ostrogradskii's formula and work out the flux through the base of the pyramid.

4463. $2\pi^2 b^2$. **4464.** $2\pi\omega R^2$.

4465. $-\pi$. Use Stokes's theorem, taking as the contour L the curve of intersection of the paraboloid with the Oxy plane.

APPENDIX

TABLES

1. Trigonometric Functions

$\alpha°$	$\sin\alpha$	$\tan\alpha$	$\operatorname{cotan}\alpha$	$\cos\alpha$	
0	0·0000	0·0000	—	1·000	90
1	0175	0175	57·3	1·000	89
2	0349	0349	28·6	0·999	88
3	0523	0524	19·1	999	87
4	0697	0699	14·3	998	86
5	0·0872	0·0875	11·4	0·996	85
6	1045	1051	9·51	995	84
7	1219	1228	8·11	993	83
8	139	141	7·11	990	82
9	156	158	6·31	988	81
10	0·174	0·176	5·67	0·985	80
11	191	194	5·145	982	79
12	208	213	4·705	978	78
13	225	231	4·331	974	77
14	242	249	4·011	970	76
15	0·259	0·268	3·732	0·966	75
16	276	287	487	961	74
17	292	306	271	956	73
18	309	325	3·078	951	72
19	326	344	2·904	946	71
20	0·342	0·364	2·747	0·940	70
21	358	384	605	934	69
22	375	404	475	927	68
23	391	424	356	921	67
24	407	445	246	914	66
25	0·423	0·466	2·145	0·906	65
26	438	488	2·050	899	64
27	454	510	1·963	891	63
28	469	532	881	883	62
29	485	554	804	875	61
30	0·500	0·577	1·732	0·866	60
31	515	601	664	857	59
32	530	625	600	848	58
33	545	649	540	839	57
34	559	675	483	829	56
35	0·574	0·700	1·428	0·819	55
36	588	727	376	809	54
37	601	754	327	799	53
38	616	781	280	788	52
39	629	810	235	777	51
40	0·643	0·839	1·192	0·766	50
41	656	869	150	755	49
42	669	900	111	743	48
43	682	933	072	731	47
44	695	966	036	719	46
45	0·707	1·000	1·000	0·707	45
	$\cos\alpha$	$\cot\alpha$	$\tan\alpha$	$\sin\alpha$	$\alpha°$

$\alpha°$	a radians	$\sin a$	$\tan a$
0	0	0·000	+0·000
5·73	0·1	0·100	+0·100
11·5	0·2	0·199	+0·203
17·2	0·3	0·296	+0·310
22·9	0·4	0·389	+0·422
28·7	0·5	0·480	+0·547
34·4	0·6	0·564	+0·68
40·1	0·7	0·644	+0·842
45·0	$\frac{\pi}{4}$	0·707	+1·000
45·8	0·8	0·717	+1·028
51·6	0·9	0·784	+1·260
57·3	1·0	0·842	+1·558
63·0	1·1	0·891	+1·963
68·8	1·2	0·932	+2·579
74·5	1·3	0·964	+3·606
80·2	1·4	0·985	+5·789
86·0	1·5	0·998	+14·30
90·0	$\frac{\pi}{2}$	1·000	—
91·7	1·6	0·999	−33·75
97·4	1·7	0·992	−7·695
103·1	1·8	0·974	−4·292
108·9	1·9	0·946	−2·921
114·6	2·0	0·909	−2·184
120·3	2·1	0·863	−1·711
126·1	2·2	0·808	−1·373
131·8	2·3	0·745	−1·118
135·0	$\frac{3\pi}{4}$	0·707	−1·000
137·5	2·4	0·676	−0·916
143·2	2·5	0·599	−0·748
149·0	2·6	0·515	−0·602
154·7	2·7	0·428	−0·472
160·4	2·8	0·336	−0·356
166·1	2·9	0·240	−0·247
171·9	3·0	0·141	−0·142
177·6	3·1	0·042	−0·042
180·0	π	0·000	0·000

$$\sin\frac{\pi}{6} = \frac{1}{2}, \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2},$$

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}, \qquad \cot\frac{\pi}{6} = \sqrt{3},$$

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

$$\tan\frac{\pi}{4} = \cot\frac{\pi}{4} = 1.$$

α degrees	1	2	3	4	5	6	7	8	9
a radians	0·017	0·035	0·052	0·070	0·087	0·105	0·122	0·140	0·157

