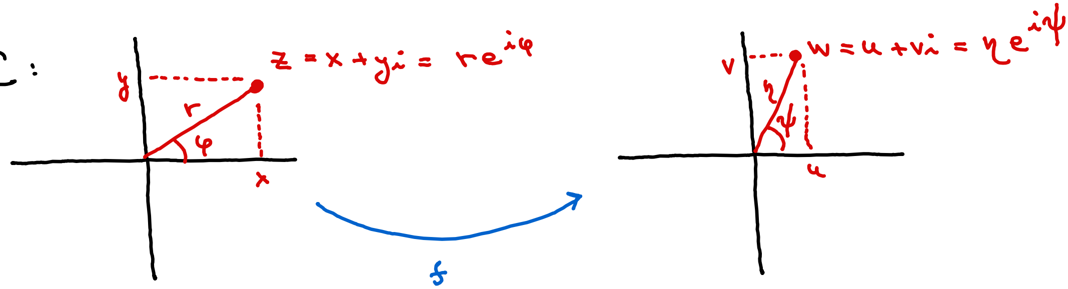
ZÁKLADNÍ VLASTNOSTI A FCE V $\mathbb{C}$

$\left(e^{i\varphi} = \cos\varphi + i\sin\varphi\right)$

CVYO

$\mathbb{C}$:



$\operatorname{Re} z := x$

$\operatorname{Im} z := y$

$|z| := r$

$\arg z := \varphi$

$\bar{z} := x - iy$

$e^z = e^{x+iy} := e^x e^{iy} = e^x(\cos y + i\sin y)$

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

OSBORNOVA PRAVIDLA:

$\cos iz = \cosh z \qquad \sin iz = i\sinh z$

$\cosh iz = \cos z \qquad \sinh iz = i\sin z$

$\boxed{V}$ TROJÚHELNÍKOVÉ NEROVNOSTI

$|z \pm w| \leq |z| + |w|$

$|z \pm w| \geq \big||z| - |w|\big|$

MOIVROVY VĚTY:

$|zw| = |z|\,|w| \qquad \arg(zw) = \arg z + \arg w$

$\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|} \qquad \arg\dfrac{z}{w} = \arg z - \arg w$

mod 2π

$\boxed{D}$ (KOMPLEXNÍ DERIVACE) $\quad f'(z) := \lim\limits_{h\to 0} \dfrac{f(z+h) - f(z)}{h}; \quad h \in \mathbb{C}$

$\boxed{D}$ (HOLOMORFIE): $\exists f'$ na $\left\{\begin{matrix} \Omega \text{ otevř.} \\ U_\delta(z_0) \end{matrix}\right\} \Leftrightarrow f$ „holomorfní“ $\left\{\begin{matrix} \text{na } \Omega \\ \text{v } z_0 \end{matrix}\right\}$

$\boxed{V}$ 1) $\exists f'$ v $z \Rightarrow$ CAUCHY–RIEMANNOVY PODMÍNKY

2) f CR v z $\wedge$ $\exists$ totální diferenciál v $z \Rightarrow \exists f'$ v z

CAUCHY – RIEMANNOVY PODMÍNKY

Předpoklad $\exists f'$ v z

I. $f(x+iy) = u + iv$

$$\partial_x : \quad f'(z) = u_{,x} + i v_{,x}$$

$$f'(z)\, i = u_{,x}\, i - v_{,x}$$

$$\partial_y : \quad f'(z)\, i = u_{,y} + i v_{,y}$$

$$\boxed{\begin{aligned} u_{,x} &= v_{,y} \\ v_{,x} &= -u_{,y} \end{aligned}}$$

II. $f(re^{i\varphi}) = u + iv$

$$\partial_r : \quad f'(z) e^{i\varphi} = u_{,r} + i v_{,r}$$

$$f'(z)\, r i e^{i\varphi} = r i u_{,r} - r v_{,r}$$

$$\partial_\varphi : \quad f'(z)\, r i e^{i\varphi} = u_{,\varphi} + i v_{,\varphi}$$

$$\boxed{\begin{aligned} r u_{,r} &= v_{,\varphi} \\ r v_{,r} &= -u_{,\varphi} \end{aligned}}$$

III. $f(x+iy) = \eta e^{i\psi}$

$$\partial_x : \quad f'(z) = (\eta_{,x} + i\eta \psi_{,x}) e^{i\psi}$$

$$f'(z)\, i = (i\eta_{,x} - \eta \psi_{,x}) e^{i\psi}$$

$$\partial_y : \quad f'(z)\, i = (\eta_{,y} + i\eta \psi_{,y}) e^{i\psi}$$

$$\boxed{\begin{aligned} \eta_{,x} &= \eta \psi_{,y} \\ \eta_{,y} &= -\eta \psi_{,x} \end{aligned}}$$

IV. $f(re^{i\varphi}) = \eta e^{i\psi}$

$$\partial_r : \quad f'(z) e^{i\varphi} = (\eta_{,r} + i\eta \psi_{,r}) e^{i\psi}$$

$$f'(z)\, r i e^{i\varphi} = (r i \eta_{,r} - r\eta \psi_{,r}) e^{i\psi}$$

$$\partial_\varphi : \quad f'(z)\, r i e^{i\varphi} = (\eta_{,\varphi} + i\eta \psi_{,\varphi}) e^{i\psi}$$

$$\boxed{\begin{aligned} r\eta_{,r} &= \eta \psi_{,\varphi} \\ r\eta \psi_{,r} &= -\eta_{,\varphi} \end{aligned}}$$

Vzorce pro derivaci:

I & III : $f'(z) = \dfrac{\partial f}{\partial x} = \dfrac{1}{i} \dfrac{\partial f}{\partial y}$

II & IV : $f'(z) = e^{-i\varphi} \dfrac{\partial f}{\partial r} = \dfrac{1}{ir} \dfrac{\partial f}{\partial \varphi}$

Př Určete, kde je fce holomorfní

a) $f(z) = |z|^2 = z\bar{z}$

- CR Varianta 1: $z\bar{z} = (x+iy)(x-iy) = x^2+y^2$

$$\left.\begin{array}{l} u = x^2+y^2 \\ v = 0 \end{array}\right\} \text{CR: } \begin{array}{l} u_{,x} = v_{,y} \\ v_{,x} = -u_{,y} \end{array} \quad :: \quad \begin{array}{l} 2x = 0 \\ 0 = -2y \end{array} \quad \times$$

$\therefore$ CR nesplněny v $\mathbb{C}/\{0\} \Rightarrow f' \not\exists$ v $\mathbb{C}/\{0\} \Rightarrow f$ není hol. nikde

- CR Varianta 2: $z\bar{z} = r^2 + 0\cdot i$

$$\left.\begin{array}{l} u = r^2 \\ v = 0 \end{array}\right\} \text{CR: } \begin{array}{l} ru_{,r} = v_{,\varphi} \\ rv_{,r} = -u_{,\varphi} \end{array} \quad :: \quad \begin{array}{l} 2r^2 = 0 \\ 0 - 0 \end{array}$$

