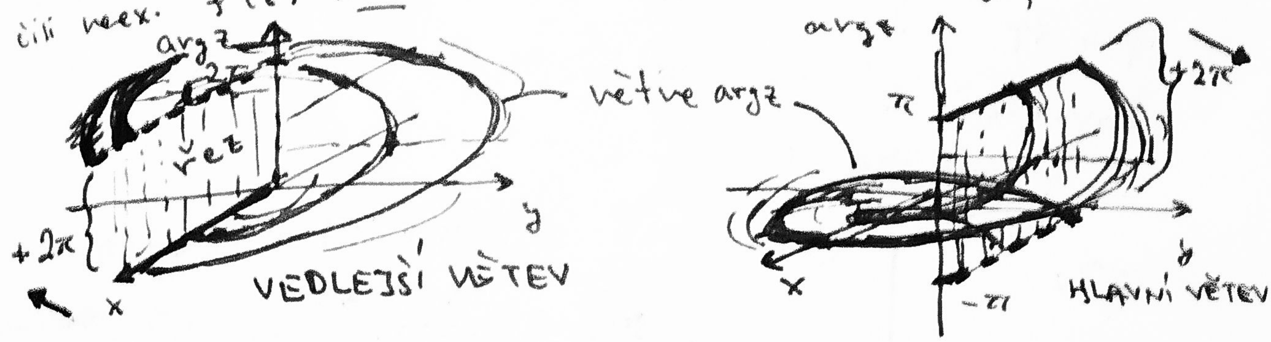
$\boxed{\text{Cv 42}}$ T.10 + KOMPLEXNÍ LOGARITMUS & OBECNÁ MOCNINA

p. 89

Motivace: $f(z) = \arg z$ $\quad (\arg(re^{i\varphi}) = \varphi + \overset{?}{0i} \in \mathbb{R})$

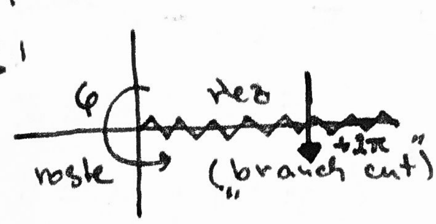
čili $\left.\begin{aligned} u &= \varphi \\ v &= 0 \end{aligned}\right\}$ CR: $\begin{aligned} r u_{,r} &= v_{,\varphi} \\ r v_{,r} &= -u_{,\varphi} \end{aligned}$ $\;::\;$ $\begin{aligned} 0 &= 1 \;\times \\ 0 &= -1 \;\times \end{aligned}$

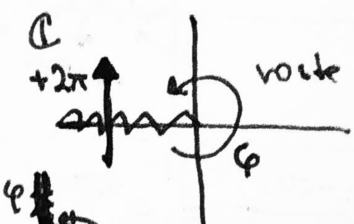
čili neex. $f'(z)$ nikde (ani v $z=0$, ověřte si)



VEDLEJŠÍ VĚTEV $\qquad$ HLAVNÍ VĚTEV

$\varphi \in [0, 2\pi)$ $\qquad$ $\varphi \in (-\pi, \pi]$





! Poloha řezu je na nás ! :  třeba levá část $y = x^2$?
čili výběr větve

(Př.) Zavedení fce $\sqrt{z}$ (čili komplexní odmocniny)

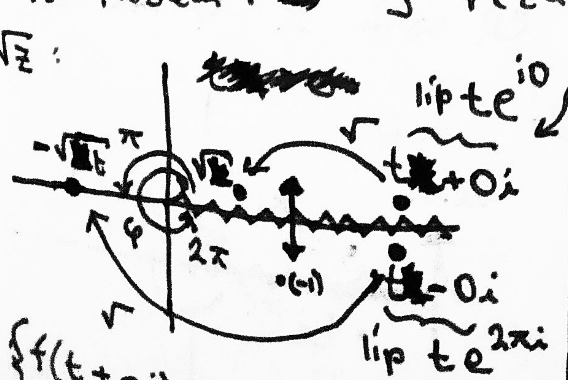
Co chci? $\sqrt{z} = \sqrt{re^{i\varphi}} \overset{\text{def}}{=} \sqrt{r}\, e^{\frac{1}{2}i\varphi}$ $\;::$