1 radian = 57°17′45″

2. Hyperbolic Functions

x	sinh x	cosh x	x	sinh x	cosh x
0	0	1			
0·1	0·100	1·005	2·1	4·022	4·289
0·2	0·201	1·020	2·2	4·457	4·568
0·3	0·304	1·045	2·3	4·937	5·037
0·4	0·411	1·081	2·4	5·466	5·557
0·5	0·521	1·128	2·5	6·050	6·132
0·6	0·637	1·185	2·6	6·695	6·769
0·7	0·759	1·255	2·7	7·407	7·474
0·8	0·888	1·337	2·8	8·192	8·253
0·9	1·026	1·433	2·9	9·060	9·115
1·0	1·175	1·543	3·0	10·02	10·07
1·1	1·336	1·669	3·1	11·08	11·12
1·2	1·509	1·811	3·2	12·25	12·29
1·3	1·698	1·971	3·3	13·54	13·58
1·4	1·904	2·151	3·4	14·97	15·00
1·5	2·129	2·352	3·5	16·54	16·57
1·6	2·376	2·578	3·6	18·29	18·32
1.7	2·646	2·828	3·7	20·21	20·24
1·8	2·942	3·107	3·8	22·34	22·36
1·9	3·268	3·418	3·9	24·69	24·71
2·0	3·627	3·762	4·0	27·29	27·31

With $x > 4$ we can take $\sinh x \sim \cosh x \sim \frac{1}{2} e^x$ to an accuracy of 0·1.

$$\sinh x = \frac{e^x - e^{-x}}{2}; \qquad \cosh x = \frac{e^x + e^{-x}}{2};$$

$$e^x = \sinh x + \cosh x; \qquad e^{xi} = \sin x + i \cos x.$$

3. Reciprocals, Square and Cube Roots, Logarithms, Exponential Functions

x	$\frac{1}{x}$	$\sqrt{x}$	$\sqrt{10x}$	$\sqrt[3]{x}$	$\sqrt[3]{10x}$	$\sqrt[3]{100x}$	log x	ln x	e^x	x
1·0	1·000	1·00	3·16	1·00	2·15	4·64	000	0·000	2·72	1·0
1·1	0·909	05	32	03	22	79	041	095	3·00	1·1
1·2	833	10	46	06	29	93	079	192	3·32	1·2
1·3	769	14	61	09	35	5·07	114	252	3·67	1·3
1·4	714	18	74	12	41	19	146	336	4·06	1·4
1·5	0·667	1·23	3·87	1·15	2·47	5·13	176	0·405	4·48	1·5
1·6	625	27	4·00	17	52	43	204	470	4·95	1·6
1·7	588	30	12	19	57	54	230	530	5·47	1·7
1·8	556	34	24	22	62	65	255	588	6·05	1·8
1·9	526	38	36	24	67	75	279	642	6·69	1·9
2·0	0·500	1·41	4·47	1·26	2·71	5·85	301	0·693	7·39	2·0
2·1	476	45	58	28	76	94	322	742	8·17	2·1
2·2	455	48	69	30	80	6·03	342	789	9·03	2·2
2·3	435	52	80	32	84	13	362	833	9·97	2·3
2·4	417	55	90	34	88	21	380	875	11·0	2·4

x	$\frac{1}{x}$	$\sqrt{x}$	$\sqrt{10x}$	$\sqrt[3]{x}$	$\sqrt[3]{10x}$	$\sqrt[3]{100x}$	$\log x$	$\ln x$	e^x	x
2·5	0·400	1·58	5·00	1·36	2·92	6·30	398	0·916	12·2	2·5
2·6	385	61	10	38	96	38	415	955	13·5	2·6
2·7	370	64	20	39	3·00	46	431	993	14·9	2·7
2·8	357	67	29	41	04	54	447	1·030	16·4	2·8
2·9	345	70	39	43	07	62	462	065	18·2	2·9
3·0	0·333	1·73	5·48	1·44	3·11	6·69	477	1·099	20·1	3·0
3·1	323	76	57	46	14	77	491	131	22·2	3·1
3·2	313	79	66	47	18	84	505	163	24·5	3·2
3·3	303	81	75	49	21	91	519	194	27·1	3·3
3·4	294	84	83	50	24	98	532	224	30·0	3·4
3·5	0·286	1·87	5·92	1·52	3·27	7·05	544	1·253	33·1	3·5
3·6	278	90	6·00	53	30	11	556	281	36·6	3·6
3·7	270	92	08	55	33	18	568	308	40·4	3·7
3·8	263	95	16	56	36	24	580	335	44·7	3·8
3·9	256	98	25	57	39	31	591	361	49·4	3·9
4.0	0·250	2·00	6·33	1·59	3·42	7·37	602	1·386	54·6	4·0
4·1	244	03	40	60	45	43	613	411	60·3	4·1
4·2	238	05	48	61	48	49	623	435	66·7	4·2
4·3	233	07	56	63	50	55	634	458	73·7	4·3
4·4	227	10	63	64	53	61	644	482	81·5	4·4
4·5	0·222	2·12	6·71	1·65	3·56	7·66	653	1·504	90·0	4·5
4·6	217	15	78	66	58	72	663	526	99·5	4·6
4·7	213	17	86	68	61	78	672	548	110	4·7
4·8	208	19	93	69	63	83	681	569	121	4·8
4·9	204	21	7·00	70	66	88	690	589	134	4·9
5·0	0·200	2·24	7·07	1·71	3·68	7·94	699	1·609	148	5·0
5·1	196	26	14	72	71	99	708	629	164	5·1
5·2	192	28	21	73	73	8·04	716	649	181	5·2
5·3	189	30	28	74	76	09	724	668	200	5·3
5·4	185	32	35	75	78	14	732	686	221	5·4
5·5	0·182	2·35	7·42	1·77	3·80	8·19	740	1·705	244	5·5
5·6	179	37	48	78	83	24	748	723	270	5·6
5·7	175	39	55	79	85	29	756	740	299	5·7
5·8	172	41	62	80	87	34	763	758	330	5·8
5·9	170	43	68	81	89	39	771	775	365	5·9
6·0	0·167	2·45	7·75	1·82	3·92	8·43	778	1·792	403	6·0
6·1	164	47	81	83	94	48	785	808	446	6·1
6·2	161	49	87	84	96	53	792	825	493	6·2
6·3	159	51	94	85	98	57	799	841	545	6·3
6·4	156	53	8·00	86	4·00	62	806	856	602	6·4
6·5	0·154	2·55	8·06	1·87	4·02	8·66	813	1·872	665	6·5
6·6	152	57	12	88	04	71	820	887	735	6·6
6·7	149	59	19	89	06	75	826	902	812	6·7
6·8	147	61	25	90	08	79	833	918	898	6·8
6·9	145	63	31	90	10	84	839	932	992	6·9
7·0	0·143	2·65	8·37	1·91	4·12	8·88	845	1·946	1097	7·0
7·1	141	67	43	92	14	92	851	960	1212	7·1
7·2	139	68	49	93	16	96	857	974	1339	7·2
7·3	137	70	54	94	18	9·00	863	982	1480	7·3
7·4	135	72	60	95	20	05	869	2·001	1636	7·4
7·5	0·133	2·74	8·66	1·96	4·22	9·09	875	2·015	1808	7·5
7·6	132	76	72	97	24	13	881	028	1998	7·6
7·7	130	78	78	98	25	17	887	041	2208	7·7
7·8	128	79	83	98	27	21	892	054	2440	7·8
7·9	127	81	89	99	29	24	898	067	2697	7·9