- CR Varianta 4: $z\bar{z} = r^2 \cdot e^{0\cdot i}$

$$\left.\begin{array}{l} \eta = r^2 \\ \psi = 0 \end{array}\right\} \text{CR: } \begin{array}{l} r\eta_{,r} = \eta\psi_{,\varphi} \\ r\eta\psi_{,r} = -\eta_{,\varphi} \end{array} \quad :: \quad \begin{array}{l} r\cdot 2r = 0 \\ 0 = 0 \end{array} \quad \times$$

b) $f(z) = z^2$

- CR Varianta 1: $z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi$

$$\left.\begin{array}{l} u = x^2 - y^2 \\ v = 2xy \end{array}\right\} \text{CR: } \begin{array}{l} u_{,x} = v_{,y} \\ v_{,x} = -u_{,y} \end{array} \quad :: \quad \begin{array}{l} 2x = 2x \\ 2y = -(-2y) \end{array} \quad \checkmark \text{ splněno v } \mathbb{C}$$

(+) u, v má totální diferenciál na $\mathbb{C}$ $\Rightarrow \exists f'$ na $\mathbb{C}$ a f zde holomorf.

- CR Varianta 4: $z^2 = (re^{i\varphi})^2 = r^2 e^{i\cdot 2\varphi}$

$$\left.\begin{array}{l} \eta = r^2 \\ \psi = 2\varphi \end{array}\right\} \text{CR: } \begin{array}{l} r\eta_{,r} = \eta\psi_{,\varphi} \\ r\eta\psi_{,r} = -\eta_{,\varphi} \end{array} \quad :: \quad \begin{array}{l} r\cdot 2r = r^2\cdot 2 \ \checkmark \\ r\cdot r^2\cdot 0 = 0 \ \checkmark \end{array}$$

<u>Platí</u>: $(z^2)' = 2z$

c) $f(z) = e^z$

- CR Varianta 1: $e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + i e^x \sin y$

$$\left.\begin{array}{l} u = e^x \cos y \\ v = e^x \sin y \end{array}\right\} \text{CR:} \quad \begin{array}{l} u_{,x} = v_{,y} \\ v_{,x} = -u_{,y} \end{array} \quad \therefore \quad \begin{array}{l} e^x \cos y = e^x \cos y \ \checkmark \\ e^x \sin y = -e^x(-\sin y) \ \checkmark \end{array}$$

(+) u, v tot. dif. na $\mathbb{C}$ $\Rightarrow$ $\exists f'$ na $\mathbb{C}$ a fce zde hol

- CR Varianta 3: $e^z = e^{x+iy} = e^x \cdot e^{iy}$

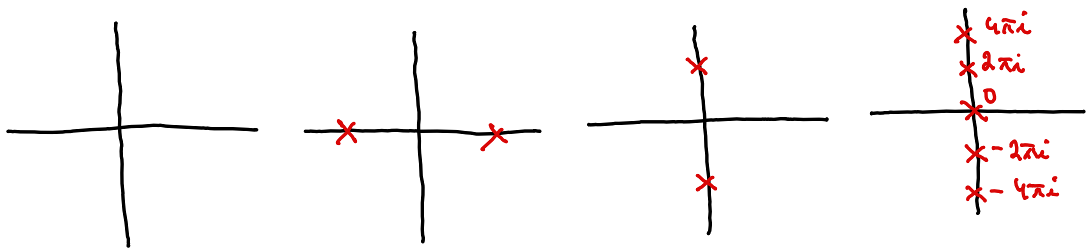
$$\left.\begin{array}{l} \eta = e^x \\ \psi = y \end{array}\right\} \text{CR:} \quad \begin{array}{l} \eta_{,x} = \eta \psi_{,y} \\ \eta_{,y} = -\eta \psi_{,x} \end{array} \quad \therefore \quad \begin{array}{l} e^x = e^x \cdot 1 \ \checkmark \\ 0 = -e^x \cdot 0 \ \checkmark \end{array}$$

<u>Platí</u>: $(e^z)' = e^z$

$\boxed{V}$ Skládání holomorfních fcí zachovává holomorfii

(Př) Určete, kde je fce holomorfní (dle skládání)

a) $e^{\sin z}$ b) $\frac{1}{1-z^2}$ c) $\frac{1}{1+z^2}$ d) $\frac{1}{e^z-1}$



KOMPLEXNÍ LOGARITMUS & OBECNÁ MOCNINA

Cv 41

I. Zavedení argumentu :

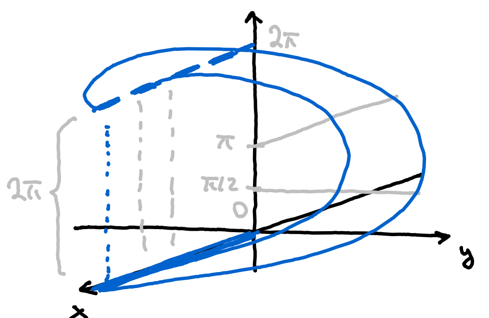
$$\boxed{\arg z = \arg(re^{i\varphi}) \overset{\text{def.}}{=} \varphi}$$

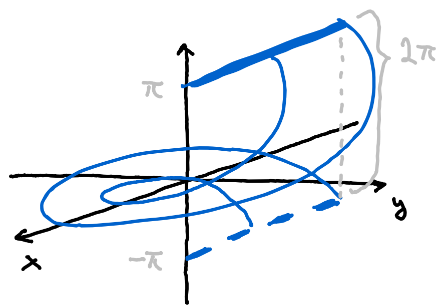
$$\left.\begin{aligned} u &= \varphi \\ v &= 0 \end{aligned}\right\} \quad CR: \quad \begin{aligned} r u_{,r} &= v_{,\varphi} \\ r v_{,r} &= -u_{,\varphi} \end{aligned} \quad \therefore \quad \begin{aligned} 0 &= 1 \quad \times \\ 0 &= -1 \quad \times \end{aligned}$$

čili neex. $f'(z)$ nikde (přestože φ má tot. dif) $\Rightarrow$ f není holomorf.