$\left.\begin{aligned} \eta &= \sqrt{r} \\ \psi &= \tfrac{1}{2}\varphi \end{aligned}\right\}$ CR: $\begin{aligned} r\eta_{,r} &= \eta\psi_{,\varphi} \\ r\eta\psi_{,r} &= -\eta_{,\varphi} \end{aligned}$ $\;::\;$ $\begin{aligned} r\tfrac{1}{2\sqrt{r}} &\equiv \sqrt{r}\cdot\tfrac{1}{2} \;✓ \\ r\sqrt{r}\cdot 0 &= 0 \;✓ \end{aligned}$

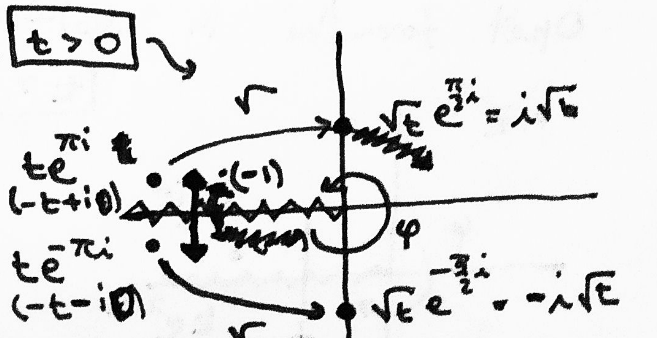
Formálně f splňuje CR všude na $\mathbb{C}/\{0\}$ (+ tot. dif.)

× Problém: $\Rightarrow$ $\exists$ řezu! (kvůli volbě φ)

$\sqrt{z}$:



$\boxed{t>0}$



$\forall t>0 \begin{cases} f(t+0i) - f(t-0i) = 2\sqrt{t} \\ f(t+0i)\,/\,f(t-0i) = -1 \end{cases}$ $\qquad$ $\begin{aligned} f(-t+0i) - f(-t-0i) &= 2i\sqrt{t} \\ \text{—"—} \;/\; \text{—"—} &= -1 \end{aligned}$

Fun: $1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i\cdot i = -1$ $\qquad$ $\boxed{(\sqrt{z})' = \frac{1}{2\sqrt{z}} \text{ pokud stejná větev}}$

Rovněž $\frac{z}{\sqrt{z}} \neq \sqrt{z} \;\wedge\; \sqrt{z}\cdot\sqrt{z} = z$ pokud $\sqrt{\,},\sqrt{\,}$ stejná větev

(Př) Zavedení obecné mocniny z^α i $\alpha \in \mathbb{C}$

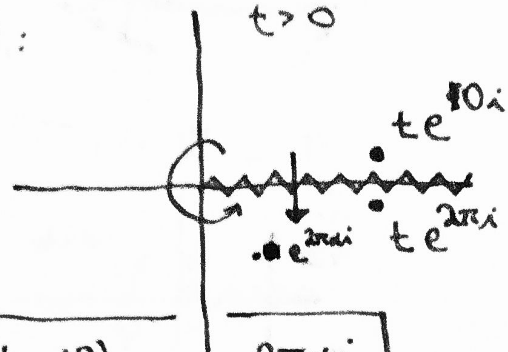
Chceme: $z^\alpha = (re^{i\varphi})^\alpha \overset{def}{=} r^\alpha e^{i\alpha\varphi}$ (pozn. $r^\alpha = e^{\dots}$)

$$\left.\begin{aligned} \eta &= r^\alpha \\ \psi &= \alpha\varphi \end{aligned}\right\} \text{CR}: \quad \begin{aligned} r\eta_{,r} &= \eta\psi_{,\varphi} \\ r\eta\psi_{,r} &= -\eta_{,\varphi} \end{aligned} \quad :: \quad \begin{aligned} r\alpha r^{\alpha-1} &= r^\alpha \cdot \alpha \ \checkmark \\ r \cdot r^\alpha \cdot 0 &= 0 \ \checkmark \end{aligned}$$

Formálně z^α CR všude na $\mathbb{C}/\{0\}$ a (+ tot. dif.)

X ~~φ má řez~~ :

z^α :



$f(t+i0) - f(t-i0) =$
$t^\alpha \cdot e^{0\cdot\alpha\cdot i} - t^\alpha \cdot e^{\alpha 2\pi\alpha i}$
$= t^\alpha(1 - e^{2\pi\alpha i})$

$$\begin{cases} = 0 \ ; \ \alpha \in \mathbb{Z} \\ \neq 0 \ ; \ \alpha \in \mathbb{C}/\mathbb{Z} \end{cases}$$

NEBO

Taky: $\boxed{\dfrac{f(t-i0)}{f(t+i0)} = e^{2\pi\alpha i}}$

Exponenciála je periodická $\Rightarrow$ pro $\alpha \in \mathbb{Z}$ řez jen zdánlivý

(a z^α holom. na celém $\mathbb{C}/\{0\}$: $(z^\alpha)' = \alpha z^{\alpha-1}$)
{ stejné vlast.

