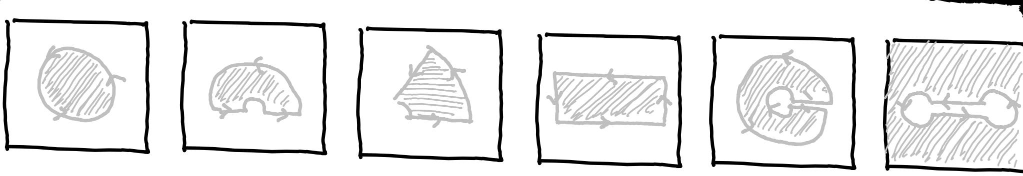
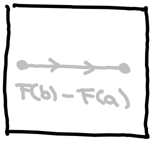
$$\oint f(z)\,dz$$

: „dle již spočítaného výsledku ...“

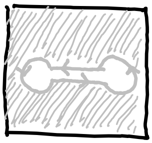


SEZNAM PŘÍKLADŮ



$$\int_0^\infty \frac{dx}{(1+x+x^2)^2}$$

$$\int_1^\infty \frac{x^2-1}{x^4+1}\,dx$$



$$\int_0^1 \sqrt{x}\sqrt{1-x}\,dx$$

$$\int_{-1}^1 \frac{1}{4+x^2}\,\frac{dx}{\sqrt[3]{(1-x)^2(1+x)}}$$

$$\int_0^1 \ln^2\left(\frac{x}{1-x}\right)\frac{dx}{\sqrt{x}\sqrt{1-x}}$$

$$\int_{-1}^1 \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{\sqrt[3]{(1-x)^2(1+x)}}$$

$$\int_0^1 \frac{2x-1}{1-x+x^2}\ln\left(\frac{x}{1-x}\right)dx$$

$$\int_0^1 \frac{dx}{(1+x+x^2)^2}$$

$$\int_0^1 \ln^2\left(\frac{x}{1-x}\right)dx$$

$$\fint_0^1 \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{x}$$

$$\fint_0^1 \ln^3\left(\frac{1+x}{1-x}\right)\frac{dx}{x}$$

$$\int_{-1}^1 \left(\ln\left(\frac{1+x}{1-x}\right)-2x\right)\frac{dx}{x^3}$$

$$\int_0^1 \ln^3\left(\frac{1+x}{1-x}\right)\frac{dx}{x^3}$$

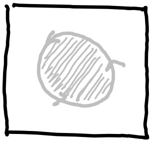
$$\int_0^1 \frac{\ln x\,dx}{\sqrt{x(1-x)}}dx$$

$$\fint_0^1 \sqrt{\frac{x}{1-x}}\,\ln x\,dx$$

$$\int_0^1 \frac{\ln^2 x\,dx}{\sqrt{x(1-x)}}dx$$

$$\int_0^1 \ln^2\frac{x}{1-x}\arccos\sqrt{x}\,dx$$

$$\fint_0^1 \ln\frac{x}{1-x}\ln x\arccos\sqrt{x}\,dx$$



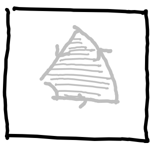
$$\int_0^{2\pi} \frac{\cos^2 x}{1+\sin^2 x}\,dx$$

$$\int_0^{2\pi} \frac{\cos^2 x\,dx}{(1+4\sin^2 x)^2}$$

$$\int_0^{2\pi} \frac{dx}{\cos^2 x-\cos x+1}$$

$$\fint_0^{\pi} \frac{dx}{\cos x-\alpha}$$

$$\int_0^{2\pi} \ln(1-2\alpha\cos x+\alpha^2)\,dx$$



$$\int_{-\infty}^\infty \frac{dx}{x^6+1}$$

$$\cancel{\int_0^\infty \frac{dx}{\sqrt{x}(x^6+1)^2}}$$

$$\int_0^\infty \frac{\ln x}{\sqrt{x}(x^6+1)}\,dx$$

$$\int_0^\infty \frac{x\ln x}{1+x^3}$$

$$\int_0^\infty \frac{\ln x}{1+x^4}\,dx$$

$$\int_0^\infty \frac{\ln^2 x}{1+x^4}\,dx$$

$$\fint_0^\infty \frac{dx}{1-x^3}$$

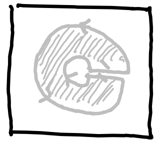
$$\fint_0^\infty \frac{\ln x\,dx}{\sqrt{x}(x^6-1)^2}$$

$$\int_0^\infty \frac{\ln^2 x}{x^2+x+1}\,dx$$

$$\int_0^\infty \frac{\ln x}{1-x+x^2}\,dx$$

$$\int_0^\infty \frac{\ln x}{x^2+2x+2}\,dx$$

$$\int_0^\infty \frac{\operatorname{arctg} x}{x(1+x^4)}\,dx$$



$$\int_0^\infty \frac{dx}{\sqrt{x}(1+x)^2}$$

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^6+1)^2}$$

$$\int_0^\infty \frac{\ln x}{\sqrt{x}(1+x)^2}\,dx$$

$$\int_0^\infty \frac{\ln x\,dx}{(x+1)^2(x+4)\sqrt{x}}$$

$$\int_0^\infty \frac{\ln x}{\sqrt[3]{x}(1+x)}\,dx$$

$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$$

$$\int_0^\infty \frac{dx}{(1+x^2)(1+x^3)}$$

$$\int_0^\infty \frac{dx}{(1+x+x^2)^2}$$

$$\int_0^\infty \frac{\ln x}{(1+x)^3}\,dx$$

$$\fint_0^\infty \frac{dx}{x^2-6x+1}$$

$$\int_0^\infty \frac{\ln x}{1+x+x^2}\,dx$$

$$\int_0^\infty \frac{\ln^2 x}{(x+1)^2}\,dx$$

$$\int_0^\infty \frac{\ln^3 x\,dx}{(1+x^2)(1+x)^2}$$

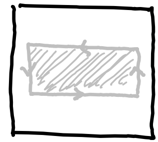
$$\int_0^1 \frac{\ln^2 x}{1-x+x^2}\,dx$$

$$\fint_0^\infty \frac{\ln x}{(x-1)^2}\,dx$$

$$\int_0^1 \frac{\ln x\,dx}{\sqrt{x(1-x)}}dx$$

$$\int_0^\infty \frac{\operatorname{arctg}^3 x}{x^3}\,dx$$

$$\int_0^1 \frac{\ln x}{1+x}\,dx$$



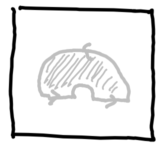
$$\int_{-\infty}^\infty \frac{\cos\alpha x}{\cosh x}\,dx$$

$$\int_{-\infty}^\infty \frac{e^{x/2}}{e^{2x}+2e^x+2}\,dx$$

$$\int_{-\infty}^\infty \frac{x}{\sinh x}\,dx$$

$$\int_0^\infty \frac{x\,dx}{1+e^x}$$

$$\int_0^\infty \frac{\sin x}{\sinh x}e^{-x}\,dx$$



$$\int_{-\infty}^\infty \frac{dx}{x^2+\alpha^2}$$

$$\int_{-\infty}^\infty \frac{\sin kx}{x(x^2+\alpha^2)}$$

$$\int_0^\infty \frac{\sin^3 x}{x^3}$$

$$\fint_{-\infty}^\infty \frac{dx}{1-x^3}$$

$$\fint_0^\infty \frac{\ln x\,dx}{(x-1)^2}$$

$$\int_0^\infty \frac{\ln^2 x}{1+x^2}\,dx$$

$$\fint_0^\infty \frac{\ln^2 x\,dx}{(x-1)^2(x-4)\sqrt{x}}$$

$$\int_{-\infty}^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx$$

$$\int_{-\infty}^\infty \frac{\operatorname{arcctg} x}{x(1+x^2)}\,dx$$

$$\int_{-\infty}^\infty \frac{\ln(1+x^2)}{1-x+x^2}\,dx$$

$$\int_0^\infty \frac{\ln(1+x^4)}{(1+x^2)^2}\,dx$$

$$\int_{-\infty}^\infty \frac{\ln^2(1+x^2)}{1+x^2}\,dx$$

$$\int_0^\infty \frac{\ln x\,\ln(1+x^2)}{1+x^2}\,dx$$

V (LAURENTOVA ŘADA) f holomorfní na $B_{a,b}(\sigma)$, pak

$$\boxed{f(z) = \sum_{\ell=-\infty}^{\infty} a_\ell (z-\sigma)^\ell} \quad ; \quad \text{kde } a_\ell = \frac{1}{2\pi i}\oint_{C_r(\sigma)} \frac{f(z)\,dz}{(z-\sigma)^{\ell+1}} \; ; \; r\in(a,b)$$

D (SINGULARITA) : σ je singularita $\Leftrightarrow$ $a=0$ (resp. $b=\infty$ pro $\sigma=\infty$)

- odstranitelná : $a_\ell = 0$; $\ell<0$ $\qquad \frac{x}{\sin x}$
- pól n-tého řádu : $a_{-n}\neq 0$, jinak $a_\ell = 0$, $\ell<-n$ $\qquad \frac{\sinh^2 x}{x^5}$
- podstatná : $a_\ell \neq 0$ pro nekonečně $\ell<0$ $\qquad \exp\left(\frac{1}{x}\right)$

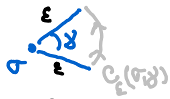
D (RESIDUUM) : $\operatorname{Res}_\sigma f(z) := a_{-1}$; resp. $\operatorname{Res}_\infty f(z) := -a_{-1}$

- σ je pól n-tého řádu: $\operatorname{Res}_\sigma f(z) = \lim_{z\to\sigma} \frac{1}{(n-1)!}\left[(z-\sigma)^n f(z)\right]^{(n-1)}$

  spec. 1. řád a g holom. v σ: $\operatorname{Res}_\sigma \frac{g(z)}{h(z)} = \frac{g(\sigma)}{h'(z)}$
- $\operatorname{Res}_\infty f(z) = -\operatorname{Res}_0 \frac{1}{z^2} f\left(\frac{1}{z}\right)$; spec. 1. řád $\operatorname{Res}_\infty f(z) = -\lim_{z\to\infty} z f(z)$

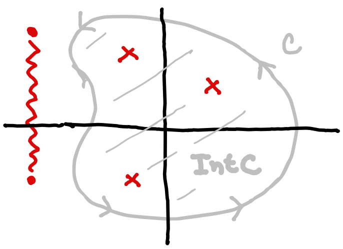
T $\lim_{\varepsilon\to 0^+} \oint_{C_\varepsilon(\sigma)} f(z) = 2\pi i \operatorname{Res}_\sigma f(z)$; resp. $\lim_{R\to\infty} \oint_{C_R(z_0)} f(z) = -2\pi i \operatorname{Res}_\infty f(z)$

Spec. pro pól 1. řádu : $\lim_{\varepsilon\to 0^+} \int_{C_\varepsilon(\sigma,\gamma)} f(z)\,dz = \gamma i \operatorname{Res}_\sigma f(z)$



D (MEROMORFIE) : f meromorfní na Ω $\Leftrightarrow$ f hol. na $\Omega \setminus S$;
kde $S = \{\sigma_1, \sigma_2, \dots \sigma_n\}$ konečná

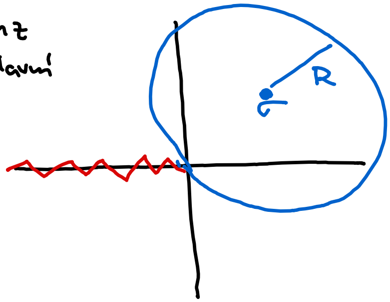
V (RESIDUOVÁ) : f meromorfní na $\operatorname{Int} C$ (resp. $\operatorname{Ext} C$) a spojitá na $\partial \operatorname{Int} C$ (resp. $\partial \operatorname{Ext} C$), pak



$$\boxed{\oint_C f(z) = \begin{cases} 2\pi i \sum\limits_{\sigma\in \operatorname{Int} C} \operatorname{Res}_\sigma f(z) \; ; \; \text{resp.} \\ -2\pi i \sum\limits_{\sigma\in \operatorname{Ext} C \cup \{\infty\}} \operatorname{Res}_\sigma f(z) \end{cases}}$$

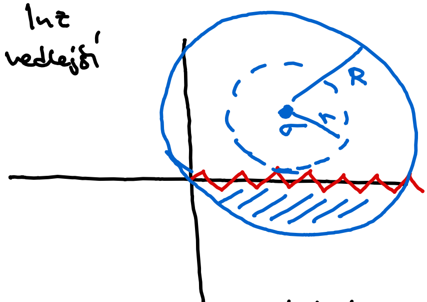
$\boxed{!}$ Radius konvergence u nejednoznačných funkcí

ln z hlavní



Laurentova řada konverguje na $C_R(\sigma)$, neboť je zde funkce ln z (hlavní) holomorfní

ln z vedlejší



Ačkoli funkce není holom. na $C_R(\sigma)$ (jen $C_r(\sigma)$), řada přesto konverguje na $C_R(\sigma)$, ale pod osou x k jiné fci

Sochotskij – Plemelj

$\boxed{T}$ Pól n-tého řádu

Dk.: $$f(z) = \frac{a_{-n}}{z^n} + \frac{a_{-n+1}}{z^{n-1}} + \dots + \frac{a_{-1}}{z} + a_0 + a_1 z + \dots$$

$$\int_{C_\varepsilon(0,\pi)} f(z)\,dz = \int_0^\pi f(\varepsilon e^{it})\,\varepsilon i e^{it}\,dt = \sum_l \int_0^\pi i a_l \varepsilon^l e^{(l+1)it}\,dt$$

$$= \varepsilon^l \sum_l \frac{a_l}{(l+1)} e^{(l+1)it}\Big|_0^\pi = \varepsilon^l \sum_l \frac{a_l}{l+1}\left[(-1)^{l+1} - 1\right]$$

$\boxed{T}$ Pól 2. řádu: $$f(z) = \frac{a_{-2}}{(z-\sigma)^2} + \frac{a_{-1}}{z-\sigma} + a_0 + a_1(z-\sigma) + \dots$$

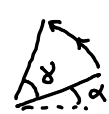
$$\int_{C_\varepsilon(\sigma,\gamma)} f(z)\,dz = \int_\alpha^{\alpha+\gamma} f(\sigma + \varepsilon e^{it})\, i\varepsilon e^{it}\,dt =$$

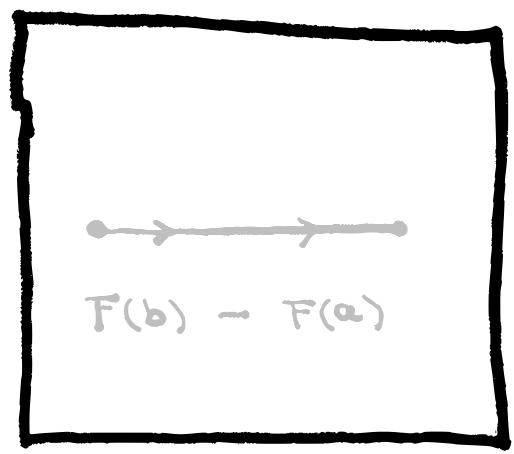
$$= \int_\alpha^{\alpha+\gamma} \frac{a_{-2}}{(\varepsilon e^{it})^2} i\varepsilon e^{it} + \frac{a_{-1}}{\varepsilon e^{it}} i\varepsilon e^{it} + a_0 i\varepsilon e^{it} + \dots\,dt =$$

$$= \left[-\frac{i}{\varepsilon} a_2 e^{-it} + i a_1 t + \varepsilon a_0 e^{it} + \dots\right]_\alpha^{\alpha+\pi} =$$

$$= \frac{2i}{\varepsilon} a_{-2} e^{-\alpha i} + 2\pi i a_{-1} + O(\varepsilon)$$

$C_\varepsilon(\sigma,\gamma)$:




F(b) - F(a)

(Př) $I = \int_0^\infty \frac{dx}{(1+x+x^2)^2} \in \mathbb{R}$

• póly: $1+x+x^2 = \frac{x^3-1}{x-1} = (x-\sigma_+)(x-\sigma_-)$; $\sigma_\pm = e^{\pm\frac{2\pi i}{3}} = -\frac12 \pm i\frac{\sqrt3}{2}$

• parciální zlomky:

$$\frac{1}{(1+x+x^2)^2} = \frac{1}{(x-\sigma_+)^2(x-\sigma_-)^2} = \left[\frac{1}{(x-\sigma_+)(x-\sigma_-)}\right]^2 = \left[\frac{\frac{1}{\sigma_+-\sigma_-}}{x-\sigma_+} + \frac{\frac{1}{\sigma_- - \sigma_+}}{x-\sigma_-}\right]^2 =$$

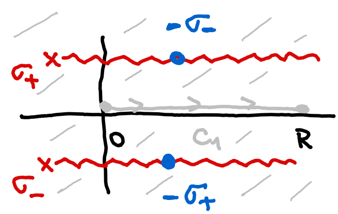
$$= \left[\frac{\frac{1}{i\sqrt3}}{x-\sigma_+} - \frac{\frac{1}{i\sqrt3}}{x-\sigma_-}\right]^2 = \left(\frac{1}{i\sqrt3}\right)^2\left[\frac{1}{x-\sigma_+}-\frac{1}{x-\sigma_-}\right]^2 =$$

$$= -\frac13\left[\frac{1}{(x-\sigma_+)^2} - \frac{2}{(x-\sigma_+)(x-\sigma_-)} + \frac{1}{(x-\sigma_-)^2}\right] =$$

$$= -\frac13\left[\frac{1}{(x-\sigma_+)^2} - \frac{2}{i\sqrt3}\frac{1}{x-\sigma_+} + \frac{2}{i\sqrt3}\frac{1}{x-\sigma_-} + \frac{1}{(x-\sigma_-)^2}\right]$$

$\therefore I = \lim_{R\to\infty}\int_{C_1} f(z)\,dz = \lim_{R\to\infty} F(z)\Big|_0^R$; [f hol. na okolí C_1]

$$F(z) = -\frac13\left[-\frac{1}{z-\sigma_+} - \frac{2}{i\sqrt3}\ln(z-\sigma_+) + \frac{2}{i\sqrt3}\ln(z-\sigma_-) - \frac{1}{z-\sigma_-}\right]$$



• dosazení mezí:

i) $F(R) = -\frac13\left(-\frac{2}{i\sqrt3}\right)\left[\ln(R-\sigma_+) - \ln(R-\sigma_-)\right] + O\left(\frac1R\right)$

$$= \frac{2}{3\sqrt3 i}\left[\ln|R-\sigma_+| + i\arg(R-\sigma_+) - \ln|R-\sigma_-| - i\arg(R-\sigma_-)\right] + O\left(\frac1R\right)$$

$$\xrightarrow{R\to\infty} \frac{2}{3\sqrt3 i}\left[2\pi i + 0i\right] = \frac{4\pi}{3\sqrt3}$$

ii) $F(0) = -\frac13\left[\frac{1}{\sigma_+} - \frac{2}{i\sqrt3}\ln(-\sigma_+) + \frac{2}{i\sqrt3}\ln(-\sigma_-) + \frac{1}{\sigma_-}\right] =$

$$= -\frac13\left[\sigma_- - \frac{2}{\sqrt3}\arg(-\sigma_+) + \frac{2}{\sqrt3}\arg(-\sigma_-) + \sigma_+\right] =$$

$$= -\frac13\left[-1 - \frac{2}{\sqrt3}\left(\frac{5\pi}{3}\right) + \frac{2}{\sqrt3}\left(\frac{\pi}{3}\right)\right] = \frac13 + \frac{8\pi}{9\sqrt3}$$

$$\therefore I = \frac{4\pi}{3\sqrt3} - \left(\frac13 + \frac{8\pi}{9\sqrt3}\right) = \frac{4\pi}{9\sqrt3} - \frac13$$

Pr $\quad I = \int_1^{\infty} \frac{x^2-1}{x^4+1}\,dx \quad \in \mathbb{R}$

 $e^{\frac{\pi}{2}i} = \frac{1}{\sqrt{2}}(1+i)$

jest $\quad \frac{1}{x^2-i} = \frac{x^2+i}{(x^2-i)(x^2+i)} = \frac{x^2+i}{x^4+1}$

$$\therefore \quad \frac{x^2-1}{x^4+1} = \mathrm{Re}\left[\frac{1}{x^2-i} + \frac{i}{x^2-i}\right] = \mathrm{Re}\,\frac{1+i}{x^2-i}$$

czyli $\quad I = \int_1^{\infty} \frac{x^2-1}{x^4+1}\,dx = \mathrm{Re}\,(1+i)\int_1^{\infty}\frac{dx}{x^2-i} =$

$$= \sqrt{2}\,\mathrm{Re}\, e^{\frac{\pi i}{4}} \int_1^{\infty} \frac{1}{(x-e^{\frac{\pi i}{4}})(x+e^{\frac{\pi i}{4}})}\,dx =$$

$$= \sqrt{2}\,\mathrm{Re}\, e^{\frac{\pi i}{4}} \int_1^{\infty} \frac{\frac{1}{2e^{\frac{\pi i}{4}}}}{x-e^{\frac{\pi i}{4}}} + \frac{-\frac{1}{2e^{\frac{\pi i}{4}}}}{x+e^{\frac{\pi i}{4}}}\,dx =$$

$$= \frac{\sqrt{2}}{2}\,\mathrm{Re} \int_1^{\infty} \frac{1}{x-e^{\frac{\pi i}{4}}} - \frac{1}{x+e^{\frac{\pi i}{4}}}\,dx =$$

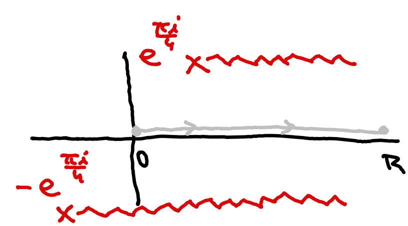
$$= \frac{\sqrt{2}}{2}\,\mathrm{Re} \lim_{R\to\infty} \Big[\underbrace{\ln(z-e^{\frac{\pi i}{4}}) - \ln(z+e^{\frac{\pi i}{4}})}_{F(z)}\Big]_1^R =$$

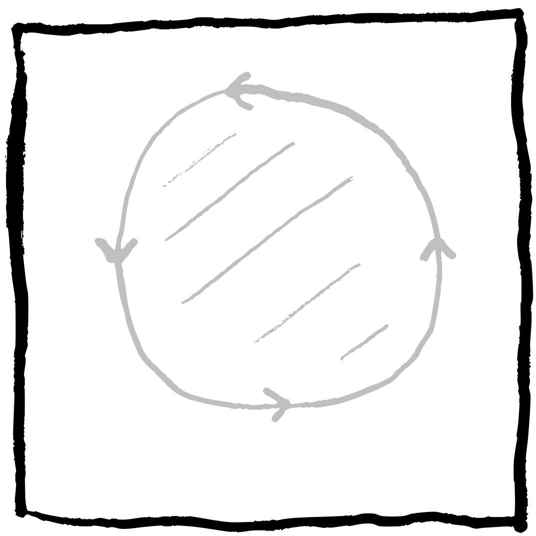
$$= \frac{\sqrt{2}}{2} \lim_{R\to\infty} \Big[\ln|z-e^{\frac{\pi i}{4}}| - \ln|z+e^{\frac{\pi i}{4}}|\Big]_1^R =$$

$$= \frac{\sqrt{2}}{2}\left[\lim_{R\to\infty} \ln\left|\frac{R-e^{\frac{\pi i}{4}}}{R+e^{\frac{\pi i}{4}}}\right| - \ln\left|\frac{1-e^{\frac{\pi i}{4}}}{1+e^{\frac{\pi i}{4}}}\right|\right] =$$

$$= -\frac{\sqrt{2}}{2} \ln \frac{\sqrt{(1-e^{\frac{\pi i}{4}})(1-e^{-\frac{\pi i}{4}})}}{\sqrt{(1+e^{\frac{\pi i}{4}})(1+e^{-\frac{\pi i}{4}})}} = -\frac{\sqrt{2}}{4} \ln \frac{1-e^{\frac{\pi i}{4}}-e^{-\frac{\pi i}{4}}+1}{1+e^{\frac{\pi i}{4}}+e^{-\frac{\pi i}{4}}+1} =$$

$$= -\frac{\sqrt{2}}{4} \ln \frac{2-2\cos\frac{\pi}{4}}{2+2\cos\frac{\pi}{4}} = \frac{\sqrt{2}}{4} \ln \frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2}\,\mathrm{arg\,tgh}\,\frac{\sqrt{2}}{2}$$



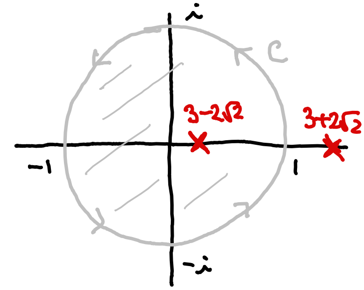


(7) $I = \int_0^{2\pi} \frac{\cos^2 x}{1+\sin^2 x}\,dx \in \mathbb{R}$

úprava : $I = \int_0^{2\pi} \frac{\frac{1+\cos 2x}{2}}{1+\frac{1-\cos 2x}{2}}\,dx = \left| \begin{array}{l|l} x = \frac{u}{2} & 2\pi \to 4\pi \\ dx = \frac{1}{2}du & 0 \to 0 \end{array} \right| =$

$= \frac{1}{2}\int_0^{4\pi} \frac{1+\cos u}{3-\cos u}\,du \overset{\text{sym.}}{=} \int_0^{2\pi} \frac{1+\overbrace{\cos u}^{-4+4}}{3-\cos u}\,du = -2\pi + 4\underbrace{\int_0^{2\pi} \frac{du}{3-\cos u}\,du}_{I' \in \mathbb{R}}$

$J = \oint_C \frac{1}{3-\frac{z+\frac{1}{z}}{2}}\,\frac{dz}{2z} = \oint_C \frac{dz}{6z-z^2-1}$



singularity : $z = \frac{-6\pm\sqrt{32}}{-2} = 3\pm 2\sqrt{2}$

• RESIDUOVÁ VĚTA :

$\longrightarrow$ $\operatorname{Res}_{3-2\sqrt{2}} f(z) = \frac{1}{6-2z}\Big|_{3-2\sqrt{2}} = \frac{1}{4\sqrt{2}}$ $\quad\therefore\quad J = 2\pi i\left(\frac{1}{4\sqrt{2}}\right) = \frac{\pi i\sqrt{2}}{4}$

• PARAMETRIZACE :

$\longrightarrow$ $C : z = e^{it}$; $dz = ie^{it}dt$

$J = \int_0^{2\pi} \frac{1}{3-\cos u}\,\frac{i\cancel{e^{it}}dt}{2\cancel{e^{it}}} = \frac{i}{2}I'$

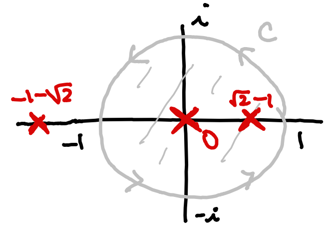
• POROVNÁNÍ : $\frac{\pi i\sqrt{2}}{4} = \frac{i}{2}I'$ $\quad\therefore\quad I' = \frac{\pi\sqrt{2}}{2}$

z čehož $\boxed{I = -2\pi + 2\pi\sqrt{2} = 2\pi(\sqrt{2}-1)}$

(Př) $I = \int_0^{2\pi} \frac{\cos^2 x}{1+\sin^2 x}\, dx \in \mathbb{R} = \operatorname{Im} \int_0^{2\pi} \frac{\cos^2 x}{\sin x - i}\, dx \in \mathbb{C}$

$$J = \oint_C \frac{\left(\frac{z+\frac{1}{z}}{2}\right)^2}{\frac{z-\frac{1}{z}}{2i} - i} \frac{dz}{iz} = \oint_C \frac{1}{2} \frac{(z^2+1)^2}{z^2-1+2z} \frac{1}{z^2}\, dz$$

singularity: $z^2 + 2z - 1 = 0 \iff z = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$



• RESIDUOVÁ VĚTA:

→ $z = 0$: $f(z) = \frac{1}{2} \frac{(z^2+1)^2}{(z^2-1+2z)} \frac{1}{z^2} = \frac{1}{-2} \frac{1+O(z^2)}{1-2z+O(z^2)} \frac{1}{z^2} =$

$= -\frac{1}{2}\left(1 + 2z + O(z^2)\right) \frac{1}{z^2} = -\frac{1}{2z^2} - \frac{1}{z} + O(1) \therefore \operatorname{Res}_0 f(z) = -1$

→ $z = \sqrt{2}-1$: $\operatorname{Res}_{\sqrt{2}-1} f(z) = \frac{1}{2} \frac{(z^2+1)^2}{2z+2} \frac{1}{z^2} = \frac{1}{2} \frac{1}{2\sqrt{2}} \frac{(4-2\sqrt{2})^2}{3-2\sqrt{2}} =$

$= \frac{1}{4\sqrt{2}} (2\sqrt{2})^2 \frac{(\sqrt{2}-1)^2}{3-2\sqrt{2}} = \sqrt{2}$

čili: $J = 2\pi i \sum_{\sigma \in \operatorname{Int} C} \operatorname{Res}_\sigma f = 2\pi i (\sqrt{2}-1)$

• PARAMETRIZACE:

→ $C: z = e^{it}$, $t \in \langle 0, 2\pi)$; $dz = ie^{it}\, dt = iz\, dt$

$$J = \int_0^{2\pi} \frac{\cos^2 t}{\sin t - i}\, dt = \int_0^{2\pi} \frac{\cos^2 t\, (\sin t + i)}{\sin^2 t + 1}\, dt =$$

$$= \underbrace{\int_0^{2\pi} \frac{\cos^2 t \sin t}{\sin^2 t + 1}\, dt}_{I_0 \in \mathbb{R}} + i \underbrace{\int_0^{2\pi} \frac{\cos^2 t}{\sin^2 t + 1}\, dt}_{I} = I_0 + iI$$

• POROVNÁNÍ: $2\pi i (\sqrt{2}-1) = I_0 + iI$

Im: $\boxed{I = 2\pi(\sqrt{2}-1)}$

Re: [Bonus] $I_0 = \int_0^{2\pi} \frac{\cos^2 t \sin t}{\sin^2 t + 1}\, dt = 0$

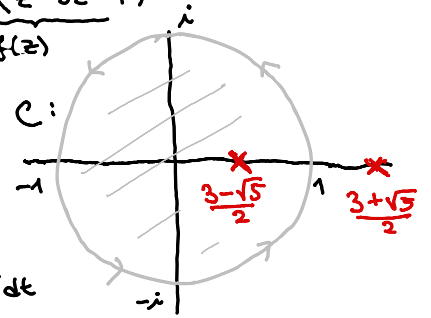
Př $I = \underbrace{\int_0^{2\pi} \frac{\cos^2 x}{(1+4\sin^2 x)^2}\,dx}_{\in \mathbb{R}} = \frac{1}{2}\int_0^{2\pi} \frac{1+\cos 2x}{(1+2(1-\cos 2x))^2}\,dx = \left| \begin{matrix} x = \frac{u}{2} & 0 \to 0 \\ dx = \frac{1}{2}du & 2\pi \to 4\pi \end{matrix} \right|$

$$= \frac{1}{4}\int_0^{4\pi} \frac{1+\cos u}{(3-2\cos u)^2}\,du \overset{\text{perio.}}{=} \frac{1}{2}\int_0^{2\pi} \frac{1+\cos u}{(3-2\cos u)^2}\,du \in \mathbb{R}$$

$$J := \oint_C \frac{2\left(1+\frac{z+\frac{1}{z}}{2}\right)}{\left(3-2\cdot\left(\frac{z+\frac{1}{z}}{2}\right)\right)^2}\frac{dz}{z} = \oint_C \underbrace{\frac{z^2+2z+1}{(z^2-3z+1)^2}}_{f(z)}\,dz$$

póly: $z^2 - 3z + 1 = 0$

$\hookrightarrow z = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2} \equiv \sigma_\pm$ 2,4



- PARAMETRIZACE :

$\longrightarrow C: z = e^{it}; \; t \in \langle 0, 2\pi); \; dz = ie^{it}dt$

$$J = \oint_C f(z)\,dz = \int_0^{2\pi} \frac{2(1+\cos t)}{(3-\cos t)^2}\frac{i\cancel{e^{it}}}{\cancel{e^{it}}}\,dt = 4iI$$

- RESIDUOVÁ VĚTA :

$\longrightarrow \operatorname{Res}_{\overset{\sigma_-}{\overbrace{\frac{3-\sqrt{5}}{2}}}} f(z) \overset{\text{pól 2.}}{=} \lim_{z\to\sigma_-} \frac{1}{(2-1)!}\left[(z-\sigma_-)^2 f(z)\right]^{(2-1)} =$

$$= \lim_{z\to\sigma_-}\left[\frac{(z+1)^2}{(z-\sigma_+)^2}\right]' = \lim_{z\to\sigma_-} 2\frac{z+1}{z-\sigma_+}\left(\frac{z+1}{z-\sigma_+}\right)' = \lim_{z\to\sigma_-} 2\frac{z+1}{z-\sigma_+}\frac{z-\sigma_+-(z+1)}{(z-\sigma_+)^2}$$

$$= 2\frac{\sigma_-+1}{\sigma_--\sigma_+}\left(-\frac{\sigma_++1}{(\sigma_--\sigma_+)^2}\right) = 2\frac{1+\sigma_++\sigma_-+\sigma_-\sigma_+}{(\sigma_+-\sigma_-)^3} \overset{\text{Viète}}{=} 2\frac{1+3+1}{5\sqrt{5}} = \frac{2}{\sqrt{5}}$$

$$\therefore\; J = 2\pi i \sum_{\sigma \in \operatorname{Int} C} \operatorname{Res}_\sigma f(z) = 2\pi i\left(\frac{2}{\sqrt{5}}\right) = \frac{4\pi i}{\sqrt{5}}$$

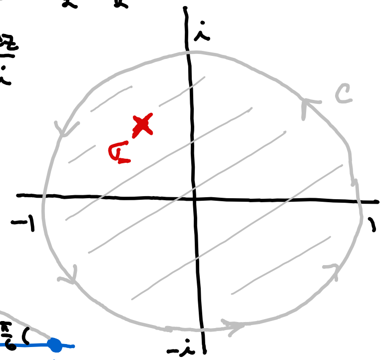
- POROVNÁNÍ :

$$\frac{4\pi i}{\sqrt{5}} = 4iI \qquad \therefore \quad \boxed{I = \frac{\pi}{\sqrt{5}}}$$

(Př) $I = \int_0^{2\pi} \frac{dx}{\cos^2 x - \cos x + 1} \in \mathbb{R}$

Trik $\frac{1}{\cos^2 x - \cos x + 1} = \frac{1}{(\cos x - \frac{1}{2})^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \operatorname{Im} \frac{1}{\cos x - \frac{1}{2} - \frac{\sqrt{3}}{2} i} = \frac{2}{\sqrt{3}} \operatorname{Im} \frac{1}{\cos x - e^{\frac{\pi}{3} i}}$

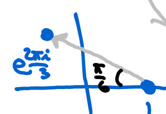
$$J = \oint_C \frac{1}{\frac{z + \frac{1}{z}}{2} - e^{\frac{\pi}{3} i}} \frac{dz}{iz} = \oint_C \frac{2}{z^2 + 1 - 2z e^{\frac{\pi}{3} i}} \frac{dz}{i}$$



Singularity: $z^2 - 2z e^{\frac{\pi}{3} i} + 1 = 0 \iff$

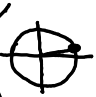
$(z - e^{\frac{\pi}{3} i})^2 = -1 + (e^{\frac{\pi}{3} i})^2 = e^{\frac{2\pi i}{3}} - 1 = \sqrt{3}\, e^{\frac{5\pi}{6} i}$

$\therefore \ \sigma_\pm = e^{\frac{\pi}{3} i} \pm \sqrt[4]{3}\, e^{\frac{5\pi}{12} i} \ \overset{?}{\in} C_1(0)\,?$



$|\sigma_\pm|^2 = \sigma_\pm \overline{\sigma_\pm} = (e^{\frac{\pi}{3} i} \pm \sqrt[4]{3}\, e^{\frac{5\pi i}{12}})(e^{-\frac{\pi}{3} i} \pm \sqrt[4]{3}\, e^{-\frac{5\pi i}{12}}) = 1 \pm 2\sqrt[4]{3} \cos\frac{\pi}{12} + \sqrt{3}$

$= 1 \pm \sqrt{2\sqrt{3} + 3} + \sqrt{3}$ čili $\sigma_- \in \operatorname{Int} C_1(0) \wedge \sigma_+ \in \operatorname{Ext} C_1(0)$

$\Big\langle$  $\cos\frac{\pi}{12} = +\sqrt{\cos^2 \frac{\pi}{16}} = \sqrt{\frac{1 + \cos\frac{\pi}{6}}{2}} = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \frac{1}{2}\sqrt{2 + \sqrt{3}}$ $\Big\rangle$

Přesněji: $\sigma_- = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3} - \sqrt[4]{3}\left(\cos\frac{5\pi}{12} + i \sin\frac{5\pi}{12}\right) =$

$= \frac{1}{2} - \sqrt[4]{3}\sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} + i\left(\frac{\sqrt{3}}{2} - \sqrt[4]{3}\sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}}\right) = \frac{1 - \sqrt{2\sqrt{3} - 3}}{2} + i\,\frac{\sqrt{3} - \sqrt{2\sqrt{3} + 3}}{2} \approx 0.16 - 0.41i$

- RESIDUOVÁ VĚTA:

$\longrightarrow \operatorname{Res}_{\sigma_-} f(z) = \frac{1}{i} \frac{2}{2z - 2e^{\frac{\pi}{3} i}}\Big|_{\sigma_-} = \frac{1}{i} \frac{1}{-\sqrt[4]{3}\, e^{\frac{5\pi}{12} i}} = \frac{i}{\sqrt[4]{3}} e^{-\frac{5\pi}{12} i}$

$\therefore \ J = -\frac{2\pi}{\sqrt[4]{3}} e^{-\frac{5\pi}{12} i}$

- PARAMETRIZACE: C: $z = e^{it}$; $t \in \langle 0, 2\pi \rangle$; $dz = i e^{it}\, dt$

$$J = \int_0^{2\pi} \frac{1}{\cos t - e^{\frac{\pi}{3} i}} \frac{\cancel{ie^{it}}}{\cancel{ie^{it}}}\, dt = \underbrace{\int_0^{2\pi} \frac{\cos t - \frac{1}{2}}{\cos^2 t - \cos t + 1}\, dt}_{I_0 \in \mathbb{R}} + \frac{\sqrt{3}}{2} i \underbrace{\int_0^{2\pi} \frac{dt}{\cos^2 t - \cos t + 1}}_{I}$$

- POROVNÁNÍ: $-\frac{2\pi}{\sqrt[4]{3}} e^{-\frac{5\pi}{12} i} = I_0 + \frac{\sqrt{3}}{2} i I$

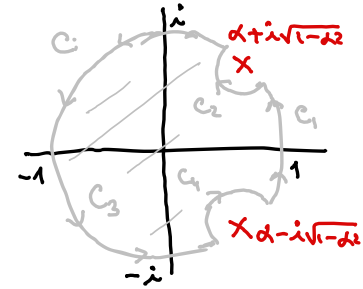
$\boxed{\operatorname{Im}:}$ $\frac{2\pi}{\sqrt[4]{3}} \sin\frac{5\pi}{12} = \frac{\sqrt{3}}{2} I$ $\quad \therefore \ \boxed{I = \frac{2\pi}{3}\sqrt{2\sqrt{3} + 3}}$