x	$\frac{1}{x}$	$\sqrt{x}$	$\sqrt{10x}$	$\sqrt[3]{x}$	$\sqrt[3]{10x}$	$\sqrt[3]{100x}$	$\log x$	$\ln x$	e^x	x
8·0	0·125	2·83	8·94	2·00	4·31	9·28	903	2·079	2981	8·0
8·1	124	85	9·00	01	33	32	909	092	3294	8·1
8·2	122	86	06	02	34	36	914	104	3641	8·2
8·3	121	88	11	03	36	40	919	116	4024	8·3
8·4	119	90	17	03	38	44	924	128	4447	8·4
8·5	0·118	2·92	9·22	2·04	4·40	9·47	929	2·140	4914	8·5
8·6	116	93	27	05	41	51	935	152	5432	8·6
8·7	115	95	33	06	43	55	940	163	6003	8·7
8·8	114	97	38	07	45	58	945	175	6634	8·8
8·9	112	98	43	07	47	62	949	186	7332	8·9
9·0	0·111	3·00	9·49	2·08	4·48	9·66	954	2·197	8103	9·0
9·1	110	02	54	09	50	69	959	208	8955	9·1
9·2	109	03	59	10	51	73	964	219	9897	9·2
9·3	108	05	64	10	53	76	969	230	10938	9·3
9·4	106	07	68	11	55	80	973	241	12088	9·4
9·5	0·105	3·08	9·75	2·12	4·56	9·83	978	2·251	13360	9·5
9·6	104	10	80	13	58	87	982	263	14765	9·6
9·7	103	11	84	13	60	90	987	272	16318	9·7
9·8	102	13	90	14	61	93	991	282	18034	9·8
9·9	101	15	95	15	63	97	996	293	19930	9·9
10·0	0·100	3·16	10·00	2·15	4·64	10·00	000	2·303	22026	10·0

The mantissae of the common logarithms are given in the $\log x$ column.

To find the natural logarithms of numbers greater than 10 or less than 1 we have to use the formula

$$\ln(x \cdot 10^k) = \ln x + k \ln 10.$$

Notice that

$$\ln 10 = 2{\cdot}303; \qquad \ln 10^2 = 4{\cdot}605$$
$$\log x = 0{\cdot}4343 \ln x; \qquad \ln x = 2{\cdot}303 \log x.$$

The formulae for approximate evaluation of roots are:

$$\text{(1)} \quad \sqrt[n]{1+x} \approx 1 + \frac{x}{n} + \frac{1-n}{2n^2} x^2 \quad \text{for } |x| < 1$$

$$\text{(2)} \quad \sqrt[n]{a^n + b} \approx a\left(1 + \frac{b}{na^n} + \frac{1-n}{2n^2} \cdot \frac{b^2}{a^{2n}}\right) \quad \text{for } \left|\frac{b}{a^n}\right| < 1$$

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