Navíc : φ lze zavést více způsoby : $\varphi \in (\varphi_0, \varphi_0 + 2\pi)$

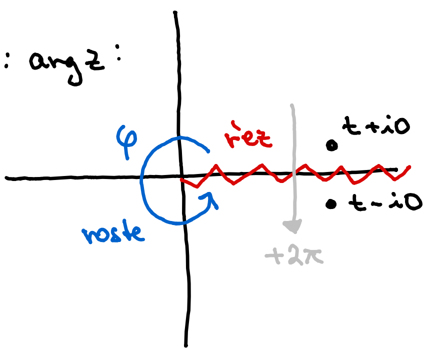
VEDLEJŠÍ VĚTEV („BRANCH") HLAVNÍ VĚTEV



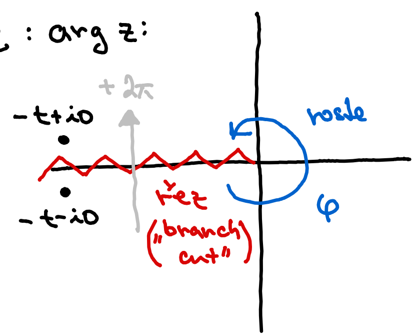


$\varphi \in [0, 2\pi)$ $\varphi \in (-\pi, \pi]$

$\mathbb{C}$: arg z :



$\mathbb{C}$: arg z :



$$\begin{aligned} \arg(t+i0) &= 0 \\ \arg(t-i0) &= 2\pi \end{aligned} \quad \leftarrow \quad \boxed{t > 0} \quad \rightarrow \quad \begin{aligned} \arg(-t+i0) &= \pi \\ \arg(-t-i0) &= -\pi \end{aligned}$$

Platí : $\arg(z_1 z_2) = \arg z_1 + \arg z_2 + 2k\pi$ pro nějaké $k \in \mathbb{Z}$

Pozn. : Poloha řezu („branch cut") φ je na nás, čili :

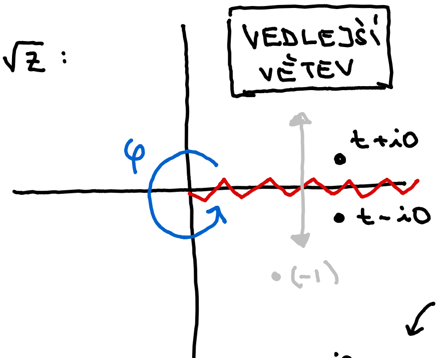
II. Zavedení druhé odmocniny : $\boxed{\sqrt{z} = \sqrt{re^{i\varphi}} \overset{def.}{=} \sqrt{r}\, e^{i\frac{1}{2}\varphi}}$

$$\left.\begin{aligned} \eta &= \sqrt{r} \\ \psi &= \tfrac{1}{2}\varphi \end{aligned}\right\} \quad CR: \quad \begin{aligned} r\eta_{,r} &= \eta\,\psi_{,\varphi} \\ r\eta\,\psi_{,r} &= -\eta_{,\varphi} \end{aligned} \quad \therefore \quad \begin{aligned} r\frac{1}{2\sqrt{r}} &= \sqrt{r}\cdot\tfrac{1}{2} \ \checkmark \\ r\sqrt{r}\cdot 0 &= 0 \ \checkmark \end{aligned}$$

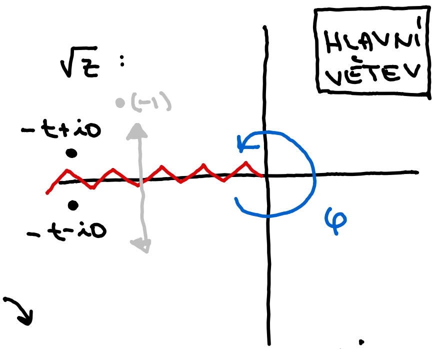
Formálně f splňuje CR všude na $\mathbb{C}/\{0\}$ ($\oplus$ totální diferenciál)

X Problém : Existence řezu ! (kvůli volbě φ)

$\sqrt{z}$:



$\sqrt{z}$:



$\boxed{t>0}$

$$\sqrt{t+i0} = \sqrt{te^{i0}} = \sqrt{t}\,e^{\frac{i0}{2}} = \sqrt{t}$$
$$\sqrt{t-i0} = \sqrt{te^{2\pi i}} = \sqrt{t}\,e^{\pi i} = -\sqrt{t}$$

$$\sqrt{-t+i0} = \sqrt{te^{\pi i}} = \sqrt{t}\,e^{\frac{\pi i}{2}} = i\sqrt{t}$$
$$\sqrt{-t-i0} = \sqrt{te^{-\pi i}} = \sqrt{t}\,e^{-\frac{\pi i}{2}} = -i\sqrt{t}$$

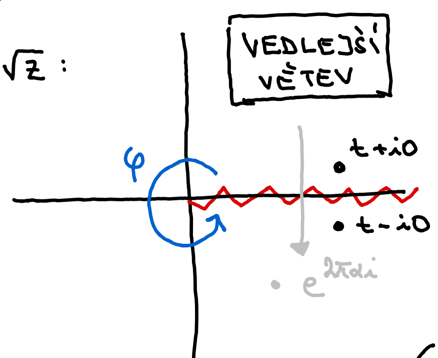
III. Zavedení obecné mocniny : $\boxed{z^\alpha = (re^{i\varphi})^\alpha \overset{def}{=} r^\alpha e^{i\alpha\varphi}}$; $\alpha \in \mathbb{C}$

$$\left.\begin{aligned} \eta &= r^\alpha \\ \psi &= \alpha\varphi \end{aligned}\right\} \quad CR: \quad \begin{aligned} r\eta_{,r} &= \eta\,\psi_{,\varphi} \\ r\eta\,\psi_{,r} &= -\eta_{,\varphi} \end{aligned} \quad \therefore \quad \begin{aligned} r\cdot\alpha r^{\alpha-1} &= r^\alpha\cdot\alpha \ \checkmark \\ r\cdot r^\alpha\cdot 0 &= 0 \ \checkmark \end{aligned}$$

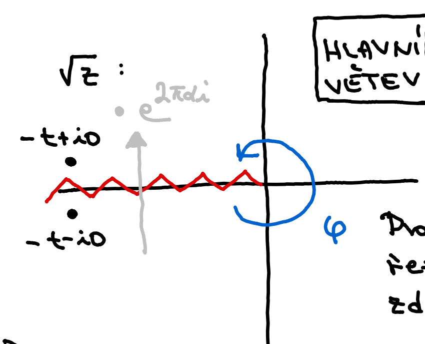
Platí: $(z^\alpha)' = \alpha z^{\alpha-1}$
(stejná větev)

Formálně f CR všude na $\mathbb{C}/\{0\}$ (+ tot. dif.) X Řez φ :

$\sqrt{z}$:



$\sqrt{z}$:



Pro $\alpha \in \mathbb{Z}$ řez jen zdánlivý!