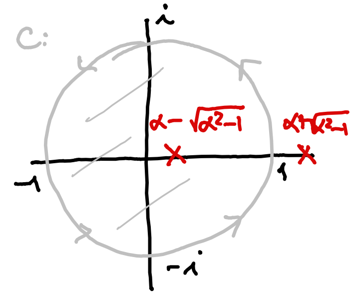
(Př) $I(\alpha)=\int_0^{\pi}\frac{dx}{\cos x-\alpha}\in\mathbb{R}$ pro $\alpha\in\mathbb{R}\setminus\{\pm1\}$; BÚNO $\alpha\geq 0$ (I liché)

sym: $I(\alpha)=\frac{1}{2}\int_0^{2\pi}\frac{dx}{\cos x-\alpha}$ $\Rightarrow$ $J=\oint_C\frac{1}{\frac{z+\frac{1}{z}}{2}-\alpha}\,\frac{dz}{2z}=\oint_C\frac{dz}{z^2-2\alpha z+1}$

póly: $z^2-2\alpha z+1=0 \iff z=\frac{2\alpha\pm\sqrt{4\alpha^2-4}}{2}=$

$$=\begin{cases}\alpha\pm i\sqrt{1-\alpha^2}\,; & \alpha\in[0,1)\\ \alpha\pm\sqrt{\alpha^2-1}\,; & \alpha\in(1,\infty)\end{cases}$$





- RESIDUOVÁ VĚTA

$$\mathrm{Res}_{\underbrace{\alpha+i\sqrt{1-\alpha^2}}_{\sigma_1}} f(z)=\left.\frac{1}{2z-2\alpha}\right|_{\sigma_1}=\frac{1}{-2i\sqrt{1-\alpha^2}}$$

$$\mathrm{Res}_{\underbrace{\alpha-i\sqrt{1-\alpha^2}}_{\sigma_2}} f(z)=\left.\frac{1}{2z-2\alpha}\right|_{\sigma_2}=\frac{1}{2i\sqrt{1-\alpha^2}}$$

i) $\alpha\in[0,1)$: $J=0$ (CAUCHYHO!)

ii) $\alpha>1$: $\mathrm{Res}_{\alpha-\sqrt{\alpha^2-1}} f(z)=\left.\frac{1}{2z-2\alpha}\right|_{\alpha-\sqrt{\alpha^2-1}}=\frac{1}{2\sqrt{\alpha^2-1}}$ $\therefore$ $J=\frac{\cancel{2}\pi i}{\cancel{2}\sqrt{\alpha^2-1}}$

- PARAMETRIZACE

i) $\alpha\in[0,1)$

$\longrightarrow C_1+C_3: z=e^{it}; t\in\overbrace{(0,\arccos\alpha-\varepsilon)\cup(\arccos\alpha+\varepsilon,2\pi-\arccos\alpha-\varepsilon)\cup(2\pi-\arccos\alpha+\varepsilon,2\pi)}^{L_\varepsilon}$

$$J_1+J_3=\int_{L_\varepsilon}\frac{1}{\cos t-\alpha}\,\frac{ie^{it}dt}{e^{it}}\xrightarrow{\varepsilon\to0^+}\int_0^{2\pi}\frac{i\,dt}{\cos t-\alpha}=iI$$

$\longrightarrow C_2: J_2\xrightarrow{\varepsilon\to0^+}-\pi i\,\mathrm{Res}_{\alpha+i\sqrt{1-\alpha^2}}f(z)=\frac{\pi}{\sqrt{1-\alpha^2}}$

$\longrightarrow C_4: J_4\xrightarrow{\varepsilon\to0^+}-\pi i\,\mathrm{Res}_{\alpha-i\sqrt{1-\alpha^2}}f(z)=-\frac{\pi}{\sqrt{1-\alpha^2}}$

ii) $\alpha>1$: $C: z=e^{it}; t\in(0,2\pi)$; $J=iI$

- POROVNÁNÍ

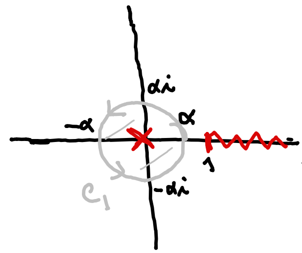
i) $\alpha\in[0,1)$: $0=\frac{\pi}{\sqrt{1-\alpha^2}}-\frac{\pi}{\sqrt{1-\alpha^2}}+iI$ $\therefore$ $\boxed{I=0}$ (stejně $\alpha\in(-1,0)$)

ii) $\alpha>1$: $\frac{\pi i}{\sqrt{\alpha^2-1}}=iI$ $\therefore$ $\boxed{I=\frac{\pi}{\sqrt{\alpha^2-1}}}$ (pro $\alpha<-1$ je $I=-\frac{\pi}{\sqrt{\alpha^2-1}}$)

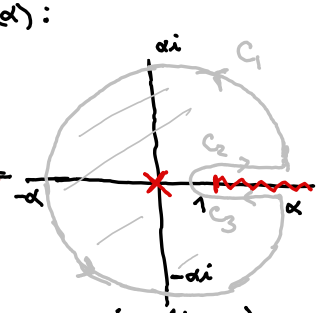
(Př) $I(\alpha) = \int_0^{2\pi} \ln(1-2\alpha\cos x+\alpha^2)\,dx \in \mathbb{R}$ [BÚNO $\alpha \geq 0$ ∵ $I(\alpha)$ sudé v α]

$$\downarrow \quad J(z) = \oint_{C(\alpha)} \frac{\ln(z-1)}{z}\,dz$$

C(α):



α ∈ (0,1]



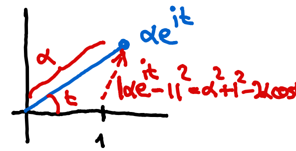
α ∈ (1,∞)

- **RESIDUOVÁ VĚTA:**

$\longrightarrow$ $\operatorname{Res}_0 f(z) = \ln(0-1) = \pi i$

$\therefore\ J = 2\pi i(\pi i) = -2\pi^2$

- **PARAMETRIZACE**



$\longrightarrow$ $C_1:\ z = \alpha e^{it};\ t\in(0,2\pi);\ dz = i\alpha e^{it}dt$

$$J_1 = \int_0^{2\pi} \frac{\ln(\alpha e^{it}-1)}{\cancel{\alpha e^{it}}}\, i\cancel{\alpha e^{it}}\,dt = i\int_0^{2\pi} \ln|\alpha e^{it}-1| + i\arg(\alpha e^{it}-1)\,dt$$

$$= \underbrace{\frac{i}{2}\int_0^{2\pi} \ln\underbrace{(\alpha e^{it}-1)(\alpha e^{-it}-1)}_{\alpha^2-\alpha(e^{it}+e^{-it})+1}\,dt}_{I(\alpha)} - \underbrace{\int_0^{2\pi}\arg(\alpha e^{it}-1)\,dt}_{I_0(\alpha)\in\mathbb{R}}$$

[čili $I(\alpha) = 2\,\mathrm{Im}\,J_1$]

$\xrightarrow{\alpha>1}$ $C_2:\ z = t+i0;\ t\in(1,\alpha);\ dz = dt$

$$J_2 = \int_1^{\alpha} \frac{\ln(t+i0-1)}{t}\,dt = \int_1^{\alpha}\frac{\ln(t-1)}{t}\,dt$$

$\xrightarrow{\alpha>1}$ $\ominus\, C_3:\ z = t-i0;\ t\in(1,\alpha);\ dz = dt$

$$J_3 = \ominus\int_1^{\alpha}\frac{\ln(t-i0-1)}{t}\,dt = -\int_1^{\alpha}\frac{\ln(t-1)+2\pi i}{t}\,dt = -\int_1^{\alpha}\frac{\ln(t-1)}{t}\,dt - 2\pi i\ln\alpha$$

- **POROVNÁNÍ**:

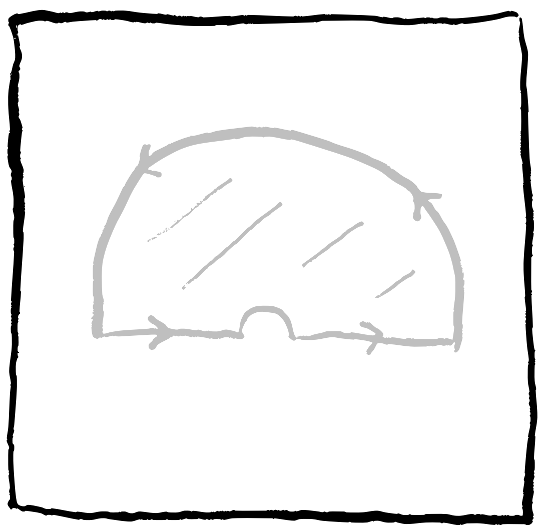
$\boxed{\mathrm{Im}:}$

i) $\alpha\in(0,1]$: $-2\pi^2 = \frac{i}{2}I(\alpha) - I_0(\alpha)$

ii) $\alpha>1$: $-2\pi^2 = \frac{i}{2}I(\alpha) - I_0(\alpha) - 2\pi i\ln\alpha$

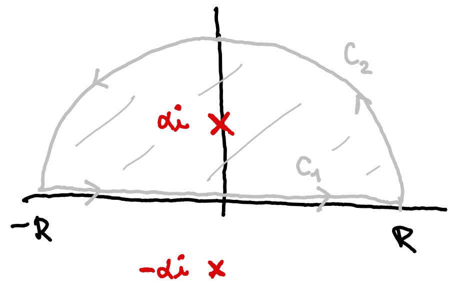
$$\boxed{I(\alpha) = \begin{cases} 0;\ \alpha\in(0,1] \\ 4\pi\ln\alpha;\ \alpha>1\end{cases}}$$

BONUS: Re: $I_0(\alpha) = \int_0^{2\pi}\arg(\alpha e^{it}-1)\,dt = 2\pi^2$; $\alpha>0$



Př $I(\alpha)=\int_{-\infty}^{\infty}\frac{dx}{x^2+\alpha^2}\,dx \in \mathbb{R}$ pokud $\alpha\in\mathbb{R}\setminus\{0\}$; BÚNO $\alpha>0$
[$I(\alpha)$ sudá v α]

$$J := \oint_C \frac{dz}{z^2+\alpha^2} \quad ; \quad C:$$



- RESIDUOVÁ VĚTA: $\sigma = \pm\alpha i$

$\longrightarrow$ $\operatorname{Res}_{\alpha i} f(z) = \frac{1}{(z^2+\alpha^2)'}\Big|_{\alpha i} = \frac{1}{2z}\Big|_{\alpha i} = \frac{1}{2\alpha i}$

$$\therefore\ J = 2\pi i \sum_{\sigma\in \operatorname{Int} C} \operatorname{Res}_\sigma f(z) = \frac{2\pi i}{2\alpha i} = \frac{\pi}{\alpha}$$

- PARAMETRIZACE:

$\longrightarrow$ C_1: $z = t$; $t\in(-R,R)$; $dz = dt$

$$J_1 = \int_{C_1} f(z)\,dz = \int_{-R}^{R}\frac{1}{t^2+\alpha^2}\,dt \xrightarrow{R\to\infty} I(\alpha)$$

$\longrightarrow$ C_2: $z = Re^{it}$; $t\in(0,\pi)$; $dz = Rie^{it}$

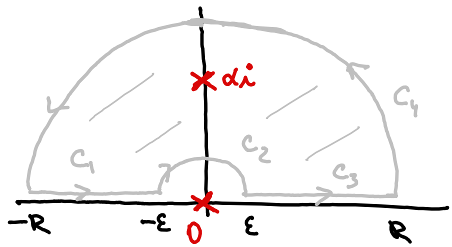
$$|J_2| = \left|\int_{C_2} f(z)\,dz\right| \overset{ML}{\leq} \frac{1}{R^2-\alpha^2}\,\pi R \xrightarrow{R\to\infty} 0 \quad \therefore\ J_2 \xrightarrow{R\to\infty} 0$$

- POROVNÁNÍ: $J = \oint_C f(z)\,dz = \sum_k \oint_{C_k} f(z)\,dz = \sum_k J_k \ /R\to\infty:$

$$\boxed{\frac{\pi}{\alpha} = I(\alpha)} \ ; \ \alpha>0$$

Př $\quad I = \underbrace{\int_{-\infty}^{\infty} \frac{\sin kx}{x(x^2+\alpha^2)}\, dx}_{\in \mathbb{R} \text{ pokud } \alpha \neq 0} = \mathrm{Im} \int_{-\infty}^{\infty} \frac{e^{ikx}}{x(x^2+\alpha^2)}\, dx$ [BÚNO $\alpha > 0$; $k \geq 0$, sudé v α, liché v k]

$$J := \oint_C \frac{e^{ikz}}{z(z^2+\alpha^2)}\, dz \quad ; \quad C:$$



- RESIDUOVÁ VĚTA

$\longrightarrow \quad \mathrm{Res}_0 f(z) = \left.\frac{e^{ikz}}{z^2+\alpha^2}\right|_0 = \frac{1}{\alpha^2}$

$\longrightarrow \quad \mathrm{Res}_{\alpha i} f(z) = \left.\frac{e^{ikz}}{z(2z)}\right|_{\alpha i} = -\frac{e^{-k\alpha}}{2\alpha^2}$

$\therefore \quad J = 2\pi i \sum_{\sigma \in \mathrm{Int}C} \mathrm{Res}_\sigma f(z) = 2\pi i \left(-\frac{e^{-k\alpha}}{2\alpha^2}\right) = -\frac{\pi i}{\alpha^2} e^{-k\alpha}$

- PARAMETRIZACE :

$\longrightarrow \quad C_1 + C_3 : \; z = t; \; t \in (-R, -\varepsilon) \cup (\varepsilon, R) \; ; \; dz = dt$

$J_1 + J_2 = \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right) \frac{e^{ikt}}{t(t^2+\alpha^2)}\, dt \xrightarrow[\varepsilon \to 0^+]{R \to \infty} \int_{-\infty}^{\infty} \frac{e^{ikt}}{t(t^2+\alpha^2)}\, dt =$

$= \underbrace{\int_{-\infty}^{\infty} \frac{\cos kt}{t(t^2+\alpha^2)}\, dt}_{0 \; \because \text{ antisym}} + i \underbrace{\int_{-\infty}^{\infty} \frac{\sin kt}{t(t^2+\alpha^2)}\, dt}_{I(\alpha)} = iI$

$\longrightarrow \quad C_2 : \; J_2 \xrightarrow{\varepsilon \to 0^+} -\pi i \, \mathrm{Res}_0 f(z) = -\frac{\pi i}{\alpha^2}$

$\longrightarrow \quad C_4 : \; z = Re^{it} \; ; \; t \in (0, \pi) \; ; \; dz = Rie^{it}\, dt$

$|J_4| \overset{ML}{\leq} \sup_{t \in (0,\pi)} \left| \left.\frac{e^{ikz}}{z^2+\alpha^2}\right| \right|_{Re^{it}} \pi R \leq \sup \frac{e^{-R\sin t}}{R^2-\alpha^2} \pi R \leq \frac{\pi R}{R^2-\alpha^2} \xrightarrow{R \to \infty} 0$

- POROVNÁNÍ :

$-\frac{\pi i}{\alpha^2} e^{-k\alpha} = iI - \frac{\pi i}{\alpha^2} \qquad \therefore \quad \boxed{I = \frac{\pi}{\alpha^2}\left(1 - e^{-k\alpha}\right)} \; ; \; k \geq 0 \, ; \, \alpha > 0$

(Př) $I = \int_0^\infty \frac{\sin^3 x}{x^3}\,dx \overset{\text{sym.}}{=} \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\,dx \in \mathbb{R}$ [srov. $\frac{1}{x^3}$ v ∞, v 0 spoj.]

jest $\sin 3x = \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x =$

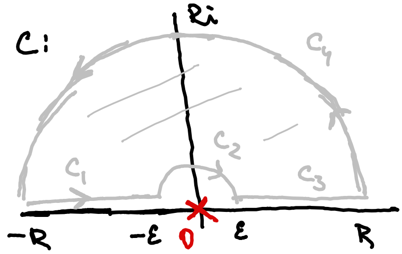
$= 2\sin x \cos^2 x + (\cos^2 x - \sin^2 x)\sin x = 2\sin x(1-\sin^2 x) + (1-2\sin^2 x)\sin x$

$= 3\sin x - 4\sin^3 x \quad \therefore \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$\therefore I = \frac{1}{8}\int_{-\infty}^{\infty} \frac{3\sin x - \sin 3x}{x^3}\,dx = \frac{1}{8}\operatorname{Im}\int_{-\infty}^{\infty} \frac{3e^{ix} - e^{3ix} - 2}{x^3}\,dx$

$\because 3e^{ix} - e^{3ix} = 3(1 + ix - \frac{x^2}{2}) - (1 + 3ix - \frac{9x^2}{2}) + O(x^3) = 2 + 3x^2 + O(x^3)$

$J := \oint_C \frac{3e^{iz} - e^{3iz} - 2}{z^3}\,dz$



- RESIDUOVÁ VĚTA:

$\frac{3e^{iz} - e^{3iz} - 2}{z^3} = \frac{3}{z} + O(1) \quad \therefore \operatorname{Res}_0 f(z) = 3$

ale $J = 0$ [CAUCHY]

- PARAMETRIZACE:

$\longrightarrow C_1 + C_3 : z = t;\ t \in (-R, -\varepsilon) \cup (\varepsilon, R);\ dz = dt$

$J_1 + J_3 = \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right) \frac{3e^{it} - e^{3it} - 2}{t^3}\,dt \xrightarrow[\varepsilon \to 0^+]{R \to \infty} \int_{-\infty}^{\infty} \frac{3e^{it} - e^{3it} - 2}{t^3}\,dt$

$= \underbrace{\int_{-\infty}^{\infty} \frac{3\cos t - \cos 3t - 2}{t^3}\,dt}_{0 \ \because \text{antisym}} + i\int_{-\infty}^{\infty} \frac{3\sin t - \sin 3t}{t^3}\,dt = 8iI$

$\longrightarrow C_2 : J_2 \xrightarrow{\varepsilon \to 0^+} -\pi i \operatorname{Res}_0 f(z) = -3\pi i$

$\longrightarrow C_4 : |J_4| \overset{ML}{\leq} \frac{3+1+2}{R^3}\pi R \xrightarrow{R \to \infty} 0$

- POROVNÁNÍ:

$0 = 8iI - 3\pi i \quad \therefore \boxed{I = \frac{3\pi}{8}}$

Př. $I = \underbrace{\int_0^\infty \frac{\sin^3 x}{x^3}\,dx}_{\in \mathbb{R}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\,dx = \langle \text{v } x=0 \text{ spoj.} \rangle$

„Divide & Conquer“

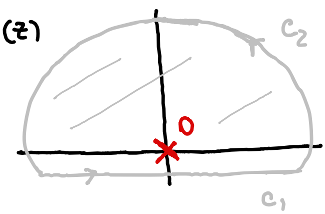
$$\frac{1}{2}\lim_{\varepsilon\to 0^+}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{\sin^3 z}{z^3}\,dz = \frac{1}{2}\lim_{\varepsilon\to 0^+}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{\left(\frac{e^{iz}-e^{-iz}}{2i}\right)^3}{z^3}\,dz =$$

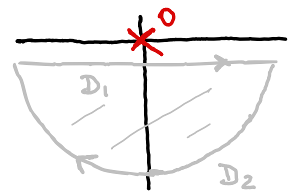
$$= -\frac{1}{16i}\lim_{\varepsilon\to 0^+}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{e^{3iz}-3e^{iz}+3e^{-iz}-e^{-3iz}}{z^3}\,dz =$$

$$= -\frac{1}{16i}\lim_{\varepsilon\to 0^+}\Bigg[\underbrace{\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{e^{3iz}-3e^{iz}}{z^3}\,dz}_{J_1(\varepsilon)} + \underbrace{\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{3e^{-iz}-e^{-3iz}}{z^3}\,dz}_{K_1(\varepsilon)}\Bigg]$$

$$J := \oint_C \frac{e^{3iz}-3e^{iz}}{z^3}\,dz \quad \leftarrow f(z)$$

$$K := \oint_D \frac{3e^{-iz}-e^{-3iz}}{z^3}\,dz \quad \leftarrow g(z)$$





- **RESIDUOVÁ VĚTA**:

$\longrightarrow$ $\operatorname{Res}_0 f(z) \overset{\text{pól 3.}}{=} \lim_{z\to 0}\frac{1}{(3-1)!}\left[z^3 f(z)\right]^{(3-1)} = \lim_{z\to 0}\frac{1}{2}\left[e^{3iz}-3e^{iz}\right]'' =$

$$= \frac{1}{2}\left[-9e^{3iz}+3e^{iz}\right]\Big|_0 = -3$$

$$\therefore \quad J = 2\pi i(-3) = -6\pi i \qquad \& \qquad K = 0$$

- **PARAMETRIZACE**:

$\longrightarrow$ $C_2: z = Re^{it};\ t\in(0,\pi);\ |e^{iz}| = e^{-R\sin t} \le 1$

$$|J_2| \le \frac{4}{R^3}\pi R \xrightarrow{R\to\infty} 0$$

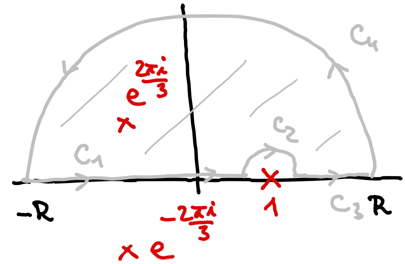
$\longrightarrow$ $D_2: z = Re^{-it};\ t\in(0,\pi);\ |e^{-iz}| = e^{-R\sin t} \le 1$

$$|K_2| \le \frac{4}{R^3}\pi R \xrightarrow{R\to\infty} 0$$

- **POROVNÁNÍ**:

$$\boxed{I = -\frac{1}{16i}\lim_{\varepsilon\to 0^+}\left[-6\pi i + 0\right] = \frac{3\pi}{8}}$$

(Př) $I = \int_{-\infty}^{\infty} \frac{dx}{1-x^3} \in \mathbb{R}$



$\longrightarrow J = \oint_C \frac{dz}{1-z^3}$; C:

póly: $1 - z^3 = -(z^3-1) = -(z-1)(z-e^{\frac{2\pi i}{3}})(z-e^{-\frac{2\pi i}{3}})$

- RESIDUOVÁ VĚTA

$\longrightarrow \operatorname{Res}_1 f(z) = \frac{1}{-3z^2}\Big|_1 = -\frac{1}{3}$

$\longrightarrow \operatorname{Res}_{e^{\frac{2\pi i}{3}}} = \frac{1}{-3z^2}\Big|_{e^{\frac{2\pi i}{3}}} = -\frac{z}{3z^3}\Big|_{e^{\frac{2\pi i}{3}}} = -\frac{1}{3} e^{\frac{2\pi i}{3}}$

$\therefore J = 2\pi i \sum_{\sigma \in \operatorname{Int} C} = 2\pi i \left(-\frac{1}{3} e^{\frac{2\pi i}{3}}\right) = -\frac{2\pi i}{3} e^{\frac{2\pi i}{3}}$

- PARAMETRIZACE

$\longrightarrow C_1 + C_3$: $z = t$; $t \in (-R, 1-\varepsilon) \cup (1+\varepsilon, R)$; $dz = dt$

$J_1 + J_3 = \left(\int_{-R}^{1-\varepsilon} + \int_{1+\varepsilon}^{R}\right) \frac{dt}{1-t^3} \xrightarrow[\varepsilon \to 0^+]{R\to\infty} \int_{-\infty}^{\infty} \frac{dt}{1-t^3} = I$

$\longrightarrow C_2 : J_2 \xrightarrow{\varepsilon \to 0^+} -\pi i \operatorname{Res}_1 f(z) = -\pi i \left(-\frac{1}{3}\right) = \frac{\pi i}{3}$

$\longrightarrow C_4 : |J_4| \le \frac{1}{R^3-1} \pi R \xrightarrow{R\to\infty} 0$

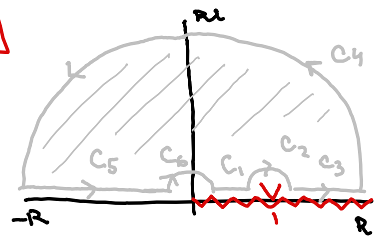
- POROVNÁNÍ

$-\frac{2\pi i}{3} \underbrace{e^{\frac{2\pi i}{3}}}_{\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}} = \frac{\pi i}{3} + I$

$\therefore \boxed{I = -\frac{2\pi i}{3} i \sin\frac{2\pi}{3} = \frac{2\pi}{3} \frac{\sqrt{3}}{2} = \frac{\pi}{\sqrt{3}}}$

$$\text{(R)}\quad I = \int_0^\infty \frac{\ln x}{(x-1)^2}\,dx \in \mathbb{R} \qquad \text{[přechod na jiný integrál]}$$

$$J = \oint_C \frac{\ln z}{(z-1)^2}\,dz$$



- **CAUCHYHO VĚTA** : $J = 0$

$$\longrightarrow \quad \operatorname{Res}_1 f(z) = \lim_{z\to 1} (z-1)\frac{\ln z}{(z-1)^2} = 1$$

- **PARAMETRIZACE** :

$\longrightarrow \; C_1 + C_3 : \quad z = t + i0 \, ; \quad t \in (\varepsilon, 1-\varepsilon) \cup (1+\varepsilon, R) \, ; \quad dz = dt$

$$J_1 + J_3 = \left(\int_\varepsilon^{1-\varepsilon} + \int_{1+\varepsilon}^{R} \right) \frac{\ln(t+i0)}{(t+i0-1)^2}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{\ln t}{(t-1)^2}\,dt = I$$

$\longrightarrow \; C_2 : \quad J_2 = -\pi i \operatorname{Res}_1 f(z) = -\pi i$

$\longrightarrow \; C_4 : \quad |J_4| \le \dfrac{\ln R + 2\pi}{(R-1)^2}\,\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow \; (-)\, C_5 : \quad z = -t \, ; \quad t \in (\varepsilon, R) \, ; \quad dz = -dt$

$$J_5 = (-)\int_\varepsilon^R \frac{\ln(-t)}{(-t-1)^2}(-dt) \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{\ln t + \pi i}{(t+1)^2}\,dt = \underbrace{\int_0^\infty \frac{\ln t\,dt}{(t+1)^2}}_{I_1\in\mathbb{R}} + \pi i \underbrace{\int_0^\infty \frac{dt}{(t+1)^2}}_{I_0\in\mathbb{R}}$$

$\longrightarrow \; C_6 : \quad |J_6| \le \dfrac{2\pi - \ln\varepsilon}{(1-\varepsilon)^2}\,\pi\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$

- **POROVNÁNÍ** :

$$0 = I - \pi i + I_1 + \pi i I_0$$

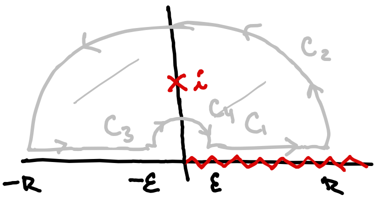
Re :

$$\boxed{I = -I_1 = -\int_0^\infty \frac{\ln t}{(t+1)^2}\,dt = 0}$$

⟨nebo symetrie $x \to \frac{1}{x}$⟩

Př $I = \int_0^\infty \frac{\ln^2 x}{1+x^2}\,dx \in \mathbb{R}$

↪ $J = \oint_C \frac{\ln^2 z}{1+z^2}\,dz$; C :



- **RESIDUOVÁ VĚTA**

$\longrightarrow$ $\operatorname{Res}_i f(z) = \frac{\ln^2 z}{2z}\Big|_i = \frac{\ln^2 e^{\frac{\pi}{2}i}}{2i} = \frac{(\frac{\pi}{2}i)^2}{2i} = \frac{\pi^2 i}{8}$

$\therefore\ J = 2\pi i\left(\frac{\pi^2 i}{8}\right) = -\frac{\pi^3}{4}$

- **PARAMETRIZACE**

$\longrightarrow$ C_1 : $z = t + i0$; $t \in (\varepsilon, R)$; $dz = dt$

$$J_1 = \int_\varepsilon^R \frac{\ln^2(t+i0)}{1+t^2}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{\ln^2 t}{1+t^2}\,dt = I$$

$\longrightarrow$ C_2: $z = Re^{it}$; $|J_2| \le \frac{(\ln R + \pi)^2}{R^2-1}\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow$ $\ominus C_3$: $z = -t$; $t \in (\varepsilon, R)$; $dz = -dt$;

$$J_3 = \ominus \int_\varepsilon^R \frac{\ln^2(-t)}{1+t^2}(-dt) = \int_\varepsilon^R \frac{(\ln t + \pi i)^2}{1+t^2} \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{(\ln t + \pi i)^2}{1+t^2}\,dt =$$

$$= \int_0^\infty \frac{\ln^2 t}{1+t^2}\,dt + 2\pi i \underbrace{\int_0^\infty \frac{\ln t}{1+t^2}\,dt}_{I_0 \in \mathbb{R}} - \pi^2 \underbrace{\int_0^\infty \frac{dt}{1+t^2}\,dt}_{\pi/2} =$$

$$= I + 2\pi i I_0 - \frac{\pi^3}{2}$$

$\longrightarrow$ C_4 : $|J_4| \le \frac{(\pi - \ln\varepsilon)^2}{1-\varepsilon^3}\pi\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$

- **POROVNÁNÍ** : $-\frac{\pi^3}{4} = 2I + 2\pi i I_0 - \frac{\pi^3}{2}$

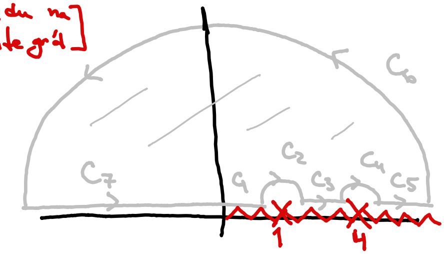
Re: $\boxed{I = \pi^3/8}$

Im: $I_0 = 0$

[BONUS]

Př:

$$I := \int_0^\infty \frac{\ln^2 x}{(x-1)^2(x-4)\sqrt{x}}\,dx \in \mathbb{R}$$ [převedu na jiný integrál]



$$J := \oint_C \frac{\ln^2 z}{(z-1)^2(z-4)\sqrt{z}}\,dz$$

- RESIDUOVÁ VĚTA :

$\longrightarrow$ C_2 : odstranitelná singularita $\therefore$ $\mathrm{Res}_1 f(z) = 0$

$\longrightarrow$ C_4 : $\mathrm{Res}_4 f(z) = \left.\frac{\ln^2 z}{(z-1)^2\sqrt{z}}\right|_4 = \frac{2\ln^2 2}{9}$; ale $J = 0$ [CAUCHY]

- PARAMETRIZACE :

$\longrightarrow$ $C_1 + C_3 + C_5$: $z = t + i0$; $t \in (0, 1-\varepsilon) \cup (1+\varepsilon, 4-\varepsilon) \cup (4+\varepsilon, R)$

$$J_1 + J_3 + J_5 = \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{4-\varepsilon} + \int_{4+\varepsilon}^{R}\right) \frac{\ln^2(t+i0)}{(t-1)^2(t-4)\underbrace{\sqrt{t+i0}}_{+\sqrt{t}}} \xrightarrow[\varepsilon\to 0^+]{R\to\infty} I$$

$\longrightarrow$ C_2 : $J_2 \xrightarrow{\varepsilon\to 0^+} -\pi i\, \mathrm{Res}_1 f(z) = 0$

$\longrightarrow$ C_4 : $J_4 \xrightarrow{\varepsilon\to 0^+} -\pi i\, \mathrm{Res}_4 f(z) = -\frac{2\pi i \ln^2 2}{9}$

$\longrightarrow$ C_6 : $|J_6| \le \frac{(\ln R + \pi)^2}{(R-1)^2(R-4)\sqrt{R}}\,\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow$ $\ominus$ C_7: $z = -t$; $t \in (0, R)$; $dz = -dt$

$$J_7 = \ominus \int_0^R \frac{\ln^2(-t)\,(-dt)}{(-t-1)^2(-t-4)\underbrace{\sqrt{-t}}_{i\sqrt{t}}} \xrightarrow{R\to\infty} -\frac{1}{i}\int_0^\infty \frac{(\ln t + \pi i)^2}{(t+1)^2(t+4)\sqrt{t}}\,dt =$$

$$= -\frac{1}{i}\underbrace{\int_0^\infty \frac{\ln^2 t\,dt}{(t+1)^2(t+4)\sqrt{t}}}_{I_0} - 2\pi\underbrace{\int_0^\infty \frac{\ln t\,dt}{(t+1)^2(t+4)\sqrt{t}}}_{I_1} - \pi^2 i\underbrace{\int_0^\infty \frac{dt}{(t+1)^2(t+4)\sqrt{t}}}_{I_2}$$

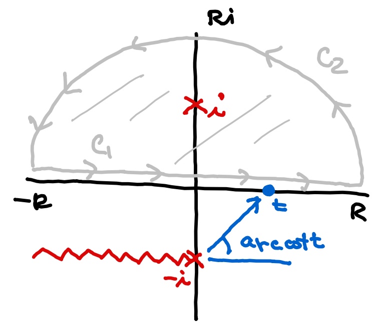
- POROVNÁNÍ : $0 = I - \frac{2\pi i \ln^2 2}{9} - \frac{1}{i} I_0 - 2\pi I_1 - \pi^2 i\, I_2$

$\boxed{\mathrm{Re}:}$ $I = 2\pi I_1 = 2\pi\left(\frac{\pi}{9}\ln 2 - \frac{\pi}{3}\right) = \boxed{\frac{2\pi^2}{9}\ln 2 - \frac{2\pi^2}{3}}$

$\boxed{\mathrm{Im}:}$ [BONUS] $I_0 = \int_0^\infty \frac{\ln^2 t\,dt}{(t+1)^2(t+4)\sqrt{t}} = \frac{2\pi\ln^2 2}{9} + \pi^2\int_0^\infty \frac{dt}{(t+1)^2(t+4)\sqrt{t}} =$

$$= \frac{2\pi\ln^2 2}{9} + \pi^2\left(\frac{\pi}{9}\right) = \frac{2\pi\ln^2 2}{9} + \frac{\pi^3}{9}$$

(9i) $I = \int_{-\infty}^{\infty} \frac{\ln(x^2+1)}{x^2+1}\,dx \in \mathbb{R}$



$$\hookrightarrow J := \oint_C \frac{\ln(z+i)}{z^2+1}\,dz$$

- **RESIDUOVÁ VĚTA**

$$\longrightarrow \operatorname{Res}_i f(z) = \left.\frac{\ln(z+i)}{2z}\right|_i = \frac{\ln(2i)}{2i} = \frac{\ln 2 + \frac{\pi}{2}i}{2i}$$

$$\therefore\ J = 2\pi i \sum_{\sigma \in \operatorname{Int} C} \operatorname{Res}_\sigma f(z) = 2\pi i \left(\frac{\ln 2 + \frac{\pi}{2}i}{2i}\right) = \pi \ln 2 + \frac{\pi^2}{2} i$$

- **PARAMETRIZACE**

$$\longrightarrow\ C_1 : z = t\ ;\ t \in \langle -R, R\rangle\ ;\ dz = dt$$

$$J_1 = \int_{-R}^{R} \frac{\ln(t+i)}{t^2+1}\,dt \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \frac{\frac{1}{2}\ln(t^2+1) + i\arg(t+i)}{t^2+1}\,dt$$

$$= \frac{1}{2}\underbrace{\int_{-\infty}^{\infty} \frac{\ln(t^2+1)}{t^2+1}\,dt}_{I} + i\underbrace{\int_{-\infty}^{\infty} \frac{\operatorname{arccot} t}{t^2+1}\,dt}_{I_0 \in \mathbb{R}} = \frac{1}{2} I + i I_0$$

$$\longrightarrow\ C_2 : |J_2| \leq \frac{\ln(R+1) + \pi}{R^2 - 1}\,\pi R \xrightarrow{R\to\infty} 0$$

- **POROVNÁNÍ**:

$$\pi \ln 2 + \frac{\pi^2}{2} i = \frac{1}{2} I + i I_0$$

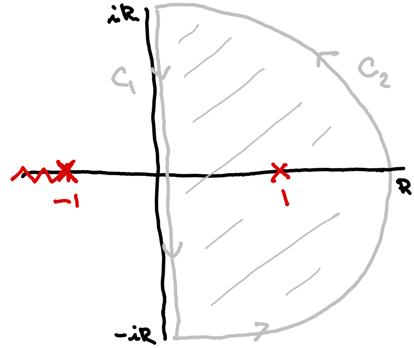
Re: $\boxed{I = 2\pi \ln 2}$

Im: $I_0 = \int_{-\infty}^{\infty} \frac{\operatorname{arccot} t}{t^2+1}\,dt = \frac{\pi^2}{2} = \left[-\frac{1}{2}\operatorname{arccot}^2 t\right]_{-\infty}^{\infty}$ triviální

[BONUS]

Př $I = \int_0^\infty \frac{\operatorname{arctg} x}{x(1+x^2)}\,dx$

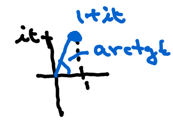
$$\hookrightarrow \quad J := \oint_C \frac{\ln(1+z)}{z(1-z^2)}\,dz \;; \quad C:$$



- **RESIDUOVÁ VĚTA** :

$$\longrightarrow \quad \mathrm{Res}_1 f(z) = \frac{\ln(1+z)}{z\cdot(-2z)}\Big|_1 = -\frac{1}{2}\ln 2$$

$$\therefore \quad J = 2\pi i \sum_{\sigma \in \mathrm{Int}\, C} \mathrm{Res}_\sigma f(z) = 2\pi i\left(-\frac{1}{2}\ln 2\right) = -\pi i \ln 2$$



- **PARAMETRIZACE** :

$$\longrightarrow \quad \ominus\; C_1:\; z = it;\; t \in (-R,R);\; dz = i\,dt;\quad 1+it = \sqrt{1+t^2}\, e^{i \operatorname{arctg} t}$$

$$J_1 = \ominus \int_{-R}^{R} \frac{\ln(1+it)}{it(1-(it)^2)}\, i\,dt \xrightarrow{R\to\infty} \ominus \int_{-\infty}^{\infty} \frac{\frac{1}{2}\ln(1+t^2) + i \operatorname{arctg} t}{t(1+t^2)}\,dt$$

$$= -\frac{1}{2}\underbrace{\int_{-\infty}^{\infty} \frac{\ln(1+t^2)}{t(1+t^2)}\,dt}_{I_0 = 0 \;\because\; \text{sym.}} - i \underbrace{\int_{-\infty}^{\infty} \frac{\operatorname{arctg} t}{t(1+t^2)}\,dt}_{2I} = -2iI$$

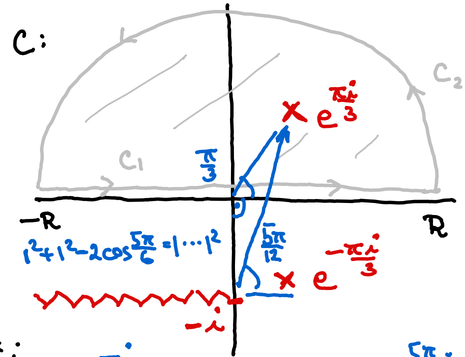
$$\longrightarrow \quad C_2:\; |J_2| \le \frac{\ln(R+1)+\pi}{R(R^2-1)}\,\pi R \xrightarrow{R\to\infty} 0$$

- **POROVNÁNÍ** :

$$-\pi i \ln 2 = -2iI \qquad \therefore \qquad \boxed{I = \frac{\pi}{2}\ln 2}$$

Př $\quad I = \int_{-\infty}^{\infty} \frac{\ln(1+x^2)}{1-x+x^2}\,dx$

$$J = \oint_C \frac{\ln(z+i)}{1-z+z^2}\,dz$$



póly: $1-z+z^2 = \dfrac{1+z^3}{1+z} \quad \therefore \sigma_\pm = e^{\pm\frac{\pi i}{3}} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