(Př) Zavedení logaritmu $\ln z$

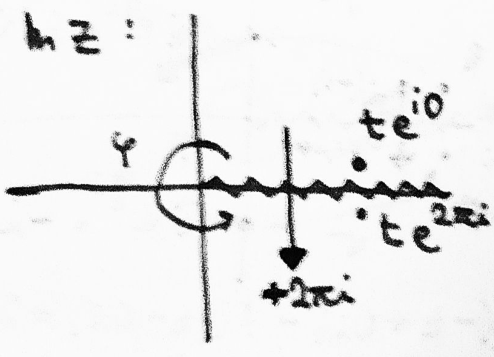
Chceme: $\ln z = \ln(re^{i\varphi}) = \underbrace{\ln r}_{u} + \underbrace{i\varphi}_{v}$

$$\left.\begin{aligned} u &= \ln r \\ v &= \varphi \end{aligned}\right\} \text{CR}: \quad \begin{aligned} r u_{,r} &= v_{,\varphi} \\ r v_{,r} &= -u_{,\varphi} \end{aligned} \quad :: \quad r \cdot \frac{1}{r} = 1 \ \checkmark$$

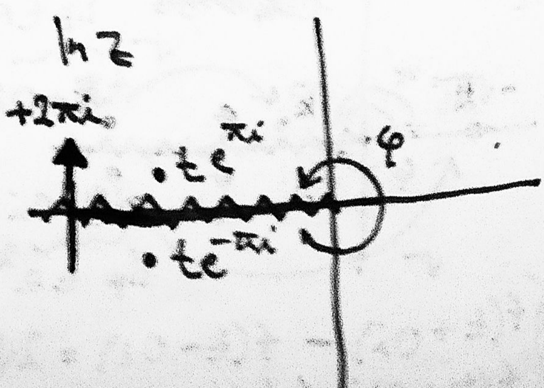
Opět formálně ln splňuje CR všude X φ má řez

$\boxed{t > 0}$

$\ln z$:



$\ln z$



$t > 0$: $f(t+i0) - f(t-i0) =$
$= \ln t + i0 - (\ln t + 2\pi i)$
$= -2\pi i$

$f(-t+i0) - f(-t-i0) =$
$= (\ln t + \pi i) - (\ln t - \pi i)$
$= 2\pi i$

Př: Určete, kde je fce holomorfní (u všech fcí uvažujeme hlavní větvi)

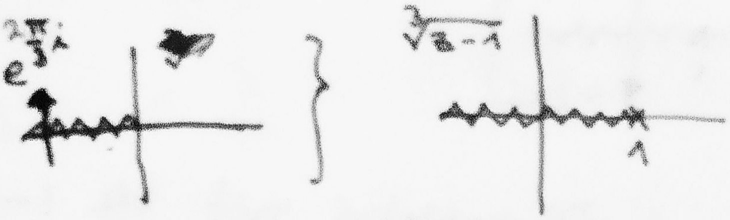
e): $\ln(1+iz)$

a) $\sqrt[3]{z-1}$ b) $\sqrt{\frac{z+1}{z-1}}$ c) $\frac{\sqrt{z+1}}{\sqrt{z-1}}$

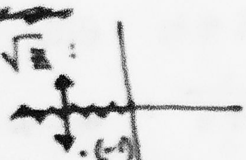
d) $\ln(\frac{\cdot}{z+1}) + \ln(z-1)$ e) $\ln(z^2-1)$

Řeš.

a) $\sqrt[3]{z}$:



b) $\sqrt{z}$:



řez: $z=-t$; $t>0$

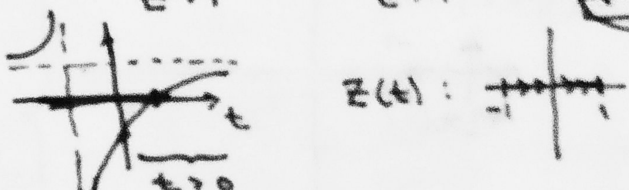
čili potřebujeme kde je řez $\sqrt{\frac{1+z}{1-z}}$?