$\boxed{t>0}$

$$(t+i0)^\alpha = (te^{i0})^\alpha = t^\alpha$$
$$(t-i0)^\alpha = (te^{2\pi i})^\alpha = t^\alpha e^{2\pi\alpha i}$$

$$(-t+i0)^\alpha = (te^{\pi i})^\alpha = t^\alpha e^{\pi\alpha i}$$
$$(-t-i0)^\alpha = (te^{-\pi i})^\alpha = t^\alpha e^{-\pi\alpha i}$$

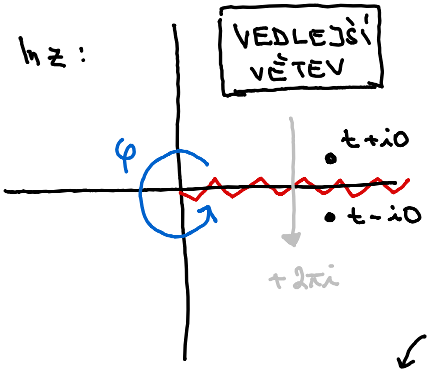
IV. Zavedení komplexního logaritmu:

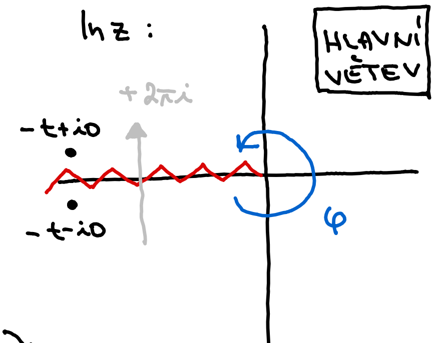
$$\boxed{\ln z = \ln(re^{i\varphi}) \overset{\text{def.}}{=} \ln r + i\varphi}$$

$$\left.\begin{aligned} u &= \ln r \\ v &= \varphi \end{aligned}\right\} \text{CR}: \quad \begin{aligned} r u_{,r} &= v_{,\varphi} \\ r v_{,r} &= -u_{,\varphi} \end{aligned} \quad \therefore \quad \begin{aligned} r\cdot\tfrac{1}{r} &= 1 \ \checkmark \\ 0 &= 0 \ \checkmark \end{aligned}$$

Platí: $(\ln z)' = \frac{1}{z}$

Opět formálně ln splňuje CR všude na $\mathbb{C}/\{0\}$ X φ má řez





$\boxed{t>0}$

$\ln(t+i0) = \ln(te^{i0}) = \ln t$

$\ln(t-i0) = \ln(te^{2\pi i}) = \ln t + 2\pi i$

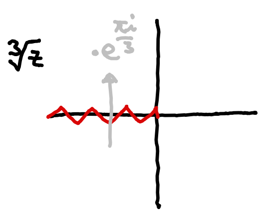
$\ln(-t+i0) = \ln(te^{\pi i}) = \ln t + \pi i$

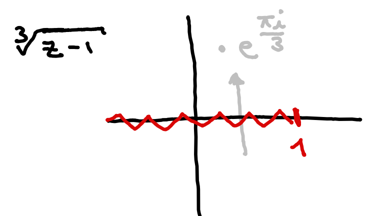
$\ln(-t-i0) = \ln(te^{-\pi i}) = \ln t - \pi i$

Platí: $\ln z = \ln|z| + i\arg z$

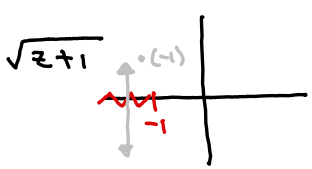
(Př) Určete, kde je fce holomorfní (u všech fcí uvažujte hl. větve)

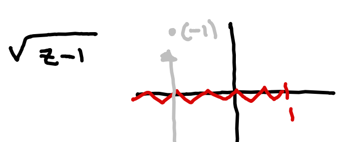
a) $\sqrt[3]{z-1}$

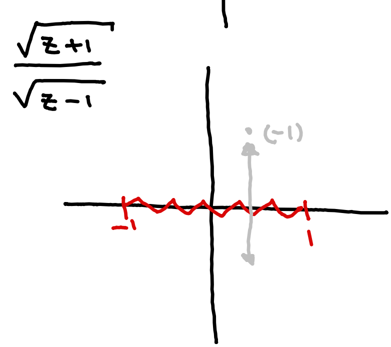




b) $\dfrac{\sqrt{z+1}}{\sqrt{z-1}}$

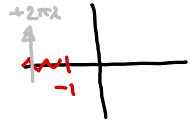






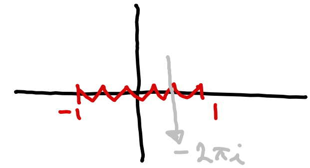
c) $\ln(z+1) - \ln(z-1)$

$\ln(z+1)$:



$\ln(z-1)$:

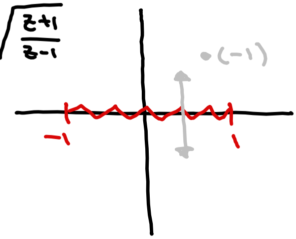
$\ln(z+1) - \ln(z-1)$:



d) $\sqrt{\dfrac{z+1}{z-1}}$ $\quad$ Poloha řezu : $\dfrac{z+1}{z-1} = -t \ ; \ t > 0$

$\Rightarrow \quad z = \dfrac{t-1}{t+1} \in (-1, 1)$

$\sqrt{\dfrac{z+1}{z-1}}$



e) $\sqrt[3]{\dfrac{z+1}{z-1}}$ $\quad$ Poloha řezu taktéž $z \in (-1, 1)$

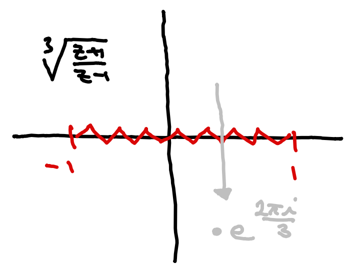
Skok na řezu : Pomocí Taylorova rozvoje :

$$\frac{z+1}{z-1}\bigg|_{t+i0} \overset{t\in(-1,1)}{=} \frac{z+1}{z-1}\bigg|_t + \left(\frac{z+1}{z-1}\right)'\bigg|_t \, i0$$

$$= \frac{t+1}{t-1} - \frac{2}{(t-1)^2}\, i0 = -\frac{1+t}{1-t} - i0 = \frac{1+t}{1-t}\, e^{-\pi i}$$

$$\therefore \quad \left\| \begin{array}{l} \sqrt[3]{\dfrac{z+1}{z-1}}\bigg|_{t+i0} \overset{t\in(-1,1)}{=} \sqrt[3]{\dfrac{1+t}{1-t}}\, e^{-\frac{\pi i}{3}} \\[2ex] \sqrt[3]{\dfrac{z+1}{z-1}}\bigg|_{t-i0} \overset{t\in(-1,1)}{=} \sqrt[3]{\dfrac{1+t}{1-t}}\, e^{\frac{\pi i}{3}} \end{array} \right.$$

$\sqrt[3]{\dfrac{z+1}{z-1}}$



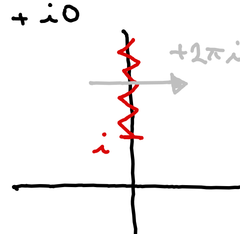
f) $\ln(1+iz)$ $\quad$ Poloha řezu : $1 + iz = -t \ ; \ t > 0 \Rightarrow z = i(1+t) \in (i, i\infty)$