- **RESIDUOVÁ VĚTA** :

$$\longrightarrow \operatorname{Res}_{\sigma_+} f(z) = \left.\frac{\ln(z+i)}{2z-1}\right|_{\sigma_+} = \frac{\frac{1}{2}\ln(2+\sqrt{3}) + \frac{5\pi}{12}i}{i\sqrt{3}}$$

$$e^{\frac{\pi i}{3}} - (-i) = \sqrt{2+\sqrt{3}}\, e^{\frac{5\pi}{12}i}$$

$$\therefore J = 2\pi i \left(\frac{\frac{1}{2}\ln(2+\sqrt{3}) + \frac{5\pi}{12}i}{i\sqrt{3}} \right) = \frac{\pi}{\sqrt{3}}\ln(2+\sqrt{3}) + \frac{5\pi^2}{6\sqrt{3}}i$$

- **PARAMETRIZACE** :

$\longrightarrow C_1:\ z = t;\ t \in (-R, R);\ dz = dt$

$$J_1 = \int_{-R}^{R} \frac{\ln(t+i)}{1-t+t^2}\,dt \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \frac{\ln|t+i| + i\underbrace{\arg(t+i)}_{\in(0,\pi)}}{1-t+t^2}\,dt =$$

$$= \frac{1}{2}\underbrace{\int_{-\infty}^{\infty} \frac{\ln(1+t^2)}{1-t+t^2}\,dt}_{I} + i\underbrace{\int_{-\infty}^{\infty} \frac{\operatorname{arccot} t}{1-t+t^2}\,dt}_{I_0 \in \mathbb{R}} = \frac{1}{2}I + iI_0$$

$$\longrightarrow C_2:\ |J_2| \le \frac{\ln(R+1)+\pi}{R^2-R-1}\,2\pi R \xrightarrow{R\to\infty} 0$$

- **POROVNÁNÍ** :

$$\frac{\pi}{\sqrt{3}}\ln(2+\sqrt{3}) + \frac{5\pi^2}{6\sqrt{3}}i = \frac{1}{2}I + iI_0$$

$\boxed{\text{Re:}}$ $\quad \boxed{I = \frac{2\pi}{\sqrt{3}}\ln(2+\sqrt{3})}$

$\boxed{\text{Im:}}$ $\quad I_0 = \int_{-\infty}^{\infty} \frac{\operatorname{arccot} x}{1-x+x^2}\,dx = \frac{5\pi^2}{6\sqrt{3}}$

[BONUS]

(Př) $I = \int_0^\infty \frac{\ln(1+x^4)}{(1+x^2)^2}\,dx \overset{sym}{=} \frac12\int_{-\infty}^\infty \frac{\ln(1+x^4)}{(1+x^2)^2}\,dx$

Trik: $x^4+1 = (x-e^{\frac{\pi}{4}i})(x-e^{\frac{3\pi}{4}i})(x-e^{\frac{5\pi}{4}i})(x-e^{\frac{7\pi}{4}i}) =$

$= (x-e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i})(x-e^{\frac{3\pi}{4}i})(x-e^{-\frac{3\pi}{4}i}) =$

$= (x^2-2x\cos\frac{\pi}{4}+1)(x^2-2x\cos\frac{3\pi}{4}+1) =$

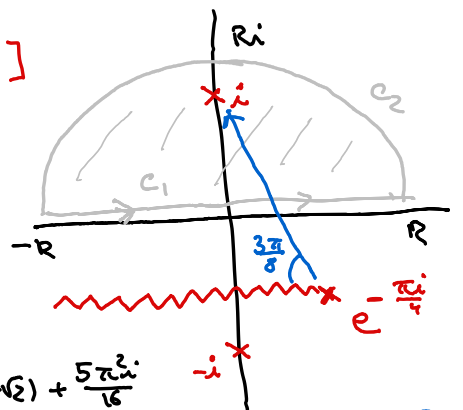
$= (x^2-\sqrt2 x+1)(x^2+\sqrt2 x+1)$

$\therefore$ ⟨sym $x \to -x$⟩ $\quad I = \int_{-\infty}^\infty \frac{\ln(x^2-\sqrt2 x+1)}{(1+x^2)}\,dx$

ale $x^2-\sqrt2 x+1 = (x-\frac{\sqrt2}{2})^2+\frac12 = (x-e^{\frac{\pi}{4}i})(x-\underbrace{e^{-\frac{\pi}{4}i}}_{\frac{1-i}{\sqrt2}})$

$J = \oint_c \frac{\ln(z-e^{-\frac{\pi}{4}i})}{(1+z^2)^2}\,dz$

• **RESIDUOVÁ VĚTA**: $[\sigma = e^{-\frac{\pi}{4}i} = \frac{1-i}{\sqrt2}]$



$\longrightarrow \operatorname{Res}_i f(z) \overset{\text{pól } 2}{=} \left(\frac{\ln(z-\sigma)}{(z+i)^2}\right)'\Big|_i =$

$= \frac{1}{z-\sigma}\,\frac{1}{(z+i)^2} - 2\,\frac{\ln(z-\sigma)}{(z+i)^3}\Big|_i =$

$= \frac{e^{-5\pi i/8}}{\sqrt{2+\sqrt2}}\left(-\frac14\right) - \frac{i}{4}\left(\ln\sqrt{2+\sqrt2}+\frac{5\pi}{8}i\right)$

$\therefore\ J = 2\pi i\sum_{\sigma\in \mathrm{Int}\,c}\operatorname{Res}_\sigma f(z) = -\frac{\pi i e^{5\pi i/8}}{2\sqrt{2+\sqrt2}} + \frac{\pi}{4}\ln(2+\sqrt2)+\frac{5\pi^2 i}{16}$

$i - e^{-\frac{\pi}{4}i} = \sqrt{2+\sqrt2}\, e^{\frac{5\pi}{8}i}$

• **PARAMETRIZACE**:

$\longrightarrow c_1: z = t$, $t\in(-R,R)$; $dz = dt$; $z - e^{-\frac{\pi}{4}i} = t - \frac{\sqrt2}{2} + i\frac{\sqrt2}{2}$

$J_1 = \int_{-R}^R \frac{\ln(t-\frac{\sqrt2}{2}+i\frac{\sqrt2}{2})}{(1+t^2)^2}\,dt \xrightarrow{R\to\infty} \underbrace{\frac12\int_{-\infty}^\infty \frac{\ln(t^2-\sqrt2 t+1)}{(1+t^2)^2}\,dt}_{I} + i\underbrace{\int_{-\infty}^\infty\frac{\operatorname{arccot}(\sqrt2 t-1)}{(1+t^2)^2}\,dt}_{I_0\in\mathbb{R}}$

$\longrightarrow c_2: |J_2| \le \frac{\ln(R+1)+\pi}{(R^2-1)^2}\,\pi R \xrightarrow{R\to\infty} 0$

• **POROVNÁNÍ**: $-\frac{\pi i e^{-5\pi i/8}}{2\sqrt{2+\sqrt2}} + \frac{\pi}{4}\ln(2+\sqrt2) + \frac{5\pi^2 i}{16} = \frac12 I + iI_0$

$\boxed{\mathrm{Re}:}$ $\boxed{I = \frac{\pi}{2}\ln(2+\sqrt2) - \frac{\pi}{2}}$ & $\boxed{\mathrm{Im}:}$ $I_0 = \int_{-\infty}^\infty \frac{\operatorname{arccot}(\sqrt2 t-1)}{(1+t^2)^2}\,dt = \frac{5\pi^2}{16}+\frac{\pi}{4}\sqrt{\frac{2-\sqrt2}{2+\sqrt2}}$

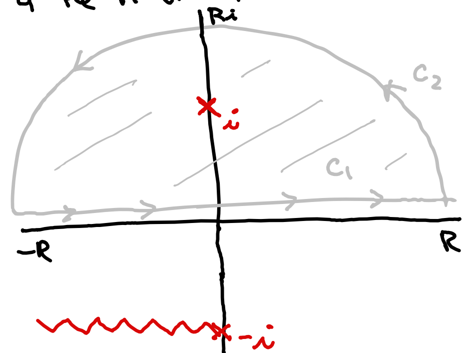
[BONUS]

(Př) $I = \int_{-\infty}^{\infty} \frac{\ln^2(1+x^2)}{1+x^2}\,dx \in \mathbb{R}$

↳ jest. mat. $\ln(x+i) = \frac{1}{2}\ln(x^2+1) + i \operatorname{arccotg} x$

$\therefore \quad \ln^2(x+i) = \frac{1}{4}\ln^2(x^2+1) + i \ln(x^2+1)\operatorname{arccotg} x - \operatorname{arccotg}^2 x$

↳ $\boxed{\mathrm{Re}}$: $\ln^2(x+1) = 4\operatorname{arccotg}^2 x + 4\,\mathrm{Re}\,\ln^2(x+i)$



mat

$$J := \oint_C \frac{\ln^2(z+i)}{1+z^2}\,dz$$

- RESIDUOVÁ VĚTA :

⟶ $\mathrm{Res}_i\, f(z) = \frac{\ln^2(z+i)}{2z}\Big|_i = \frac{\left(\ln 2 + \frac{\pi}{2}i\right)^2}{2i}$

$\therefore\ J = 2\pi i \sum_{\sigma \in \mathrm{Int}\,C} \mathrm{Res}_\sigma f(z) = \pi\left(\ln^2 2 + \pi i \ln 2 - \frac{\pi^2}{4}\right)$

- PARAMETRIZACE :

⟶ C_1: $z = t;\ t \in (-R,R);\ dz = dt$

$$J_1 = \int_{-R}^{R} \frac{\ln^2(t+i)}{1+t^2}\,dt \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \frac{\ln^2(t+i)}{1+t^2}\,dt =$$

$$= \frac{1}{4}\underbrace{\int_{-\infty}^{\infty} \frac{\ln^2(t^2+1)}{1+t^2}\,dt}_{I} + i\underbrace{\int_{-\infty}^{\infty} \frac{\ln(t^2+1)\operatorname{arccot} t}{1+t^2}\,dt}_{I_0 \in \mathbb{R}} - \int_{-\infty}^{\infty} \frac{\operatorname{arccot}^2 t}{1+t^2}\,dt$$

$$= \frac{1}{4}I + iI_0 + \left[\tfrac{1}{3}\operatorname{arccot}^3 t\right]_{-\infty}^{\infty} = \frac{1}{4}I + iI_0 - \frac{\pi^3}{3}$$

⟶ C_2: $|J_2| \le \frac{(\ln(R+1)+\pi)^2}{R^2-1}\,\pi R \xrightarrow{R\to\infty} 0$

- POROVNÁNÍ :

$$\pi\left(\ln^2 2 + \pi i \ln 2 - \frac{\pi^2}{4}\right) = \frac{1}{4}I + iI_0 - \frac{\pi^3}{3}$$

$\boxed{\mathrm{Re:}}$ $\pi\ln^2 2 - \frac{\pi^3}{4} = \frac{1}{4}I - \frac{\pi^3}{3} \qquad \therefore\ I = 4\pi\ln^2 2 + \frac{\pi^3}{3}$

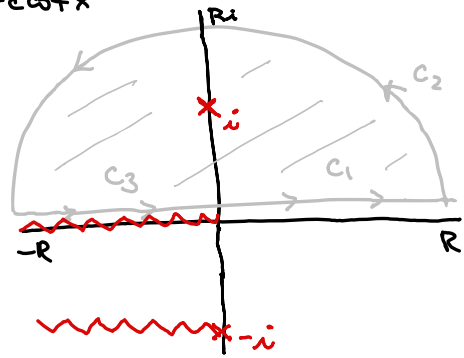
$\boxed{\mathrm{Im:}}$ $I_0 = \int_{-\infty}^{\infty} \frac{\ln(t^2+1)\operatorname{arccot} t}{1+t^2}\,dt = \pi^2 \ln 2$

Bonus

(Př) $I = \int_0^\infty \frac{\ln x \, \ln(1+x^2)}{1+x^2}\,dx \in \mathbb{R}$

jest $\ln(x+i) = \frac{1}{2}\ln(x^2+1) + i\,\mathrm{arccot}\, x$

$\hookrightarrow \; J = \oint_C \frac{\ln z \, \ln(z+i)}{1+z^2}\,dz \;;$



- **RESIDUOVÁ VĚTA**

$\longrightarrow \operatorname{Res}_i f(z) = \frac{\ln i \, \ln(2i)}{2i} = \frac{\frac{\pi}{2}i\left(\ln 2 + \frac{\pi}{2}i\right)}{2i}$

$\therefore \; J = 2\pi i \sum_{\sigma \in \mathrm{Int}\, C} \operatorname{Res}_\sigma f(z) = \frac{\pi^2}{2} i \ln 2 - \frac{\pi^3}{4}$

- **PARAMETRIZACE** :

$\longrightarrow \; C_1 : \; z = t;\; t \in (0,R) \;;\; dz = dt \,;$

$J_1 = \int_0^R \frac{\ln t \, \ln(t+i)}{1+t^2}\,dt \xrightarrow{R\to\infty} \underbrace{\frac{1}{2}\int_0^\infty \frac{\ln t \, \ln(1+t^2)}{1+t^2}\,dt}_{I} + i\int_0^\infty \frac{\ln t \;\mathrm{arccot}\, t}{1+t^2}\,dt$

$\longrightarrow \; C_2 : \; |J_2| \le \frac{(\ln R + \pi)(\ln(R+1) + \pi)}{R^2 - 1}\,\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow \; \ominus C_3 : \; z = -t + i0 \;;\; t \in (0,R) \;;\; dz = -dt$

$J_3 = \ominus\int_0^R \frac{\ln(-t+i0)\,\ln(-t+i)}{1+t^2}(-dt) \xrightarrow{R\to\infty} \int_0^\infty \frac{(\ln t + \pi i)\left(\frac{1}{2}\ln(1+t^2) + i\,\mathrm{arccot}(-t)\right)}{1+t^2}\,dt$

$= \frac{1}{2}\int_0^\infty \frac{\ln t \, \ln(1+t^2)}{1+t^2}\,dt + i\int_0^\infty \frac{\ln t \;\overbrace{\mathrm{arccot}(-t)}^{\pi - \mathrm{arccot}\, t}}{1+t^2}\,dt + \frac{\pi i}{2}\underbrace{\int_0^\infty \frac{\ln(1+t^2)}{1+t^2}\,dt}_{I_0} - \pi\int_0^\infty \frac{\mathrm{arccot}(-t)}{1+t^2}\,dt$

$= \frac{1}{2} I + \pi i \underbrace{\int_0^\infty \frac{\ln t}{1+t^2}\,dt}_{0} - i\int_0^\infty \frac{\ln t \;\mathrm{arccot}\, t}{1+t^2}\,dt + \frac{\pi}{2} i\, I_0 - \pi\left[\frac{\mathrm{arccot}^2(-t)}{2}\right]_0^\infty$

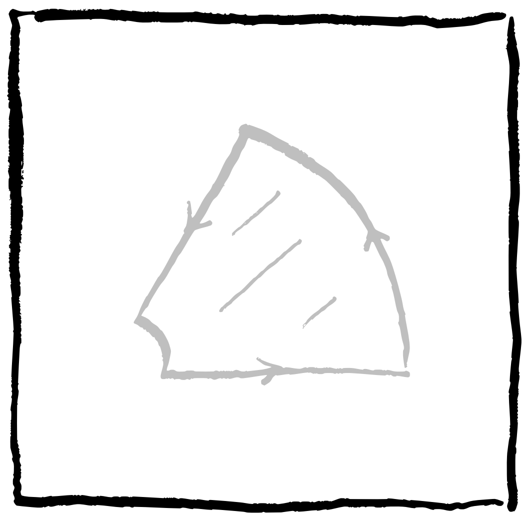
- **POROVNÁNÍ** :

$\frac{\pi^2}{2} i \ln 2 - \frac{\pi^3}{4} = I + \frac{\pi}{2} i\, I_0 - \frac{3}{8}\pi^3$

Re : $\boxed{I = \frac{\pi^3}{8}}$

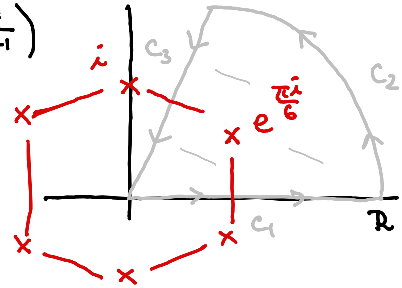
Im : $I_0 = \int_0^\infty \frac{\ln(1+t^2)}{1+t^2}\,dt = \pi \ln 2$

[BONUS]



(Př) $I=\int_{-\infty}^{\infty}\frac{dx}{x^6+1} \in \mathbb{R} \quad \left(=2\int_0^{\infty}\frac{dx}{x^6+1}\right)$

↳ $J=\oint_C \frac{dz}{z^6+1}$; C:



póly $z^6=-1 \Rightarrow \sigma \in \{e^{\frac{\pi i}{6}}, e^{\frac{3\pi i}{6}}, \dots\}$

• RESIDUOVÁ VĚTA:

$$\longrightarrow \operatorname{Res}_{e^{\frac{\pi i}{6}}} f(z) = \frac{1}{6z^5}\Big|_{e^{\frac{\pi i}{6}}} = \frac{z}{6z^6}\Big|_{e^{\frac{\pi i}{6}}} = -\frac{1}{6}e^{\frac{\pi i}{6}}$$

$$\therefore\ J = 2\pi i\left(-\frac{1}{6}e^{\frac{\pi i}{6}}\right) = -\frac{\pi i}{3}e^{\frac{\pi i}{6}}$$

• PARAMETRIZACE:

$$\longrightarrow C_1:\ z=t;\ t\in(0,R);\ dz=dt$$

$$J_1=\int_0^R \frac{dt}{t^6+1} \xrightarrow{R\to\infty} \frac{I}{2}$$

$$\longrightarrow C_2:\ |J_2| \overset{ML}{\le} \frac{1}{R^6-1}\,\frac{2\pi}{6}R \xrightarrow{R\to\infty} 0$$

$$\longrightarrow \ominus C_3:\ z=te^{\frac{\pi i}{3}};\ t\in(0,R);\ dz=e^{\frac{\pi i}{3}}dt$$

$$J_3=\ominus\int_0^R \frac{e^{\frac{\pi i}{3}}dt}{t^6+1} \xrightarrow{R\to\infty} -e^{\frac{\pi i}{3}}\frac{I}{2}$$

• POROVNÁNÍ:

$$-\frac{\pi i}{3}e^{\frac{\pi i}{6}} = \frac{I}{2} - e^{\frac{\pi i}{3}}\frac{I}{2} \qquad /\cdot e^{-\frac{\pi i}{6}}$$

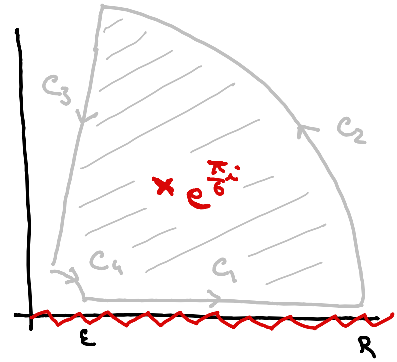
↳ $$-\frac{\pi i}{3} = \left(e^{-\frac{\pi}{6}i} - e^{\frac{\pi}{6}i}\right)\frac{I}{2} = -2i\sin\frac{\pi}{6}\,\frac{I}{2} = -2i\cdot\frac{1}{2}\,\frac{I}{2}$$

$$\therefore\ \boxed{I = \frac{2\pi}{3}}$$

Př: $I = \int_0^\infty \frac{dx}{\sqrt{x}(x^6+1)^2}$

$\downarrow$

$$J = \oint_C \frac{1}{\sqrt{z}\,(z^6+1)^2}\,dz$$



singularity:

$$z^6+1 = (z^2)^3+1 = (z^2+1)(z^4-z^2+1) =$$

$$= (z^2+1)\left((z^2+1)^2-3z^2\right) =$$

$$= \underbrace{(z^2+1)}_{(z-i)(z+i)}\ \underbrace{(z^2+\sqrt{3}z+1)}_{(z-e^{\frac{5\pi}{6}i})(z-e^{-\frac{5\pi}{6}i})}\ \underbrace{(z^2-\sqrt{3}z+1)}_{(z-e^{\frac{\pi}{6}i})(z-e^{-\frac{\pi}{6}i})}$$

$\nwarrow -\frac{\sqrt{3}}{2} \pm \frac{i}{2}$ $\qquad$ $\nwarrow \frac{\sqrt{3}}{2} \pm \frac{i}{2}$

- RESIDUOVÁ VĚTA :

$$\longrightarrow \operatorname{Res}_{e^{\frac{\pi}{6}i}} f(z) = \lim_{z\to e^{\frac{\pi}{6}i}} \frac{1}{(2-1)!}\left[(z-e^{\frac{\pi}{6}i})^2 f(z)\right]^{(2-1)} =$$

$$= \left(\frac{1}{\sqrt{z}\,(z^2+1)^2(z^2+\sqrt{3}z+1)^2(z-e^{-\frac{\pi}{6}i})^2}\right)'\Bigg|_{e^{\frac{\pi}{6}i}} =$$

$$= \ldots \quad [1000 \text{ years later} \ldots]$$

[Dá se spočítat, ale nevhodná metoda $\longrightarrow$ ukázáno jinak v]

$J = \ldots$

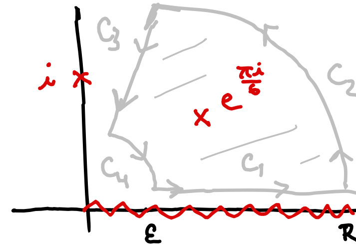
- PARAMETRIZACE :

$\longrightarrow C_1 : z = t + i0 \,;\ t \in (\varepsilon, R)\,;\ dz = dt$

$$J_1 = \int_\varepsilon^R \frac{dt}{\sqrt{t+i0}\,(t^6+1)^2} \xrightarrow[\varepsilon\to 0+]{R\to\infty} \int_0^\infty \frac{dt}{\sqrt{t}\,(t^6+1)} = I$$

...

Př $I = \int_0^\infty \frac{\ln x}{\sqrt{x}\,(x^6+1)}\,dx \in \mathbb{R}$



fun

$$J = \oint_C \frac{\ln z}{\sqrt{z}\,(z^6+1)}\,dz \quad ; \quad C:$$

fun

póly : $\sigma \in \{ e^{\frac{\pi i}{6}}, e^{\frac{3\pi i}{6}}, \dots, e^{\frac{11}{6}\pi i} \}$

- RESIDUOVÁ VĚTA :

$\longrightarrow$ $$\mathrm{Res}_{e^{\frac{\pi i}{6}}} f(z) = \frac{\ln z}{\sqrt{z}\,(6z^5)}\Big|_{e^{\frac{\pi i}{6}}} = \frac{z \ln z}{6z^6\sqrt{z}}\Big|_{e^{\frac{\pi i}{6}}} = \frac{e^{\frac{\pi i}{6}} \ln e^{\frac{\pi i}{6}}}{-6\, e^{\frac{\pi i}{12}}} =$$

$$= -\frac{\pi i}{6} e^{\frac{\pi i}{12}} \qquad \therefore \; J = 2\pi i \left(-\frac{\pi i}{6} e^{\frac{\pi i}{12}}\right) = \frac{\pi^2}{6} e^{\frac{\pi i}{12}}$$

- PARAMETRIZACE :

$\longrightarrow$ $C_1: \; z = t + i0 \; ; \; t \in (\varepsilon, R) \; ; \; dz = dt \; ;$

$$J_1 = \int_\varepsilon^R \frac{\ln(t+i0)}{\sqrt{t+i0}\,(t^6+1)}\,dt \xrightarrow[R\to\infty]{\varepsilon\to 0^+} \int_0^\infty \frac{\ln t}{\sqrt{t}\,(t^6+1)}\,dt = I$$

$\longrightarrow$ $C_2: \; |J_2| \le \frac{\ln R + \frac{\pi}{3}}{\sqrt{R}\,(R^6-1)} \frac{\pi}{3} R \xrightarrow{R\to\infty} 0$

$\longrightarrow$ $\ominus C_3: \; z = t e^{\frac{\pi i}{3}} ; \; t \in (\varepsilon, R) \; : \; dz = e^{\frac{\pi i}{3}}\,dt$

$$J_3 = \ominus \int_\varepsilon^R \frac{\ln(t e^{\frac{\pi}{3}i}) e^{\frac{\pi}{3}i}\,dt}{\sqrt{t e^{\frac{\pi}{3}i}}\,(t^6+1)} \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \ominus e^{\frac{\pi}{6}i} \int_0^\infty \frac{\ln t + \frac{\pi}{3}i}{\sqrt{t}\,(t^6+1)}\,dt = -e^{\frac{\pi}{6}i} I - \frac{\pi i}{3} e^{\frac{\pi}{6}i} \underbrace{\int_0^\infty \frac{dt}{\sqrt{t}\,(t^6+1)}}_{I_0 \in \mathbb{R}}$$

$\longrightarrow$ $C_4: \; |J_4| \le \frac{\frac{\pi}{3} - \ln\varepsilon}{\sqrt{\varepsilon}\,(1-\varepsilon^6)} \frac{\pi}{3}\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$

- POROVNÁNÍ :

$$\frac{\pi^2}{6} e^{\frac{\pi i}{12}} = I - e^{\frac{\pi}{6}i} I - \frac{\pi i}{3} e^{\frac{\pi}{6}i} I_0 \qquad / \cdot e^{-\frac{\pi}{6}i}$$

$$\frac{\pi^2}{6} e^{-\frac{\pi}{12}i} = I e^{-\frac{\pi}{6}i} - I - \frac{\pi i}{3} I_0$$

$\cdot e^{-\frac{\pi}{12}i}$

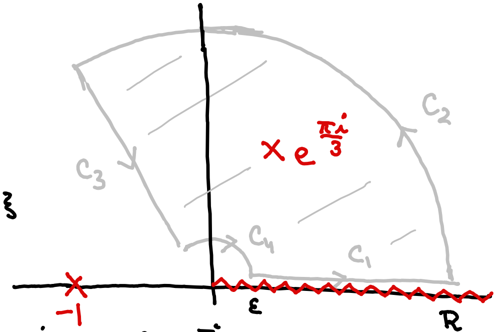
Re: $$-\frac{\pi^2}{6}\cos\frac{\pi}{12} = I\cos\frac{\pi}{6} - I \Rightarrow \boxed{I = \frac{\frac{\pi^2}{6}\cos\frac{\pi}{12}}{1-\cos\frac{\pi}{6}} = \frac{\frac{\pi^2}{6}\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}}{1-\frac{\sqrt{3}}{2}}}$$

$= \frac{\pi^2}{12}(7\sqrt{2} + 4\sqrt{6})$

$$\frac{\pi^2}{6} = I(\underbrace{e^{-\frac{\pi}{12}i} - e^{\frac{\pi}{12}i}}_{-2i\sin\frac{\pi}{12}}) - \frac{\pi i}{3} e^{\frac{\pi}{12}i} I_0$$

Re: BONUS $\quad I_0 = \int_0^\infty \frac{dx}{\sqrt{x}\,(x^6+1)} = \frac{\pi}{2\sin\frac{\pi}{12}}$

Př: $I = \int_0^\infty \frac{x \ln x}{1+x^3}\, dx \in \mathbb{R}$



tan

$$J = \oint_C \frac{z \ln z}{1+z^3}\, dz \;;\quad C:$$

$\hookrightarrow$ póly $\sigma \in \{e^{\frac{\pi i}{3}}, -1, e^{-\frac{\pi i}{3}}\}$

- **RESIDUOVÁ VĚTA**:

$$\longrightarrow \operatorname{Res}_{e^{\frac{\pi i}{3}}} f(z) = \frac{z \ln z}{3z^2}\Big|_{e^{\frac{\pi i}{3}}} = \frac{1}{3} e^{-\frac{\pi i}{3}} \ln\left(e^{\frac{\pi i}{3}}\right) = \frac{\pi i}{9} e^{-\frac{\pi i}{3}}$$

$$\therefore\; J = 2\pi i \left(\frac{\pi i}{9} e^{-\frac{\pi i}{3}}\right) = -\frac{2\pi^2}{9} e^{-\frac{\pi i}{3}}$$

- **PARAMETRIZACE**:

$$\longrightarrow C_1 : z = t + i0 \;;\; t \in (\varepsilon, R) \;;\; dz = dt$$

$$J_1 = \int_\varepsilon^R \frac{t \ln(t+i0)}{1+t^3}\, dt \xrightarrow[\varepsilon \to 0^+]{R \to \infty} \int_0^\infty \frac{t \ln t}{1+t^3} = I$$

$$\longrightarrow C_2 : |J_2| \le \frac{R\left(\ln R + \frac{2\pi}{3}\right)}{R^3 - 1} \frac{2\pi}{3} R \xrightarrow{R \to \infty} 0$$

$$\longrightarrow \ominus C_3 : z = e^{\frac{2\pi i}{3}} t \;;\; t \in (\varepsilon, R) \;;\; dz = e^{\frac{2\pi i}{3}} dt$$

$$J_3 = \ominus \int_\varepsilon^R \frac{e^{\frac{2\pi i}{3}} t \,\ln\left(e^{\frac{2\pi i}{3}} t\right)}{1 + \left(e^{\frac{2\pi i}{3}} t\right)^3} e^{\frac{2\pi i}{3}} dt \xrightarrow[\varepsilon \to 0^+]{R \to \infty} -e^{\frac{4\pi i}{3}} \int_0^\infty t \frac{\ln t + \frac{2\pi i}{3}}{1+t^3}\, dt =$$

$$= -e^{\frac{4\pi i}{3}} \underbrace{\int_0^\infty \frac{t \ln t}{1+t^3}\, dt}_{I} - \frac{2\pi i}{3} e^{\frac{4\pi i}{3}} \underbrace{\int_0^\infty \frac{t\, dt}{1+t^3}}_{I_0 \in \mathbb{R}} = -e^{\frac{4\pi i}{3}} I - \frac{2\pi i}{3} e^{\frac{4\pi i}{3}} I_0$$

$$\longrightarrow C_4 : |J_4| \le \frac{\varepsilon\left(\frac{2\pi}{3} - \ln \varepsilon\right)}{1 - \varepsilon^3} \frac{2\pi}{3} \varepsilon \xrightarrow{\varepsilon \to 0^+} 0$$

- **POROVNÁNÍ**:

$$-\frac{2\pi^2}{9} e^{-\frac{\pi i}{3}} = I - e^{\frac{4\pi i}{3}} I - \frac{2\pi i}{3} e^{\frac{4\pi i}{3}} I_0 = e^{\frac{2\pi i}{3}} \underbrace{\left(e^{-\frac{2\pi i}{3}} - e^{\frac{2\pi i}{3}}\right)}_{-2i \sin\frac{2\pi}{3}} I - \frac{2\pi i}{3} e^{\frac{4\pi i}{3}} I_0$$

$\cdot e^{\frac{2\pi i}{3}}$

$$-\frac{2\pi^2}{9} e^{\frac{\pi i}{3}} = -i\sqrt{3}\, e^{\frac{4\pi i}{3}} I - \frac{2\pi i}{3} I_0$$

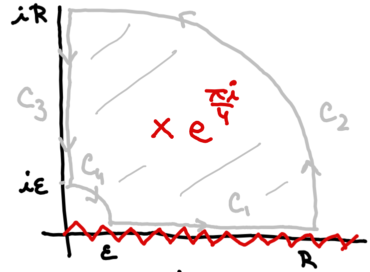
$\boxed{\text{Re:}}$ $$-\frac{2\pi^2}{9} \underbrace{\cos\frac{\pi}{3}}_{1/2} = \sqrt{3} \underbrace{\sin\frac{4\pi}{3}}_{-\sqrt{3}/2} I \qquad \therefore\; \boxed{I = \frac{2\pi^2}{27}} \qquad \& \quad I_0 = \int_0^\infty \frac{t\, dt}{1+t^3} = \frac{2\pi}{3\sqrt{3}}$$

[BONUS]

$$\cdot e^{-\frac{4\pi i}{3}} : \quad \frac{2\pi^2}{9} = -i\sqrt{3}\, I - \frac{2\pi i}{3} e^{-\frac{4\pi i}{3}} I_0 \;\Big\}\; \boxed{\text{Re:}} \;\; \frac{2\pi^2}{9} = -\frac{2\pi}{3} \sin\frac{4\pi}{3} I_0$$

Př $I = \int_0^\infty \frac{\ln x}{1+x^4}\,dx \in \mathbb{R}$

$J = \oint_C \frac{\ln z}{1+z^4}\,dz$; C:



- RESIDUOVÁ VĚTA :

$\longrightarrow \operatorname{Res}_{e^{\frac{\pi i}{4}}} f(z) = \frac{\ln z}{4z^3}\Big|_{e^{\frac{\pi i}{4}}} = \frac{z\ln z}{4z^4}\Big|_{e^{\frac{\pi i}{4}}} = -\frac{1}{4} e^{\frac{\pi i}{4}}\frac{\pi i}{4} = -\frac{\pi i}{16} e^{\frac{\pi i}{4}}$

$\therefore J = 2\pi i\left(-\frac{\pi i}{16} e^{\frac{\pi i}{4}}\right) = \frac{\pi^2}{8} e^{\frac{\pi i}{4}}$

- PARAMETRIZACE :

$\longrightarrow C_1 : z = t + i0$; $t \in (\varepsilon, R)$; $dz = dt$

$J_1 = \int_\varepsilon^R \frac{\ln(t+i0)}{1+t^4}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{\ln t}{1+t^4}\,dt = I$

$\longrightarrow C_2 : |J_2| \leq \frac{\ln R + \frac{\pi}{2}}{R^4 - 1}\frac{\pi}{2}R \xrightarrow{R\to\infty} 0$

$\longrightarrow \ominus C_3 : z = it$; $t \in (\varepsilon, R)$; $dz = i\,dt$

$J_3 = \ominus\int_\varepsilon^R \frac{\ln(it)}{1+(it)^4}\,i\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} -i\int_0^\infty \frac{\ln t + i\frac{\pi}{2}}{1+t^4}\,dt =$

$= -i\underbrace{\int_0^\infty \frac{\ln t}{1+t^4}\,dt}_{I} + \frac{\pi}{2}\underbrace{\int_0^\infty \frac{dt}{1+t^4}}_{I_0} = -iI + \frac{\pi}{2}I_0$

$\longrightarrow C_4 : |J_4| \leq \frac{\frac{\pi}{2} - \ln\varepsilon}{1-\varepsilon^4}\frac{\pi}{2}\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$

- POROVNÁNÍ :

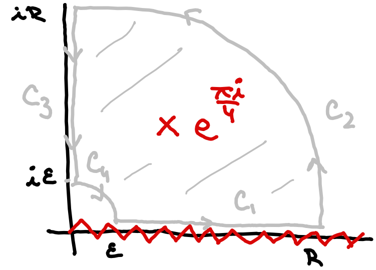
$\frac{\pi^2}{8} e^{\frac{\pi i}{4}} = I - iI + \frac{\pi}{2}I_0$

$\boxed{\text{Im:}}$ $\frac{\pi^2}{8}\sin\frac{\pi}{4} = -I$ $\therefore$ $\boxed{I = -\frac{\pi^2}{8\sqrt{2}}}$

$\boxed{\text{Re:}}$ $\frac{\pi^2}{8}\cos\frac{\pi}{4} = I + \frac{\pi}{2}I_0$ $\therefore$ $I_0 = \frac{\pi}{2\sqrt{2}}$

[BONUS]

(Př) $I = \int_0^\infty \frac{\ln^2 x}{1+x^4}\,dx \in \mathbb{R}$



$\boxed{\equiv}\int_0^\infty \frac{dx}{1+x^4}$

$+\pi i$

$$J := \oint_C \frac{\ln^2 z}{1+z^4}\,dz \;;\quad C:$$

- **RESIDUOVÁ VĚTA** :

$$\longrightarrow \operatorname{Res}_{e^{\frac{\pi i}{4}}} f(z) = \left.\frac{\ln^2 z}{4z^3}\right|_{e^{\frac{\pi i}{4}}} = \left.\frac{z\ln^2 z}{4z^4}\right|_{e^{\frac{\pi i}{4}}} = -\frac{1}{4}e^{\frac{\pi i}{4}}\left(\frac{\pi i}{4}\right)^2 = \frac{\pi^2}{64}e^{\frac{\pi i}{4}}$$

$$\therefore\ J = 2\pi i\left(\frac{\pi^2}{64}e^{\frac{\pi i}{4}}\right) = \frac{\pi^3 i}{32}e^{\frac{\pi i}{4}}$$

- **PARAMETRIZACE** :

$$\longrightarrow C_1 : z = t + i0\,;\ t\in(\varepsilon, R)\,;\ dz = dt$$

$$J_1 = \int_\varepsilon^R \frac{\ln^2(t+i0)}{1+t^4}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{\ln^2 t}{1+t^4}\,dt = I$$

$$\longrightarrow C_2 : |J_2| \le \frac{(\ln R + \frac{\pi}{2})^2}{R^4 - 1}\,\frac{\pi}{2}R \xrightarrow{R\to\infty} 0$$

$$\longrightarrow \ominus C_3 : z = it\,;\ t\in(\varepsilon, R)\,;\ dz = i\,dt$$

$$J_3 = \ominus\int_\varepsilon^R \frac{\ln^2(it)}{1+t^4}\,i\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} -i\int_0^\infty \frac{(\ln t + i\frac{\pi}{2})^2}{1+t^4}\,dt =$$

$$= -i\underbrace{\int_0^\infty \frac{\ln^2 t}{1+t^4}\,dt}_{I} + \pi\underbrace{\int_0^\infty \frac{\ln t}{1+t^4}\,dt}_{I_1\in\mathbb{R}} + i\frac{\pi^2}{4}\underbrace{\int_0^\infty \frac{dt}{1+t^4}}_{\frac{\pi}{2\sqrt{2}}} =$$

$$= -iI + \pi I_1 + i\frac{\pi^3}{8\sqrt{2}}$$

$$\longrightarrow C_4 : |J_4| \le \frac{(\pi/2 - \ln\varepsilon)^2}{1-\varepsilon^4}\,\frac{\pi}{2}\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$$

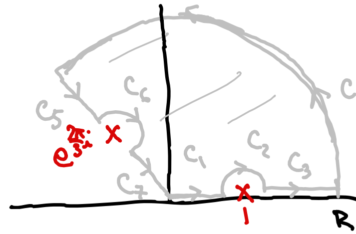
- **POROVNÁNÍ** : $\frac{\pi^3 i}{32}e^{\frac{\pi i}{4}} = I - iI + \pi I_1 + i\frac{\pi^3}{8\sqrt{2}}$

$\boxed{\text{Im}:}$ $\frac{\pi^3}{32}\cos\frac{\pi}{4} = -I + \frac{\pi^3}{8\sqrt{2}}$ $\quad\therefore\quad$ $\boxed{I = \frac{3\pi^3}{32\sqrt{2}}}$

$\boxed{\text{Re}:}$ [BONUS] $-\frac{\pi^3}{32}\sin\frac{\pi}{4} = I + \pi I_1$ $\quad\therefore\quad$ $I_1 = -\frac{\pi^2}{8\sqrt{2}}$

Pf $I = \int_0^\infty \frac{dx}{1-x^3} \in \mathbb{R}$

$$J = \oint_C \frac{dz}{1-z^3} \quad ; \qquad C:$$



póly: $1-z^3 = 0 \Rightarrow \sigma \in \{1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\}$