: $\frac{z+1}{z-1} = -t$; řešme

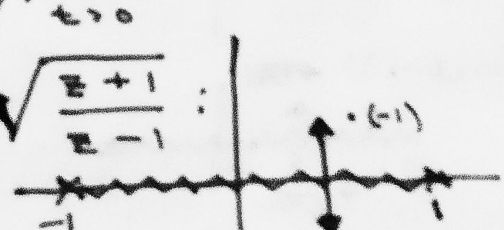
$1+z = -zt+t$

$z(1+t) = t-1$

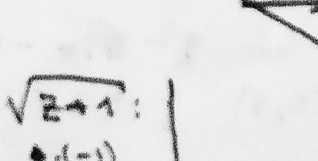
$z = \frac{t-1}{t+1} = 1 - \frac{2}{t+1}$



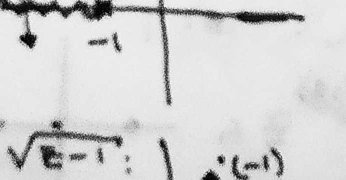
∴ $\sqrt{\frac{z+1}{z-1}}$:

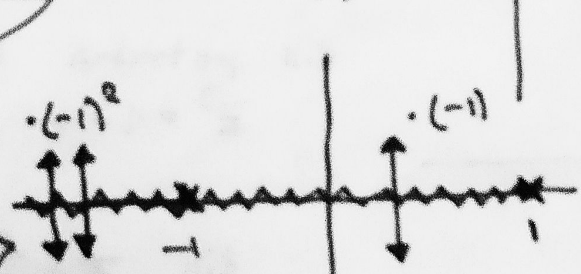


c) $\sqrt{z+1}$:

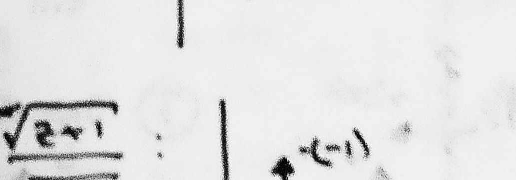


$\sqrt{z-1}$:





∴ $\frac{\sqrt{z+1}}{\sqrt{z-1}}$:

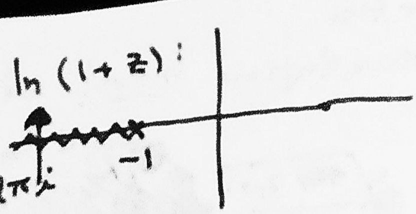


e) $\ln(1+iz)$: stačí vyřešit řez $1+iz = -t$ → $z = i(1+t)$; $t>0$

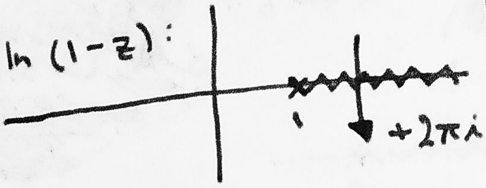
skok: [illegible]

VYNECHAT!

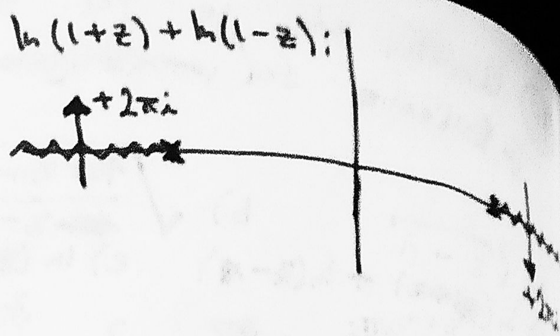
d) $\ln(1+z)$:



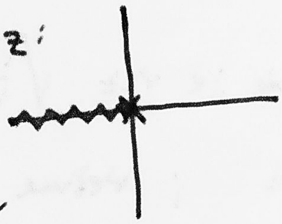
$\ln(1-z)$:



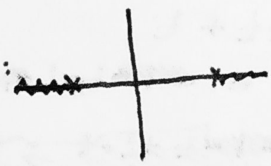
$\ln(1+z)+\ln(1-z)$:



e) $\ln z$:



$\ln(1-z^2)$:



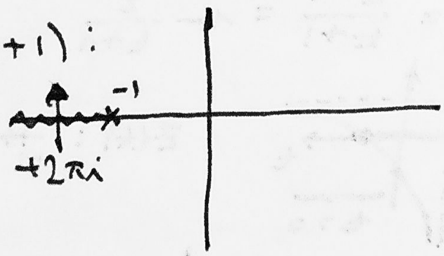
čili potřebuji najít kde $1-z^2=-t$, $t\in\mathbb{R}^+$

řeším tedy $z^2 = 1+t$ čili

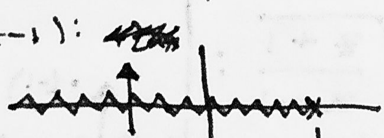
$z = \pm\sqrt{1+t}$:

LEPŠÍ:

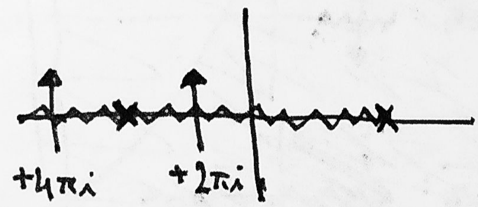
d) $\ln(z+1)$:



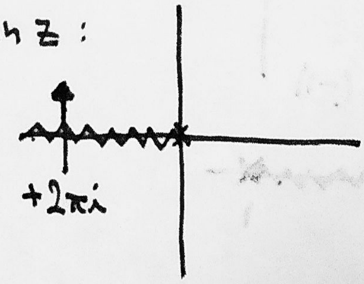
$\ln(z-1)$:



$\ln(z+1)+\ln(z-1)$:



e) $\ln z$:

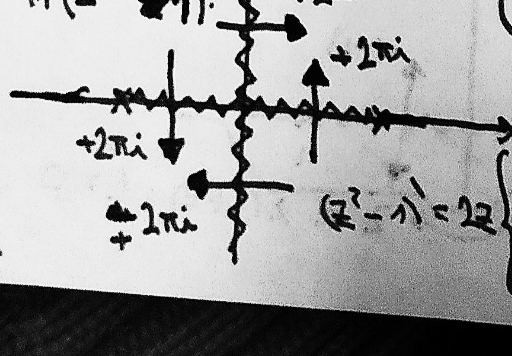


čili potřebuji najít kde $z^2-1=-t$

$$z^2 = 1-t \in \begin{cases}(0,1) & ;\ t\in(0,1)\\ (-\infty,0) & ;\ t\in(1,\infty)\end{cases}$$

čili $z \in$

$\ln(z^2-1)$:



? Jak zjistím $\pm 2\pi i$???

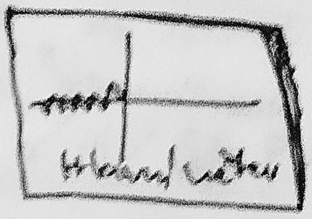
$\xrightarrow{t}$ $\quad \frac{\partial f}{\partial t} = f'(z)\frac{\partial z}{\partial t} = \begin{cases}2ci & \\ -2ci & \end{cases}$

$(z^2-1)' = 2z$ $\begin{cases} \text{i): } ci+t : f_{,t} = 2(ci+t)\cdot 1\big|_{t=0} = 2ci \\ \text{ii): } c+it : f_{,t} = 2(c+it)\cdot i\big|_{t=0} = 2ci \end{cases}$

ZAVEDENÍ ELEMENTÁRNÍCH ~~fcí~~ INVERZ. FCÍ

[všechny fce hlavní větev]

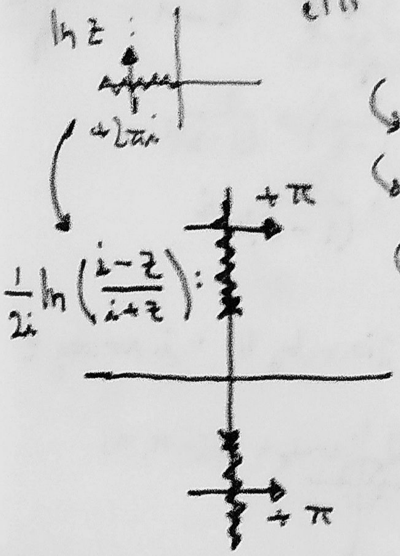
{LÉPE: P104-105



a) $\arcsin z \overset{\text{def}}{=} \frac{1}{i} \ln\left(iz + \sqrt{1-z^2}\right)$

b) $\arccos z \overset{\text{def}}{=} \frac{1}{i} \ln\left(z + \sqrt{z^2-1}\right)$

c) $\operatorname{arctg} z := \frac{1}{2i} \ln\left(\frac{i-z}{i+z}\right)$



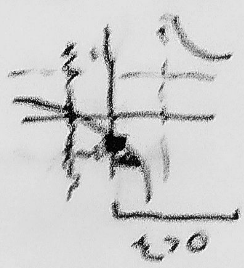
čili $\frac{i-z}{i+z} = -t \quad ; \; t > 0$ je poloha ~~fce~~ řezu

$\hookrightarrow i - z = -it - zt$

$\hookrightarrow z(t-1) = -i(1+t)$

$\hookrightarrow z = i \frac{t+1}{t-1}$

$\frac{t+1}{t-1}$:

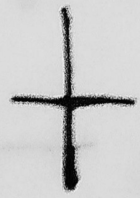


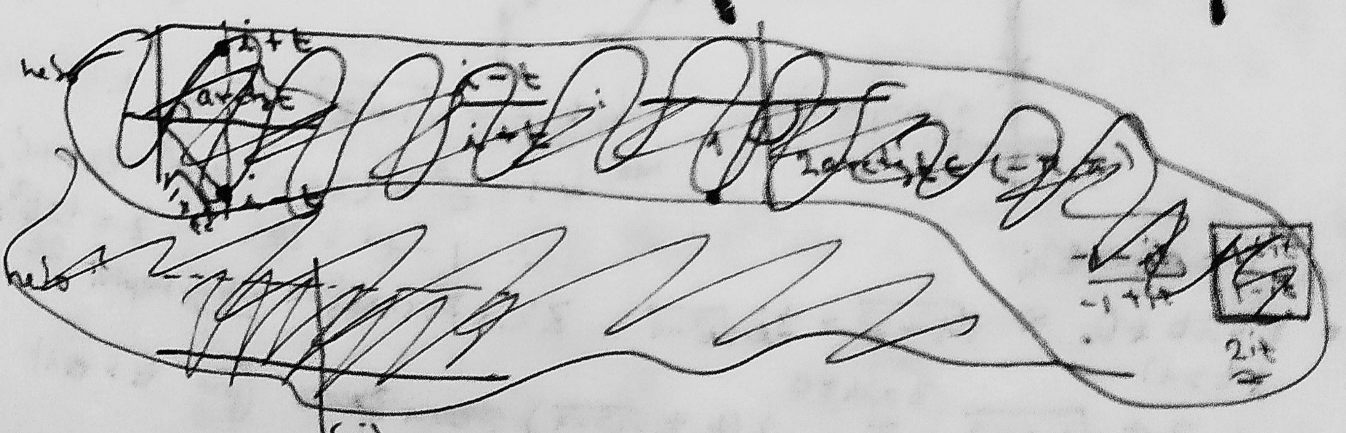
?: $\left(\frac{i-z}{i+z}\right)' = \left(-1 + \frac{2i}{i+z}\right)' = -\frac{2i}{(i+z)^2}$

$z = ci + t \; : \; f_{,t} = -\frac{2i}{(i+ci)^2} \cdot 1 = \frac{2i}{(1+c)^2}$

Na reálné ose: $z = t \,; \; t \in (-\infty, \infty)$, potom

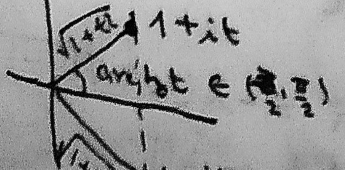
$\frac{i-z}{i+z} = \frac{i-t}{i+t} = \frac{-(i-t)^2}{1+t^2} = \frac{1-t^2}{1+t^2} + \frac{2ti}{1+t^2}$:

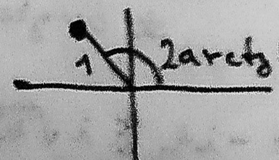




nebo: $\frac{i-z}{i+z} \overset{(-i)/(-i)}{=} \frac{1+iz}{1-iz} = \frac{1+it}{1-it} = 1 \cdot e^{2i \operatorname{arctg} t}$

$\therefore \ln \frac{i-t}{i+t} = 2i \operatorname{arctg} t$





$\therefore \frac{1}{2i} \ln\left(\frac{i-t}{i+t}\right) = \operatorname{arctg} t$

BONUS: $f(it) = \frac{1}{2i} \ln\left(\frac{i-it}{i+it}\right) = \frac{i}{2} \ln\left(\frac{1+t}{1-t}\right) = i \operatorname{argtgh} t$

Motivace:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}, \quad \operatorname{tgh} z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

<u>Jest</u>: $\sinh(it) = i \sin t$, $\cosh(it) = \cos t$, $\operatorname{tgh}(it) = i \operatorname{tg} t$

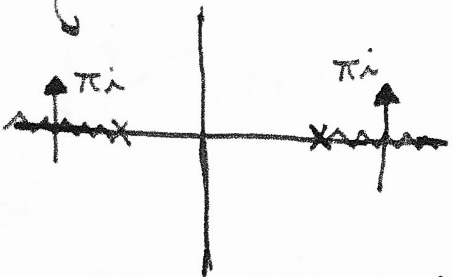
Inverzní fce — NEJPŘIROZENĚJŠÍ !!(! !)!!