Skok na řezu

$$1+iz\Big|_{it+0} \overset{t>1}{=} 1+iz\Big|_{it} + (1+iz)'\Big|_{it}(+0) =$$

$$= 1 - t + i(+0) = -(t-1) + i0$$

$$\therefore \quad \left\| \begin{array}{l} \ln(1+iz)\Big|_{it+0} \overset{t>1}{=} \ln(t-1) + \pi i \\[1ex] \ln(1+iz)\Big|_{it-0} \overset{t>1}{=} \ln(t-1) - \pi i \end{array} \right.$$

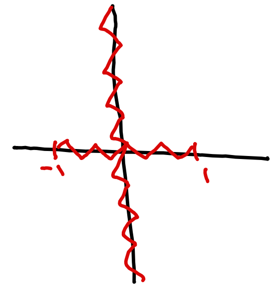


g) $\ln(z^2-1)$

Poloha řezu: $z^2-1 = -t\ ;\ t>0$, čili $\ln(z^2-1)$:

$$z^2 = 1-t \quad \Rightarrow \quad z = \begin{cases} \pm\sqrt{1-t} & ;\ t\in(0,1) \\ \pm i\sqrt{t-1} & ;\ t>1 \end{cases}$$

$\therefore$ Řezy $z \in (-1,1) \cup (-i\infty, i\infty)$



Skoky na řezu

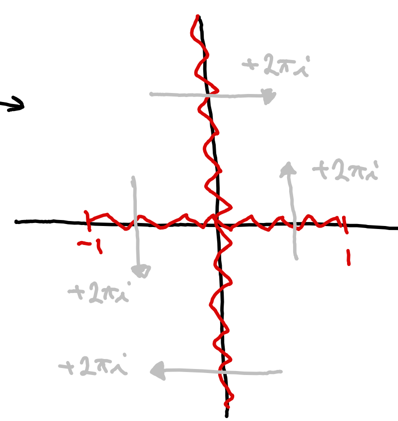
a) $z^2-1\big|_{t+i0} = z^2-1\big|_t + (z^2-1)'\big|_t\, i0 = t^2-1+2t\cdot i0$

$$\therefore\ z^2-1\big|_{t+i0} \begin{cases} \overset{t\in(0,1)}{=} -(1-t^2)+i0 \Rightarrow \ln(z^2-1)\big|_{t+i0} \overset{t\in(0,1)}{=} \ln(1-t^2)+\pi i \\ \overset{t\in(-1,0)}{=} -(1-t^2)-i0 \Rightarrow \ln(z^2-1)\big|_{t+i0} \overset{t\in(-1,0)}{=} \ln(1-t^2)-\pi i \end{cases}$$

b) $z^2-1\big|_{it+0} = z^2-1\big|_{it} + (z^2-1)'\big|_{it}(+0) = -t^2-1+2it(+0)$

$$\therefore\ z^2-1\big|_{it+0} \begin{cases} \overset{t>0}{=} -(t^2+1)+i0 \Rightarrow \ln(z^2-1)\big|_{it+0} \overset{t>0}{=} \ln(t^2+1)+\pi i \\ \overset{t<0}{=} -(t^2+1)-i0 = \ln(z^2-1)\big|_{it+0} \overset{t<0}{=} \ln(t^2+1)-\pi i \end{cases}$$

Celkově:

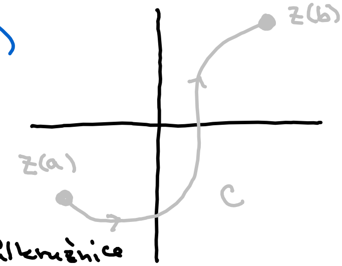


h) $\ln\left(\frac{iz-i}{z+1}\right)$

KOMPLEXNÍ KŘIVKOVÝ INTEGRÁL

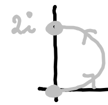
cv 12

(„contour integral")



D $\int_C f(z)\,dz \overset{def.}{=} \int_a^b f(z(t))\, z'(t)\,dt$;

kde $C: z = z(t);\ t\in(a,b)$ („contour")

Př Spočtěte $J = \int_C f(z)\,dz$; C:  ~ půlkružnice

a) $f(z) = |z|^2 = z\bar{z}$ b) $f(z) = z^2$

Sol: Nejprve parametrizace:

$\longrightarrow$ $C: z = i + e^{it};\ t\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$; $dz = ie^{it}\,dt$

a) $J = \int_{-\pi/2}^{\pi/2} (i+e^{it})(-i+e^{-it})\, ie^{it}\,dt = \int_{-\pi/2}^{\pi/2} 2ie^{it} + e^{2it} - 1\,dt =$

$= \left[2e^{it} - \frac{i}{2}e^{2it} - t\right]_{-\pi/2}^{\pi/2} = 2\left(e^{\frac{\pi}{2}i} - e^{-\frac{\pi}{2}i}\right) - \frac{i}{2}\left(e^{\pi i} - e^{-\pi i}\right) - \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$

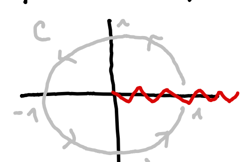
$= 2(i-(-i)) - \frac{i}{2}(-1-(-1)) - \pi = 4i - \pi$

b) $J = \int_{-\pi/2}^{\pi/2} (i+e^{it})^2\, ie^{it}\,dt = \int_{-\pi/2}^{\pi/2} ie^{3it} - 2e^{2it} - ie^{it}\,dt$

$= \left[\frac{1}{3}e^{3it} + ie^{2it} - e^{it}\right]_{-\pi/2}^{\pi/2} = \frac{1}{3}(-i-i) + i(-1-(-1)) - (i-(-i))$

$= -\frac{2}{3}i - 2i = -\frac{8}{3}i$

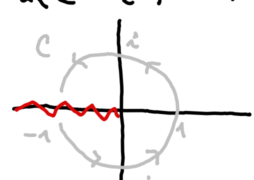
Př Spočtěte $I_1 = \oint_{|z|=1} \frac{dz}{\sqrt{z}}$; $I_2 = \oint_{|z|=1} \ln z\,dz$ pro a) b)



Sol: a) $C: z = e^{it};\ t\in(0,2\pi)$;

(I_1) $\sqrt{z} = e^{\frac{it}{2}}$; $dz = ie^{it}\,dt$

$\therefore\ I_1 = \int_0^{2\pi} \frac{1}{e^{\frac{it}{2}}}\, ie^{it}\,dt = \int_0^{2\pi} ie^{\frac{it}{2}}\,dt = \left[2e^{\frac{it}{2}}\right]_0^{2\pi} = 2\left(e^{\pi i} - e^{0}\right) = -4$



b) $C: z = e^{it};\ t\in(-\pi,\pi)$;

$\sqrt{z} = e^{\frac{it}{2}}$; $dz = ie^{it}\,dt$

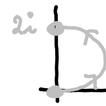
$\therefore\ I_1 = \int_{-\pi}^{\pi} \frac{1}{e^{\frac{it}{2}}}\, ie^{it}\,dt = \int_{-\pi}^{\pi} ie^{\frac{it}{2}}\,dt = \left[2e^{\frac{it}{2}}\right]_{-\pi}^{\pi} = 2(i-(-i)) = 4i$