- RESIDUOVÁ VĚTA:

$$\longrightarrow \operatorname{Res}_1 f(z) = \frac{1}{-3z^2}\Big|_1 = -\frac{1}{3}$$

$$\longrightarrow \operatorname{Res}_{e^{\frac{2\pi i}{3}}} f(z) = \frac{1}{-3z^2}\Big|_{e^{\frac{2\pi i}{3}}} = -\frac{z}{3z^3}\Big|_{e^{\frac{2\pi i}{3}}} = -\frac{1}{3} e^{\frac{2\pi i}{3}}$$

ale $J = 0$ [CAUCHY]

- PARAMETRIZACE:

$$\longrightarrow C_1 + C_3 : z = t ; t \in (0, 1-\varepsilon) \cup (1+\varepsilon, R) ; dz = dt$$

$$J_1 + J_3 = \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^R\right) \frac{dt}{1-t^3} \xrightarrow[\varepsilon \to 0^+]{R\to\infty} \int_0^\infty \frac{dt}{1-t^3} = I$$

$$\longrightarrow C_2 : J_2 \xrightarrow{\varepsilon \to 0^+} -\pi i \operatorname{Res}_1 f(z) = \frac{\pi i}{3}$$

$$\longrightarrow C_4 : |J_4| \leq \frac{1}{R^3 - 1} \frac{2\pi}{3} R \xrightarrow{R\to\infty} 0$$

$$\longrightarrow \ominus(C_5 + C_7) : z = t e^{\frac{2\pi i}{3}} ; t \in (0, 1-\varepsilon) \cup (1+\varepsilon, R) ; dz = e^{\frac{2\pi i}{3}} dt$$

$$J_5 + J_7 = \ominus\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^R\right) \frac{1}{1-t^3} e^{\frac{2\pi i}{3}} dt \xrightarrow[\varepsilon \to 0^+]{R\to\infty} -e^{\frac{2\pi i}{3}} I$$

$$\longrightarrow C_6 : J_4 \xrightarrow{\varepsilon \to 0^+} -\pi i \operatorname{Res}_{e^{\frac{2\pi i}{3}}} = \frac{\pi i}{3} e^{\frac{2\pi i}{3}}$$

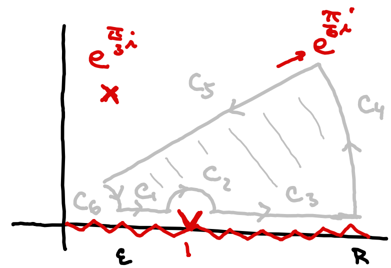
- POROVNÁNÍ:

$$0 = I + \frac{\pi i}{3} - e^{\frac{2\pi i}{3}} I + \frac{\pi i}{3} e^{\frac{2\pi i}{3}} \quad / \cdot e^{-\frac{\pi i}{3}}$$

$$0 = I \underbrace{(e^{-\frac{\pi i}{3}} - e^{\frac{\pi i}{3}})}_{-2i \sin\frac{\pi}{3}} + \frac{\pi i}{3} \underbrace{(e^{-\frac{\pi i}{3}} + e^{\frac{\pi i}{3}})}_{2\cos\frac{\pi}{3}}$$

$$\therefore \boxed{I = \frac{\pi}{3 \operatorname{tg} \frac{\pi}{3}} = \frac{\pi}{\sqrt{3}}}$$

(Př) $I = \int_0^\infty \frac{\ln x}{\sqrt{x}(x^6-1)^2}\,dx \in \mathbb{R}$ [na jiný integrál]



$$J = \oint_C \frac{\ln z}{\sqrt{z}(z^6-1)^2}\,dz$$

- **RESIDUOVÁ VĚTA** : $J = 0$ [CAUCHY]

$\ln z \doteq z-1$

pól 1.

$$\longrightarrow \text{Res}_1 f(z) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{1}{\sqrt{z}}\left(\frac{z-1}{z^6-1}\right)^2 \overset{\text{L'H}}{=} \frac{1}{\sqrt{1}}\left(\frac{1}{6}\right)^2 = \frac{1}{36}$$

- **PARAMETRIZACE** :

$\longrightarrow C_1 + C_3 : z = t+i0;\ t\in(\varepsilon, 1-\varepsilon)\cup(1+\varepsilon, R)$

$$J_1 + J_3 = \left(\int_\varepsilon^{1-\varepsilon} + \int_{1+\varepsilon}^{R}\right)\frac{\ln(t+i0)\,dt}{\sqrt{t+i0}\,((t+i0)^6-1)^2} \xrightarrow{\varepsilon\to 0} \int_0^\infty \frac{\ln t}{\sqrt{t}(t^6-1)^2}\,dt = I$$

$\longrightarrow C_2 : J_2 = -\pi i\,\text{Res}_{1+i0} = -\frac{\pi i}{36}$

$\longrightarrow C_4 : |J_4| \le \frac{\ln R + \frac{\pi}{6}}{\sqrt{R}(R^6-1)^2}\,\frac{\pi R}{6} \xrightarrow{R\to\infty} 0$

$\longrightarrow \ominus\, C_5 : z = t e^{\frac{\pi}{6}i};\ t\in(\varepsilon, R);\ dz = e^{\frac{\pi}{6}i}\,dt$

$$J_5 = \ominus \int_\varepsilon^R \frac{\ln(e^{\frac{\pi}{6}i}t)\,e^{\frac{\pi}{6}i}}{e^{\frac{\pi}{12}i}\sqrt{t}\,(-t^6-1)^2}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to+\infty} -e^{\frac{\pi}{12}i}\underbrace{\int_0^\infty \frac{\ln t\,dt}{\sqrt{t}(t^6+1)^2}}_{I_1\in\mathbb{R}} - \frac{\pi}{6}i\,e^{\frac{\pi}{12}i}\underbrace{\int_0^\infty \frac{dt}{\sqrt{t}(t^6+1)^2}}_{I_0\in\mathbb{R}}$$

$\longrightarrow C_6 : |J_6| \le \frac{\frac{\pi}{6} - \ln\varepsilon}{\sqrt{\varepsilon}(1-\varepsilon^6)^2}\,\frac{\pi}{6}\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$

- **POROVNÁNÍ** :

$$0 = I - \frac{\pi i}{36} - e^{\frac{\pi}{12}i} I_1 - \frac{\pi}{6}i\,e^{\frac{\pi}{12}i} I_0$$

$\cdot e^{-\frac{\pi}{12}i}$

úprava: $I = -\frac{\pi}{108}(3+11\pi)(2+\sqrt{3})$

$\boxed{\text{Im:}}$ $0 = -I\sin\frac{\pi}{12} - \frac{\pi}{36}\cos\frac{\pi}{12} - \frac{\pi}{6}I_0$ $\therefore$ $\boxed{I = \frac{-\pi}{6\sin\frac{\pi}{12}}\left(\frac{1}{6}\cos\frac{\pi}{12} + I_0\right)}$

$I_0 = \frac{11\pi}{72\sin\frac{\pi}{12}}$

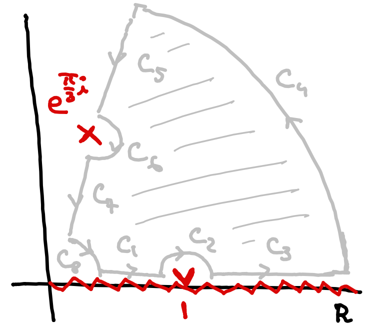
JINÁ METODA: substituce $x = \sqrt[6]{t}$

BONUS: $I_1 = \frac{-\pi}{\sin\frac{\pi}{12}}\left(\frac{1}{36} + \frac{1}{6}\cos\frac{\pi}{12}\, I_0\right)$
Im:

(Př) $I = \int_0^\infty \frac{\ln x}{\sqrt{x}(x^6-1)^2}\,dx \in \mathbb{R}$

$\downarrow$

$J = \oint_C \frac{\ln z}{\sqrt{z}(z^6-1)^2}\,dz$; C:



- **RESIDUOVÁ VĚTA** : $J = 0$ [CAUCHY]

$\longrightarrow$ $z \approx 1+i0$: $f(z) \doteq \frac{(z-1)}{\sqrt{z}(z^6-1)^2}$ pól 1. řádu!

$\therefore \mathrm{Res}_{1+i0} f(z) = \lim_{z\to1}(z-1)f(z) = \lim_{z\to1}\frac{1}{\sqrt{z}}\left(\frac{z-1}{z^6-1}\right)^2 \overset{\text{l'H}}{=} \frac{1}{\sqrt{1}}\left(\frac{1}{6}\right)^2 = \frac{1}{36}$

- **PARAMETRIZACE**

$\longrightarrow$ C_2 : $J_2 = -\pi i\, \mathrm{Res}_{1+i0} f = -\pi i/36$

$\longrightarrow$ $C_1 + C_3$: $z = t+i0$; $t\in(\varepsilon,1-\varepsilon)\cup(1+\varepsilon,R)$; $dz = dt$

$$J_1+J_3 = \left(\int_\varepsilon^{1-\varepsilon} + \int_{1+\varepsilon}^{R}\right)\frac{\ln(t+i0)\,dt}{\sqrt{t+i0}\,((t+i0)^6-1)^2} \xrightarrow[\varepsilon\to0^+]{R\to\infty} \int_0^\infty \frac{\ln t}{\sqrt{t}(t^6-1)^2}\,dt = I$$

$\longrightarrow$ C_4: $|J_4| \le \frac{\ln R + 2\pi}{\sqrt{R}(R^6-1)^2}\,\frac{\pi R}{3} \xrightarrow{R\to\infty} 0$; podobně C_8: $|J_8| \le \frac{2\pi - \ln\varepsilon}{\sqrt{\varepsilon}(1-\varepsilon^6)^2}\,\frac{\pi\varepsilon}{3} \xrightarrow{\varepsilon\to0^+} 0$

$\longrightarrow$ $\ominus(C_5+C_7)$: $z = e^{\frac{\pi}{3}i}t$; $t\in(\varepsilon,1-\varepsilon)\cup(1+\varepsilon,R)$; $dz = e^{\frac{\pi}{3}i}dt$

$$J_5+J_7 = \ominus\left(\int_\varepsilon^{1-\varepsilon} + \int_{1+\varepsilon}^{R}\right)\frac{\ln(e^{\frac{\pi}{3}i}t)}{e^{\frac{\pi}{6}i}\sqrt{t}(t^6-1)}\,e^{\frac{\pi}{3}i}dt \longrightarrow \ominus e^{\frac{\pi}{6}i}\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty}\right)\frac{\frac{\pi}{3}i + \ln t}{\sqrt{t}(t^6-1)^2}\,dt$$

$$= -e^{\frac{\pi}{6}i}\,\frac{\pi}{3}i\,\underbrace{\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty}\right)\frac{dt}{\sqrt{t}(t^6-1)}}_{I_0(\varepsilon)\in\mathbb{R}\text{ ale diverguje}} - e^{\frac{\pi}{6}i}I + O(\varepsilon)$$

$\longrightarrow$ $\ominus C_6$: $z = e^{\frac{\pi}{3}i}(1+\varepsilon e^{it})$; $t\in(\pi,2\pi)$ $\Leftarrow$ tady platí $\ln(wz) = \ln w + \ln z$

$$J_6 = \ominus\int_\pi^{2\pi}\frac{\ln e^{\frac{\pi}{3}i} + \ln(1+\varepsilon e^{it})}{e^{\frac{\pi}{6}i}\sqrt{1+\varepsilon e^{it}}\,((1+\varepsilon e^{it})^6-1)^2}\,e^{\frac{\pi}{3}i}\varepsilon i e^{it}\,dt = -\varepsilon i e^{\frac{\pi}{6}i}\int_\pi^{2\pi}\frac{\left(\frac{\pi}{3}i + \varepsilon e^{it} + O(\varepsilon^2)\right)e^{it}\,dt}{\left(1+\frac{\varepsilon e^{it}}{2}\right)\left(6\varepsilon e^{it} + 15\varepsilon^2 e^{2it}\right)^2}$$

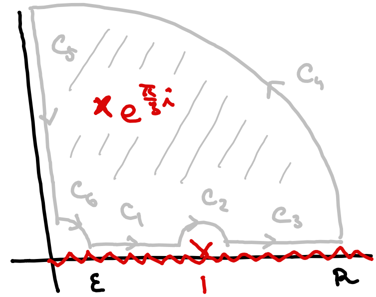
$$= -\frac{i e^{\frac{\pi}{6}i}}{36\varepsilon}\int_\pi^{2\pi}\left(\frac{\pi}{3}i + \varepsilon e^{it}\right)\left(1-\frac{\varepsilon e^{it}}{2}\right)\left(1-5\varepsilon e^{it}\right)e^{-it} + O(\varepsilon^2)\,dt \overset{\dots}{=} \frac{i e^{\frac{\pi}{6}i}}{36\varepsilon}\left(\frac{2\pi}{3} + \pi\varepsilon\left(1-\frac{11\pi}{3}\right)\right) + O(\varepsilon)$$

- **POROVNÁNÍ**: $0 = -\frac{\pi i}{36} + I(1-e^{\frac{\pi}{6}i}) - e^{\frac{\pi}{6}i}\frac{\pi}{3}i\,I_0(\varepsilon) + \frac{i e^{\frac{\pi}{6}i}}{36\varepsilon}\left(\frac{2\pi}{3} + \frac{11}{6}\pi^2\varepsilon i - \varepsilon\pi\right) + O(\varepsilon)$

$\hookrightarrow \cdot e^{-\frac{\pi}{6}i}$ / $\boxed{\mathrm{Re}}$: $0 = -\frac{\pi}{36}\sin\frac{\pi}{6} + I\left(\cos\frac{\pi}{6}-1\right) - \frac{\pi}{36}\cdot\frac{11\pi}{6}$

$$\therefore I = -\frac{\pi}{36}\,\frac{\sin\frac{\pi}{6} + \frac{11\pi}{6}}{1-\cos\frac{\pi}{6}} = -\frac{\pi}{36}\,\frac{\frac{1}{2}+\frac{11\pi}{6}}{1-\frac{\sqrt{3}}{2}} = -\frac{\pi}{108}\,\frac{3+11\pi}{2-\sqrt{3}} = -\frac{\pi}{108}(3+11\pi)(2+\sqrt{3})$$

(Př) $I = \int_0^\infty \frac{\ln x}{\sqrt{x}(x^6-1)^2}\,dx \in \mathbb{R}$ [na jiný integrál] Ř:



$$J = \oint_C \frac{\ln z}{\sqrt{z}(z^6-1)^2}\,dz$$

Singularity: $z^6-1 = (z^3-1)(z^3+1) = (z-1)(z-e^{\frac{2\pi i}{3}})(z-e^{-\frac{2\pi i}{3}})\cdot$
$\cdot (z+1)(z-e^{\frac{\pi i}{3}})(z-e^{-\frac{\pi i}{3}})$

- RESIDUOVÁ VĚTA:

→ $z \approx 1+i0$: $f(z) \,\hat{=}\, \frac{(z-1)}{\sqrt{z}(z^6-1)^2}$ pól 1. řádu!

$\therefore\ \mathrm{Res}_{1+i0} f(z) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{1}{\sqrt{z}}\left(\frac{z-1}{z^6-1}\right)^2 \overset{\text{L'H}}{=} \frac{1}{\sqrt{1}}\left(\frac{1}{6}\right)^2 = \frac{1}{36}$

→ $\mathrm{Res}_{e^{\frac{\pi}{3}i}} f \overset{2.\,\text{řád}}{=} \lim_{z\to e^{\frac{\pi}{3}i}} \frac{1}{(2-1)!}\left[(z-e^{\frac{\pi}{3}i})^2 f(z)\right]^{(2-1)} = \left(\frac{\ln z}{\sqrt{z}(z^3-1)^2(z+1)^2(z-e^{-\frac{\pi}{3}i})^2}\right)'\Bigg|_{e^{\frac{\pi}{3}i}}$

$g' = g(\ln g)'$

$$= \frac{\ln z}{\sqrt{z}(z^3-1)^2(z+1)^2(z-e^{-\frac{\pi}{3}i})^2}\left(\frac{1}{z\ln z} - \frac{1}{2z} - \frac{6z^2}{z^3-1} - \frac{2}{z+1} - \frac{2}{z-e^{-\frac{\pi}{3}i}}\right)\Bigg|_{e^{\frac{\pi}{3}i}}$$

= @ [1000 years later]

- PARAMETRIZACE:

→ C_2: $J_2 = -\pi i\,\mathrm{Res}_{1+i0} = -\frac{\pi i}{36}$

→ $C_1 + C_3$: $z = t+i0$; $t \in (\varepsilon, 1-\varepsilon)\cup(1+\varepsilon, R)$

$$J_1 + J_3 = \left(\int_\varepsilon^{1-\varepsilon} + \int_{1+\varepsilon}^R\right)\frac{\ln(t+i0)\,dt}{\sqrt{t+i0}\,((t+i0)^6-1)^2} \xrightarrow{\varepsilon\to 0} \int_0^\infty \frac{\ln t}{\sqrt{t}(t^6-1)^2}\,dt = I$$

→ C_4: $|J_4| \le \frac{\ln R + 2\pi}{\sqrt{R}(R^6-1)^2}\,\frac{\pi R}{2} \xrightarrow{R\to\infty} 0$

→ C_6: $|J_6| \le \frac{2\pi - \ln\varepsilon}{\sqrt{\varepsilon}(1-\varepsilon^6)^2}\,\frac{\pi\varepsilon}{2} \xrightarrow{\varepsilon\to 0^+} 0$

→ $\ominus C_5$: $z = it$; $t \in (\varepsilon, R)$; $dz = i\,dt$

jednodušší integrál

$$J_5 = \ominus\int_\varepsilon^R \frac{\ln(it)}{\sqrt{it}(-t^6-1)^2}\,i\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} -e^{\frac{\pi}{4}i}\int_0^\infty \frac{\ln x + 2\pi i}{\sqrt{t}(t^6+1)}\,dt = -e^{\frac{\pi}{4}i}\left(\underbrace{\int_0^\infty \frac{\ln t\,dt}{\sqrt{t}(t^6+1)^2}}_{E:\ I_1\in\mathbb{R}} 2\pi i \underbrace{\int_0^\infty \frac{dt}{\sqrt{t}(t^6+1)^2}}_{I_0\in\mathbb{R}}\right)$$

- POROVNÁNÍ: $2\pi i$ @ $= -\frac{\pi i}{36} + I - e^{\frac{\pi}{4}i}I_1 - e^{\frac{\pi}{4}i}2\pi i\, I_0 \;/\cdot e^{-\frac{\pi}{4}i}\;/\boxed{\mathrm{Im}}$

{Jako jo jde to ale $\mathrm{Res}_{e^{\frac{\pi}{3}i}}$ extrémně pracné}

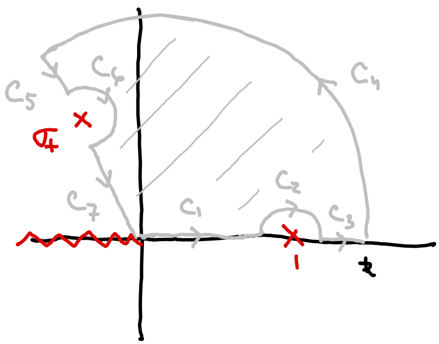
Př $I = \int_0^\infty \frac{\ln^2 x}{x^2+x+1}\,dx \in \mathbb{R}$

$\boxed{=} \int_0^\infty \frac{dx}{1-x^3}$

subst. $x \to \frac{1}{x}$

trik : $I = \int_0^\infty \frac{x-1}{x^3-1} \ln^2 x = \int_0^\infty \frac{x \ln^2 x}{x^3-1}\,dx + \int_0^\infty \frac{\ln^2 x}{1-x^3}\,dx = 2\int_0^\infty \frac{\ln^2 x}{1-x^3}\,dx$

$J = \oint_C \frac{\ln^2 z}{1-z^3}\,dz$; C:



Singularity: $1-z^3 = -(z-1)(z-\sigma_+)(z-\sigma_-)$;

$$\sigma_\pm = e^{\pm\frac{2\pi i}{3}}$$

- RESIDUOVÁ VĚTA :

$\longrightarrow$ $\mathrm{Res}_1 f(z) = \frac{\ln^2 z}{-3z^2}\Big|_1 = 0$

$\longrightarrow$ $\mathrm{Res}_{\sigma_+} f(z) = \frac{\ln^2 z}{-3z^2}\Big|_{\sigma_+} = -\frac{1}{3} e^{-\frac{4\pi i}{3}} \ln^2\left(e^{\frac{2\pi i}{3}}\right) = \frac{4\pi^2}{9} e^{\frac{2\pi i}{3}}$

ale $J = 0$ [CAUCHY]

- PARAMETRIZACE :

$\longrightarrow$ $C_1 + C_3$: $z = t$; $t \in (0, 1-\varepsilon) \cup (1+\varepsilon, R)$

$$J_1 + J_2 = \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{R}\right) \frac{\ln^2 t}{1-t^3}\,dt \xrightarrow[\varepsilon \to 0^+]{R\to\infty} \int_0^\infty \frac{\ln^2 t}{1-t^3}\,dt = \frac{I}{2}$$

$\longrightarrow$ $C_2 : J_2 \xrightarrow{\varepsilon \to 0^+} -\pi i\, \mathrm{Res}_1 f(z) = 0$

$\longrightarrow$ $C_4 : |J_4| \le \frac{(\ln R + \pi)^2}{R^3 - 1} \frac{2\pi}{3} R \xrightarrow{R\to\infty} 0$

$\longrightarrow$ $\ominus (C_5 + C_7)$: $z = e^{\frac{2\pi i}{3}} t$; $t \in (0, 1-\varepsilon) \cup (1+\varepsilon, R)$; $dz = e^{\frac{2\pi i}{3}}\,dt$

$$J_5 + J_7 = \ominus\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{R}\right) \frac{\ln^2\left(e^{\frac{2\pi i}{3}} t\right)}{1-t^3} e^{\frac{2\pi i}{3}}\,dt \xrightarrow[\varepsilon \to 0^+]{R\to\infty} \ominus e^{\frac{2\pi i}{3}} \int_0^\infty \frac{\left(\ln t + \frac{2\pi i}{3}\right)^2}{1-t^3}\,dt$$

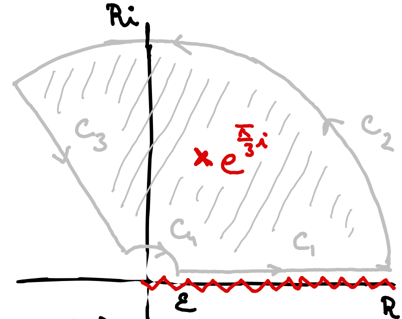
$$= -e^{\frac{2\pi i}{3}} \underbrace{\int_0^\infty \frac{\ln^2 t}{1-t^3}\,dt}_{I/2} - \frac{4\pi i}{3} e^{\frac{2\pi i}{3}} \underbrace{\int_0^\infty \frac{\ln t}{1-t^3}\,dt}_{I_0 \in \mathbb{R}} + \frac{4\pi^2}{9} e^{\frac{2\pi i}{3}} \underbrace{\int_0^\infty \frac{dt}{1-t^3}}_{I_1 \in \mathbb{R}}$$

$\longrightarrow$ $C_6 : J_6 \xrightarrow{\varepsilon \to 0^+} -\pi i\, \mathrm{Res}_{\sigma_+} f(z) = -\frac{4\pi^3 i}{9} e^{2\pi i/3}$

- POROVNÁNÍ : $0 = \frac{I}{2}\left(1 - e^{\frac{2\pi i}{3}}\right) - \frac{4\pi i}{3} e^{\frac{2\pi i}{3}} I_0 + \frac{4\pi^2}{9} e^{\frac{2\pi i}{3}} I_1 - \frac{4\pi^3 i}{9} e^{\frac{2\pi i}{3}}$

- $e^{-\frac{2\pi i}{3}} / \boxed{\mathrm{Re}}$: $0 = \frac{I}{2}\big(\underbrace{\cos\tfrac{2\pi}{3} - 1}_{-3/2}\big) + \frac{4\pi^2}{9} I_1$ $\therefore$ $\boxed{I = \frac{16\pi^2}{27} I_1 \overset{\boxed{=}}{=} \frac{16\pi^3}{27\sqrt{3}}}$

(Př) $I = \int_0^\infty \frac{\ln x}{1-x+x^2}\,dx \in \mathbb{R}$



Trik: $\frac{1}{x-e^{\frac{\pi i}{3}}} = \frac{x-e^{-\frac{\pi i}{3}}}{1-2x\cos\frac{\pi}{3}+x^2} = \frac{x-\frac12+\frac{\sqrt3}{2}i}{1-x+x^2}$

$$J = \oint_C \frac{\ln z}{z-e^{\frac{\pi i}{3}}}\,dz$$

- RESIDUOVÁ VĚTA:

→ $\mathrm{Res}_{e^{\frac{\pi}{3}i}} f = \ln z\Big|_{e^{\frac{\pi}{3}i}} = \frac{\pi}{3}i \quad \therefore J = 2\pi i\left(\frac{\pi}{3}i\right) = -\frac{2\pi^2}{3}$

- PARAMETRIZACE:

→ $C_1: z = t+i0;\ t\in(\varepsilon,R);\ dz = dt$

$$J_1 = \int_\varepsilon^R \frac{\ln(t+i0)}{t-e^{\frac{\pi i}{3}}}dt \xrightarrow{\varepsilon\to0^+} \int_0^R \frac{\ln t}{t-e^{\frac{\pi i}{3}}}dt = \int_0^R \frac{\ln t}{1-t+t^2}\left(t-\frac12+\frac{\sqrt3}{2}i\right)dt =$$

$$= \underbrace{\int_0^R \frac{t-\frac12}{1-t+t^2}\ln t\,dt}_{I_0(R)\in\mathbb{R}\text{ ale diverguje}} + \frac{\sqrt3}{2}i\underbrace{\int_0^R\frac{\ln t}{1-t+t^2}dt}_{I+O(\frac1R)\ \because\ \xrightarrow{R\to\infty} I}$$

→ $C_2: z = Re^{it};\ t\in(0,\frac{2\pi}{3});\ dz = Rie^{it}dt$

$$J_2 = \int_0^{\frac{2\pi}{3}} \frac{\ln(Re^{it})}{Re^{it}-e^{\frac{\pi i}{3}}}Rie^{it}dt = i\int_0^{\frac{2\pi}{3}}(\ln R+it)\left(1-\frac1R e^{-it}e^{\frac{\pi i}{3}}\right)^{-1}dt =$$

$$= i\int_0^{\frac{2\pi}{3}}(\ln R+it)\left(1+\frac1R e^{-it}+O\left(\frac1{R^2}\right)\right)dt = \frac{2\pi i}{3}\ln R - \frac{2\pi^2}{9}+O\left(\frac{\ln R}{R}\right)$$

→ $\ominus C_3: z = e^{\frac{2\pi i}{3}}t;\ t\in(\varepsilon,R);\ dz = e^{\frac{2\pi i}{3}}dt$

$$J_3 = \ominus\int_\varepsilon^R \frac{\ln(e^{\frac{2\pi i}{3}}t)}{e^{\frac{2\pi i}{3}}t-e^{\frac{\pi}{3}i}}e^{\frac{2\pi i}{3}}dt \xrightarrow{\varepsilon\to0^+} -\int_0^R \frac{\frac{2\pi i}{3}+\ln t}{t-e^{-\frac{\pi}{3}i}}dt =$$

$$= -\int_0^R \frac{\frac{2\pi i}{3}+\ln t}{1-t+t^2}\left(t-\frac12-\frac{\sqrt3}{2}i\right)dt = -I_0(R)+\frac{\sqrt3}{2}iI - \frac{2\pi i}{3}\underbrace{\int_0^R\frac{t-\frac12}{1-t+t^2}dt}_{I_1(R)\in\mathbb{R}\text{ diver.}} - \frac{\pi}{\sqrt3}\underbrace{\int_0^R\frac{dt}{1-t+t^2}}_{\xrightarrow{R\to\infty} I_2\in\mathbb{R}}$$

→ $C_4: |J_4| \le \frac{\frac{2\pi}{3}-\ln\varepsilon}{1-\varepsilon}\,\frac{2\pi\varepsilon}{3} \xrightarrow{\varepsilon\to0^+} 0$

- POROVNÁNÍ: $-\frac{2\pi^2}{3} = \sqrt3 iI + \frac{2\pi i}{3}\ln R - \frac{2\pi^2}{9} - \frac{2\pi i}{3}I_1(R) - \frac{\pi}{\sqrt3}I_2 + O\left(\frac{\ln R}{R}\right)$

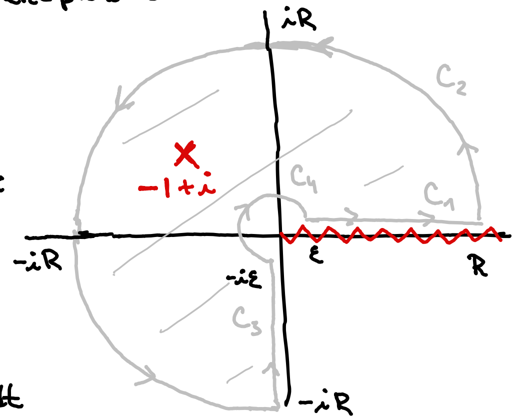
$\boxed{\text{Im:}}$ $0 = \sqrt3 I + \frac{2\pi i}{3}\ln R - \frac{2\pi i}{3}I_1(R)+O\left(\frac{\ln R}{R}\right)$ ale $I_1(R) = \int_0^R\frac{t-\frac12}{1-t+t^2}dt = \frac{\ln(1-t+t^2)}{2}\Big|_0^R$

$\therefore I = \frac{2\pi}{3\sqrt3}\lim_{R\to\infty}\left(\frac12\ln(1-R+R^2)-\ln R\right) = 0$ (Triviální protože $I = |x\to\frac1x| = -I$)

(Př) $I = \int_0^\infty \frac{\ln x}{x^2+2x+2}\,dx = \lim_{R\to\infty} \operatorname{Im} \int_0^R \frac{\ln x}{x+1-i}\,dx$

$\in \mathbb{R}$ — pro $R\to\infty$ diverguje! — nelze prohodit ∴

$$J = \oint_C \frac{\ln z}{z+1-i}\,dz$$



- <u>RESIDUOVÁ VĚTA:</u>

→ $\operatorname{Res}_{-1+i} f(z) = \ln(-1+i) = \frac{1}{2}\ln 2 + \frac{3\pi i}{4}$ ($\sqrt{2}e^{\frac{3\pi i}{4}}$)

∴ $J = 2\pi i\left(\frac{1}{2}\ln 2 + \frac{3\pi i}{4}\right) = \pi i \ln 2 - \frac{3\pi^2}{2}$

- <u>PARAMETRIZACE:</u>

→ $C_1:\ z = t + i0;\ t\in(\varepsilon,R);\ dz = dt$

$$J_1 = \int_\varepsilon^R \frac{\ln(t+i0)}{t+1-i}\,dt \xrightarrow{\varepsilon\to0^+} \int_0^R \frac{\ln t}{t+1-i}\,dt = \int_0^R \frac{(t+1+i)\ln t\,dt}{(t+1)^2+1^2} =$$

$$= \underbrace{\int_0^R \frac{(t+1)\ln t}{t^2+2t+2}\,dt}_{\mathbb{R}\ni I_0(R) \xrightarrow{R\to\infty} \infty} + i\underbrace{\int_0^R \frac{\ln t}{t^2+2t+1}\,dt}_{\mathbb{R}\ni I(R) \xrightarrow{R\to\infty} I \text{ (konv.)}} = I_0(R) + iI(R)$$

→ $C_2:\ z = Re^{it};\ t\in\left(0,\frac{3\pi}{4}\right);\ dz = Rie^{it}\,dt$

$$J_2 = \int_0^{\frac{3\pi}{4}} \frac{\ln(Re^{it})}{Re^{it}+1-i}Rie^{it}\,dt = i\int_0^{\frac{3\pi}{4}} (\ln R + it)\left(1+\frac{1-i}{R}e^{-it}\right)^{-1}dt$$

$$= i\int_0^{\frac{3\pi}{4}} (\ln R + it)\left(1-\frac{1-i}{R}e^{-it}+O\left(\frac{1}{R^2}\right)\right)dt = \frac{3\pi i}{4}\ln R - \underbrace{\int_0^{\frac{3\pi}{4}} t\,dt}_{3\pi^2/16} + O\left(\frac{\ln R}{R}\right)$$

→ ⊖ $C_3:\ z = -it;\ t\in(\varepsilon,R);\ dz = -i\,dt$

$$J_3 = \ominus\int_\varepsilon^R \frac{\ln(-it)}{-it+1-i}(-i)\,dt \xrightarrow{\varepsilon\to0^+} -\int_0^R \frac{\ln t + \frac{3\pi}{4}i}{t+i+1}\,dt =$$

$$= -\frac{3\pi i}{4}\int_0^R \frac{dt}{t+i+1} - \int_0^R \frac{(t+1-i)\ln t\,dt}{(t+1)^2+1^2} = -\frac{3\pi i}{4}\ln(t+1+i)\Big|_0^R - I_0(R) + iI(R)$$

→ $C_4:\ |J_4| \le \frac{\frac{3\pi}{4}-\ln\varepsilon}{\sqrt{2}-\varepsilon}\ \frac{3\pi}{4}\varepsilon \xrightarrow{\varepsilon\to0^+} 0$

- <u>POROVNÁNÍ:</u> $\pi i\ln 2 - \frac{3\pi^2}{2} = 2iI(R) + \frac{3\pi i}{4}\ln R - \frac{3\pi^2}{16} - \frac{3\pi i}{4}\ln(t+1+i)\Big|_0^R$

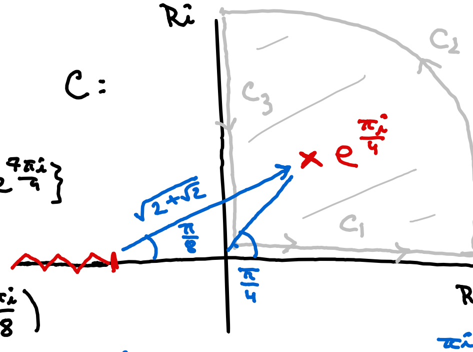
↳ $\boxed{\text{Im:}}$ $\pi\ln 2 = 2I(R) + \frac{3\pi}{4}\ln R - \frac{3\pi}{4}\left(\ln\sqrt{(R+1)^2+1} - \ln\sqrt{2}\right)$

$R\to\infty$: ↳ $\pi\ln 2 = 2I + \frac{3\pi}{8}\ln 2$ ∴ $\boxed{I = \frac{5\pi}{16}\ln 2}$

$$\text{Př} \quad I = \int_0^\infty \frac{\operatorname{arctg} x}{x(1+x^4)}\,dx \in \mathbb{R}$$

$$J = \oint_C \frac{\cancel{\operatorname{arct}}\ \ln(z+1)}{z(1+z^4)}\,dz$$

$C:$



póly: $1+z^4=0$; $\sigma \in \{e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{5\pi i}{4}}, e^{\frac{7\pi i}{4}}\}$

- **RESIDUOVÁ VĚTA**:

$$\longrightarrow \operatorname{Res}_{e^{\frac{\pi i}{4}}} f(z) = \frac{\ln(z+1)}{z\,4z^3} = -\frac{1}{4}\left(\frac{1}{2}\ln(2+\sqrt2)+\frac{\pi i}{8}\right)$$

$$e^{\frac{\pi i}{4}} - (-1) = \sqrt{2+\sqrt2}\, e^{\frac{\pi i}{8}}$$

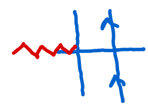
$$\therefore J = 2\pi i \sum_{\sigma \in \operatorname{Int} C} \operatorname{Res}_\sigma f(z) = -\frac{\pi i}{4}\ln(2+\sqrt2) + \frac{\pi^2}{16}$$

- **PARAMETRIZACE**:

$\longrightarrow C_1: \ z = t;\ t \in (0,R);\ dz = dt$

$$J_1 = \int_0^R \frac{\ln(t+1)}{t(1+t^4)}\,dt \xrightarrow{R\to\infty} \underbrace{\int_0^\infty \frac{\ln(t+1)}{t(1+t^4)}\,dt}_{I_0 \in \mathbb{R}}$$

$\longrightarrow C_2: \ |J_2| \le \frac{\ln(R+1)+\pi}{R(R^4-1)}\,2\pi R \xrightarrow{R\to\infty} 0$



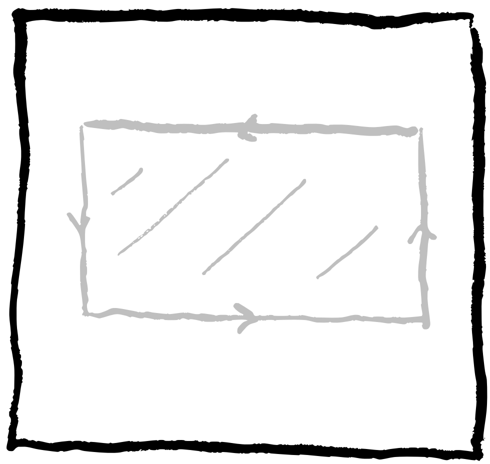
$\longrightarrow \ominus C_3: \ z = it;\ t \in (0,R);\ dz = i\,dt$

$$J_3 = \ominus \int_0^R \frac{\ln(it+1)}{it(1+t^4)}\,i\,dt \xrightarrow{R\to\infty} -\int_0^\infty \frac{\ln|it+1| + i\arg(it+1)}{t(1+t^4)}\,dt$$

$$= -\frac{1}{2}\underbrace{\int_0^\infty \frac{\ln(1+t^2)}{t(1+t^4)}\,dt}_{I_1 \in \mathbb{R}} - i\underbrace{\int_0^\infty \frac{\operatorname{arctg} t}{t(1+t^4)}\,dt}_{I} = -\frac{1}{2}I_1 - iI$$

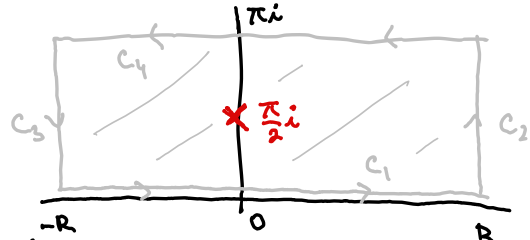
- **POROVNÁNÍ**: $-\frac{\pi i}{4}\ln(2+\sqrt2) + \frac{\pi^2}{16} = I_0 - \frac{1}{2}I_1 - iI$

$\boxed{\text{Im:}}$ $$\boxed{I = \frac{\pi}{4}\ln(2+\sqrt2)}$$



(Př) $I(\alpha)=\int_{-\infty}^{\infty}\frac{\cos\alpha x}{\cosh x}\,dx \in \mathbb{R}$ pro libov. $\alpha\in\mathbb{R}$ [BÚNO $\alpha\geq 0$ ∵ I sudé v α]

$J := \oint_C \frac{\cos\alpha z}{\cosh z}\,dz$; C:



- RESIDUOVÁ VĚTA :

$\longrightarrow$ $\mathrm{Res}_{\pi i} f(z) = \frac{\cos\alpha z}{\sinh z}\Big|_{\frac{\pi i}{2}} = \frac{\cos\frac{\alpha\pi i}{2}}{\sinh\frac{\pi i}{2}} = \frac{\cosh\frac{\alpha\pi}{2}}{i\sin\frac{\pi}{2}} = -i\cosh\frac{\alpha\pi}{2}$

∴ $J = 2\pi i\left(-i\cosh\frac{\alpha\pi}{2}\right) = 2\pi\cosh\frac{\alpha\pi}{2}$

- PARAMETRIZACE :