$$\left.\begin{aligned} \operatorname{arg\,sinh} z &= \ln\left(z + \sqrt{1+z^2}\right) \\ \operatorname{arg\,cosh} z &= \ln\left(z + \cancel{\sqrt{z^2-1}}\ \sqrt{z-1}\sqrt{z+1}\right) \\ \operatorname{arg\,tgh} z &= \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) \end{aligned}\right\}$$

<u>Komplexní rozšíření</u>: aby fungovalo na $\mathbb{R}$, bereme $\ln$, $\sqrt{\ }$ hlavní větve

<u>Vlastnosti</u>:

a) $\operatorname{arg\,tgh} z$: $\frac{1+z}{1-z} \overset{\text{B.C.}}{=} -t \Rightarrow z = \frac{t+1}{t-1}$

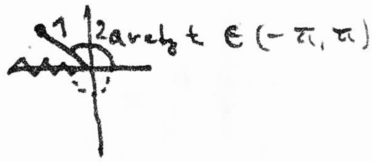


? $\left(\frac{1+z}{1-z}\right)' = \left(-1 + \frac{2}{1-z}\right)' = \frac{2}{(1-z)^2}$

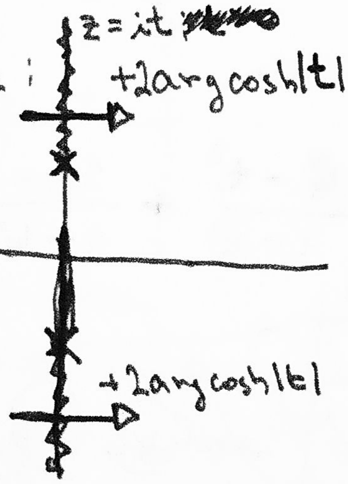
$z = c + it$

$z' = i$ $\quad \therefore f_t' = \frac{2}{(1-c)^2} \cdot i$

Na imag. ose: $z = it$: $\frac{1}{2}\ln \overbrace{\frac{1+it}{1-it}}^{e^{2i\operatorname{arctg} t}} = \frac{1}{2} \cdot 2i\operatorname{arctg} t = i \operatorname{arctg} t$



b) $\operatorname{arg\,sh} z$:



nejdřív $\sqrt{1+z^2}$: $1 + z^2 = -t$; $t > 0$

$z^2 = -(t+1)$

$\hookrightarrow z = \pm i\sqrt{t+1}$

$\ln$: $z + \sqrt{1+z^2} = -t$; $t > 0$

nech. $\hookrightarrow 1 + z^2 = (z+t)^2 = \cancel{z^2} + 2zt + t^2$

$\therefore 1 - t^2 = 2zt \Rightarrow z = \frac{1-t^2}{2t}$!

~~zmiňovali~~

- $z = it \pm 0$ $\Rightarrow$ $\sqrt{1+z^2} = \pm i\sqrt{t^2-1}$ $\quad$ Zk.: Dosazení → nikdy neplatí

($t > 1$)

$\therefore z + \sqrt{1+z^2} \overset{z = it \pm 0}{=} i\left(t \pm \sqrt{t^2-1}\right)$ (pro $t > 1$)

$\hookrightarrow \ln\left(z + \sqrt{1+z^2}\right) \overset{z = it \pm 0}{=} \underbrace{\ln\left(t \pm \sqrt{t^2-1}\right)}_{\pm \operatorname{argcosh} t} + i\frac{\pi}{2}$

$\left\langle \ln\left(\frac{t+\sqrt{t^2-1}}{t-\sqrt{t^2-1}}\right) = \ln\left(\frac{(t+\sqrt{t^2-1})^2}{t^2-(t^2-1)}\right) \right\rangle$

- $z = -it \pm 0 \Rightarrow z + \sqrt{1+z^2} = -it \mp i\sqrt{t^2-1} = -i\left(t \pm \sqrt{t^2-1}\right)$

$\therefore \ln\left(z + \sqrt{1+z^2}\right) \overset{z = -it \pm 0}{=} \pm \operatorname{argcosh} t - i\frac{\pi}{2}$

c) $\operatorname{argcosh} z = \ln\left(z + \sqrt{z-1}\sqrt{z+1}\right)$; $\sqrt{z-1}\sqrt{z+1}$:

B.C. argcosh: $z + \sqrt{z^2-1} = -t$; $t > 0$

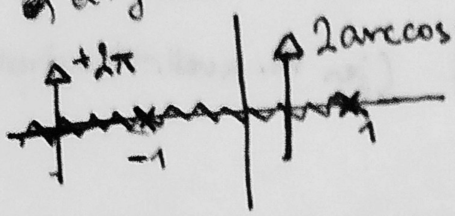
nech. $\Rightarrow$ $z^2 - 1 = z^2 + 2zt + t^2 \Rightarrow z = -\frac{1+t^2}{2t}$:

díl $z \in (-\infty,-1) \cup (1,\infty) \xRightarrow{\text{D}^{\text{k}}} z + \sqrt{z^2-1} \in (-\infty, 0)$

když $z \in (-\infty,-1)$

LÉPE do Dohromady:

argcosh z:



B.C. vals: • $z = t \pm i0$: $\underbrace{\sqrt{z-1}}_{\pm i\sqrt{1-t}} \underbrace{\sqrt{z+1}}_{\sqrt{t+1}} = \pm i \sqrt{1-t^2}$ $\in \mathbb{R}^+$

$(t \in (-1,1))$

$\therefore \operatorname{argcosh} z \overset{z=t\pm i0}{=} \ln\left(t \pm i\sqrt{1-t^2}\right) = \pm \arccos t$

$t \in (0,1)$; $t \in (-1,0)$; D^{k}

• $z = t \pm i0$; $t < -1$:

$\sqrt{z-1}\sqrt{z+1} = -\sqrt{t^2-1}$

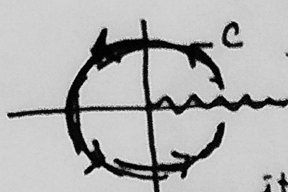
$\therefore z + \sqrt{z-1}\sqrt{z+1} = \underbrace{t \pm i0 - \sqrt{t^2-1}}_{\ominus}$ $\Rightarrow \operatorname{argcosh} z = \pm \pi i + \operatorname{argcosh} t$ ($z = t \pm i0$; $t < -1$)

Př: Spočtěte $\int_{|z|=1} \frac{1}{\sqrt{z}}\, dz$:

a) $\sqrt{\ }$: ; b) $\sqrt{\ }$:

Sol a) $C: z = e^{it}$; $t \in (0, 2\pi)$ $\rightarrow \sqrt{z} = e^{\frac{it}{2}}$

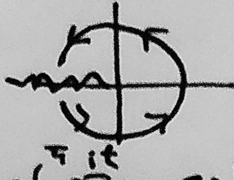
$dz = i e^{it} dt$



$\therefore \int_C \frac{1}{\sqrt{z}}\, dz = \int_0^{2\pi} \frac{1}{e^{\frac{it}{2}}}\, i e^{it}\, dt = i \int_0^{2\pi} e^{\frac{it}{2}} dt = 2\left[e^{\frac{it}{2}}\right]_0^{\pi} = -4$

b) $C: z = e^{it}$; $t \in (-\pi, \pi)$

$dz = i e^{it}$; $\sqrt{z} = e^{\frac{it}{2}}$



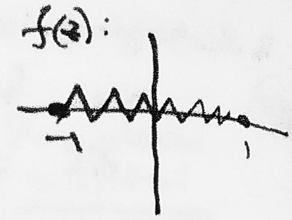
$\therefore \int_C \frac{1}{\sqrt{z}}\, dz = \int_{-\pi}^{\pi} \frac{1}{e^{\frac{it}{2}}}\, i e^{it}\, dt = i \int_{-\pi}^{\pi} e^{\frac{it}{2}} dt = 2\left[e^{\frac{it}{2}}\right]_{-\pi}^{\pi} = 4i$

Př: Spočtěte $\int_{|z|=1} \ln z\, dz$; a) ln: b) ln

DÚ Spočtěte $\lim\limits_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)\, dz$ a $\lim\limits_{R \to \infty} \int_{C_R} f(z)$; $f(z) = \frac{1}{\sqrt{z}(z+1)^2}$

C_ε, C_R

DÚ) 1) $f(z) = \dfrac{\ln(1+z) - \ln(1-z)}{(1-z)^{2/3}\,(1+z)^{1/3}}$



Navolte větve fcí ln, ln, $(\ )^{2/3}$, $(\ )^{1/3}$ tak, aby $f(z)$ měla řez na $[-1,1]$, čili

2) $f(z) = \dfrac{\ln(z^2+1)}{\sqrt{z}\,(z+1)}$; jak můžou vypadat řezy f volbami větví ln a $\sqrt{\ }$? (jen hl. a vedl. → 4 možnosti)