PRIMITIVNÍ FUNKCE

D (PRIMITIVNÍ FCE): $F'(z) = f(z)$

V Existuje-li F na Ω otevřená $\supset C$, pak $\int_C f(z)\,dz = F(z(b)) - F(z(a))$

Dk $\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt = \int_a^b (F(z(t)))'\,dt = F(z(b)) - F(z(a))$

Př Spočtěte $I = \int_C z^2\,dz$; C:  ~ půlkružnice

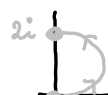
Sol : $I = \left[\frac{z^3}{3}\right]_0^{2i} = \frac{(2i)^3}{3} = -\frac{8i}{3}$

CAUCHYHO VĚTA

V (CAUCHYHO) f je holom. na $\operatorname{Int} C$ a spoj. na $\overline{\operatorname{Int} C}$, pak

$$\boxed{\oint_C f(z)\,dz = 0}$$

Př Doplněním na vhodnou uzavřenou křivku

spočtěte $I = \int_{C_1} z^2\,dz$; C_1:  ~ půlkružnice

$J = \oint_C z^2\,dz$: C:  $C = C_1 + C_2$

- Cauchyho věta : $J = 0$
- Parametrizace : $J = \oint_{C_1} z^2\,dz + \oint_{C_2} z^2\,dz = J_1 + J_2$

$\longrightarrow$ C_1 : $J_1 = \oint_{C_1} z^2\,dz = I$

$\longrightarrow$ $\ominus\, C_2$: $z = it$; $t \in (0,2)$; $dz = i\,dt$

$J_2 = \int_{C_2} z^2\,dz = \ominus \int_{\ominus C_2} z^2\,dz = \ominus \int_0^2 (it)^2\, i\,dt = \frac{8i}{3}$

- Porovnání $J = J_1 + J_2$ čili $0 = I + \frac{8i}{3} \Rightarrow \boxed{I = -\frac{8i}{3}}$

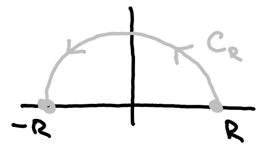
$\boxed{\text{ODHADY INTEGRÁLŮ}}$

$\boxed{\text{V}}$ (ML-LEMMA) : $\left| \int_C f(z)\,dz \right| \le ML$;

$M = \sup_{z \in C} |f(z)|$; $L = |C|$ (délka křivky C)

Dk $\left| \int_C f(z)\,dz \right| \le \underbrace{\int_C |f(z)|\,|dz|}_{\text{JORDAN}} \le \underbrace{\sup_{z\in C} f(z)}_{M} \underbrace{\int_C |dz|}_{L}$

Př Spočtěte $\lim_{R\to\infty} \int_{C_R} f(z)\,dz$; C_R :



a) $f(z) = \frac{1}{1+z^2}$; b) $f(z) = \frac{e^{iz}}{z}$

Sol : Parametrizace :

$C: z = Re^{it}$; $t \in (0, \pi)$: $dz = Rie^{it}$; $|z| = R$; $L = \pi R$

a) $|f(z)| = \frac{1}{|1+z^2|} \le \frac{1}{||z|^2 - 1|} = \frac{1}{R^2-1}$

$\therefore \left| \int_{C_R} f(z)\,dz \right| \overset{ML}{\le} \frac{\pi R}{R^2 - 1} \xrightarrow{R\to\infty} 0$ $\therefore$ $\boxed{\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0}$

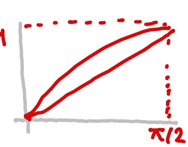
b) $|f(z)| = \frac{|e^{iz}|}{|z|} = \frac{1}{R} |e^{i(x+iy)}| = \frac{1}{R} e^{-y} = \frac{1}{R} e^{-R\sin t} \le \frac{1}{R}$

$\therefore \left| \int_{C_R} f(z)\,dz \right| \overset{ML}{\le} \frac{1}{R} \pi R \ \cancel{\xrightarrow{R\to\infty}}\ 0$ ML ODHAD SLABÝ !

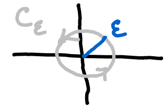
JORDAN: $\left| \int_{C_R} f(z)\,dz \right| \overset{\text{JORDAN}}{\le} \int_{C_R} |f(z)|\,|dz| = \int_0^\pi \frac{1}{R} e^{-R\sin t} R\,dt \overset{\text{sym.}}{=}$

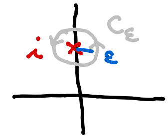
$= 2\int_0^{\frac{\pi}{2}} e^{-R\sin t}\,dt \le 2\int_0^{\frac{\pi}{2}} e^{-R\frac{2t}{\pi}}\,dt = 2\left[-\frac{\pi}{2R} e^{-R\frac{2t}{\pi}} \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2R}(1 - e^{-R}) \xrightarrow{R\to\infty} 0$

$\therefore$ $\boxed{\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0}$



$\sin t \ge \frac{2t}{\pi}$; $t \in (0, \frac{\pi}{2})$

Př: Spočítejte $\lim_{\varepsilon\to0^+}\oint_{C_\varepsilon}\frac{dz}{1+z^2}$; a) 

; b) 

Sol: a) $C:\ z=\varepsilon e^{it};\ t\in(0,2\pi);\ dz=\varepsilon i e^{it};\ |z|=\varepsilon;\ L=2\pi\varepsilon$

$$ML:\ \left|\oint_{C_\varepsilon}\frac{1}{1+z^2}\,dz\right|\le\frac{1}{1-\varepsilon^2}\,2\pi\varepsilon\xrightarrow{\varepsilon\to0^+}0\quad\therefore\ \lim_{\varepsilon\to0^+}\oint_{C_\varepsilon}\frac{dz}{1+z^2}=0$$

b) $C:\ z=i+\varepsilon e^{it};\ t\in(0,2\pi);\ dz=\varepsilon i e^{it}$

$$\oint_{C_\varepsilon}\frac{1}{1+z^2}\,dz=\int_0^{2\pi}\frac{1}{1+(i+\varepsilon e^{it})^2}\,\varepsilon i e^{it}\,dt=$$

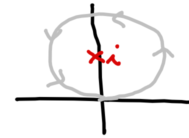
$$=\int_0^{2\pi}\frac{1}{2i\varepsilon e^{it}+\varepsilon^2e^{2it}}\,\varepsilon i e^{it}\,dt=\int_0^{2\pi}\frac{dt}{2-\varepsilon i e^{2it}}$$

$$\left\langle \text{Majoranta}\ \left|\int_0^{2\pi}\frac{dt}{2-\varepsilon i e^{2it}}\right|\le\int_0^{2\pi}\frac{dt}{2-\varepsilon}=\frac{2\pi}{2-\varepsilon}\le\pi;\ \varepsilon\in[0,1]\right\rangle$$

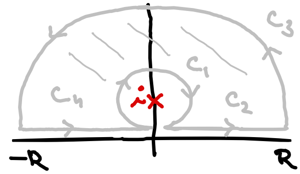
$$\therefore\ \lim_{\varepsilon\to0^+}\oint_{C_\varepsilon}\frac{1}{1+z^2}\,dz=\int_0^{2\pi}\lim_{\varepsilon\to0^+}\frac{dt}{2-\varepsilon i e^{2it}}=\int_0^{2\pi}\frac{dt}{2}=\pi$$

Aplikace Cauchyho věty:

(Př) Užitím Cauchyho věty spočtěte $I = \oint_{|z-i|=1} \frac{dz}{1+z^2}$; tj.