$\longrightarrow$ $C_1:\ z=t;\ t\in(-R,R);\ dz=dt$

$$J_1 = \int_{-R}^{R}\frac{\cos\alpha t}{\cosh t}\,dt \xrightarrow{R\to\infty} \int_{-\infty}^{\infty}\frac{\cos\alpha t}{\cosh t}\,dt = I$$

$\longrightarrow$ $C_2:\ z=R+it;\ t\in(0,\pi);\ dz=i\,dt$

$$|J_2| \overset{ML}{\leq} \sup_{t\in(0,\pi)}\left|\frac{e^{i\alpha z}+e^{-i\alpha z}}{e^{z}+e^{-z}}\right|_{R+it}\pi \leq \sup\frac{e^{-\alpha t}+e^{\alpha t}}{e^{R}-e^{-R}}\pi \leq \frac{\cosh\alpha\pi}{\sinh R}\pi \xrightarrow{R\to\infty} 0$$

$\longrightarrow$ $(-)$ $C_3:\ z=\pi i+t;\ t\in(-R,R);\ dz=dt$

$$J_3 = (-)\int_{-R}^{R}\frac{\cos\alpha(\pi i+t)}{\cosh(\pi i+t)}\,dt \xrightarrow{R\to\infty} -\int_{-\infty}^{\infty}\frac{\cos\alpha\pi i\cos\alpha t-\sin\alpha\pi i\sin\alpha t}{\cosh\pi i\cosh t+\sinh\pi i\sinh t}\,dt$$

$$= -\int_{-\infty}^{\infty}\frac{\cosh\alpha\pi\cos\alpha t-i\sinh\alpha\pi\sin\alpha t}{\cos\pi\cosh t+i\underbrace{\cancel{\sin\pi}}_{0}\sinh t}\,dt =$$

$$= \cosh\alpha\pi\int_{-\infty}^{\infty}\frac{\cos\alpha t}{\cosh t}\,dt + i\sinh\alpha\pi\underbrace{\int_{-\infty}^{\infty}\frac{\sin\alpha t}{\cosh t}\,dt}_{=0\ ∵\ \text{liché}} = I\cosh\alpha\pi$$

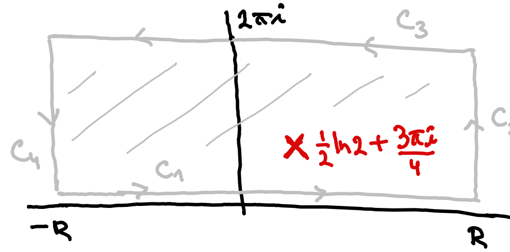
$\longrightarrow$ $C_4:\ f(-z)=f(z)\Rightarrow |J_4| \xrightarrow{R\to\infty} 0$

- POROVNÁNÍ :

$2\pi\cosh\frac{\alpha\pi}{2} = I + I\cosh\alpha\pi$ ∴ $\boxed{I = \frac{2\pi\cosh\frac{\alpha\pi}{2}}{1+\cosh\alpha\pi}\ ;\ \alpha\in\mathbb{R}}$

(Př) $$I = \int_{-\infty}^{\infty} \frac{e^{\frac{x}{2}}}{e^{2x}+2e^{x}+2}\,dx \underset{\in\mathbb{R}}{=} \operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{\frac{x}{2}}}{e^{x}+1-i}\,dx \in \mathbb{C}$$

$$\hookrightarrow \quad J := \oint_C \frac{e^{\frac{z}{2}}}{e^{z}+1-i}\,dz \;;\; C:$$



póly: $e^{z} = -1+i = \sqrt{2}\,e^{\frac{3\pi i}{4}} = e^{\frac{1}{2}\ln 2 + \frac{3\pi i}{4}}$

$\hookrightarrow \sigma_k = \frac{1}{2}\ln 2 + \frac{3\pi i}{4} + 2k\pi \;;\; k \in \mathbb{Z}.$

- RESIDUOVÁ VĚTA :

$$\longrightarrow \operatorname{Res}_{\sigma_0} f(z) = \left.\frac{e^{\frac{z}{2}}}{e^{z}}\right|_{\sigma_0} = \left.e^{-\frac{z}{2}}\right|_{\sigma_0} = e^{-\frac{1}{4}\ln 2 - \frac{3\pi i}{8}} = \frac{1}{\sqrt[4]{2}}\,e^{-\frac{3\pi i}{8}}$$

$$\therefore\; J = 2\pi i \sum_{\sigma \in \operatorname{Int} C} \operatorname{Res}_{\sigma} f(z) = \frac{2\pi i}{\sqrt[4]{2}}\,e^{-\frac{3\pi i}{8}}$$

- PARAMETRIZACE :

$\longrightarrow C_1 : z = t \;;\; t \in (-R, R) \;;\; dz = dt$

$$J_1 = \int_{-R}^{R} \frac{e^{\frac{t}{2}}}{e^{t}+1-i}\,dt \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \frac{e^{\frac{t}{2}}}{e^{t}+1-i}\,dt =$$

$$= \int_{-\infty}^{\infty} \frac{e^{\frac{t}{2}}(e^{t}+1+i)}{(e^{t}+1)^2+1^2}\,dt = \underbrace{\int_{-\infty}^{\infty} \frac{e^{\frac{t}{2}}(e^{t}+1)}{e^{2t}+2e^{t}+2}\,dt}_{I_0} + i\underbrace{\int_{-\infty}^{\infty} \frac{e^{\frac{t}{2}}}{e^{2t}+2e^{t}+2}\,dt}_{I}$$

$$= I_0 + iI$$

$\longrightarrow C_2 : z = R + it \;;\; t \in (0, 2\pi) : \; |J_2| \leq \dfrac{e^{R/2}}{e^{R}-2}\,2\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow \ominus\, C_3 : z = 2\pi i + t \;;\; t \in (-R, R) \;;\; dz = dt$

$$J_3 = \ominus \int_{-R}^{R} \frac{e^{\frac{t}{2}+\pi i}}{e^{t+2\pi i}+1-i}\,dt \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \frac{e^{\frac{t}{2}}}{e^{t}+1-i}\,dt = I_0 + iI$$

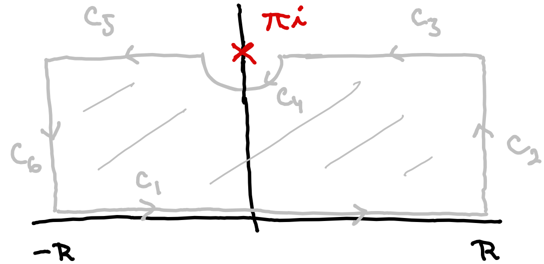
$\longrightarrow C_4 : z = -R + it \;;\; t \in (0, 2\pi) : \; |J_4| \leq \dfrac{e^{-R/2}}{\sqrt{2}-e^{-R}}\,2\pi \xrightarrow{R\to\infty} 0$

- POROVNÁNÍ :

$$\frac{2\pi i}{\sqrt[4]{2}}\,e^{-\frac{3\pi i}{8}} = 2I_0 + 2iI$$

$\boxed{\operatorname{Im}:}$ $$\boxed{I = \frac{\pi}{\sqrt[4]{2}}\cos\frac{3\pi}{8} = \frac{\pi}{\sqrt[4]{2}}\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \frac{\pi}{2}\sqrt{\sqrt{2}-1}}$$

(Př) $I = \int_{-\infty}^{\infty} \frac{x}{\sinh x}\,dx \in \mathbb{R}$



$\hookrightarrow \; J := \oint_C \frac{z}{\sinh z}\,dz \; ; \; C:$

- **RESIDUOVÁ VĚTA:**

$\longrightarrow \operatorname{Res}_{\pi i} f(z) = \left.\frac{z}{\cosh z}\right|_{\pi i} = -\pi i$

ale $J = 0$ [CAUCHY]

- **PARAMETRIZACE:**

$\longrightarrow C_1: z = t; \; t \in (-R, R) \; ; \; dz = dt$

$J_1 = \int_{-R}^{R} \frac{t}{\sinh t}\,dt \xrightarrow{R\to\infty} \int_{-\infty}^{\infty} \frac{t}{\sinh t}\,dt = I$

$\longrightarrow C_2: z = R + it \; ; \; t \in (0, \pi)$

$|J_2| \leq \sup_{t\in(0,\pi)} \left| \left. \frac{2z}{e^z - e^{-z}} \right| \right|_{z = R+it} \cdot \pi \leq \frac{2(R+\pi)}{e^R - e^{-R}}\,\pi \xrightarrow{R\to\infty} 0$

$\longrightarrow \ominus (C_3 + C_5): z = \pi i + t \; ; \; t \in (-R, -\varepsilon) \cup (\varepsilon, R) \; ; \; dz = dt$

$J_3 + J_5 = \ominus \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) \frac{\pi i + t}{\sinh(\pi i + t)}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} -\int_{-\infty}^{\infty} \frac{\pi i + t}{\sinh(\pi i + t)}\,dt$

$= -\int_{-\infty}^{\infty} \frac{\pi i + t}{\underbrace{\sinh \pi i}_{i \sin \pi} \cosh t + \underbrace{\cosh \pi i}_{\cos \pi} \sinh t}\,dt = \pi i \underbrace{\int_{-\infty}^{\infty} \frac{dt}{\sinh t}}_{0} + \underbrace{\int_{-\infty}^{\infty} \frac{t\,dt}{\sinh t}}_{I}$

$\longrightarrow C_4: J_4 \xrightarrow{\varepsilon\to 0^+} -\pi i \operatorname{Res}_{\pi i} f(z) = -\pi^2$

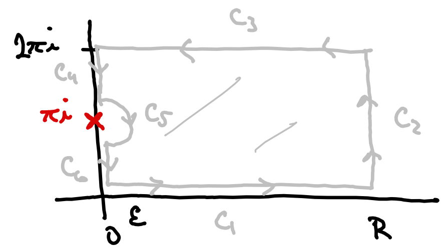
$\longrightarrow \ominus C_6: z = -R + it \; ; \; t \in (0, \pi)$

$|J_6| \leq \sup_{t\in(0,\pi)} \left| \left. \frac{2z}{e^z - e^{-z}} \right| \right|_{-R+it} \pi = \frac{2(R+\pi)}{e^R - e^{-R}}\,\pi \xrightarrow{R\to\infty} 0$

- **POROVNÁNÍ:**

$0 = I + I - \pi^2 \quad \therefore \quad \boxed{I = \frac{\pi^2}{2}}$

(Př) $I = \int_0^\infty \frac{x}{1+e^x}\,dx \in \mathbb{R}$



$\hookrightarrow J = \oint_C \frac{z^2\,dz}{1+e^z}$; C:

- **RESIDUOVÁ VĚTA** :

$\longrightarrow \operatorname{Res}_{\pi i} f(z) = \frac{z^2}{e^z}\Big|_{\pi i} = \frac{-\pi^2}{e^{\pi i}} = \pi^2$

$J = 0$ [CAUCHY]

- **PARAMETRIZACE** :

$\longrightarrow C_1: z = t;\ t \in (0,R);\ dz = dt;$

$J_1 \xrightarrow{R\to\infty} \int_0^\infty \frac{t^2\,dt}{1+e^t}$ K

$\longrightarrow C_2: z = R + it;\ t \in (0, 2\pi);\ |J_2| \leq \frac{(R+2\pi)^2}{e^R - 1} 2\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow \ominus C_3: z = 2\pi i + t;\ t \in (0,R);\ dz = dt$

$J_3 = \ominus \int_0^R \frac{(t+2\pi i)^2}{1+e^t}dt \xrightarrow{R\to\infty} -\int_0^\infty \frac{t^2 dt}{1+e^t} - 4\pi i \underbrace{\int_0^\infty \frac{t\,dt}{1+e^t}}_{I} + 4\pi^2 \underbrace{\int_0^\infty \frac{dt}{1+e^t}}_{I_0}$

$\longrightarrow \ominus(C_4 + C_6): z = it;\ t \in (0, \pi-\varepsilon) \cup (\pi+\varepsilon, 2\pi);\ dz = i\,dt$

$J_4 = \ominus\left(\int_0^{\pi-\varepsilon} + \int_{\pi+\varepsilon}^{2\pi}\right) \frac{(it)^2}{1+e^{it}}(i\,dt) \xrightarrow{\varepsilon\to 0^+} i \int_0^{2\pi} \frac{t^2\,dt}{1+e^{it}} =$

$= i \int_0^{2\pi} \frac{t^2(1+e^{-it})}{(1+e^{it})(1+e^{-it})}dt = \frac{i}{2}\int_0^{2\pi} \frac{t^2(1+\cos t - i\sin t)}{1+\cos t}dt$

$\longrightarrow C_5: J_5 \xrightarrow{\varepsilon\to 0^+} -\pi i \operatorname{Res}_{\pi i} f(z) = -\pi^3 i$

- **POROVNÁNÍ** :

$0 = -4\pi i I + 4\pi^2 I_0 + \frac{i}{2}\int_0^{2\pi} t^2 dt + \frac{1}{2}\int_0^{2\pi} \frac{t^2 \sin t}{1+\cos t}dt - \pi^3 i$

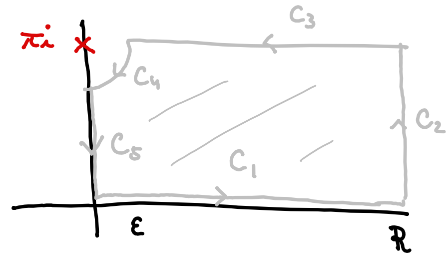
$\boxed{\mathrm{Im}:}$ $I = \frac{1}{8\pi}\int_0^{2\pi} t^2\,dt - \frac{\pi^2}{4} = \frac{1}{8\pi}\frac{(2\pi)^3}{3} - \frac{\pi^2}{4} = \frac{\pi^2}{3} - \frac{\pi^2}{4} = \boxed{\frac{\pi^2}{12} \equiv I}$

$\boxed{\mathrm{Re}:}$ $\int_0^{2\pi} \frac{t^2 \sin t}{1+\cos t}dt = -8\pi^2 \int_0^\infty \frac{dt}{1+e^t} = -8\pi^2\left[t - \ln(1+e^t)\right]_0^\infty = -8\pi^2 \ln 2$

[BONUS]

Př: $I = \int_0^\infty \frac{\sin x}{\sinh x} e^{-x}\,dx \in \mathbb{R}$

$$J := \oint_C \frac{\sin z}{\sinh z} e^{-z}\,dz \quad ; \quad C:$$



• REZIDUOVÁ VĚTA

$$\longrightarrow \quad \mathrm{Res}_{\pi i} f(z) = \frac{\sin z}{\cosh z} e^{-z}\Big|_{\pi i} = \frac{\sin(\pi i)}{\cosh(\pi i)} e^{-\pi i} = \frac{i\sinh\pi}{\cancel{\cos(\pi)}} \cancel{e^{-\pi i}} = i\sinh\pi$$

$$\therefore \quad J = 0$$

• PARAMETRIZACE

$$\longrightarrow \quad C_1: \; z = t; \; t \in (0,R); \; dz = dt$$

$$J_1 = \int_0^R \frac{\sin t}{\sinh t} e^{-t}\,dt \xrightarrow{R\to\infty} I$$

$$\longrightarrow \quad C_2: \; z = R + it; \; t \in (0,\pi); \; |J_2| \overset{ML}{\leq} \sup_{t\in(0,\pi)} \left|\frac{e^{iz}-e^{-iz}}{e^{z}-e^{-z}}\right| \left|e^{-z}\right| \Big|_{R+it} \pi \leq$$

$$\leq \sup \frac{|e^{iz}|+|e^{-iz}|}{\big||e^{z}|-|e^{-iz}|\big|} \left|e^{-z}\right| \Big|_{R+it} \pi = \sup \underbrace{\frac{e^{-t}+e^{t}}{e^{R}-e^{-R}}}_{\frac{\cosh\pi}{\sinh R}} e^{-R}\pi \xrightarrow{R\to\infty} 0$$

$$\longrightarrow \quad \ominus C_3: \; z = \pi i + t; \; t \in (\varepsilon, R)$$

$$J_3 = \ominus\int_\varepsilon^R \frac{\sin(\pi i + t)}{\sinh(\pi i + t)} e^{-\pi i - t}\,dt = \int_\varepsilon^R \frac{(\sin\pi i\cos t + \cos\pi i\sin t)\, e^{-t}}{\underbrace{\sinh(\pi i)}_{0}\cosh t + \cosh(\pi i)\sinh t}\,dt =$$

$$= \int_\varepsilon^R \frac{i\sinh\pi\cos t + \cosh\pi\sin t}{\cos(\pi)\sinh t} e^{-t}\,dt = -i\sinh\pi \underbrace{\int_\varepsilon^R \frac{\cos t}{\sinh t} e^{-t}\,dt}_{\xrightarrow{R\to\infty} \int_\varepsilon^\infty \frac{\cos t\, e^{-t}}{\sinh t}\,dt} - \cosh\pi \underbrace{\int_\varepsilon^R \frac{\sin t\, e^{-t}}{\sinh t}\,dt}_{\xrightarrow[\varepsilon\to 0^+]{R\to\infty} I}$$

$$\longrightarrow \quad C_4: \; J_4 \xrightarrow{\varepsilon\to 0^+} -\frac{\pi}{2} i\, \mathrm{Res}_{\pi i} f(z) = \frac{\pi}{2}\sinh\pi$$

$$\longrightarrow \quad \ominus C_5: \; z = it; \; t \in (0, \pi-\varepsilon); \; dz = i\,dt$$

$$J_5 = \ominus \int_0^{\pi-\varepsilon} \frac{\sin it}{\sinh it} e^{it}\, i\,dt = -i\int_0^{\pi-\varepsilon} \frac{\sinh t}{\sin t} e^{-it}\,dt \quad (\text{nelze } \varepsilon \to 0^+)$$

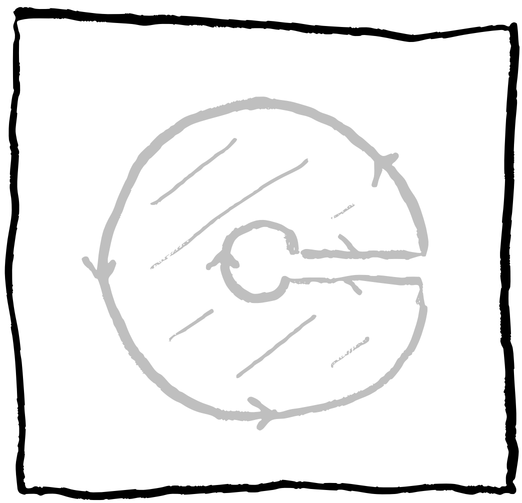
• POROVNÁNÍ:

$$0 \overset{\varepsilon\to 0^+}{=} I - i\sinh\pi \int_\varepsilon^\infty \frac{\cos t\, e^{-t}}{\sinh t}\,dt - I\cosh\pi + \frac{\pi}{2}\sinh\pi - i\int_0^{\pi-\varepsilon} \frac{\sinh t\, e^{-it}}{\sin t}\,dt$$

Re: $0 \overset{\varepsilon\to 0^+}{=} I - I\cosh\pi + \frac{\pi}{2}\sinh\pi - \int_0^{\pi-\varepsilon} \frac{\sinh t}{\cancel{\sin t}} \cancel{\sin t}\,dt$

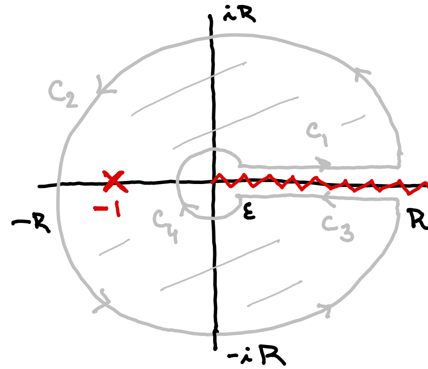
$$\varepsilon \to 0 \Rightarrow I(\cosh\pi - 1) = \frac{\pi}{2}\sinh\pi - \int_0^\pi \sinh t\,dt = \frac{\pi}{2}\sinh\pi - (\cosh\pi - 1)$$

$$\therefore \quad \boxed{I = \frac{\pi}{2}\frac{\sinh\pi}{\cosh\pi - 1} - 1}$$



Př $I = \int_0^\infty \frac{dx}{\sqrt{x}(1+x)^2} \in \mathbb{R}$

$$J = \oint_C \frac{dz}{\sqrt{z}(1+z)^2} \quad ; \quad C:$$



• RESIDUOVÁ VĚTA :

$\longrightarrow$ $$\operatorname{Res}_{-1} f(z) \overset{\text{pól 2.}}{=} \lim_{z\to -1} \frac{1}{(2-1)!}\left[(z+1)^2 f(z)\right]^{(2-1)} = \lim_{z\to -1}\left(\frac{1}{\sqrt{z}}\right)' =$$

$$= -\frac{1}{2} z^{-\frac{3}{2}}\Big|_{e^{\pi i}} = -\frac{1}{2} e^{-\frac{3}{2}\pi i} = -\frac{i}{2}$$

$$\therefore\ J = 2\pi i \sum_{\sigma \in \operatorname{Int} C} \operatorname{Res}_\sigma f(z) = 2\pi i\left(-\frac{i}{2}\right) = \pi$$

• PARAMETRIZACE :

$\longrightarrow$ $C_1: z = t + i0\ ;\ t \in (\varepsilon, R)\ ;\ dz = dt$

$$J_1 = \int_\varepsilon^R \frac{dt}{\sqrt{t+i0}\,(1+t)^2} \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{dt}{\sqrt{t}(1+t)^2} = I$$

$\longrightarrow$ $C_2: \ |J_2| \le \frac{1}{\sqrt{R}(R-1)^2} 2\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow$ $\ominus C_3: z = t - i0\ ;\ t \in (\varepsilon, R)\ ;\ dz = dt$

$$J_3 = \ominus \int_\varepsilon^R \frac{dt}{\underbrace{\sqrt{t-i0}}_{-\sqrt{t}}\,(1+t)^2} \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{dt}{\sqrt{t}(1+t)^2} = I$$

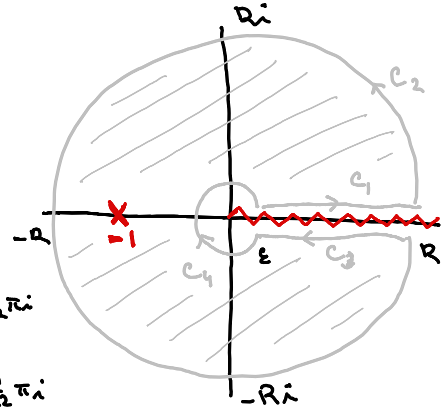
$\longrightarrow$ $C_4: \ |J_4| \le \frac{1}{\sqrt{\varepsilon}(1-\varepsilon)^2} 2\pi\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$

• POROVNÁNÍ : $J = J_1 + J_2 + J_3 + J_4 \ / \ R\to\infty$

$$\pi = 2I \quad \therefore \quad \boxed{I = \frac{\pi}{2}}$$

Př: $I = \int_0^\infty \frac{dx}{\sqrt{x}(x^6+1)^2} \overset{\text{subst.}}{=} \left| \begin{array}{l} x = t^{1/6} \\ dx = \frac{1}{6} t^{-5/6}\,dt \end{array} \right| = \frac{1}{6}\int_0^\infty \frac{t^{-11/12}}{(t+1)^2}\,dt$

$$J = \oint_C \frac{z^{-11/12}\,dz}{(z+1)^2}$$



- **RESIDUOVÁ VĚTA**

$\rightarrow$ $\operatorname{Res}_{-1} f = \lim_{z\to -1} \frac{1}{(2-1)!}\left[(z+1)^2 f(z)\right]^{(2-1)} =$

$= \left(z^{-11/12}\right)'\Big|_{-1} = -\frac{11}{12} z^{-\frac{11}{12}-1}\Big|_{e^{\pi i}} = \frac{11}{12} e^{-\frac{11}{12}\pi i}$

$\therefore\ J = 2\pi i \sum_{\sigma \in \operatorname{Int} C} \operatorname{Res}_\sigma f(z) = \frac{11}{6}\pi i\, e^{-\frac{11}{12}\pi i}$

- **PARAMETRIZACE** :

$\longrightarrow$ $C_1 : z = t + i0;\ t \in (\varepsilon, R);\ dz = dt$

$$J_1 = \int_\varepsilon^R \frac{(t+i0)^{-11/12}}{(t+1)^2}\,dt \xrightarrow[\varepsilon \to 0^+]{R\to+\infty} \int_0^\infty \frac{t^{-11/12}}{(t+1)^2}\,dt = 6I$$

$\longrightarrow$ $C_2 : |J_2| \le \frac{R^{-11/12}}{(R-1)^2} 2\pi R \xrightarrow{R\to+\infty} 0$

$\longrightarrow$ $\ominus C_3 : z = t - i0;\ t \in (\varepsilon, R) : dz = dt$

$$J_3 = \ominus \int_\varepsilon^R \frac{(t-i0)^{-11/12}}{(t+1)^2}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to+\infty} \ominus\, e^{-\frac{11}{6}\pi i}\int_0^\infty \frac{t^{-11/12}}{(t+1)^2}\,dt = -6e^{-\frac{11}{6}\pi i} I$$

$\longrightarrow$ $C_4 : |J_4| \le \frac{\varepsilon^{-11/12}}{(1-\varepsilon)^2} 2\pi\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$

- **POROVNÁNÍ** :

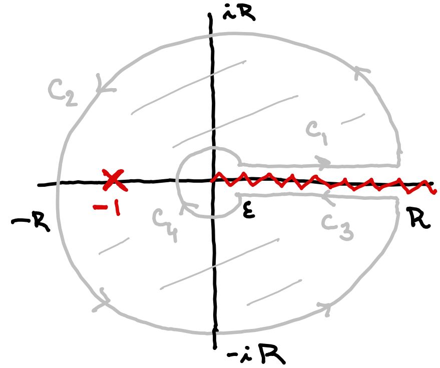
$$\frac{11}{6}\pi i\, e^{-\frac{11}{12}\pi i} = 6I\left(1 - e^{-\frac{11}{6}\pi i}\right) = 6I e^{-\frac{11}{12}\pi i}\underbrace{\left(e^{\frac{11}{12}\pi i} - e^{-\frac{11}{12}\pi i}\right)}_{2i \sin\frac{11}{12}\pi \,=\, 2i\sin\frac{\pi}{12}}$$

$$\boxed{\therefore\ I = \frac{11\pi}{72 \sin\frac{\pi}{12}}} = \frac{11\pi\sqrt{2}}{36(\sqrt{3}-1)} = \frac{11\pi}{72}\sqrt{2}(\sqrt{3}+1)$$

úprava: $\frac{11}{12} = \frac{8+3}{12} = \frac{2}{3} + \frac{1}{4}\ \therefore\ \sin\frac{11}{12}\pi = \cos\frac{2\pi}{3}\sin\frac{\pi}{4} + \sin\frac{2\pi}{3}\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$

Př $\quad I = \int_0^\infty \frac{\ln x}{\sqrt{x}\,(1+x)^2}\,dx \in \mathbb{R}$

$$J = \oint_C \frac{\ln z}{\sqrt{z}\,(1+z)^2}\,dz \quad ; \quad C:$$



- RESIDUOVÁ VĚTA :

$$\longrightarrow \operatorname{Res}_{-1} f(z) \overset{\text{poř. 2}}{=} \lim_{z\to -1} \frac{1}{(2-1)!}\left[(z+1)^2 f(z)\right]^{(2-1)} = \lim_{z\to -1}\left[\frac{\ln z}{\sqrt{z}}\right]' = \left(z^{-\frac{1}{2}}\ln z\right)'\Big|_{-1}$$

$$= -\frac{1}{2} z^{-\frac{3}{2}}\ln z + z^{-\frac{1}{2}}\frac{1}{z}\Big|_{e^{\pi i}} = -\frac{1}{2}\underbrace{e^{-\frac{3}{2}\pi i}}_{i}(\pi i) + \underbrace{e^{-\frac{\pi i}{2}}}_{-i} e^{-\pi i} = \frac{\pi}{2} + i$$

$$\therefore\ J = 2\pi i\left(\frac{\pi}{2} + i\right) = \pi^2 i - 2\pi$$

- PARAMETRIZACE :

$\longrightarrow C_1 : z = t + i0 \ ;\ t \in (\varepsilon, R) : \ dz = dt$

$$J_1 = \int_\varepsilon^R \frac{\ln(t+i0)}{\sqrt{t+i0}\,(1+t)^2}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{\ln t}{\sqrt{t}\,(1+t)^2}\,dt = I$$

$$\longrightarrow C_2 : \ |J_2| \le \frac{\ln R + 2\pi}{\sqrt{R}\,(R-1)^2}\,2\pi R \xrightarrow{R\to\infty} 0$$

$\longrightarrow \ominus C_3 : \ z = t - i0 \ ;\ t \in (\varepsilon, R) \ ; \ dz = dt$

$$J_3 = \ominus\int_\varepsilon^R \frac{\ln(t-i0)}{\sqrt{t-i0}\,(1+t)^2}\,dt = \ominus\int_\varepsilon^R \frac{\ln t + 2\pi i}{(-\sqrt{t})(1+t)^2}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{\ln t + 2\pi i}{\sqrt{t}\,(1+t)^2}\,dt$$

$$= \underbrace{\int_0^\infty \frac{\ln t}{\sqrt{t}\,(1+t)^2}\,dt}_{I} + 2\pi i \underbrace{\int_0^\infty \frac{dt}{\sqrt{t}\,(1+t)^2}}_{I_0 \in \mathbb{R}} = I + 2\pi i\, I_0$$

$$\longrightarrow C_4 : \ |J_4| \le \frac{2\pi - \ln\varepsilon}{\sqrt{\varepsilon}\,(1-\varepsilon)^2}\,2\pi\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$$

- POROVNÁNÍ : $\quad \pi^2 i - 2\pi = 2I + 2\pi i\, I_0$

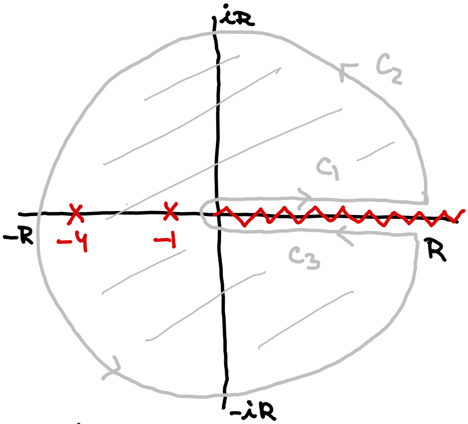
$\boxed{\text{Re :}}$ $\quad \boxed{I = -\pi}$

$\boxed{\text{Im :}}$ $\quad I_0 = \int_0^\infty \frac{dt}{\sqrt{t}\,(1+t)^2} = \frac{\pi}{2}$

[BONUS]

Př: $I = \int_0^\infty \frac{\ln x}{(x+1)^2(x+4)\sqrt{x}}\,dx$

$$J := \oint \frac{\ln z}{(z+1)^2(z+4)\sqrt{z}}\,dz$$



- RESIDUOVÁ VĚTA:

$\longrightarrow \mathrm{Res}_{-4} f(z) \overset{\text{pól 1.}}{=} \frac{\ln z}{(z+1)^2\sqrt{z}}\Big|_{-4} = \frac{\ln 4 + \pi i}{18 i}$

$\longrightarrow \mathrm{Res}_{-1} f(z) \overset{\text{pól 2.}}{=} \left(\frac{\ln z}{(z+4)\sqrt{z}}\right)'\Big|_{-1} = \frac{1}{z(z+4)\sqrt{z}} - \frac{\ln z}{(z+4)^2\sqrt{z}} - \frac{\ln z}{2(z+4)z^{3/2}}\Big|_{-1} =$

$$= -\frac{1}{3i} - \frac{\pi i}{9i} + \frac{\pi i}{6i} = \frac{\pi}{18} - \frac{1}{3i}$$

$\therefore\ J = 2\pi i \sum_{\sigma \in \mathrm{Int}\, C} \mathrm{Res}_\sigma f(z) = 2\pi i \left(\frac{\ln 2}{9i} + \frac{\pi}{18} + \frac{\pi}{18} - \frac{1}{3i}\right) = \frac{2\pi \ln 2}{9} - \frac{2\pi}{3} + \frac{2\pi^2 i}{9}$

- PARAMETRIZACE:

$\longrightarrow C_1:\ z = t + i0;\ t \in (0, R);\ dz = dt$

$$J_1 = \int_0^R \frac{\ln(t+i0)\,dt}{(t+1)^2(t+4)\sqrt{t+i0}} \xrightarrow{R\to\infty} \int_0^\infty \frac{\ln t}{(t+4)^2(t+1)\sqrt{t}}\,dt = I$$

$\longrightarrow C_2:\ |J_2| \le \frac{\ln R + 2\pi}{(R-1)^2(R-4)\sqrt{R}}\, 2\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow \ominus C_3:\ z = t - i0;\ t \in (0, R);\ dz = dt$

$$J_3 = \ominus \int_0^R \frac{\ln(t-i0)\,dt}{(t+1)^2(t+4)\sqrt{t-i0}} \xrightarrow{R\to\infty} \ominus \int_0^\infty \frac{\ln t + 2\pi i}{(t+1)^2(t+4)(-\sqrt{t})}\,dt =$$

$$= \underbrace{\int_0^\infty \frac{\ln t\,dt}{(t+1)^2(t+4)\sqrt{t}}}_{I} + 2\pi i \underbrace{\int_0^\infty \frac{dt}{(t+1)^2(t+4)\sqrt{t}}\,dt}_{I_0} = I + 2\pi i I_0$$

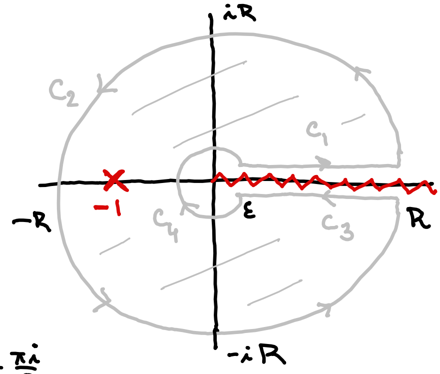
- POROVNÁNÍ: $\frac{2\pi \ln 2}{9} - \frac{2\pi}{3} + \frac{2\pi^2 i}{9} = 2I + 2\pi i I_0$

$\boxed{\text{Re:}}$ $\boxed{I = \frac{\pi}{9}\ln 2 - \frac{\pi}{3}}$

$\boxed{\text{Im:}}$ $I_0 = \int_0^\infty \frac{dx}{(x+1)^2(x+4)\sqrt{x}} = \frac{\pi}{9}$

[BONUS]

Př $I = \int_0^\infty \frac{\ln x}{\sqrt[3]{x}\,(1+x)}\,dx \in \mathbb{R}$



$$J = \oint_C \frac{\ln z}{\sqrt[3]{z}\,(1+z)}\,dz$$

- **RESIDUOVÁ VĚTA**:

$\longrightarrow$ $\operatorname{Res}_{-1} f(z) = \frac{\ln z}{\sqrt[3]{z}}\Big|_{e^{\pi i}} = \frac{\pi i}{e^{\frac{\pi i}{3}}} = \pi i\, e^{-\frac{\pi i}{3}}$

$\therefore\ J = 2\pi i \left(\pi i\, e^{-\frac{\pi i}{3}}\right) = -2\pi^2 e^{-\frac{\pi i}{3}}$

- **PARAMETRIZACE**:

$\longrightarrow$ $C_1: z = t + i0;\ t \in (\varepsilon, R);\ dz = dt$

$$J_1 = \int_\varepsilon^R \frac{\ln(t+i0)}{\sqrt[3]{t+i0}\,(1+t)}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{\ln t}{\sqrt[3]{t}\,(1+t)}\,dt = I$$

$\longrightarrow$ $C_2: |J_2| \le \frac{\ln R + 2\pi}{\sqrt[3]{R}\,(R-1)}\, 2\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow$ $\ominus\, C_3: z = t - i0;\ t \in (\varepsilon, R);\ dz = dt$

$$J_3 = \ominus \int_\varepsilon^R \frac{\ln(t e^{2\pi i})\,dt}{\sqrt[3]{t e^{2\pi i}}\,(1+t)} \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \ominus \int_0^\infty \frac{\ln t + 2\pi i}{e^{\frac{2\pi i}{3}}\sqrt[3]{t}\,(1+t)}\,dt =$$

$$= -e^{-\frac{2\pi i}{3}} \underbrace{\int_0^\infty \frac{\ln t\,dt}{\sqrt[3]{t}\,(1+t)}}_{I} - 2\pi i\, e^{-\frac{2\pi i}{3}} \underbrace{\int_0^\infty \frac{dt}{\sqrt[3]{t}\,(1+t)}}_{I_0 \in \mathbb{R}} = -e^{-\frac{2\pi i}{3}} I - 2\pi i\, e^{-\frac{2\pi i}{3}} I_0$$

$\longrightarrow$ $C_4: |J_4| \le \frac{2\pi - \ln\varepsilon}{\sqrt[3]{\varepsilon}\,(1-\varepsilon)}\, 2\pi\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$

- **POROVNÁNÍ**:

$$-2\pi^2 e^{-\frac{\pi i}{3}} = I - e^{-\frac{2\pi i}{3}} I - 2\pi i\, e^{-\frac{2\pi i}{3}} I_0 = e^{-\frac{\pi i}{3}} \underbrace{\left(e^{\frac{\pi i}{3}} - e^{-\frac{\pi i}{3}}\right)}_{2i \sin\frac{\pi}{3}} I - 2\pi i\, e^{-\frac{2\pi i}{3}} I_0$$

$\cdot e^{\frac{2\pi i}{3}}$ $\quad -2\pi^2 e^{\frac{\pi i}{3}} = i\sqrt{3}\, e^{\frac{\pi i}{3}} I - 2\pi i\, I_0$

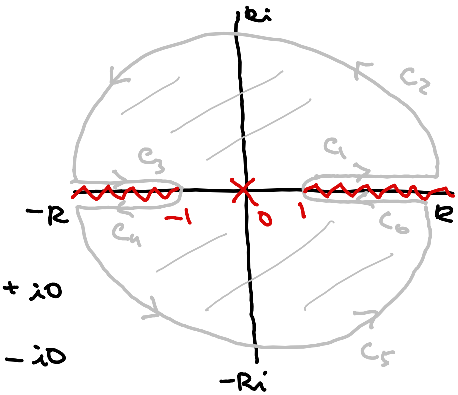
$\boxed{\text{Re:}}$ $\quad -2\pi^2 \cos\frac{\pi}{3} = -\sqrt{3}\sin\frac{\pi}{3}\, I \quad \therefore \boxed{I = \frac{2\pi^2}{3}} = \int_0^\infty \frac{\ln x}{\sqrt[3]{x}\,(1+x)}\,dx$

$\boxed{\text{Im:}}$ $\quad -2\pi^2 \sin\frac{\pi}{3} = \sqrt{3}\cos\frac{\pi}{3}\, I - 2\pi I_0 \quad \therefore\ I_0 = \int_0^\infty \frac{dt}{\sqrt[3]{t}\,(1+t)} = \frac{2\pi}{\sqrt{3}}$

[BONUS]

(Př) $I = \int_1^\infty \frac{dx}{x\sqrt{x^2-1}}$

$J := \oint_C \frac{dz}{z\sqrt{z^2-1}}$



řez: $z^2-1 = t \Rightarrow z = \pm\sqrt{t+1} \in (-\infty,-1)\cup(1,\infty)$

skok: $z^2-1\big|_{t+i0} = z^2-1\big|_t + (z^2-1)'\big|_t\, i0 = t^2-1+i0$

podobně $z^2-1\big|_{-t+i0} = t^2-1+(-2t)i0 = t^2-1-i0$

• RESIDUOVÁ VĚTA:

$\longrightarrow \mathrm{Res}_0 f(z) = \frac{1}{\sqrt{z^2-1}}\Big|_0 = \frac{1}{\sqrt{e^{\pi i}}} = e^{-\frac{\pi i}{2}} = -i$

$\therefore\ J = 2\pi i \sum_{\sigma\in \mathrm{Int}\,C} \mathrm{Res}_\sigma f(z) = 2\pi i(-i) = 2\pi$

• PARAMETRIZACE:

$\longrightarrow C_1:\ z = t+i0;\ t\in(1,R);\ dz = dt$

$J_1 = \int_0^R \frac{dt}{t\sqrt{t^2-1+i0}} \xrightarrow{R\to\infty} \int_0^\infty \frac{dt}{t\sqrt{t^2-1}} = I$

$\longrightarrow C_2:\ |J_2| \le \frac{1}{R\sqrt{R^2-1}}\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow \ominus C_3:\ z = -t+i0;\ t\in(1,R);\ dz = -dt$

$J_3 = \ominus\int_0^R \frac{-dt}{-t\underbrace{\sqrt{t^2-1-i0}}_{-\sqrt{t^2-1}}} \xrightarrow{R\to\infty} \int_0^\infty \frac{dt}{t\sqrt{t^2-1}} = I$

$\longrightarrow C_4:\ z = -t-i0;\ t\in(1,R);\ dz = -dt$

$J_4 = \int_0^R \frac{-dt}{-t\sqrt{t^2-1+i0}} \xrightarrow{R\to\infty} I$

$\longrightarrow C_5:\ |J_5| \le \frac{1}{R\sqrt{R^2-1}}\pi R \xrightarrow{R\to\infty} 0$

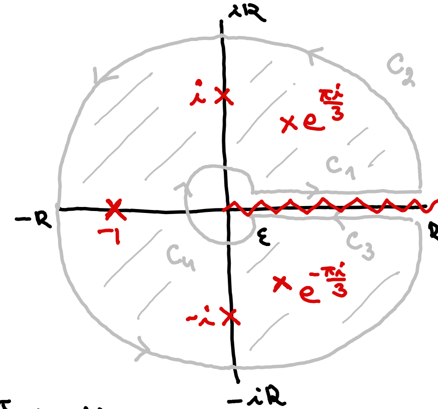
$\longrightarrow \ominus C_6:\ z = t-i0;\ t\in(1,R)$

$J_6 = \ominus\int_1^R \frac{dt}{t\sqrt{t^2-1-i0}} \xrightarrow{R\to\infty} I$

• POROVNÁNÍ: $2\pi = 4I$ $\quad\boxed{\therefore\ I = \frac{\pi}{2}}$

(Př) $I = \int_0^\infty \frac{dx}{(1+x^2)(1+x^3)} \in \mathbb{R}$

$$J = \oint_C \frac{\ln z \, dz}{(1+z^2)(1+z^3)}$$



• **RESIDUOVÁ VĚTA**

$\longrightarrow$ $\operatorname{Res}_{-1} f = \frac{\ln z}{(1+z^2)\,3z^2}\Big|_{e^{\pi i}} = \frac{\ln(e^{\pi i})}{6} = \frac{\pi i}{6}$

$\longrightarrow$ $\operatorname{Res}_{i} f = \frac{\ln z}{2z\,(1+z^3)}\Big|_{i} = \frac{\ln(e^{\frac{\pi}{2}i})}{2i(1-i)} = \frac{\pi}{12}(1+i)$

$\longrightarrow$ $\operatorname{Res}_{-i} f = \frac{\ln z}{2z\,(1+z^3)}\Big|_{-i} = \frac{\ln(e^{\frac{5\pi}{3}i})}{-2i(1+i)} = -\frac{5\pi}{12}(1-i)$

$\longrightarrow$ $\operatorname{Res}_{e^{\frac{\pi i}{3}}} f = \frac{\ln z}{(1+z^2)\,3z^2}\cdot\frac{z}{z}\Big|_{e^{\frac{\pi}{3}i}} = -\frac{1}{3}\,\frac{e^{\frac{\pi}{3}i}}{1+e^{\frac{2\pi}{3}i}}\ln(e^{\frac{\pi}{3}i}) = -\frac{1}{3}\,\frac{\frac{\pi}{3}i}{\underbrace{e^{-\frac{\pi}{3}i}+e^{\frac{\pi}{3}i}}_{2\cos\frac{\pi}{3}}} = -\frac{\pi}{9}i$

$\longrightarrow$ $\operatorname{Res}_{e^{\frac{\pi i}{3}}} f = \frac{\ln z}{(1+z^2)\,3z^2}\cdot\frac{z}{z}\Big|_{e^{-\frac{\pi}{3}i}} = -\frac{1}{3}\,\frac{e^{-\frac{\pi}{3}i}}{1+e^{-\frac{2\pi}{3}i}}\underbrace{\ln(e^{-\frac{\pi}{3}i})}_{\frac{5\pi}{3}i} = -\frac{1}{3}\,\frac{\frac{5\pi}{3}i}{2\cos\frac{\pi}{3}} = -\frac{5\pi}{9}i$

$$\therefore J = 2\pi i \sum_{\sigma \in \operatorname{Int} C} \operatorname{Res}_\sigma f = 2\pi i\left(\frac{\pi i}{6} + \frac{\pi}{12} + \frac{\pi}{12}i - \frac{5\pi}{12} + \frac{5\pi}{12}i - \frac{\pi}{9}i - \frac{5\pi}{9}i\right) = -\frac{2\pi^2 i}{3}$$

• **PARAMETRIZACE**

$\longrightarrow$ $C_1 : z = t + i0;\ t \in (\varepsilon, R);\ dz = dt$

$$J_1 = \int_\varepsilon^R \frac{\ln(t+i0)}{(1+t^2)(1+t^3)}\,dt \xrightarrow[\varepsilon\to0^+]{R\to\infty} \int_0^\infty \frac{\ln t \, dt}{(1+t^2)(1+t^3)} = I_0 \in \mathbb{R}$$

$\longrightarrow$ $C_2 : \ |J_2| \le \frac{\ln R + 2\pi}{(R^2-1)(R^3-1)}\,2\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow$ $\ominus\, C_3 : \ z = t - i0;\ t \in (\varepsilon, R);\ dz = dt$

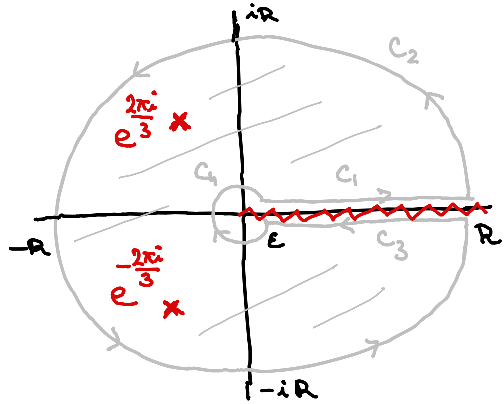
$$J_3 = \ominus \int_\varepsilon^R \frac{\ln(t-i0)}{(1+t^2)(1+t^3)}\,dt \xrightarrow[\varepsilon\to0^+]{R\to\infty} -\int_0^\infty \frac{\ln t + 2\pi i}{(1+t^2)(1+t^3)}\,dt = -I_0 - 2\pi i\, I$$

$\longrightarrow$ $C_4 : \ |J_4| \le \frac{2\pi - \ln \varepsilon}{(1-\varepsilon^2)(1-\varepsilon^3)}\,2\pi\varepsilon \xrightarrow{\varepsilon\to0^+} 0$

• **POROVNÁNÍ** : $-\frac{2\pi^2 i}{3} = I_0 - I_0 - 2\pi i I_0 \quad \therefore \boxed{I = \frac{\pi}{3}}$

(Př) $I = \int_0^\infty \frac{dx}{(1+x+x^2)^2} \in \mathbb{R}$

$J := \oint_C \frac{\ln z}{(1+z+z^2)^2}\,dz$; C:



singularity: $1+z+z^2 = 0$

tj. $\frac{1-z^3}{1-z} = 0 \Rightarrow \sigma_\pm = e^{\pm\frac{2\pi i}{3}} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

- RESIDUOVÁ VĚTA :

$\longrightarrow$ $\operatorname{Res}_{\sigma_+} f(z) \overset{\text{pól 2.}}{=} \lim_{z\to\sigma_+} \frac{1}{(2-1)!}\left[(z-\sigma_+)^2 f(z)\right]^{(2-1)} = \left(\frac{\ln z}{(z-\sigma_-)^2}\right)'\Big|_{\sigma_+} =$

$= \frac{\frac{1}{z}(z-\sigma_-)^2 - \ln z\; 2(z-\sigma_-)}{(z-\sigma_-)^4}\Big|_{\sigma_-} = \frac{\frac{1}{\sigma_-}(\sigma_+-\sigma_-) - 2\ln\sigma_+}{(\sigma_+-\sigma_-)^3} =$

$= \frac{e^{\frac{2\pi i}{3}}(\sqrt{3}i) - 2\left(\frac{2\pi i}{3}\right)}{(\sqrt{3}i)^3} = -\frac{1}{3}e^{\frac{2\pi i}{3}} + \frac{4\pi}{9\sqrt{3}}$

$\longrightarrow$ $\operatorname{Res}_{\sigma_-} f(z) \overset{\dots}{=} \frac{\frac{1}{\sigma_+}(\sigma_--\sigma_+) - 2\ln\sigma_-}{(\sigma_--\sigma_+)^3} = \frac{e^{-\frac{2\pi i}{3}}(-\sqrt{3}i) - 2\left(\frac{4\pi i}{3}\right)}{(-\sqrt{3}i)^3} = -\frac{1}{3}e^{-\frac{2\pi i}{3}} - \frac{8\pi}{9\sqrt{3}}$

$\therefore\; J = 2\pi i \sum_{\sigma\in \mathrm{Int}\,C} \operatorname{Res}_\sigma f(z) = 2\pi i\left[-\frac{1}{3}\underbrace{\left(e^{\frac{2\pi i}{3}} + e^{-\frac{2\pi i}{3}}\right)}_{2\cos\frac{2\pi}{3}} - \frac{4\pi}{9\sqrt{3}}\right] = \frac{2\pi i}{3} - \frac{8\pi^2 i}{9\sqrt{3}}$

- PARAMETRIZACE :

$\longrightarrow$ C_1: $z = t + i0$; $t \in (\varepsilon, R)$; $dz = dt$

$J_1 = \int_\varepsilon^R \frac{\ln(t+i0)}{(1+t+t^2)^2}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{\ln t}{(1+t+t^2)^2}\,dt = I$

$\longrightarrow$ C_2 : $|J_2| \le \frac{\ln R + 2\pi}{(R^2-R-1)^2}\, 2\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow$ $\ominus\, C_3$: $z = t - i0$; $t \in (\varepsilon, R)$; $dz = dt$

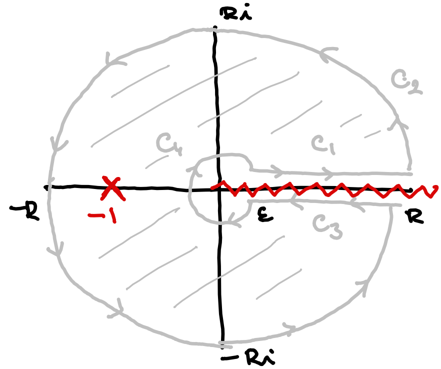
$J_3 = \ominus \int_\varepsilon^R \frac{\ln(t-i0)}{(1+t+t^2)^2}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} -\int_0^\infty \frac{\ln t + 2\pi i}{(1+t+t^2)^2}\,dt = -\int_0^\infty \frac{\ln t\, dt}{(1+t+t^2)^2} - 2\pi i I$

$\longrightarrow$ C_4 : $|J_4| \le \frac{2\pi - \ln\varepsilon}{(1-\varepsilon-\varepsilon^2)}\, 2\pi\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$

- POROVNÁNÍ : $\frac{2\pi i}{3} - \frac{8\pi^2 i}{9\sqrt{3}} = -2\pi i I$ $\;\therefore\;$ $\boxed{I = \frac{4\pi}{9\sqrt{3}} - \frac{1}{3}}$

Př $I = \int_0^\infty \frac{\ln x}{(1+x)^3}\,dx$

$$J = \oint \frac{\ln^2 z}{(1+z)^3}\,dz\ ; \quad C:$$



• **RESIDUOVÁ VĚTA**:

$$\to \operatorname{Res}_{-1} f(z) \overset{\text{pól 3.}}{=} \lim_{z\to -1} \frac{1}{(3-1)!}\left[(z+1)^3 f(z)\right]^{(3-1)} =$$

$$= \frac{1}{2!}\left(\ln^2 z\right)''\Big|_{-1} = \frac{1}{\cancel{2}}\cancel{2}\left(\ln z \cdot \frac{1}{z}\right)' = \frac{1}{z^2} - \frac{1}{z^2}\ln z\Big|_{-1} = 1 - \pi i$$

$$\therefore\ J = 2\pi i \sum_{\sigma \in \operatorname{Int} C} \operatorname{Res}_\sigma f(z) = 2\pi i\,(1-\pi i) = 2\pi i + 2\pi^2$$

• **PARAMETRIZACE**:

$\to C_1:\ z = t + i0;\ t \in (\varepsilon, R);\ dz = dt$

$$J_1 = \int_\varepsilon^R \frac{\ln^2(t+i0)}{(1+t)^3}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{\ln^2 t}{(1+t)^3}\,dt \in \mathbb{R}$$

$$\to C_2:\ |J_2| \leq \frac{(\ln R + 2\pi)^2}{(R-1)^3}\,2\pi R \xrightarrow{R\to\infty} 0$$

$\to \ominus C_3:\ z = t - i0;\ t \in (\varepsilon, R);\ dz = dt$

$$J_3 = (-)\int_\varepsilon^R \frac{\ln^2(t-i0)}{(1+t)^3}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} -\int_0^\infty \frac{(\ln t + 2\pi i)^2}{(1+t)^3}\,dt =$$

$$= -\int_0^\infty \frac{\ln^2 t}{(1+t)^3}\,dt - 4\pi i \underbrace{\int_0^\infty \frac{\ln t}{(1+t)^3}\,dt}_{I} + 4\pi^2 \underbrace{\int_0^\infty \frac{dt}{(1+t)^3}}_{I_0 \in \mathbb{R}}$$

$$\to C_4:\ |J_4| \leq \frac{(2\pi - \ln\varepsilon)^2}{(1-\varepsilon)^3}\,2\pi\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$$

• **POROVNÁNÍ**:

$$2\pi i + 2\pi^2 = -4\pi i\, I + 4\pi^2 I_0$$

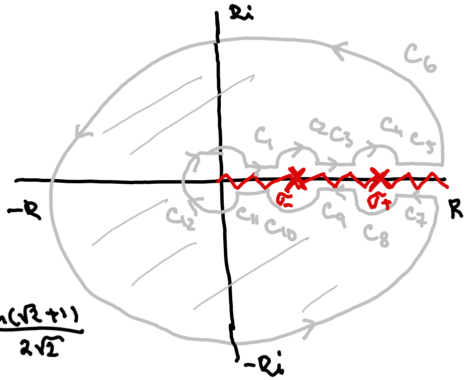
$\boxed{\text{Im:}}$ $\boxed{I = -\frac{1}{2}}$

$\boxed{\text{Re:}}$ $I_0 = \int_0^\infty \frac{dt}{(1+t)^3} = \frac{1}{2}$

[BONUS, leč triviální]

Př: $I = \int_0^\infty \frac{dx}{x^2-6x+1} \in \mathbb{R}$

$$J = \oint_C \frac{\ln z}{z^2-6z+1}\, dz$$



singularity: $\sigma_\pm = 3 \pm 2\sqrt{2} = (\sqrt{2} \pm 1)^2$

• RESIDUOVÁ VĚTA

→ $\mathrm{Res}_{\sigma_+ + i0} f = \frac{\ln z}{2z-6}\Big|_{\sigma_+ + i0} = \frac{2\ln(\sqrt{2}+1)}{4\sqrt{2}} = \frac{\ln(\sqrt{2}+1)}{2\sqrt{2}}$

→ $\mathrm{Res}_{\sigma_- + i0} f = \frac{\ln z}{2z-6}\Big|_{\sigma_- + i0} = \frac{2\ln(\sqrt{2}-1)}{-4\sqrt{2}} = \frac{\ln(\sqrt{2}+1)}{2\sqrt{2}}$

→ $\mathrm{Res}_{\sigma_+ - i0} f = \frac{\ln(\sigma_+ - i0)}{2\sigma_+ - 6} = \frac{2\ln(\sqrt{2}+1)+2\pi i}{4\sqrt{2}} = \frac{\ln(\sqrt{2}+1)}{2\sqrt{2}} + \frac{\pi i}{2\sqrt{2}}$

→ $\mathrm{Res}_{\sigma_- - i0} f = \frac{\ln(\sigma_- - i0)}{2\sigma_- - 6} = \frac{2\ln(\sqrt{2}-1)+2\pi i}{-4\sqrt{2}} = \frac{\ln(\sqrt{2}+1)}{2\sqrt{2}} - \frac{\pi i}{2\sqrt{2}}$

ale $J = 0$ [CAUCHY]

• PARAMETRIZACE :

→ $C_1 + C_3 + C_5$: $J_1 + J_3 + J_5 \xrightarrow[\varepsilon \to 0^+]{R\to\infty} \int_0^\infty \frac{\ln t}{t^2-6t+1}\, dt =: I_0 \in \mathbb{R}$

→ $C_7 + C_9 + C_{11}$: $J_7 + J_9 + J_{11} \xrightarrow[\varepsilon \to 0^+]{R\to\infty} \ominus \int_0^\infty \frac{\ln t + 2\pi i}{t^2-6t+1}\, dt = -I_0 - 2\pi i I$

→ C_6 : $|J_6| \le \frac{\ln R + 2\pi}{R^2 - 6R - 1}\, 2\pi R \xrightarrow{R\to\infty} 0$

→ C_{12} : $|J_{12}| \le \frac{2\pi - \ln\varepsilon}{1 - 6\varepsilon - \varepsilon^2}\, 2\pi\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$

→ $C_2 + C_4 + C_8 + C_{10}$: $J_2 + J_4 + J_8 + J_{10} \xrightarrow{\varepsilon\to 0^+} -\pi i \sum \mathrm{Res} =$

$= -\pi i \left(4 \frac{\ln(\sqrt{2}+1)}{2\sqrt{2}} \right) = -\pi i \sqrt{2} \ln(\sqrt{2}+1)$

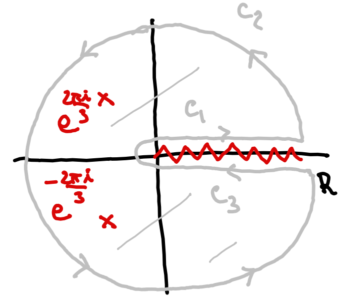
• POROVNÁNÍ : $0 = I_0 - I_0 - 2\pi i I - \pi i \sqrt{2} \ln(\sqrt{2}+1)$

$$\therefore \quad \boxed{I = -\frac{\ln(\sqrt{2}+1)}{\sqrt{2}}}$$

(Př) $I = \int_0^\infty \frac{\ln x}{1+x+x^2}\,dx \in \mathbb{R}$

$\hookrightarrow J = \oint_C \frac{\ln^2 z}{1+z+z^2}\,dz$ fun

C:



pól: $1+z+z^2 = \frac{1-z^3}{1-z} \Rightarrow \sigma \in \{e^{\frac{2\pi}{3}i}, e^{-\frac{2\pi}{3}i}\}$

$e^{\frac{2\pi}{3}i} = -\frac{1}{2}+\frac{i\sqrt{3}}{2}$, $\quad e^{-\frac{2\pi}{3}i} = -\frac{1}{2}-\frac{i\sqrt{3}}{2}$

- RESIDUOVÁ VĚTA:

$\longrightarrow \operatorname{Res}_{e^{\frac{2\pi i}{3}}} f(z) = \frac{\ln^2 z}{2z+1}\Big|_{e^{\frac{2\pi}{3}i}} = \frac{\left(\frac{2\pi}{3}i\right)^2}{i\sqrt{3}} = \frac{4\pi^2 i}{9\sqrt{3}}$

$\longrightarrow \operatorname{Res}_{e^{-\frac{2\pi i}{3}}} f(z) = \frac{\ln^2 z}{2z+1}\Big|_{e^{-\frac{2\pi}{3}i}} = \frac{\left(\frac{4\pi}{3}i\right)^2}{-i\sqrt{3}} = -\frac{16\pi^2 i}{9\sqrt{3}}$

$\therefore\ J = 2\pi i \sum_{\sigma \in \operatorname{Int} C} \operatorname{Res}_\sigma f(z) = 2\pi i\left(\frac{4\pi^2 i}{9\sqrt{3}} - \frac{16\pi^2 i}{9\sqrt{3}}\right) = \frac{12\pi^3}{9\sqrt{3}} = \frac{4\pi^3}{3\sqrt{3}}$

- PARAMETRIZACE:

$\int_0^\infty \frac{\ln x}{(1+x)^3}\,dx$

$\int_0^\infty \frac{dx}{x^2-6x+1}$

$\int_0^\infty \frac{\ln x}{1+x+x^2}$

$\int_0^\infty \frac{\ln^2 x}{(x+1)^2}\,dx$

$\longrightarrow c_1:\ z = t+i0;\ t \in (0,R);\ dz = dt;$

$J_1 = \int_0^R \frac{\ln^2(t+i0)}{1+t+t^2}\,dt \xrightarrow{R\to\infty} \int_0^\infty \frac{\ln^2 t}{1+t+t^2}\,dt$

$\longrightarrow c_2:\ |J_2| \le \frac{(\ln R + 2\pi)^2}{R^2-R-1}\,2\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow \ominus c_3:\ z = t - i0;\ t \in (0,R);\ dz = dt$

$J_3 = \ominus \int_0^R \frac{\ln^2(t-i0)}{1+t+t^2}\,dt = -\int_0^R \frac{(\ln t + 2\pi i)^2}{1+t+t^2}\,dt \xrightarrow{R\to\infty}$

$\xrightarrow{R\to\infty} -\int_0^\infty \frac{\ln^2 t}{1+t+t^2}\,dt - 4\pi i \underbrace{\int_0^\infty \frac{\ln t}{1+t+t^2}\,dt}_{I} + 4\pi^2 \underbrace{\int_0^\infty \frac{dt}{1+t+t^2}}_{I_0 \in \mathbb{R}}$

- POROVNÁNÍ:

$\frac{4\pi^3}{3\sqrt{3}} = -4\pi i\, I + 4\pi^2 I_0 \quad \therefore \boxed{I = 0}$ $\left[\&\ \int_0^\infty \frac{dt}{1+t+t^2} = \frac{\pi}{3\sqrt{3}}\right]$ BONUS

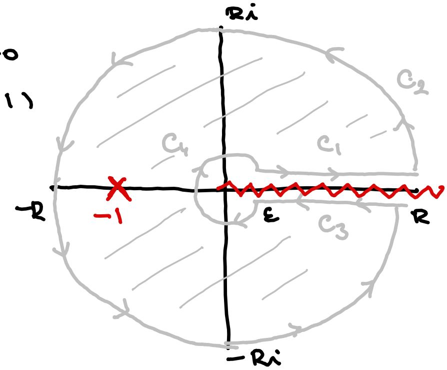
Jinak: $I = \int_0^\infty \frac{\ln x}{1+x+x^2}\,dx = \left|\begin{matrix} x = \frac{1}{y} & ;\ 0 \to \infty \\ dx = -\frac{1}{y^2} & ;\ \infty \to 0 \end{matrix}\right| = -\int_0^\infty \frac{-\ln y}{1+\frac{1}{y}+\frac{1}{y^2}}\left(-\frac{1}{y^2}\right)dy = -I$

$\Rightarrow 2I = 0$ a tedy $I = 0$

Př

$$I = \int_0^\infty \frac{\ln^2 x}{(x+1)^2}\,dx = \frac{\partial^2}{\partial\alpha^2} \underbrace{\int_0^\infty \frac{x^\alpha}{(x+1)^2}\,dx}_{I(\alpha)\in\mathbb{R}\,;\ \alpha\in(-1,1)} \Big|_{\alpha=0}$$

$$J = \oint_C \frac{z^\alpha}{(z+1)^2}\,dz \ ; \ C$$



- **RESIDUOVÁ VĚTA**

$$\longrightarrow \operatorname{Res}_{-1} f \overset{\text{pól 2.}}{=} \lim_{z\to -1} \frac{1}{(2-1)!}\left[(z+1)^2 f(z)\right]^{(2-1)}$$

$$= (z^\alpha)'\big|_{-1} = \alpha z^{\alpha-1}\big|_{e^{\pi i}} = -\alpha e^{\pi\alpha i}$$

$$\therefore\ J = 2\pi i \sum_{\sigma\in \operatorname{Int} C} \operatorname{Res}_\sigma f(z) = -2\pi i\alpha e^{\pi\alpha i}$$

- **PARAMETRIZACE :**

$\longrightarrow C_1:\ z = t + i0\ ;\ t\in(\varepsilon, R)\ ;\ dz = dt$

$$J_1 = \int_\varepsilon^R \frac{(t+i0)^\alpha}{(t+1)^2}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{t^\alpha}{(t+1)^2}\,dt = I(\alpha)$$

$$\longrightarrow C_2:\ |J_2| \le \frac{R^\alpha}{(R-1)^2}\,2\pi R \xrightarrow{R\to\infty} 0\ ;\ \alpha<1$$

$\longrightarrow \ominus C_3:\ z = t - i0\ ;\ t\in(\varepsilon,R)\ ;\ dz = dt$

$$J_3 = \ominus \int_\varepsilon^R \frac{(t-i0)^\alpha}{(t+1)^2}\,dt \xrightarrow[R\to\infty]{\varepsilon\to 0^+} \ominus\, e^{2\pi i\alpha}\int_0^\infty \frac{t^\alpha}{(t+1)^2}\,dt = -e^{2\pi i\alpha} I(\alpha)$$

$$\longrightarrow C_4:\ |J_4| \le \frac{\varepsilon^\alpha}{(1-\varepsilon)^2}\,2\pi\varepsilon \xrightarrow{\varepsilon\to 0^+} 0\ ;\ \alpha>-1$$

- **POROVNÁNÍ** :

$$-2\pi i\alpha e^{\pi\alpha i} = I(\alpha)\left(1 - e^{2\pi i\alpha}\right) = I(\alpha)\, e^{\pi\alpha i}\underbrace{\left(e^{-\pi\alpha i} - e^{\pi\alpha i}\right)}_{-2i\sin\pi\alpha}$$

$$\therefore\ I(\alpha) = \frac{\pi\alpha}{\sin\pi\alpha} = \frac{\pi\alpha}{\pi\alpha - \frac{1}{3!}(\pi\alpha)^3 + O(\alpha^5)} = \left(1 - \frac{\pi^2\alpha^2}{6} + O(\alpha^4)\right)^{-1} =$$

$$= 1 + \frac{\pi^2\alpha^2}{6} + O(\alpha^4) \qquad \therefore \quad \boxed{I = \frac{\partial^2 I(\alpha)}{\partial\alpha^2}\Big|_0 = \frac{\pi^2}{3}}$$

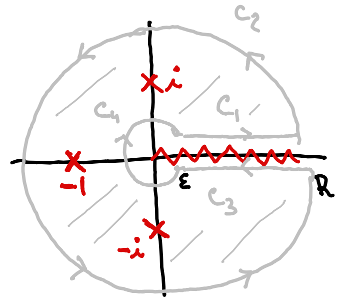
BONUS :

$$\frac{\partial I}{\partial\alpha}\Big|_0 = \int_0^\infty \frac{\ln x}{(x+1)^2}\,dx = 0$$

ale to je triviální :

$$\left[\int_0^\infty \frac{\ln x}{(1+x)^2}\,dx = \left|x\to\tfrac{1}{x}\right| = \int_0^\infty \frac{\ln\frac{1}{x}}{\left(\frac{1}{x}+1\right)^2}\,\frac{1}{x^2}\,dx = -\int_0^\infty \frac{\ln x}{(1+x)^2}\,dx\ \therefore\ \int_0^\infty \frac{\ln x}{(1+x)^2}\,dx = 0\right]$$

(Př) $I = \int_0^\infty \frac{\ln^3 x}{(1+x^2)(1+x)^2}\,dx = \frac{\partial^3 I(\alpha)}{\partial \alpha^3}\Big|_{\alpha=0}$



<u>trik</u>: $I(\alpha) := \int_0^\infty \frac{x^\alpha}{(1+x^2)(1+x)^2}\,dx \in \mathbb{R};\ \alpha \in (-1,3)$

$J := \oint_C \frac{\cancel{\tan}\; z^\alpha}{(1+z^2)(1+z)}\,dz \quad ; \quad C:$

- <u>RESIDUOVÁ VĚTA</u>:

$\longrightarrow \operatorname{Res}_{-1} f \overset{\text{pól 2.}}{=} \left(\frac{z^\alpha}{1+z^2}\right)'\Big|_{-1} = \frac{\alpha z^{\alpha-1}}{1+z^2} - \frac{2z^{\alpha+1}}{(1+z^2)^2}\Big|_{e^{\pi i}} = -\frac{\alpha e^{\pi\alpha i}}{2} + \frac{2e^{\pi\alpha i}}{4}$

$\longrightarrow \operatorname{Res}_{i} f = \frac{z^\alpha}{2z(1+z)^2}\Big|_{i} = \frac{e^{\frac{\pi}{2}\alpha i}}{2i(1+i)^2} = -\frac{1}{4}e^{\frac{\pi}{2}\alpha i}$

$\longrightarrow \operatorname{Res}_{-i} f = \frac{z^\alpha}{2z(1+z)}\Big|_{-i} = \frac{e^{\frac{3}{2}\pi\alpha i}}{-2i(1-i)^2} = -\frac{1}{4}e^{\frac{3}{2}\pi\alpha i}$

$\therefore\ J = 2\pi i \sum_{\sigma \in \mathrm{Int}\,C} \operatorname{Res}_\sigma f = -\alpha\pi i\, e^{\pi\alpha i} + \pi i\, e^{\pi\alpha i} - \frac{\pi i}{2} e^{\frac{\pi}{2}\alpha i} - \frac{\pi i}{2} e^{\frac{3}{2}\pi\alpha i}$

- <u>PARAMETRIZACE</u>

$\longrightarrow C_1: z = t + i0;\ t \in (\varepsilon, R);\ dz = dt$

$J_1 = \int_\varepsilon^R \frac{(t+i0)^\alpha}{(1+t^2)(1+t)^2}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{t^\alpha}{(1+t^2)(1+t)}\,dt = I(\alpha)$

$\longrightarrow C_2: |J_2| \le \frac{R^\alpha}{(R^2-1)(R-1)^2}\, 2\pi R \xrightarrow{R\to\infty} 0$ pro $\alpha < 3$

$\longrightarrow \ominus\, C_3: z = t - i0;\ t \in (\varepsilon, R);\ dz = dt$

$J_3 = \ominus \int_\varepsilon^R \frac{(t-i0)^\alpha}{(1+t^2)(1+t)^2}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} -e^{2\pi\alpha i}\int_0^\infty \frac{t^\alpha\,dt}{(1+t^2)(1+t)^2} = -e^{2\pi\alpha i} I(\alpha)$

$\longrightarrow C_4: |J_4| \le \frac{\varepsilon^\alpha}{(1-\varepsilon^2)(1-\varepsilon)^2}\, 2\pi\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$ pro $\alpha > -1$

- <u>POROVNÁNÍ</u>: $-\alpha\pi i\, e^{\pi\alpha i} + \pi i\, e^{\pi\alpha i} - \frac{\pi i}{2} e^{\frac{\pi}{2}\alpha i} - \frac{\pi i}{2} e^{\frac{3}{2}\pi\alpha i} = I(\alpha)\left(1 - e^{2\pi\alpha i}\right)$

- $e^{-\pi\alpha i}$: $-\alpha\pi i + \pi i - \frac{\pi i}{2}\Big(\underbrace{e^{\frac{\pi}{2}\alpha i} + e^{-\frac{\pi}{2}\alpha i}}_{2\cos\frac{\pi}{2}\alpha}\Big) = -I(\alpha)\Big(\underbrace{e^{\pi\alpha i} - e^{-\pi\alpha i}}_{2i\sin\pi\alpha}\Big)$

$\therefore\ I(\alpha) = \frac{\pi}{2}\,\frac{\alpha - 1 + \cos\frac{\pi}{2}\alpha}{\sin\pi\alpha} = \frac{\pi}{2}\,\frac{\alpha - 1 + 1 - \frac{\pi^2\alpha^2}{8} + \frac{1}{4!}\left(\frac{\pi}{2}\alpha\right)^4 + O(\alpha^5)}{\pi\alpha - \frac{\pi^3\alpha^3}{6} + O(\alpha^5)} =$

$= \frac{1}{2}\left(1 - \frac{\pi^2}{8}\alpha + \frac{\pi^4}{4!\,2^4}\alpha^3\right)\left(1 + \frac{\pi^2\alpha^2}{6}\right) + O(\alpha^4) = \frac{1}{2}\left[1 - \frac{\pi^2}{8}\alpha + \frac{\pi^2\alpha^2}{6} + \left(\frac{\pi^4}{4!\,2^4} - \frac{\pi^4}{6\cdot 8}\right)\alpha^3 + O(\alpha^4)\right]$

$\therefore\ \boxed{I = \frac{3!\,\pi^4}{2}\left(\frac{1}{4!\,2^4} - \frac{1}{6\cdot 8}\right) = -\frac{7\pi^4}{128}}$

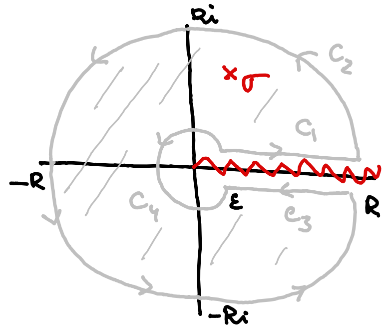
Př $\quad I = \int_0^1 \frac{\ln^2 x}{1-x+x^2}\,dx \in \mathbb{R} \overset{x\mapsto\frac1x}{\Longrightarrow} I = \int_1^\infty \frac{\ln^2\left(\frac1x\right)}{1-\frac1x+\frac1{x^2}}\,\frac{1}{x^2}\,dx$

$$\therefore\ I = \frac{I+I}{2} = \frac12\int_0^\infty \frac{\ln^2 x}{1-x+x^2}\,dx$$

Trik $\quad I(\alpha) = \int_0^\infty \frac{x^\alpha}{1-x+x^2}\,dx \in \mathbb{R}$ pro $\alpha\in(-1,1) \Rightarrow I = \frac12\frac{\partial^2 I(\alpha)}{\partial\alpha^2}\Big|_{\alpha=0}$

dále: $\frac{1}{1-x+x^2} = \frac{1}{(x-\frac12)^2+\frac34} = \frac{2}{\sqrt3}\operatorname{Im}\frac{1}{x-\frac12-\frac{\sqrt3}{2}i}$; $\sigma := \frac12+\frac{\sqrt3}{2}i = e^{\frac\pi3 i}$

$\downarrow\ J := \oint_C \frac{\overset{\text{tan}}{z^\alpha}}{z-\sigma}\,dz$; C:



- RESIDUOVÁ VĚTA:

$\longrightarrow \operatorname{Res}_\sigma f(z) = z^\alpha\big|_\sigma = e^{\frac\pi3\alpha i}\quad \therefore J = 2\pi i\, e^{\frac\pi3\alpha i}$

- PARAMETRIZACE

$\longrightarrow C_1:\ z = t+i0;\ t\in(\varepsilon,R);\ dz = dt$

$$J_1 = \int_\varepsilon^R \frac{(t+i0)^\alpha}{t-\sigma}\,dt \xrightarrow{\varepsilon\to0^+} \int_0^R \frac{t^\alpha}{t-\sigma}\,dt = \int_0^R \frac{t^\alpha(t-\frac12+\frac{\sqrt3}{2}i)}{t^2-t+1}\,dt =$$

$$= \underbrace{\int_0^R \frac{t^\alpha(t-\frac12)}{t^2-t+1}\,dt}_{I_0(R)\in\mathbb{R}\text{ ale diverguje}} + \frac{\sqrt3}{2}i\underbrace{\int_0^R \frac{t^\alpha}{t^2-t+1}\,dt}_{\xrightarrow{R\to\infty} I(\alpha)} = I_0(R)+\frac{\sqrt3}{2}i\,I(\alpha)+O\left(\frac{1}{R^{1-\alpha}}\right)$$

$$\left[\int_R^\infty \frac{t^\alpha}{t^2}\,dt \sim R^{\alpha-1}\right]$$

$\longrightarrow C_2:\ z = Re^{it};\ t\in(0,2\pi);\ dz = Rie^{it}$

$$J_2 = \int_0^{2\pi} \frac{R^\alpha e^{\alpha t i}}{Re^{it}-\sigma}\,Rie^{it}\,dt = i\int_0^{2\pi} R^\alpha e^{\alpha t i}\left(1+O\left(\tfrac1R\right)\right)dt = iR^\alpha\,\frac{e^{2\pi\alpha i}-1}{\alpha i}+O\left(\frac{1}{R^{1-\alpha}}\right)$$

$\longrightarrow \ominus C_3:\ z = t-i0;\ t\in(\varepsilon,R);\ dz = dt$

$$J_3 = \ominus\int_\varepsilon^R \frac{(t-i0)^\alpha}{t-\sigma}\,dt \xrightarrow{\varepsilon\to0^+} -e^{2\pi\alpha i}\int_0^R \frac{t^\alpha}{t-\sigma}\,dt = -e^{2\pi\alpha i}\left(\ -\text{"}-\ \right)$$

$\longrightarrow C_4:\ |J_4| \le \frac{\varepsilon^\alpha}{1-\varepsilon}\,2\pi\varepsilon \xrightarrow{\varepsilon\to0^+} 0$ pro $\alpha > -1$

- POROVNÁNÍ: $\quad 2\pi i\, e^{\frac\pi3\alpha i} = \left(1-e^{2\pi\alpha i}\right)\left(-\frac{R^\alpha}{\alpha}+I_0(R)+\frac{\sqrt3}{2}i\,I(\alpha)\right)\ /\cdot e^{-\pi\alpha i}$

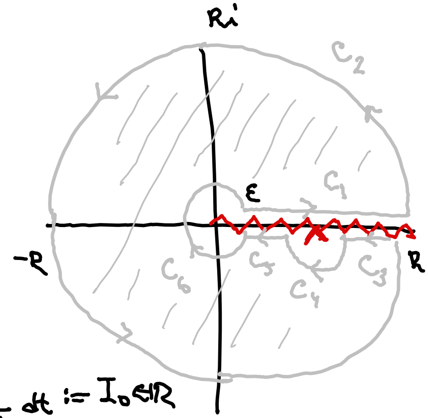
$$2\pi i\, e^{-\frac{2\pi}{3}\alpha i} = -2i\sin(\pi\alpha)\left(-\frac{R^\alpha}{\alpha}+I_0(R)+\frac{\sqrt3}{2}i\,I(\alpha)\right)\quad /\!/\ \operatorname{Re}$$

$$I(\alpha) = \frac{2\pi}{\sqrt3}\,\frac{\sin\frac{2\pi}{3}\alpha}{\sin\pi\alpha} = \frac{2\pi}{\sqrt3}\,\frac{\frac{2\pi}{3}\alpha-\frac{1}{3!}\left(\frac{2\pi}{3}\alpha\right)^3+O(\alpha^5)}{\pi\alpha-\frac{1}{3!}(\pi\alpha)^3+O(\alpha^5)} = \frac{2\pi}{\sqrt3}\,\frac23\,\frac{1-\frac{1}{3!}\left(\frac{2\pi}{3}\alpha\right)^2+O(\alpha^4)}{1-\frac{1}{3!}(\pi\alpha)^2+O(\alpha^4)} =$$

$$= \frac{4\pi}{3\sqrt3}\left(1-\frac{4\pi^2}{54}\alpha^2\right)\left(1+\frac{\pi^2}{6}\alpha^2\right)+O(\alpha^4) = \frac{4\pi}{3\sqrt3}\left(1+\frac{5\pi^2}{54}\alpha^2\right)+O(\alpha^4)$$

$$\therefore \boxed{I = \frac{4\pi}{3\sqrt3}\,\frac{5\pi^2}{54} = \frac{10\pi^3}{81\sqrt3}}$$

(Př) $I = \fint_0^\infty \frac{\ln x}{(x-1)^2}\,dx \in \mathbb{R}$ ∵ $x=1$: $\ln x \approx (x-1)$



↳ $J = \oint_C \frac{-\ln^2 z}{(z-1)^2}\,dz$; C:

- RESIDUOVÁ VĚTA: $J = 0$ [CAUCHY]
- PARAMETRIZACE:

$\longrightarrow C_1: z = t + i0;\ t \in (\varepsilon, R);\ dz = dt$

$$J_1 = \int_\varepsilon^R \frac{\ln^2(t+i0)}{(t+i0-1)^2}\,dt \xrightarrow[\varepsilon \to 0^+]{R\to\infty} \int_0^\infty \frac{\ln^2 t}{(t-1)^2}\,dt := I_0 \in \mathbb{R}$$

$\longrightarrow C_2: |J_2| \le \frac{(\ln R + 2\pi)^2}{(R-1)^2} 2\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow \ominus(C_3 + C_5): z = (\varepsilon, 1-\varepsilon) \cup (1+\varepsilon, R)$:

$$J_3 + J_5 = \ominus\left(\int_\varepsilon^{1-\varepsilon} + \int_{1+\varepsilon}^R\right) \frac{\ln^2(t-i0)}{(t-1)^2}\,dt \longrightarrow \ominus\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^\infty\right) \frac{(\ln t + 2\pi i)^2}{(t-1)^2}\,dt$$

$$= \underbrace{-\int_0^\infty \frac{\ln^2 t}{(t-1)^2}\,dt}_{I_0} - 2\pi i \underbrace{\fint_0^\infty \frac{\ln t\,dt}{(t-1)^2}}_{I} + 4\pi^2 \underbrace{\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^\infty\right) \frac{dt}{(t-1)^2}}_{I(\varepsilon) \in \mathbb{R} \text{ (ale diverguje)}} + O(\varepsilon)$$

$\longrightarrow \ominus C_4: z = 1 + \varepsilon e^{it};\ t \in (\pi, 2\pi);\ dz = \varepsilon i e^{it}\,dt$

$$J_4 = \ominus \int_\pi^{2\pi} \frac{(\ln(1+\varepsilon e^{it}))^2}{(\varepsilon e^{it})^2} \varepsilon i e^{it}\,dt = -\frac{i}{\varepsilon} \int_\pi^{2\pi} (2\pi i + \varepsilon e^{it} + O(\varepsilon^2))^2 e^{-it}\,dt$$

$$= -\frac{i}{\varepsilon} \int_\pi^{2\pi} -4\pi^2 e^{-it} + 4\pi i \varepsilon + O(\varepsilon^2)\,dt = -\frac{i}{\varepsilon}\left[\frac{4\pi^2}{i} e^{-it} + 4\pi i \varepsilon t + O(\varepsilon^2)\right]_\pi^{2\pi}$$

$$= -\frac{i}{\varepsilon}\left[\frac{8\pi^2}{i} + 4\pi^2 i \varepsilon + O(\varepsilon^2)\right] = -\frac{8\pi^2}{\varepsilon} + 4\pi^2 + O(\varepsilon)$$

$\longrightarrow C_6: |J_6| \le \frac{(2\pi - \ln\varepsilon)^2}{(1-\varepsilon)^2} 2\pi\varepsilon \xrightarrow{\varepsilon \to 0^+} 0$

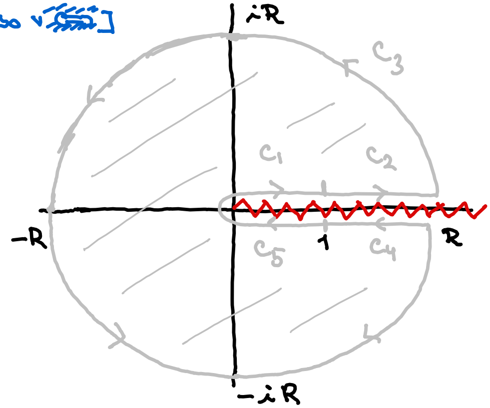
- POROVNÁNÍ: $0 = I_0 - I_0 - 2\pi i I + 4\pi^2 I(\varepsilon) - \frac{8\pi^2}{\varepsilon} + 4\pi^2 + O(\varepsilon)$

[Im]: $\boxed{I = 0}$ & [Re:] [BONUS] $I(\varepsilon) = \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^\infty\right) \frac{dt}{(t-1)^2}\,dt = \frac{2}{\varepsilon} - 1 + O(\varepsilon)$

$$\boxed{I = \fint_0^\infty \frac{\ln x}{(x-1)^2}\,dx \in \mathbb{R} \therefore I = \left| x \to \tfrac{1}{x} \right| = \int_0^\infty \frac{\ln \frac{1}{x}}{(\frac{1}{x}-1)^2} \frac{1}{x^2}\,dx = -I \Rightarrow I = 0}$$

Př: $I = \int_0^1 \frac{\ln x}{\sqrt{x(1-x)}}\,dx \in \mathbb{R}$ [nebo v [illegible]]

$$J := \oint_C \frac{\ln z}{\sqrt{z}\,\sqrt{z-1}}\,dz\ ; \qquad C:$$



- CAUCHYHO VĚTA:

$$J = 0$$

- PARAMETRIZACE:

$\longrightarrow$ C_1: $z = t + i0$; $t \in (0,1)$; $dz = dt$

$$J_1 = \int_0^1 \frac{\ln(t+i0)}{\sqrt{t+i0}\,\underbrace{\sqrt{t-1}}_{i\sqrt{1-t}}}\,dt = \frac{1}{i}\int_0^1 \frac{\ln t}{\sqrt{t}\,\sqrt{1-t}}\,dt = -iI$$

$\longrightarrow$ C_2: $z = t + i0$; $t \in (1,R)$; $dz = dt$

$$J_2 = \int_1^R \frac{\ln(t+i0)}{\sqrt{t+i0}\,\sqrt{t-1+i0}}\,dt = \int_1^R \frac{\ln t}{\sqrt{t}\,\sqrt{t-1}}\,dt \qquad (\text{diverguje pro } R \to \infty)$$

$\longrightarrow$ C_3: $z = Re^{it}$; $t \in (0, 2\pi)$; $dz = Rie^{it}\,dt$

$$J_3 = \int_0^{2\pi} \frac{\ln(Re^{it})}{\sqrt{Re^{it}}\,\sqrt{Re^{it}-1}}\,Rie^{it}\,dt = i\int_0^{2\pi} (\ln R + it)\left(1 - \frac{1}{R}e^{-it}\right)^{-\frac{1}{2}} dt =$$

$$= i\int_0^{2\pi} (\ln R + it)\left(1 + \frac{1}{R}e^{-it} + O\!\left(\frac{1}{R^2}\right)\right) dt = 2\pi i \ln R - \underbrace{\int_0^{2\pi} t\,dt}_{2\pi^2} + O\!\left(\frac{\ln R}{R}\right)$$

$\longrightarrow$ $\ominus$ C_4: $z = t - i0$; $t \in (1,R)$; $dz = dt$

$$J_4 = \ominus \int_1^R \frac{\ln(t-i0)\,dt}{\underbrace{\sqrt{t-i0}}_{-\sqrt{t}}\,\underbrace{\sqrt{t-1-i0}}_{-\sqrt{t-1}}} = -\int_1^R \frac{\ln t + 2\pi i}{\sqrt{t}\sqrt{t-1}}\,dt = -\int_1^R \frac{\ln t}{\sqrt{t}\sqrt{t-1}}\,dt - 2\pi i \underbrace{\int_1^R \frac{dt}{\sqrt{t}\sqrt{t-1}}}_{I_1(R) \in \mathbb{R}}$$

$\longrightarrow$ $\ominus$ C_5: $z = t - i0$; $t \in (0,1)$; $dz = dt$

$$J_5 = \ominus \int_0^1 \frac{\ln(t-i0)}{\underbrace{\sqrt{t-i0}}_{-\sqrt{t}}\,\underbrace{\sqrt{t-1}}_{i\sqrt{1-t}}}\,dt = \frac{1}{i}\int_0^1 \frac{\ln t + 2\pi i}{\sqrt{t}\sqrt{1-t}}\,dt = -iI + 2\pi \underbrace{\int_0^1 \frac{dt}{\sqrt{t}\sqrt{1-t}}}_{I_0 \in \mathbb{R}}$$

- POROVNÁNÍ:

$$0 = -2iI + 2\pi i \ln R - 2\pi^2 - 2\pi i\, I_1(R) + 2\pi I_0 + O\!\left(\frac{\ln R}{R}\right); \quad R \to \infty$$

$\boxed{\text{Im:}}$ $I = \lim\limits_{R\to\infty} \pi \ln R - \pi \int_1^R \frac{dt}{\sqrt{t}\sqrt{t-1}} = \pi \ln R - 2\pi \underbrace{\operatorname{argcosh}\sqrt{R}}_{\ln(\sqrt{R}+\sqrt{R-1})} \xrightarrow{R\to\infty} -2\pi \ln 2$

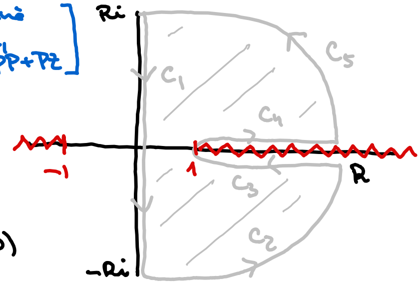
$\boxed{\text{Re:}}$ $I_0 = \int_0^1 \frac{dt}{\sqrt{t}\sqrt{1-t}} = \pi = \left[-\arccos^2\sqrt{t}\right]_0^1$

[BONUS]

$$\boxed{I = -2\pi \ln 2}$$

(Př) $I = \int_0^\infty \frac{\operatorname{arctg}^3 x}{x^3}\,dx \in \mathbb{R}$ [zbytečně složité, lepší PP+PZ]

$$J := \oint_C \ln^3\left(\frac{1+z}{1-z}\right)\frac{dz}{z^3}$$



řezy: $\frac{1+z}{1-z} = -t;\ t>0;\ z = \frac{t+1}{t-1} \in (-\infty,-1)\cup(1,\infty)$

skoky: $\frac{1+z}{1-z}\Big|_{t+i0} \overset{t>1}{=} \frac{1+z}{1-z}\Big|_t + \left(\frac{1+z}{1-z}\right)'\Big|_t i0 = \frac{1+t}{1-t} + \frac{2}{(1-t)^2}\, i0 = -\frac{t+1}{t-1} + i0$

• CAUCHYHO VĚTA : $J = 0$

• PARAMETRIZACE :

$\longrightarrow \ominus C_1$: $z = it;\ t\in(-R,R);\ dz = i\,dt;\ \frac{1+z}{1-z} = \frac{1+it}{1-it} = \frac{\sqrt{1+t^2}\,e^{i\operatorname{arctg} t}}{\sqrt{1+t^2}\,e^{-i\operatorname{arctg} t}}$

$$J_1 = \ominus\int_{-R}^{R} \ln^3\left(e^{2i\operatorname{arctg} t}\right)\frac{i\,dt}{(it)^3} \xrightarrow{R\to\infty} -8i\int_{-\infty}^{\infty}\frac{\operatorname{arctg}^3 t}{t^3}\,dt \overset{\text{sym.}}{=} -16iI$$

$\longrightarrow C_2$: $|J_2| \le \frac{1}{R^3}\left(\ln\left(\frac{R+1}{R-1}\right)+\pi\right)^3 \frac{\pi}{2}R \xrightarrow{R\to\infty} 0$

$\longrightarrow \ominus C_3$: $z = t - i0;\ t\in(1,R):\ dz = dt$

$$J_3 = \ominus\int_1^R \ln^3\left(-\frac{t+1}{t-1} - i0\right)\frac{dt}{t^3} \xrightarrow{R\to\infty} \ominus\int_1^\infty \left(\ln\left(\frac{t+1}{t-1}\right) - \pi i\right)^3 \frac{dt}{t^3}$$

$\longrightarrow C_4$: $z = t + i0;\ t\in(1,R):\ dz = dt$

$$J_4 = \int_1^R \ln^3\left(-\frac{t+1}{t-1} + i0\right)\frac{dt}{t^3} \xrightarrow{R\to\infty} \int_1^\infty \left(\ln \frac{t+1}{t-1} + \pi i\right)^3 \frac{dt}{t^3}$$

$\longrightarrow C_5$: $|J_5| \le \frac{1}{R^3}\left(\ln\left(\frac{R+1}{R-1}\right)+\pi\right)^3 \frac{\pi}{2}R \xrightarrow{R\to\infty} 0$

• POROVNÁNÍ : $[(A+B)^3-(A-B)^3 = 2B((A+B)^2+(A+B)(A-B)+(A-B)^2) = 2B(3A^2+B^2)]$

$$0 = -16iI + \int_1^\infty \left(\ln\frac{t+1}{t-1}+\pi i\right)^3 - \left(\ln\frac{t+1}{t-1}-\pi i\right)^3 \frac{dt}{t^3} =$$

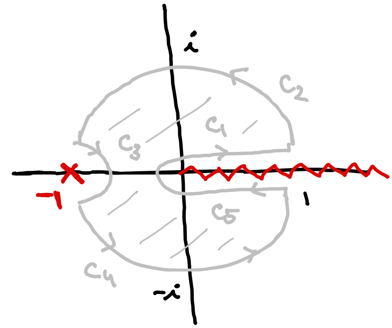
$$= -16iI + 2\pi i\left[3\int_1^\infty \ln^2\frac{t+1}{t-1}\,\frac{dt}{t^3} - \pi^2\underbrace{\int_1^\infty\frac{dt}{t^3}}_{1/2}\right] = \left|t\to\frac{1}{t}\right|$$

$$= -16iI - \pi^3 i + 6\pi i\int_0^1 t\ln^2\frac{1+t}{1-t}\,dt = \text{[primitivní funkce!]}$$

$$= -16iI - \pi^3 i + 6\pi i\,(4\ln 2) \quad \therefore \quad \boxed{I = \frac{3\pi}{2}\ln 2 - \frac{\pi^3}{16}}$$

$$\left\langle \int t\ln^2\frac{1+t}{1-t}\,dt \overset{PP}{=} \frac{t^2-1}{2}\ln\frac{1+t}{1-t} - \int\frac{t^2-1}{2}\,2\ln\frac{1+t}{1-t}\,\frac{2}{1-t^2}\,dt = \frac{t^2-1}{2}\ln\frac{1+t}{1-t} - 2(t-1)\ln\left(\frac{1+t}{1-t}\right) - 4\ln(1+t)\right\rangle$$

(97) $I = \int_0^1 \frac{\ln x}{1+x}\,dx \in \mathbb{R}$



$$J = \oint_C \frac{\ln^2 z}{1+z}\,dz$$

- **RESIDUOVÁ VĚTA**:

$\longrightarrow$ $\operatorname{Res}_{-1} f(z) = \left.\frac{\ln^2 z}{1}\right|_{-1} = (\pi i)^2 = -\pi^2$

ale $J = 0$ [CAUCHY]

- **PARAMETRIZACE**:

$\longrightarrow$ C_1: $z = t + i0$; $t \in (0,1)$; $dz = dt$

$$J_1 = \int_0^1 \frac{\ln^2(t+i0)}{1+t}\,dt = \int_0^1 \frac{\ln^2 t}{1+t}$$

$\longrightarrow$ $C_2 + C_4$: $z = e^{it}$; $t \in (0, \pi-\varepsilon) \cup (\pi+\varepsilon, 2\pi)$; $dz = ie^{it}\,dt$

$$J_2 = \left(\int_0^{\pi-\varepsilon} + \int_{\pi+\varepsilon}^{2\pi}\right) \frac{\ln^2(e^{it})}{1+e^{it}}\, ie^{it}\,dt =$$

$$= -i \fint_0^{2\pi} \frac{t^2 e^{\frac{it}{2}}}{e^{\frac{it}{2}} + e^{-\frac{it}{2}}}\,dt = -i \fint_0^{2\pi} \frac{t^2(\cos\frac{t}{2} + i\sin\frac{t}{2})}{2\cos\frac{t}{2}}\,dt =$$

$$= -\frac{i}{2}\fint_0^{2\pi} t^2\,dt + \underbrace{\frac{1}{2}\fint_0^{2\pi} t^2 \operatorname{tg}\frac{t}{2}\,dt}_{I_0 \in \mathbb{R}} = -\frac{4\pi^3 i}{3} + \frac{1}{2} I_0$$

$\longrightarrow$ C_3: $J_3 = -\pi i \operatorname{Res}_{-1} f(z) = \pi^3 i$

$\longrightarrow$ $\ominus C_5$: $z = t - i0$; $t \in (0,1)$; $dz = dt$

$$J_5 = \ominus \int_0^1 \frac{\ln^2(t-i0)}{1+t}\,dt = -\int_0^1 \frac{(\ln t + 2\pi i)^2}{1+t}\,dt =$$

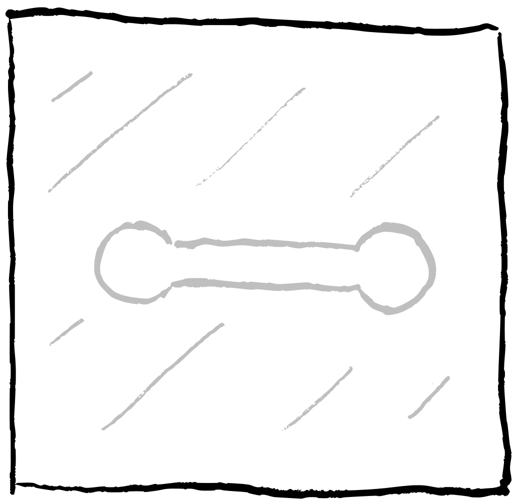
$$= -\int_0^1 \frac{\ln^2 t}{1+t}\,dt - 4\pi i \underbrace{\int_0^1 \frac{\ln t}{1+t}\,dt}_{I} + 4\pi^2 \underbrace{\int_0^1 \frac{dt}{1+t}}_{\ln 2}$$

- **POROVNÁNÍ**:

$$0 = -\frac{4\pi^3 i}{3} + \frac{1}{2} I_0 + \pi^3 i - 4\pi i I + 4\pi^2 \ln 2$$

$\therefore$ $\boxed{\operatorname{Im}}$: $\boxed{I = -\frac{\pi^2}{3}}$ & $\boxed{\operatorname{Re}:}$ $I_0 = \fint_0^{2\pi} t^2 \operatorname{tg}\frac{t}{2}\,dt = -8\pi^2 \ln 2$

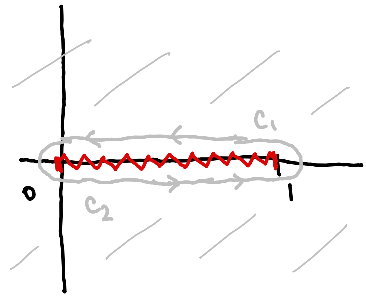
[BONUS]



Př $\quad I = \int_0^1 \sqrt{x}\sqrt{1-x}\,dx \in \mathbb{R}$

$$J := \oint_C \sqrt{z}\sqrt{z-1}\,dz$$

C:



- **RESIDUOVÁ VĚTA** : $f(z) = \dots + \frac{a_{-1}}{z-\sigma} + \dots$

Res_∞ : $\quad f(t+i0) \overset{t\gg 1}{=} \sqrt{t+i0}\sqrt{t+i0-1} = \sqrt{t}\sqrt{t-1} = t\sqrt{1-\frac{1}{t}} =$

$(\exists : f \text{ hol } B_{R,\infty}(0))$

$$= t\left(1-\tfrac{1}{t}\right)^{1/2} = t\left(1 + \left(-\tfrac{1}{t}\right)\tfrac{1}{2} + \left(-\tfrac{1}{t}\right)^2 \tfrac{1}{2}\left(-\tfrac{1}{2}\right)\tfrac{1}{2!} + \dots\right)$$

$$= t\left(1 - \tfrac{1}{2t} - \tfrac{1}{8t^2} + \dots\right) = t - \tfrac{1}{2} - \tfrac{1}{8t} + O\left(\tfrac{1}{t^2}\right)$$

$$\therefore \quad \mathrm{Res}_\infty f(z) = \frac{1}{8} \quad \Rightarrow \quad J = -2\pi i \sum_{\sigma \in \mathrm{Ext}\,C \cup \{\infty\}} \mathrm{Res}_\sigma f(z) = -\frac{\pi i}{4}$$

- **PARAMETRIZACE** :

$\longrightarrow \ominus C_1 : z = t + i0 ; \; t \in (0,1) ; \; dz = dt$

$$J_1 = \ominus \int_0^1 \sqrt{t+i0}\,\underbrace{\sqrt{t-1}}_{i\sqrt{1-t}}\,dt = -i\int_0^1 \sqrt{t}\sqrt{1-t}\,dt = -iI$$

$\longrightarrow C_2 : z = t - i0 ; \; t \in (0,1) ; \; dz = dt$

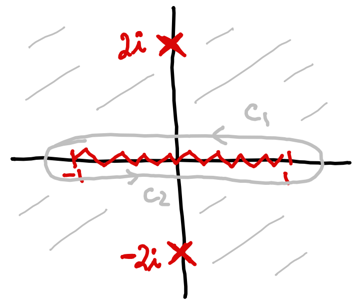
$$J_2 = \int_0^1 \underbrace{\sqrt{t-i0}}_{-\sqrt{t}}\,\underbrace{\sqrt{t-1}}_{i\sqrt{1-t}}\,dt = -i\int_0^1 \sqrt{t}\sqrt{1-t}\,dt = -iI$$

- **POROVNÁNÍ** :

$$-\frac{\pi i}{4} = -2iI \qquad \therefore \quad \boxed{I = \frac{\pi}{8}}$$

(P^c_r) $I = \int_{-1}^{1} \frac{1}{4+x^2} \frac{dx}{\sqrt[3]{(1-x)^2(1+x)}} \in \mathbb{R}$

$J = \oint_C \frac{1}{4+z^2} \frac{1}{(z-1)^{2/3}(z+1)^{1/3}} dz$



• RESIDUOVÁ VĚTA

$\longrightarrow z \to \infty: \quad f(t+i0) \overset{t \gg 1}{=} \frac{1}{4+t^2} \frac{1}{(t-1)^{2/3}(t+1)^{1/3}} \approx \frac{1}{t^3} + O\left(\frac{1}{t^4}\right)$

$\therefore \quad \operatorname{Res}_\infty f(z) = 0$

$\longrightarrow \operatorname{Res}_{2i} f(z) = \frac{1}{2z} \frac{1}{(z-1)^{2/3}(z+1)^{1/3}} \Big|_{2i} = \frac{1}{4i} \frac{1}{(2i-1)^{2/3}} \frac{1}{(2i+1)^{1/3}}$

$= \frac{1}{4i} \frac{1}{\left(\sqrt{5}\, e^{i(\pi - \operatorname{arctg} 2)}\right)^{2/3}} \frac{1}{\left(\sqrt{5}\, e^{i \operatorname{arctg} 2}\right)^{1/3}} = \frac{1}{4\sqrt{5}\, i} e^{i\left(-\frac{2\pi}{3} + \frac{1}{3}\operatorname{arctg} 2\right)}$

$\longrightarrow \operatorname{Res}_{-2i} f(z) = -\frac{1}{4i} \frac{1}{(-2i-1)^{2/3}} \frac{1}{(-2i+1)^{1/3}}$

$= -\frac{1}{4i} \frac{1}{\left(\sqrt{5}\, e^{i(-\pi + \operatorname{arctg} 2)}\right)^{2/3}} \frac{1}{\left(\sqrt{5}\, e^{-i \operatorname{arctg} 2}\right)^{1/3}} = -\frac{1}{4\sqrt{5}\, i} e^{i\left(\frac{2\pi}{3} - \frac{1}{3}\operatorname{arctg} 2\right)}$

$\therefore \quad J = -2\pi i \sum_{\sigma \in \operatorname{Ext} C \cup \{\infty\}} \operatorname{Res}_\sigma f(z) = -2\pi i \frac{1}{4\sqrt{5}\, i} \left(e^{i\left(-\frac{2\pi}{3} + \frac{1}{3}\operatorname{arctg} 2\right)} - e^{i\left(\frac{2\pi}{3} - \frac{1}{3}\operatorname{arctg} 2\right)}\right)$

$= \frac{\pi i}{\sqrt{5}} \sin\left(\frac{2\pi}{3} - \frac{1}{3}\operatorname{arctg} 2\right)$

• PARAMETRIZACE :

$\longrightarrow \ominus C_1: \ z = t + i0; \ t \in (-1,1) \ ; \ dz = dt$

$J_1 = \ominus \int_0^1 \frac{1}{4+t^2} \frac{1}{(\underbrace{t+i0-1}_{(1-t)e^{\pi i}})^{2/3}(t+1)^{1/3}} dt = -e^{-\frac{2\pi}{3}i} I$

$\longrightarrow C_2: \ z = t - i0; \ t \in (-1,1) \ ; \ dz = dt$

$J_2 = \int_0^1 \frac{1}{4+t^2} \frac{1}{(\underbrace{t-i0-1}_{(1-t)e^{-\pi i}})^{2/3}(t+1)^{1/3}} dt = e^{\frac{2\pi i}{3}} I$

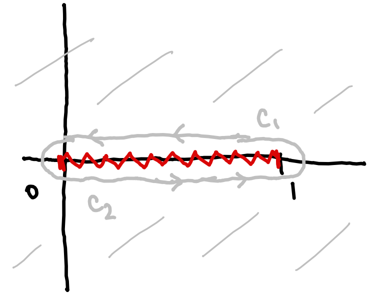
• POROVNÁNÍ :

$\frac{\pi i}{\sqrt{5}} \sin\left(\frac{2\pi}{3} - \frac{1}{3}\operatorname{arctg} 2\right) = I \underbrace{\left(e^{\frac{2\pi i}{3}} - e^{-\frac{2\pi i}{3}}\right)}_{2i \sin \frac{2\pi}{3}} = i\sqrt{3}\, I$

$\therefore \quad \boxed{I = \frac{\pi \sin\left(\frac{2\pi - \operatorname{arctg} 2}{3}\right)}{\sqrt{15}}}$

(Př) $I = \int_0^1 \ln^2\left(\frac{x}{1-x}\right)\frac{dx}{\sqrt{x}\sqrt{1-x}} \in \mathbb{R}$

C:



↳ $J = \oint_C \ln^2\left(\frac{z}{z-1}\right)\frac{dz}{\sqrt{z}\sqrt{z-1}}$

řezy: $\boxed{\ln:}$ $\frac{z}{z-1} = -t;\ t>0$

↳ $z = t - tz \Rightarrow z(1+t) = t$

$\therefore\ z = \frac{t}{1+t} \in (0,1)$

skoky: $z = t + i0;\ t\in(0,1)$, poté

$$\therefore\ \frac{z}{1-z}\Big|_{t+i0} \overset{\text{Taylor}}{=} \frac{z}{1-z}\Big|_t + \left(\frac{z}{1-z}\right)'\Big|_t i0 = \frac{t}{1-t} + \frac{1}{(1-t)^2}i0 = -\frac{t}{1-t} + i0$$

- **RESIDUOVÁ VĚTA**:

Res$_\infty$: $f(t+i0) \overset{t\to\infty}{=} \ln^2\left(\frac{t}{t-1}\right)\frac{1}{\sqrt{t+i0}\sqrt{t+i0-1}} = \ln^2\left(1-\frac{1}{t}\right)\frac{1}{\sqrt{t}\sqrt{t-1}} =$

$$= \left[-\frac{1}{t} + O\left(\frac{1}{t^2}\right)\right]^2 \frac{1}{t}\left(1-\frac{1}{t}\right)^{-\frac{1}{2}} = \frac{1}{t^3} + O\left(\frac{1}{t^4}\right)$$

↑ totéž v z ∵ f hol. na $B_{1,\infty}(0)$

$\therefore\ \mathrm{Res}_\infty f(z) = 0$

& $J = 0$

- **PARAMETRIZACE**:

⟶ $\ominus\ C_1:\ z = t+i0;\ t\in(0,1);\ dz = dt$

$$J_1 = \ominus\int_0^1 \ln^2\left(-\frac{t}{1-t}+i0\right)\frac{dt}{\sqrt{t}\underbrace{\sqrt{t-1+i0}}_{i\sqrt{1-t}}} = i\int_0^1\left[\ln\left(\frac{t}{1-t}\right)+\pi i\right]^2\frac{dt}{\sqrt{t}\sqrt{1-t}}$$

A + B

⟶ $C_2:\ z = t-i0;\ t\in(0,1);\ dz = dt$

$$J_2 = \int_0^1 \ln^2\left(-\frac{t}{1-t}-i0\right)\frac{dt}{\sqrt{t}\underbrace{\sqrt{t-1-i0}}_{-i\sqrt{1-t}}} = i\int_0^1\left[\ln\left(\frac{t}{1-t}\right)-\pi i\right]^2\frac{dt}{\sqrt{t}\sqrt{1-t}}$$

A − B

- **POROVNÁNÍ**:

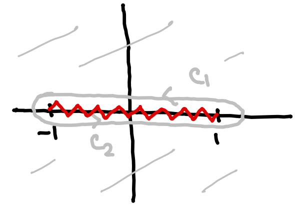
$$0 = i\int_0^1\left[2\ln^2\left(\frac{t}{1-t}\right) - 2\pi^2\right]\frac{dt}{\sqrt{t}\sqrt{1-t}} = 2iI - 2\pi^2 i\underbrace{\int_0^1\frac{dt}{\sqrt{t}\sqrt{1-t}}}_{\pi}$$

∵ $\left[-\arccos\sqrt{t}\right]_0^\pi$

$\therefore\ \boxed{I = \pi^3}$

Př $\quad I = \int_{-1}^{1} \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{\sqrt[3]{(1-x)^2(1+x)}} \in \mathbb{R}$

$\hookrightarrow J = \oint_C \ln\left(\frac{1+z}{1-z}\right)\frac{dz}{(z-1)^{2/3}(z+1)^{1/3}}$; C:

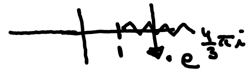


řezy:

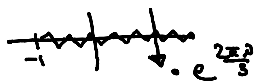
$\boxed{\ln:}$ $\quad \frac{1+z}{1-z} = t>0;\ 1+z = t - tz \quad \therefore\ z = \frac{t-1}{t+1} \in (-1,1)$

$\boxed{(z-1)^{2/3}(z+1)^{1/3}}$:

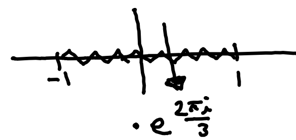
$(z-1)^{2/3}$:



$(z+1)^{1/3}$:



$(z-1)^{2/3}(z+1)^{1/3}$:



skoky:

$\boxed{\ln:}$ $\quad z = t + i0;\ t\in(-1,1)$:

$$\left.\frac{1+z}{1-z}\right|_{t+i0} \overset{\text{Tayl.}}{=} \left.\frac{1+z}{1-z}\right|_t + \left.\left(\frac{1+z}{1-z}\right)'\right|_t i0 = \frac{1+t}{1-t} + \frac{2}{(1-t)^2}\, i0 = \frac{1+t}{1-t} + i0$$

• RESIDUOVÁ VĚTA:

$z\to\infty$: $\quad f(t+i0) \overset{t\gg1}{=} \ln\left(\frac{1+t}{1-t}\right)\frac{1}{(t-1+i0)^{2/3}(t+1+i0)^{1/3}} =$

$$= \Big[\underbrace{\ln\left(\frac{t+1}{t-1}\right)}_{\to 0} + \pi i\Big]\frac{1}{(t-1)^{2/3}(t+1)^{1/3}} = \frac{\pi i}{t} + O\left(\frac{1}{t^2}\right)$$

$\therefore\ \operatorname{Res}_\infty f(z) = -\pi i \quad \& \quad J = -2\pi i \sum_{\sigma\in \operatorname{Ext} C\cup\{\infty\}} \operatorname{Res}_\sigma f(z) = -2\pi i(-\pi i) = -2\pi^2$

$+2\pi^2\sqrt{3} + \frac{2\pi^2}{3}\sqrt{3}$

• PARAMETRIZACE:

$\longrightarrow \ominus C_1$: $\quad z = t+i0;\ t\in(-1,1);\ dz = dt$

$$J_1 = \ominus\int_{-1}^{1}\ln\left(\frac{1+t}{1-t}+i0\right)\frac{dt}{\underbrace{(t-1)^{2/3}}_{(1-t)e^{\pi i}}\underbrace{(t+1+i0)^{1/3}}_{(1+t)e^{0i}}} = -e^{-\frac{2\pi i}{3}}\int_{-1}^1 \ln\left(\frac{1+t}{1-t}\right)\frac{dt}{(1-t)^{2/3}(1+t)^{1/3}} = -e^{-\frac{2\pi i}{3}} I$$

$\longrightarrow C_2$: $\quad z = t - i0;\ t\in(-1,1);\ dz = dt$

$$J_2 = \int_{-1}^1 \ln\left(\frac{1+t}{1-t} - i0\right)\frac{dt}{(t-1)^{2/3}\underbrace{(t+1-i0)^{1/3}}_{(1+t)e^{2\pi i}}} = e^{-\frac{4\pi i}{3}}\int_{-1}^1\left[\ln\frac{1+t}{1-t}+2\pi i\right]\frac{dt}{(1-t)^{2/3}(1+t)^{1/3}} =$$

$$= e^{-\frac{4\pi i}{3}} I + 2\pi i\, e^{-\frac{4\pi i}{3}} I_0$$

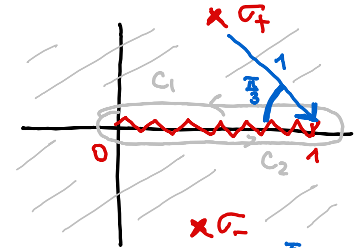
• POROVNÁNÍ: $\quad -2\pi^2 = -e^{-\frac{2\pi i}{3}} I + e^{-\frac{4\pi i}{3}} I + 2\pi i\, e^{-\frac{4\pi i}{3}} I_0$

• $e^{\frac{4\pi i}{3}}$ / $\boxed{\text{Re:}}$ $\quad -2\pi^2\cos\frac{4\pi}{3} = -I\cos\frac{2\pi}{3} + I \quad \therefore\ \boxed{I = 2\pi^2/3}$

[BONUS] $\boxed{\text{Im:}}$ $\quad -2\pi^2 \sin\frac{4\pi}{3} = -I\sin\frac{2\pi}{3} + 2\pi I_0 \quad \therefore\ I_0 = \int_{-1}^1\frac{dt}{(1-t)^{2/3}(1+t)^{1/3}} = \frac{2\pi}{\sqrt{3}}$

Př: $I = \int_0^1 \frac{2x-1}{1-x+x^2} \ln\left(\frac{x}{1-x}\right) dx \in \mathbb{R}$

$$J = \oint_C \frac{2z-1}{1-z+z^2} \ln^2\left(\frac{z}{1-z}\right) dz \quad ; \quad C:$$



singularity: $1 - z + z^2 = \frac{z^3+1}{z+1} = (z-\sigma_+)(z-\sigma_-)$;

$$\sigma_\pm = e^{\pm \pi i/3} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$1 - \sigma_\pm = e^{\mp\frac{\pi}{3}i}$$

řezy: $\frac{z}{1-z} = t > 0$; $z = \frac{t}{1+t} \in (0,1)$

$$\hookrightarrow \frac{z}{1-z}\Big|_{t+i0} \overset{t\in(0,1)}{=} \frac{z}{1-z}\Big|_t + \left(\frac{z}{1-z}\right)'\Big|_t i0 = \frac{t}{1-t} + \overbrace{\frac{1}{(1-t)^2} i0}^{i0}$$

- **RESIDUOVÁ VĚTA**:

$\longrightarrow$ $z \to \infty$: $f(z) \simeq \frac{2}{z}\ln^2(-1) = -\frac{2\pi^2}{z}$ $\therefore$ $\mathrm{Res}_\infty f(z) = 2\pi^2$

$$\longrightarrow \mathrm{Res}_{\sigma_+} f(z) = \frac{2z-1}{z-\sigma_-} \ln^2 \frac{z}{1-z}\Big|_{\sigma_+} = \frac{2\sigma_+-1}{\sigma_+-\sigma_-}\ln^2\frac{\sigma_+}{1-\sigma_+} =$$

$$= \frac{i\sqrt{3}}{i\sqrt{3}} \ln^2\left(\frac{e^{\pi i/3}}{e^{-\pi i/3}}\right) = \left(\frac{2\pi i}{3}\right)^2 = -\frac{4\pi^2}{9}$$

$$\longrightarrow \mathrm{Res}_{\sigma_-} f(z) = \frac{2\sigma_- - 1}{\sigma_- - \sigma_+} \ln^2\left(\frac{e^{-\pi i/3}}{e^{\pi i/3}}\right) = \frac{-i\sqrt{3}}{-i\sqrt{3}}\left(\frac{4\pi i}{3}\right)^2 = -\frac{16\pi^2}{9}$$

$$\therefore \quad J = -2\pi i \sum_{\sigma \in \{\sigma_\pm\} \cup \{\infty\}} \mathrm{Res}_\sigma f(z) = -2\pi i\left(2\pi^2 - \frac{4\pi^2}{9} - \frac{16\pi^2}{9}\right) = \frac{4\pi^3 i}{9}$$

- **PARAMETRIZACE**:

$\longrightarrow$ $\ominus$ C_1: $z = t + i0$; $t \in (0,1)$; $dz = dt$

$$J_1 = \ominus \int_0^1 \frac{2t-1}{1-t+t^2} \ln^2\left(\frac{t}{1-t} + i0\right) dt = \ominus \int_0^1 \frac{2t-1}{1-t+t^2} \ln^2\left(\frac{t}{1-t}\right) dt$$

$\longrightarrow$ C_2: $z = t - i0$; $t \in (0,1)$: $dz = dt$

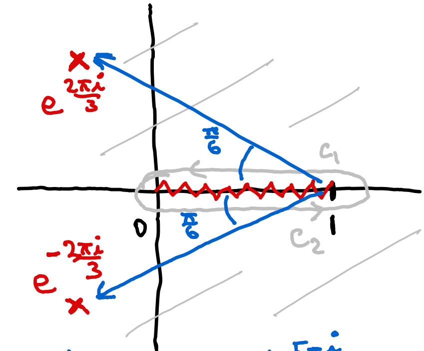
$$J_2 = \int_0^1 \frac{2t-1}{1-t+t^2} \ln^2\left(\frac{t}{1-t} - i0\right) dt = \int_0^1 \frac{2t-1}{1-t+t^2}\left(\ln\frac{t}{1-t} + 2\pi i\right)^2 dt =$$

$$= \int_0^1 \frac{2t-1}{1-t+t^2} \ln^2\frac{t}{1-t}\, dt + 4\pi i \underbrace{\int_0^1 \frac{2t-1}{1-t+t^2} \ln\frac{t}{1-t}}_{I} - 4\pi^2 \underbrace{\int_0^1 \frac{2t-1}{1-t+t^2}\, dt}_{I_0 \in \mathbb{R}}$$