Metoda I: $J := \oint_C \frac{dz}{1+z^2}$; C:



- **Cauchyho věta**: $J = 0$
- **Parametrizace**: $J = \sum J_k$; $J_k = \oint_{C_k} f(z)\,dz$

$\longrightarrow$ C_1: $J_1 = \ominus I$

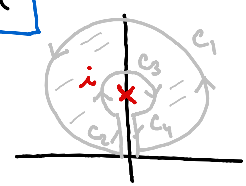
$\longrightarrow$ $C_4 + C_2$: $z = t$; $t \in (-R,R)$; $dz = dt$

$$\therefore\ J_4 + J_2 = \int_{-R}^{R} \frac{1}{1+t^2}\,dt \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \frac{dt}{1+t^2} = [\operatorname{arctg} t]_{-\infty}^{\infty} = \pi$$

$\longrightarrow$ C_3: $|J_3| \underset{ML}{\leq} \frac{1}{R^2-1}\pi R \xrightarrow{R\to\infty} 0 \quad \therefore\ J_3 \xrightarrow{R\to\infty} 0$

- **Porovnání**: $0 = -I + \pi \Rightarrow \boxed{I = \pi}$

Metoda II: $J := \oint_C \frac{dz}{1+z^2}$; C:



- **Cauchyho věta**: $J = 0$
- **Parametrizace**:

$\longrightarrow$ C_1: $J_1 = I$

$\longrightarrow$ $C_2 = \ominus C_4 \quad \therefore\ J_2 + J_4 = 0$

vizte předchozí strana

$\longrightarrow$ C_3: $J_3 = -\lim_{\varepsilon\to 0^+} \oint_{|z-i|=\varepsilon} \frac{dz}{1+z^2} = -\pi$

- **Porovnání**: $0 = I - \pi \quad \therefore\ \boxed{I = \pi}$

LAURENTOVA ŘADA

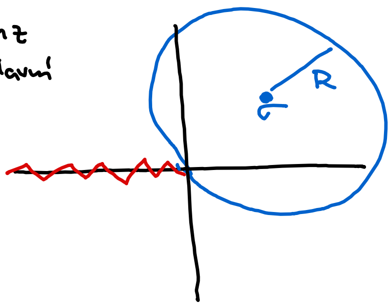
Cv 43

V (LAURENTOVA ŘADA) f holomorfní na $B_{a,b}(\sigma)$, pak

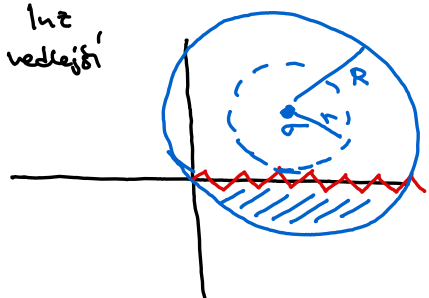
$$\boxed{f(z) = \sum_{l=-\infty}^{\infty} a_l (z-\sigma)^l} \quad ; \quad \text{kde } a_l = \frac{1}{2\pi i}\oint_{C_r(\sigma)} \frac{f(z)\,dz}{(z-\sigma)^{l+1}} \; ; \; r \in (a,b)$$

! Radius konvergence u nejednoznačných funkcí

ln z hlavní



ln z vedlejší



Laurentova řada konverguje na $C_R(\sigma)$, neboť je zde funkce ln z (hlavní) holomorfní

Ačkoli funkce není holom. na $C_R(\sigma)$ (jen $C_r(\sigma)$), řada přesto konverguje na $C_R(\sigma)$, ale pod osou x k jiné fci

D (SINGULARITA) : σ je singularita $\iff$ $a = 0$ (resp. $b = \infty$ pro $\sigma = \infty$)

- odstranitelná : $a_l = 0$; $l < 0$ — $\frac{x}{\sin x}$
- pól n-tého řádu : $a_{-n} \neq 0$, jinak $a_l = 0$, $l < -n$ — $\frac{\sinh^2 x}{x^5}$
- podstatná : $a_l \neq 0$ pro nekonečně $l < 0$ — $\exp\left(\frac{1}{x}\right)$

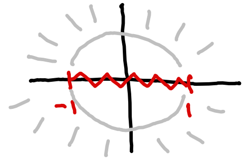
D (RESIDUUM) : $\mathrm{Res}_\sigma f(z) := a_{-1}$; resp. $\mathrm{Res}_\infty f(z) := -a_{-1}$

- σ je pól n-tého řádu: $\mathrm{Res}_\sigma f(z) = \lim_{z\to\sigma} \frac{1}{(n-1)!}\left[(z-\sigma)^n f(z)\right]^{(n-1)}$

  spec 1. řád a g holom. v σ: $\mathrm{Res}_\sigma \frac{g(z)}{h(z)} = \frac{g(\sigma)}{h'(z)}$
- $\mathrm{Res}_\infty f(z) = -\mathrm{Res}_0 \frac{1}{z^2} f\left(\frac{1}{z}\right)$; spec. 1. řád $\mathrm{Res}_\infty f(z) = -\lim_{z\to\infty} z f(z)$

($\mathbb{P}_r^c$) Určete, zda má fce singularitu v nekonečnu a pokud ano, určete residuum (všechny fce hlavní větve)

a) $\ln\left(\frac{z+1}{z-1}\right)$ <u>Holomorfie</u>:

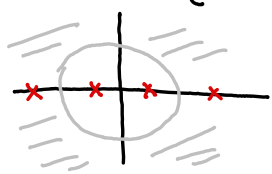


- Fce je holomorfní na $B_{1,\infty}(0)$ => má LŘ v ∞ $\wedge$ má v ∞ singularitu?
- Laurentova řada je jednoznačná => volím $z = t \gg 1$ $(t \in \mathbb{R})$

$$f(t) = \ln\left(\frac{t+1}{t-1}\right) = \ln\left(\frac{t-1+2}{t-1}\right) = \ln\left(1+\frac{2}{t-1}\right) = \frac{2}{t-1} + O\left(\frac{1}{(t-1)^2}\right) =$$

$$= \frac{2}{t} + O\left(\frac{1}{t^2}\right) \Rightarrow f(z) = \frac{2}{z} + O\left(\frac{1}{z^2}\right) \quad \therefore \quad \underline{\mathrm{Res}_\infty f = -2}$$

b) $\operatorname{tg} z$ <u>Holomorfie</u>:



- Fce není holomorfní na žádném $B_{R,\infty}(z_0)$ okolí nekonečna

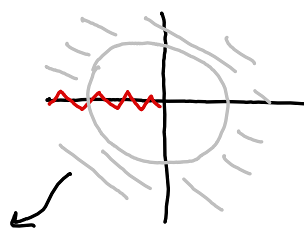
=> <u>Nemá</u> Laurentovu řadu ; $\sigma = \infty$ není singularita

c) $\sqrt{1+\sqrt{z}}$ <u>Holomorfie</u>: Celkově $\sqrt{1+\sqrt{z}}$:

Řez na $-t ; t > 0$ kvůli vnitřní $\sqrt{\ }$

Vnější $\sqrt{\ }$: $1+\sqrt{z} = -t \Rightarrow \sqrt{z} = -t-1$

(toto ale nemá řešení! $\because \arg\sqrt{z} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$)



- Opět fce není holom. na okolí ∞ => nemá singularitu

d) e^z <u>Holomorfie</u>:

(na celém $\mathbb{C}$)

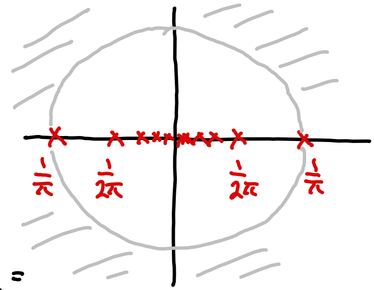
- Fce má singularitu v ∞ $\because$ Holom. na $B_{R,\infty}(z_0)$ libov z_0
- Jest $e^z = 1 + z + \frac{z^2}{2} + \dots \Rightarrow \mathrm{Res}_\infty f = 0$

(jedná se o podstatnou singularitu v ∞ $\because$ LŘ ∞ členů $z^k; k > 0$)

e) $\frac{1}{\sin\frac{1}{z}}$ **Holomorfie:**

$z \neq \frac{1}{\pi k};\ k\in\mathbb{Z}/\{0\}$

$z\neq 0$

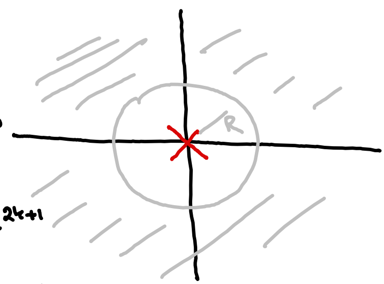


- Fce má singularitu v ∞ ∵ Hol. na $B_{\frac{1}{\pi},\infty}(0)$
- $\frac{1}{\sin\frac{1}{z}} = \frac{1}{\frac{1}{z} - \frac{1}{6z^3}+O(\frac{1}{z^5})} = z\frac{1}{1-\frac{1}{6z^2}+O(\frac{1}{z^4})} =$

$= z\left(1+\frac{1}{6z^2}+O(\frac{1}{z^4})\right) = z+\frac{1}{6z}+O(\frac{1}{z^3})$ ∴ $\underline{\operatorname{Res}_\infty f = -\frac{1}{6}}$

- Pozn: Fce nemá v Laurentovu řadu v 0.

f) $z\sin z\ \ \sin\frac{1}{z}$ **Holomorfie:**



- Fce hol. na okolí ∞ ⇒ Má singularitu v ∞

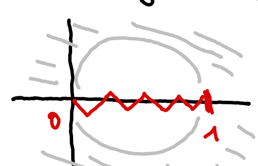
$z\sin z\ \sin\frac{1}{z} = z\sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!} z^{2l+1}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\frac{1}{z^{2k+1}}$

$= \dots + \frac{1}{z^2}c_2 + \frac{1}{z}\sum_{2l-2k+1=-1} \frac{(-1)^l}{(2l+1)!}\frac{(-1)^k}{(2k+1)!} + c_0 + c_1 z+\dots$

∴ $\operatorname{Res}_\infty f(z) = -\sum_{l=0}^{\infty} \frac{1}{(2l+1)!(2l+3)!}$

- Pozn: Funkce má Laurentovu řadu v 0 (je tam podstatná singul.)

g) $\sqrt{z}\sqrt{z-1}$ **Holomorfie:**



- Fce hol. na $B_{\frac{1}{2},\infty}(\frac{1}{2})$ ⇒ má singularitu v ∞: $z=t;\ t>>1$ volím

$f(t)=\sqrt{t}\sqrt{t-1} = t\sqrt{1-\frac{1}{t}} = t\left(1+(\frac{1}{2})(-\frac{1}{t})+\frac{1}{2!}(\frac{1}{2})(-\frac{1}{2})(-\frac{1}{t})^2+O(\frac{1}{t^3})\right)$

$= t\left(1-\frac{1}{2t}-\frac{1}{8t^2}+O(\frac{1}{t^3})\right) = t-\frac{1}{2}-\frac{1}{8t}+O(\frac{1}{t^2})$

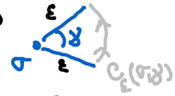
⇒ Jednoznačnost LŘ: $f(z)=z-\frac{1}{2}-\frac{1}{8z}+O(\frac{1}{z^2})$ ∴ $\operatorname{Res}_\infty f(z) = \frac{1}{8}$

RESIDUOVÁ VĚTA

Cv 44

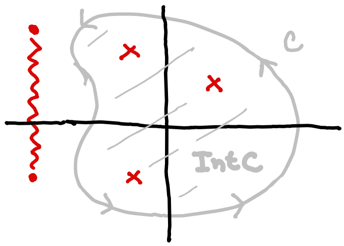
T $\lim\limits_{\varepsilon\to 0^+} \oint_{C_\varepsilon(\sigma)} f(z) = 2\pi i \operatorname{Res}_\sigma f(z)$; resp. $\lim\limits_{R\to\infty} \oint_{C_R(z_0)} f(z) = -2\pi i \operatorname{Res}_\infty f(z)$

Spec. pro pól 1. řádu: $\lim\limits_{\varepsilon\to 0^+} \int_{C_\varepsilon(\sigma,\gamma)} f(z)\,dz = \gamma i \operatorname{Res}_\sigma f(z)$



D (MEROMORFIE): f meromorfní na Ω $\Leftrightarrow$ f hol. na Ω / S ;
kde $S = \{\sigma_1, \sigma_2, \dots \sigma_n\}$ konečná

V (RESIDUOVÁ): f meromorfní na Int C (resp. Ext C)
a spojitá na $\overline{\mathrm{Int}C} / S$ (resp. $\overline{\mathrm{Ext}C} / S$), pak



$$\oint_C f(z) = \begin{cases} 2\pi i \sum\limits_{\sigma \in \mathrm{Int}C} \operatorname{Res}_\sigma f(z) \text{ ; resp.} \\ -2\pi i \sum\limits_{\sigma \in \mathrm{Ext}C \cup \{\infty\}} \operatorname{Res}_\sigma f(z) \end{cases}$$