- **POROVNÁNÍ**: $\frac{4\pi^3 i}{9} = 4\pi i I - 4\pi^2 I_0$ $\therefore$ $\boxed{\mathrm{Re}:}$ $\boxed{I = \frac{\pi^2}{9}}$

Př: $I = \int_0^1 \frac{dx}{(1+x+x^2)^2}$

$\hookrightarrow \quad J := \oint_C \frac{\ln z - \ln(z-1)}{(1+z+z^2)^2}\,dz$



singularity: $1+z+z^2 = \frac{z^3-1}{z-1} = (z-\sigma_+)(z-\sigma_-)$

kde $\sigma_\pm = e^{\pm\frac{2\pi i}{3}} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$e^{\pm\frac{2\pi i}{3}} - 1 = \sqrt{3}\, e^{\pm\frac{5\pi i}{6}}$

- RESIDUOVÁ VĚTA

$\longrightarrow \operatorname{Res}_{\sigma_+} f(z) = \lim_{z\to\sigma_+} \frac{1}{(2-1)!}\left[(z-\sigma_+)^2 f(z)\right]^{(2-1)} = \lim_{z\to\sigma_+}\left[\frac{\ln z - \ln(z-1)}{(z-\sigma_-)^2}\right]' =$

$\frac{\frac{1}{z} - \frac{1}{z-1}}{(z-\sigma_-)^2} - 2\,\frac{\ln z - \ln(z-1)}{(z-\sigma_-)^3}\Bigg|_{\sigma_+} = \frac{e^{-\frac{2\pi i}{3}} - \frac{1}{\sqrt{3}}e^{-\frac{5\pi i}{6}}}{(i\sqrt{3})^2} - 2\,\frac{\ln e^{\frac{2\pi i}{3}} - \ln(\sqrt{3}e^{\frac{5\pi i}{6}})}{(i\sqrt{3})^3}$

$= -\frac{1}{3}e^{-\frac{2\pi i}{3}} + \frac{1}{3\sqrt{3}}e^{-\frac{5\pi i}{3}} + \frac{4\pi}{9\sqrt{3}} - \frac{\ln 3}{3i\sqrt{3}} - \frac{5\pi}{9\sqrt{3}}$

$\longrightarrow \operatorname{Res}_{\sigma_-} f(z) = \frac{e^{\frac{2\pi i}{3}} - \frac{1}{\sqrt{3}}e^{\frac{5\pi i}{6}}}{(-i\sqrt{3})^2} - 2\,\frac{\ln e^{-\frac{2\pi i}{3}} - \ln(\sqrt{3}e^{-\frac{5\pi i}{3}})}{(-i\sqrt{3})^3} =$

$= -\frac{1}{3}e^{\frac{2\pi i}{3}} + \frac{1}{3\sqrt{3}}e^{\frac{5\pi i}{6}} + \frac{4\pi}{9\sqrt{3}} + \frac{\ln 3}{3i\sqrt{3}} - \frac{5\pi}{9\sqrt{3}}$

$\longrightarrow \operatorname{Res}_\infty: \; f(t+i0) = \frac{\ln t - \ln(t-1)}{(1+t+t^2)^2} \simeq \frac{1}{t^4}\ln\left(\frac{t}{t-1}\right) \simeq \frac{1}{t^5} \quad \therefore \operatorname{Res}_\infty f(z) = 0$

$\therefore J = -2\pi i \sum_{\sigma\in \mathrm{Ext}\,C\cup\{\infty\}} \operatorname{Res}_\sigma f(z) = -2\pi i\left(-\frac{2}{3}\cos\frac{2\pi}{3} + \frac{2}{3\sqrt{3}}\cos\frac{5\pi}{6} - \frac{2\pi}{9\sqrt{3}}\right) = \frac{4\pi^2 i}{9\sqrt{3}}$

- PARAMETRIZACE

$\longrightarrow \ominus C_1: \; z = t+i0; \; t\in(0,1); \; dz = dt$

$J_1 = \ominus\int_0^1 \frac{\ln t - \overbrace{\ln(t+i0-1)}^{\ln(1-t)+\pi i}}{(1+t+t^2)^2}\,dt = -\int_0^1 \frac{\ln t - \ln(1-t)}{(1+t+t^2)^2}\,dt + \pi i I$

$\longrightarrow C_2: \; z = t-i0; \; t\in(0,1); \; dz = dt$

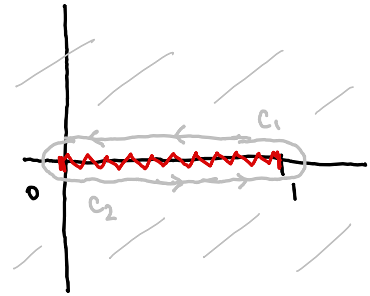
$J_2 = \int_0^1 \frac{\ln t - \overbrace{\ln(t-i0-1)}^{\ln(1-t)-\pi i}}{(1+t+t^2)^2}\,dt = \int_0^1 \frac{\ln t - \ln(1-t)}{(1+t+t^2)^2}\,dt + \pi i I$

- POROVNÁNÍ: $\frac{4\pi^2 i}{9\sqrt{3}} = 2\pi i I \quad \therefore \quad \boxed{I = \frac{2\pi}{9\sqrt{3}}}$

Př: $I = \int_0^1 \ln^2\left(\frac{x}{1-x}\right) dx \in \mathbb{R}$

↳ $J = \oint_C \ln^3\left(\frac{z}{z-1}\right)$

C:



řezy: $\boxed{\ln:}$ $\frac{z}{z-1} = -t;\ t > 0$

↳ $z = t - tz \Rightarrow z(1+t) = t$

$\therefore\ z = \frac{t}{1+t} \in (0,1)$

skoky: $z = t + i0;\ t \in (0,1)$, poté

$$\therefore \frac{z}{1-z}\Big|_{t+i0} \overset{\text{Taylor}}{=} \frac{z}{1-z}\Big|_t + \left(\frac{z}{1-z}\right)'\Big|_t i0 = \frac{t}{1-t} + \frac{1}{(1-t)^2} i0 = -\frac{t}{1-t} + i0$$

- RESIDUOVÁ VĚTA:

$\operatorname{Res}_\infty$: $f(t+i0) \overset{t\to\infty}{=} \ln^3\left(\frac{t}{t-1}\right) = \ln^3\left(1-\frac{1}{t}\right) = \left(-\frac{1}{t}\right)^3 + O\left(\frac{1}{t^3}\right)$

$\therefore\ \operatorname{Res}_\infty f(z) = 0$ & $J = 0$

- PARAMETRIZACE:

⟶ ⊖ C_1: $z = t + i0;\ t \in (0,1);\ dz = dt$

$$J_1 = \ominus \int_0^1 \ln^3\left(-\frac{t}{1-t} + i0\right) = -\int_0^1 \Big[\underbrace{\ln\left(\frac{t}{1-t}\right)}_{A} + \underbrace{\pi i}_{B}\Big]^3 dt$$

⟶ C_2: $z = t - i0;\ t \in (0,1);\ dz = dt$

$$J_2 = \int_0^1 \ln^3\left(-\frac{t}{1-t} - i0\right) = \int_0^1 \Big[\underbrace{\ln\left(\frac{t}{1-t}\right)}_{A} - \underbrace{\pi i}_{B}\Big]^3 dt$$

- POROVNÁNÍ:

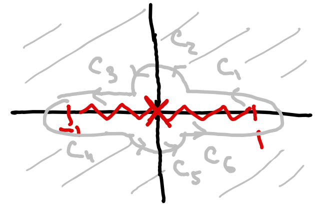
$$\Big[(A-B)^3 - (A+B)^3 = ((A-B)-(A+B))((A-B)^2 + (A-B)(A+B) + (A+B)^2)$$
$$= -2B(A^2 - \cancel{2AB} + B^2 + A^2 - B^2 + A^2 + \cancel{2AB} + B^2) = -2B(3A^2+B^2)\Big]$$

$$0 = \int_0^1 \left[\ln\frac{t}{1-t} - \pi i\right]^3 - \left[\ln\frac{t}{1-t} + \pi i\right]^3 dt =$$

$$= -2\pi i \int_0^1 3\ln^2\frac{t}{1-t} - \pi^2\, dt = -2\pi i\,(3I - \pi^2)\, dt$$

$$\therefore\ \boxed{I = \frac{\pi^2}{3}}$$

Př: $I = \int_0^1 \ln\left(\frac{1+t}{1-t}\right)\frac{dt}{t} \overset{\text{sym.}}{=} \frac{1}{2}\int_{-1}^1 \ln\left(\frac{1+t}{1-t}\right)\frac{dt}{t} \in \mathbb{R}$



$J = \oint_C \ln^2\left(\frac{z+1}{z-1}\right)\frac{dz}{z}$; C:

<u>řez</u>: $\frac{z+1}{z-1} = -t;\ t>0;\ z = \frac{t-1}{t+1} \in (-1,1)$

skoky: $\left.\frac{z+1}{z-1}\right|_{t+i0} \overset{t\in(-1,1)}{=} \left.\frac{z+1}{z-1}\right|_t + \left(\frac{z+1}{z-1}\right)'\Big|_t i0 = -\frac{1+t}{1-t} - \frac{2}{(1-t)^2}\, i0 = -\frac{1+t}{1-t} - i0$

- <u>RESIDUOVÁ VĚTA</u>:

→ $\operatorname{Res}_\infty$: $f(z) = \frac{\ln^2 1}{z} + \dots \quad \therefore \quad \operatorname{Res}_\infty f(z) = 0 \Rightarrow \boxed{J = 0}$

→ C_2 : $\operatorname{Res}_{0+i0} f(z) = \ln^2(-1-i0) = \ln^2(e^{-\pi i}) = -\pi^2$

→ C_5 : $\operatorname{Res}_{0-i0} f(z) = \ln^2(-1+i0) = \ln^2(e^{\pi i}) = -\pi^2$

- <u>PARAMETRIZACE</u>:

→ $\ominus(C_1+C_3)$: $z = t+i0$; $t \in (-1,-\varepsilon) \cup (\varepsilon,1)$

$\therefore\ J_1 = \ominus\left(\int_{-1}^{-\varepsilon} + \int_\varepsilon^1\right) \ln^2\left(-\frac{1+t}{1-t} - i0\right)\frac{dt}{t} \xrightarrow{\varepsilon\to 0^+} \ominus \fint_0^1 \left(\ln\frac{1+t}{1-t} - \pi i\right)^2 \frac{dt}{t}$

→ $C_4 + C_6$: $z = t - i0$; $t \in (-1,1)$; $dz = dt$

$J_2 = \left(\int_{-1}^{-\varepsilon} + \int_\varepsilon^1\right) \ln^2\left(-\frac{1+t}{1-t} + i0\right)\frac{dt}{t} \xrightarrow{\varepsilon\to 0^+} \fint_0^1 \left(\ln\frac{1+t}{1-t} + \pi i\right)^2 \frac{dt}{t}$

→ C_2 : $J_2 \xrightarrow{\varepsilon\to 0^+} \pi i \operatorname{Res}_0^{C_2} f(z) = -\pi^3 i$

→ C_5 : $J_5 \xrightarrow{\varepsilon\to 0^+} \pi i \operatorname{Res}_0^{C_5} f(z) = -\pi^3 i$

- <u>POROVNÁNÍ</u> : $\langle (A+B)^2 - (A-B)^2 = 4AB \rangle$

$0 = -2\pi^3 i + \fint_{-1}^1 \left(\ln\frac{1+t}{1-t} + \pi i\right)^2 - \left(\ln\frac{1+t}{1-t} - \pi i\right)^2 \frac{dt}{t} =$

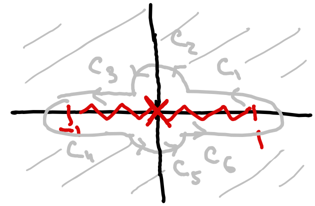
$= -2\pi^3 i + 4\pi i \underbrace{\fint_{-1}^1 \ln\frac{1+t}{1-t}\frac{dt}{t}}_{2I}$

$\therefore\ \boxed{I = \frac{\pi^2}{4}}$

P_r: $I = \int_0^1 \ln^3\left(\frac{1+t}{1-t}\right)\frac{dt}{t} \overset{\text{sym.}}{=} \frac{1}{2}\int_{-1}^{1} \ln^3\left(\frac{1+t}{1-t}\right)\frac{dt}{t} \in \mathbb{R}$

předchozí

$$J = \oint_C \ln^4\left(\frac{z+1}{z-1}\right)\frac{dz}{z} \quad ; \quad C:$$



řez: $\frac{z+1}{z-1} = -t;\ t>0;\ z = \frac{t-1}{t+1} \in (-1,1)$

skoky: $\left.\frac{z+1}{z-1}\right|_{t+i0} \overset{t\in(-1,1)}{=} \left.\frac{z+1}{z-1}\right|_t + \left.\left(\frac{z+1}{z-1}\right)'\right|_t i0 = -\frac{1+t}{1-t} - \frac{2}{(1-t)^2}\, i0 = -\frac{1+t}{1-t} - i0$

- **REZIDUOVÁ VĚTA:**

$\longrightarrow$ Res_∞: $f(z) = \frac{\ln^4 1}{z} + \dots \quad \therefore \quad \text{Res}_\infty f(z) = 0 \quad \Rightarrow \quad \boxed{J = 0}$

$\longrightarrow$ C_2: $\text{Res}_{0+i0} f(z) = \ln^4(-1-i0) = \ln^4(e^{-\pi i}) = \pi^4$

$\longrightarrow$ C_5: $\text{Res}_{0-i0} f(z) = \ln^4(-1+i0) = \ln^4(e^{\pi i}) = \pi^4$

- **PARAMETRIZACE:**

$\longrightarrow \ominus(C_1+C_3)$: $z = t+i0;\ t\in(-1,-\varepsilon)\cup(\varepsilon,1)$

$\therefore\ J_1 = \ominus\left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1}\right)\ln^4\left(-\frac{1+t}{1-t} - i0\right)\frac{dt}{t} \xrightarrow{\varepsilon\to 0^+} \ominus \fint_0^1 \left(\ln\frac{1+t}{1-t} - \pi i\right)^4 \frac{dt}{t}$

$\longrightarrow C_4 + C_6$: $z = t - i0;\ t\in(-1,1);\ dz = dt$

$J_2 = \left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1}\right)\ln^4\left(-\frac{1+t}{1-t} + i0\right)\frac{dt}{t} \xrightarrow{\varepsilon\to 0^+} \fint_0^1 \left(\ln\frac{1+t}{1-t} + \pi i\right)^4 \frac{dt}{t}$

$\longrightarrow C_2$: $J_2 \xrightarrow{\varepsilon\to 0^+} \pi i\, \text{Res}_0^{C_2} f(z) = \pi^5 i$

$\longrightarrow C_5$: $J_5 \xrightarrow{\varepsilon\to 0^+} \pi i\, \text{Res}_0^{C_5} f(z) = \pi^5 i$

- **POROVNÁNÍ:** $\left[\begin{aligned}&(A+B)^4 - (A-B)^4 = [(A+B)^2 - (A-B)^2][(A+B)^2 + (A-B)^2] = \\ &= 4AB\,(2A^2 + 2B^2) = 8B\,(A^3 + AB^2)\end{aligned}\right]$

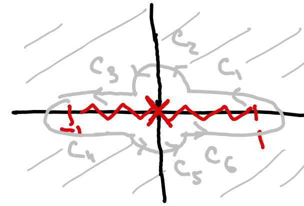
$$0 = 2\pi^5 i + \fint_{-1}^{1} \left(\underset{A}{\ln\frac{1+t}{1-t}} + \underset{B}{\pi i}\right)^4 - \left(\underset{A}{\ln\frac{1+t}{1-t}} - \underset{B}{\pi i}\right)^4 \frac{dt}{t} =$$

$$= 2\pi^5 i + 8\pi i\left[\underbrace{\fint_{-1}^{1} \ln^3\frac{1+t}{1-t}\frac{dt}{t}}_{2I} - \pi^2 \underbrace{\fint_{-1}^{1} \ln\frac{1+t}{1-t}\frac{dt}{t}}_{\text{předchozí: } I_0 = \frac{\pi^2}{2}}\right]$$

$$\therefore\ \boxed{I = \pi^4/8}$$

(Př) $I = \int_{-1}^{1} \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{x} = \frac{\partial}{\partial\alpha}\underbrace{\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^\alpha \frac{dx}{x}}_{I(\alpha)\in\mathbb{R};\ \alpha\in(-1,1)}\Bigg|_{\alpha=0}$

$$J = \oint_C \left(\frac{z+1}{z-1}\right)^\alpha \frac{dz}{z};\quad C:$$



řezy: $\frac{z+1}{z-1} = -t<0$; $z = \frac{t-1}{t+1} \in (-1,1)$

skoky: $\frac{z+1}{z-1}\Big|_{t+i0} \overset{t\in(-1,1)}{=} \frac{z+1}{z-1}\Big|_t + \left(\frac{z+1}{z-1}\right)'\Big|_t i0 = \frac{t+1}{t-1} - \frac{2}{(t-1)^2} i0 = -\frac{1+t}{1-t} - i0$

- RESIDUOVÁ VĚTA

→ pro $z\to\infty$ je $f(z) \simeq \left(1+\frac{2}{z}\right)^\alpha \frac{1}{z} = \frac{1}{z} + \frac{2\alpha}{z^2} + O\left(\frac{1}{z^3}\right)$

∴ $\operatorname{Res}_\infty f = -1$ a $J = -2\pi i \sum_{\sigma\in \mathrm{Ext}\,C\cup\{\infty\}} \operatorname{Res}_\sigma f = -2\pi i(-1) = 2\pi i$

→ $\operatorname{Res}_{0+i0} f = \left(\frac{1+z}{1-z}\right)^\alpha\Big|_{0+i0} = (-1-i0)^\alpha = e^{-\pi i\alpha}$

→ $\operatorname{Res}_{0-i0} f = \left(\frac{1+z}{1-z}\right)^\alpha\Big|_{0-i0} = (-1+i0)^\alpha = e^{\pi i\alpha}$

- PARAMETRIZACE

→ $\ominus(C_1+C_3)$: $z = t+i0$; $t\in(-1,-\varepsilon)\cup(\varepsilon,1)$; $dz=dt$

$$J_1+J_3 = \ominus \fint_{-1}^{1} \left(-\frac{1+t}{1-t} - i0\right)^\alpha \frac{1}{t}\,dt = -e^{-\pi\alpha i}\fint_{-1}^1 \left(\frac{1+t}{1-t}\right)^\alpha \frac{dt}{t} = -e^{-\pi\alpha i} I(\alpha)$$

→ C_4+C_6: $z=t-i0$; $t\in(-1,-\varepsilon)\cup(\varepsilon,1)$; $dz=dt$

$$J_4+J_6 = \fint_{-1}^1\left(-\frac{1+t}{1-t}+i0\right)^\alpha \frac{dt}{t} = e^{\pi\alpha i}\fint_{-1}^1 \left(\frac{1+t}{1-t}\right)^\alpha \frac{dt}{t} = e^{\pi\alpha i}I(\alpha)$$

→ C_2: $J_2 \xrightarrow{\varepsilon\to0^+} \pi i \operatorname{Res}_{0+i0} = \pi i e^{-\pi i\alpha}$

→ C_5: $J_2 \xrightarrow{\varepsilon\to0^+} \pi i\operatorname{Res}_{0-i0} = \pi i e^{\pi i\alpha}$

- POROVNÁNÍ: $2\pi i = I(\alpha)(\underbrace{e^{\pi\alpha i} - e^{-\pi\alpha i}}_{2i\sin\pi\alpha}) + \pi i(\underbrace{e^{\pi\alpha i}+e^{-\pi\alpha i}}_{2\cos\pi\alpha})$

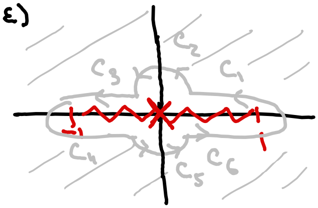
∴ $I(\alpha) = \pi\frac{1-\cos\pi\alpha}{\sin\pi\alpha} = \pi \frac{\frac12(\pi\alpha)^2 - \frac{1}{4!}(\pi\alpha)^4 + O(\alpha^6)}{\pi\alpha - \frac{1}{3!}(\pi\alpha)^3 + O(\alpha^5)} =$

$$= \frac{\pi}{\pi\alpha}\frac12(\pi\alpha)^2\left(1-\frac{\pi^2\alpha^2}{12}+O(\alpha^4)\right)\left(1+\frac{\pi^2\alpha^2}{6}+O(\alpha^4)\right) = \frac{\pi^2\alpha}{2} + \frac{\pi^4\alpha^3}{12} + O(\alpha^5)$$

čili $\boxed{I = \frac{\pi^2}{2}}$ BONUS: $\int_{-1}^1 \ln^3\left(\frac{1+x}{1-x}\right)\frac{dx}{x} = \frac{\partial^3 I}{\partial\alpha^3}\Big|_0 = \frac{\pi^4}{2}$

Př: $I = \int_0^1 \left(\ln\left(\frac{1+t}{1-t}\right) - 2t\right)\frac{dt}{t^3} = \lim_{\varepsilon\to 0^+} \underbrace{\int_\varepsilon^1 \ln\left(\frac{1+t}{1-t}\right)\frac{dt}{t^3}}_{I_1(\varepsilon)} - 2\underbrace{\int_\varepsilon^1 \frac{1}{t^2}\,dt}_{1/\varepsilon}$ $\in\mathbb{R}$!

$J = \oint_C \ln^2\left(\frac{z+1}{z-1}\right)\frac{dz}{z^3}$; C:



řez: $\frac{z+1}{z-1} = -t;\ t>0;\ z = \frac{t-1}{t+1} \in (-1,1)$

skoky: $\frac{z+1}{z-1}\Big|_{t+i0} \overset{t\in(-1,1)}{=} \frac{z+1}{z-1}\Big|_t + \left(\frac{z+1}{z-1}\right)'\Big|_t\, i0 = -\frac{1+t}{1-t} - \frac{2}{(1-t)^2}\, i0 = -\frac{1+t}{1-t} - i0$

- RESIDUOVÁ VĚTA: $f(z) \overset{z\to\infty}{\approx} \ln^2(-1)\frac{1}{z^3} + O\left(\frac{1}{z^4}\right)$ ∴ $\mathrm{Res}_\infty f(z) = 0 \Rightarrow \boxed{J = 0}$
- PARAMETRIZACE:

$\longrightarrow \ominus(C_1+C_3)$: $z = t+i0$; $t\in(-1,-\varepsilon)\cup(\varepsilon,1)$

$$\therefore\ J_1 = \ominus\left(\int_{-1}^{-\varepsilon}+\int_\varepsilon^1\right)\ln^2\left(-\frac{1+t}{1-t}-i0\right)\frac{dt}{t^3} = \ominus\left(\int_{-1}^{-\varepsilon}+\int_\varepsilon^1\right)\left(\ln\frac{1+t}{1-t} - \pi i\right)^2\frac{dt}{t^3}$$

$\longrightarrow C_4+C_6$: $z = t - i0$; $t\in(-1,1)$; $dz = dt$

$$J_2 = \left(\int_{-1}^{-\varepsilon}+\int_\varepsilon^1\right)\ln^2\left(-\frac{1+t}{1-t}+i0\right)\frac{dt}{t^3} = \left(\int_{-1}^{-\varepsilon}+\int_\varepsilon^1\right)\left(\ln\frac{1+t}{1-t}+\pi i\right)^2\frac{dt}{t^3}$$

$\longrightarrow$ ROZVOJ

C_2: [pól 3. ř]: $f(t+i0) = \ln^2\left(-\frac{1+t}{1-t} - i0\right)\frac{1}{t^3} = \left(\ln\frac{1+t}{1-t}-\pi i\right)^2\frac{1}{t^3} =$

$= \left(-\pi i + 2t + O(t^3)\right)^2\frac{1}{t^3} = \frac{-\pi^2}{t^3} - \frac{4\pi i}{t^2} + \frac{4}{t} + O(1)$

C_5: [podobně] $f(t-i0) = \left(\pi i + 2t + O(t^3)\right)^2\frac{1}{t^3} = -\frac{\pi^2}{t^3} + \frac{4\pi i}{t^2} + \frac{4}{t} + O(1)$

∴ C_2+C_5: $z = \varepsilon e^{it}$; $t\in(0,2\pi)$; $dz = i\varepsilon e^{it}\,dt$

$$\Rightarrow J_2+J_5 = \oint_{C_\varepsilon(0)} -\frac{\pi^2}{z^3} + \frac{4}{z} + O(1)\,dz + 2\int_0^\pi \frac{-4\pi i}{(\varepsilon e^{it})^2}\, i\varepsilon e^{it}\,dt = 8\pi i - \frac{16\pi i}{\varepsilon} + O(\varepsilon)$$

- POROVNÁNÍ: $\langle (A+B)^2 - (A-B)^2 = 4AB \rangle$

$$0 = 8\pi i - \frac{16\pi i}{\varepsilon} + \left(\int_{-1}^{-\varepsilon}+\int_\varepsilon^1\right)\left(\ln\frac{1+t}{1-t}+\pi i\right)^2 - \left(\ln\frac{1+t}{1-t}-\pi i\right)^2\frac{dt}{t^3} \overset{\text{sym.}}{=}$$

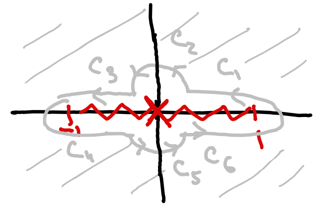
$$= 8\pi i - \frac{16\pi i}{\varepsilon} + 8\pi i\int_\varepsilon^1 \ln\left(\frac{1+t}{1-t}\right)\frac{dt}{t^3} \Rightarrow I_1(\varepsilon) = -1 + \frac{2}{\varepsilon} + O(\varepsilon)$$

$$\therefore\ \boxed{I = \lim_{\varepsilon\to 0^+} I_1(\varepsilon) - \frac{2}{\varepsilon} = -1}$$

(Př) $I = \int_0^1 \ln^3\left(\frac{1+t}{1-t}\right)\frac{dt}{t^3} \overset{\text{sym.}}{=} \frac{1}{2}\int_{-1}^1 \ln^3\left(\frac{1+t}{1-t}\right)\frac{dt}{t^3} \in \mathbb{R}$

[E] předchozí

$J = \oint_C \ln^4\left(\frac{z+1}{z-1}\right)\frac{dz}{z^3}$; C:



řez : $\frac{z+1}{z-1} = -t$; $t>0$; $z = \frac{t-1}{t+1} \in (-1,1)$

skoky: $\left.\frac{z+1}{z-1}\right|_{t+i0} \overset{t\in(-1,1)}{=} \left.\frac{z+1}{z-1}\right|_t + \left.\left(\frac{z+1}{z-1}\right)'\right|_t i0 = -\frac{1+t}{1-t} - \frac{2}{(1-t)^2}\, i0 = -\frac{1+t}{1-t} - i0$

- RESIDUOVÁ VĚTA: $f(z) \overset{z\to\infty}{\approx} \ln^4(-1)\frac{1}{z^3} + O\left(\frac{1}{z^4}\right)$ ∴ $\mathrm{Res}_\infty f(z) = 0$ ⇒ $\boxed{J = 0}$
- PARAMETRIZACE :

→ $\ominus(C_1+C_3)$: $z = t+i0$; $t\in(-1,-\varepsilon)\cup(\varepsilon,1)$

∴ $J_1 = \ominus\left(\int_{-1}^{-\varepsilon}+\int_\varepsilon^1\right)\ln^4\left(-\frac{1+t}{1-t}-i0\right)\frac{dt}{t^3} = \ominus\left(\int_{-1}^{-\varepsilon}+\int_\varepsilon^1\right)\left(\ln\frac{1+t}{1-t}-\pi i\right)^4\frac{dt}{t^3}$

→ C_4+C_6 : $z = t-i0$; $t\in(-1,1)$; $dz = dt$

$J_2 = \left(\int_{-1}^{-\varepsilon}+\int_\varepsilon^1\right)\ln^4\left(-\frac{1+t}{1-t}+i0\right)\frac{dt}{t^3} = \left(\int_{-1}^{-\varepsilon}+\int_\varepsilon^1\right)\left(\ln\frac{1+t}{1-t}+\pi i\right)^4\frac{dt}{t^3}$

→ [Rozvoj] C_2 : [pól 3.] : $f(t+i0) = \ln^4\left(-\frac{1+t}{1-t}-i0\right)\frac{1}{t^3} = \left(\ln\frac{1+t}{1-t}-\pi i\right)^4\frac{1}{t^3} =$

$= \left(-\pi i + 2t + O(t^3)\right)^4\frac{1}{t^3} = \frac{\pi^4}{t^3} + \frac{8\pi^3 i}{t^2} - \frac{24\pi^2}{t} + O(1)$

C_5 : [podobně] $f(t-i0) = \left(\pi i + 2t + O(t^3)\right)^4\frac{1}{t^3} = \frac{\pi^4}{t^3} - \frac{8\pi^3 i}{t^2} - \frac{24\pi^2}{t} + O(1)$

∴ C_2+C_5 : $z = \varepsilon e^{it}$; $t\in(0,2\pi)$; $dz = i\varepsilon e^{it}\,dt$

⇒ $J_2+J_5 = \oint_{C_\varepsilon(0)} \frac{\pi^4}{z^3} - \frac{24\pi^2}{z} + O(1)\,dz + 2\int_0^\pi \frac{8\pi^3 i}{(\varepsilon e^{it})^2}\, i\varepsilon e^{it}\,dt = -48\pi^3 i + \frac{32\pi^3 i}{\varepsilon} + O(\varepsilon)$

- POROVNÁNÍ : $\left[(A+B)^4 - (A-B)^4 = [(A+B)^2-(A-B)^2][(A+B)^2+(A-B)^2] = 4AB(2A^2+2B^2) = 8B(A^3+AB^2)\right]$

$0 = -48\pi^3 i + \frac{32\pi^3 i}{\varepsilon} + \left(\int_{-1}^{-\varepsilon}+\int_\varepsilon^1\right)\left(\Big(\underset{A}{\ln\frac{1+t}{1-t}}+\underset{B}{\pi i}\Big)^2 - \Big(\underset{A}{\ln\frac{1+t}{1-t}}-\underset{B}{\pi i}\Big)^2\right)\frac{dt}{t^3} \overset{\text{sym.}}{=}$

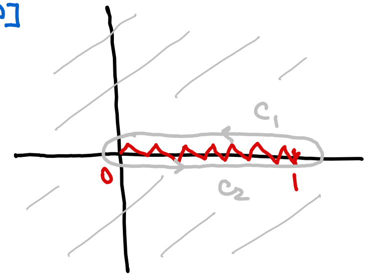
$= -48\pi^3 i + \frac{32\pi^3 i}{\varepsilon} + 16\pi i\left[\underbrace{\int_\varepsilon^1 \ln^3\left(\frac{1+t}{1-t}\right)\frac{dt}{t^3}}_{\xrightarrow{\varepsilon\to0^+}\ I} - \pi^2\underbrace{\int_\varepsilon^1 \ln\left(\frac{1+t}{1-t}\right)\frac{dt}{t^3}}\right] =$

[E] : $I_1(\varepsilon) = -1 + \frac{2}{\varepsilon} + O(\varepsilon)$

$= -48\pi^3 i + \frac{32\pi^3 i}{\varepsilon} + 16\pi i I + 16\pi^3 i - \frac{32\pi^3 i}{\varepsilon} + O(\varepsilon)$ ∴ $\boxed{I = 2\pi^2}$

(P) $I = \int_0^1 \frac{\ln x}{\sqrt{x(1-x)}}\,dx \in \mathbb{R}$ [jiná než (P)]

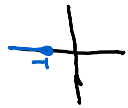
$$J = \oint_C \frac{\ln\frac{z}{z-1} - 2\ln\left(\sqrt{\frac{z}{z-1}}+1\right)}{\sqrt{z}\,\sqrt{z-1}}\,dz$$



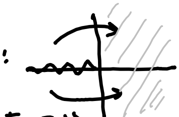
řezy: i) $\frac{z}{z-1} = -t;\ t>0 \Rightarrow z\in(0,1)$; stejně $\sqrt{\frac{z}{1-z}}$

ii) $\sqrt{\frac{z}{z-1}} + 1 = -t$ nemá řez.

Důkaz: $\sqrt{\frac{z}{z-1}} = -t-1$:



ale $\sqrt{w}$:



$\arg(-t-1) = \pm\pi$ vs. $(\arg\sqrt{w} \in (-\frac{\pi}{2}, \frac{\pi}{2}))$

skoky: $\left(\frac{z}{z-1}\right)\Big|_{t+i0} \overset{t\in(0,1)}{=} \frac{z}{z-1}\Big|_t + \left(\frac{z}{z-1}\right)'\Big|_t\, i0 = -\frac{t}{1-t} - \frac{1}{(1-t)^2}\, i0 = -\frac{t}{1-t} - i0$

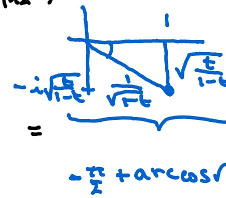
• RESIDUOVÁ VĚTA :

$\longrightarrow z\to\infty$: $f(t) \overset{t\gg1}{=} \frac{\ln(1) - 2\ln 2}{t} + O\left(\frac{1}{t^2}\right)$ ∴ $\mathrm{Res}_\infty f(z) = 2\ln 2$

• PARAMETRIZACE : $\Rightarrow J = -2\pi i\,(2\ln 2) = -4\pi i \ln 2$

$\longrightarrow \ominus C_1$: $z = t + i0$; $t\in(0,1)$; $dz = dt$

$$J_1 = \ominus\int_0^1 \frac{\ln\left(-\frac{t}{1-t} - i0\right) - 2\ln\left(\sqrt{-\frac{t}{1-t} - i0} + 1\right)}{\sqrt{t}\,\sqrt{t+i0-1}}\,dt =$$



$$= \ominus\int_0^1 \frac{\ln\frac{t}{1-t} - \pi i - 2\ln\left(-i\sqrt{\frac{t}{1-t}}+1\right)}{\sqrt{t}\ i\sqrt{1-t}}\,dt = i\int_0^1 \frac{\ln\frac{t}{1-t} - \pi i - 2\ln\sqrt{\frac{t}{1-t}+1} - 2i\arg\left(-i\sqrt{\frac{t}{1-t}}+1\right)}{\sqrt{t}\,\sqrt{1-t}}\,dt$$

$$= i\int_0^1 \frac{\ln\frac{t}{1-t} - \ln\frac{1}{1-t} - 2i\arccos\sqrt{t}}{\sqrt{t}\,\sqrt{1-t}}\,dt = i\underbrace{\int_0^1 \frac{\ln t}{\sqrt{t}\,\sqrt{1-t}}\,dt}_{I} + 2\underbrace{\int_0^1 \frac{\arccos\sqrt{t}}{\sqrt{t}\,\sqrt{1-t}}\,dt}_{\left[-\arccos^2\sqrt{t}\right]_0^1} = iI + 2\pi^2$$

$\longrightarrow C_2$: $z = t - i0$; $t\in(0,1)$; $dz = dt$

$$J_2 = \int_0^1 \frac{\ln\left(-\frac{t}{1-t} + i0\right) - 2\ln\left(\sqrt{-\frac{t}{1-t} + i0} + 1\right)}{\sqrt{t}\,\sqrt{t-i0-1}}\,dt =$$

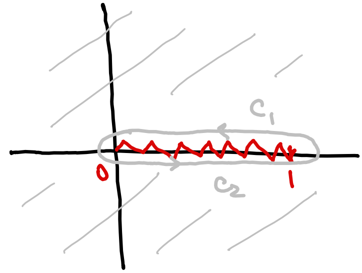
$$= \int_0^1 \frac{\ln\frac{t}{1-t} + \pi i - 2\ln\left(i\sqrt{\frac{t}{1-t}}+1\right)}{\sqrt{t}\,\left(-i\sqrt{1-t}\right)}\,dt = i\int_0^1 \frac{\ln\frac{t}{1-t} + \pi i - 2\ln\sqrt{\frac{t}{1-t}+1} - 2i\arg\left(i\sqrt{\frac{t}{1-t}}+1\right)}{\sqrt{t}\,\sqrt{1-t}}\,dt$$

$$= i\int_0^1 \frac{\ln\frac{t}{1-t} - \ln\frac{1}{1-t} + 2i\arccos\sqrt{t}}{\sqrt{t}\,\sqrt{1-t}}\,dt = i\underbrace{\int_0^1 \frac{\ln t}{\sqrt{t}\,\sqrt{1-t}}\,dt}_{I} - 2\underbrace{\int_0^1 \frac{\arccos\sqrt{t}}{\sqrt{t}\,\sqrt{1-t}}\,dt}_{\pi^2} = iI - 2\pi^2$$

• POROVNÁNÍ : $-4\pi i\ln 2 = 2iI$ ∴ $\boxed{I = -2\pi\ln 2}$

$$\text{(9r)}\quad I = \int_0^1 \sqrt{\frac{x}{1-x}}\,\ln x\,dx \in \mathbb{R}$$

$$\hookrightarrow\; J = \oint_C \sqrt{\frac{z}{z-1}}\left[\ln\frac{z}{z-1} - 2\ln\left(\sqrt{\frac{z}{z-1}}+1\right)\right]dz$$



[řezy a skoky viz předchozí strana]

- RESIDUOVÁ VĚTA

$\longrightarrow z\to\infty:\quad f(z) = \sqrt{1+\frac{1}{z}+O\left(\frac{1}{z^2}\right)}\left[\ln\left(1+\frac{1}{z}+O\left(\frac{1}{z^2}\right)\right) - 2\ln\left(1+\frac{1}{2z}+1+O\left(\frac{1}{z^2}\right)\right)\right] =$

$$= \left(1+\frac{1}{2z}\right)\left(\frac{1}{z} - 2\ln 2 - \frac{1}{2z}\right) + O\left(\frac{1}{z^2}\right) = -2\ln 2 + \frac{\frac{1}{2}-\ln 2}{z} + O\left(\frac{1}{z^2}\right)$$

$$\therefore\; \operatorname{Res}_\infty f = \ln 2 - \frac{1}{2} \quad\Rightarrow\quad J = -2\pi i\left(\ln 2 - \frac{1}{2}\right) = \pi i(1-2\ln 2)$$

- PARAMETRIZACE

$\longrightarrow \ominus C_1:\; z = t+i0;\; t\in(0,1);\; dz = dt$

$$J_1 = \ominus\int_0^1 \sqrt{-\frac{t}{1-t}-i0}\left[\ln\left(-\frac{t}{1-t}-i0\right) - 2\ln\left(\sqrt{-\frac{t}{1-t}-i0}+1\right)\right]dt =$$

$$= \ominus\int_0^1 (-i)\sqrt{\frac{t}{1-t}}\left[\ln\left(\frac{t}{1-t}\right) - \pi i - 2\ln\sqrt{\frac{t}{1-t}+1} - 2\arg\left(-i\sqrt{\frac{t}{1-t}}+1\right)\right]dt$$

$$= i\int_0^1 \sqrt{\frac{t}{1-t}}\left[\ln t - 2i\arccos\sqrt{t}\right]dt$$

$\longrightarrow C_2:\; z = t-i0;\; t\in(0,1);\; dz = dt$

$$J_2 = \int_0^1 \sqrt{-\frac{t}{1-t}+i0}\left[\ln\left(-\frac{t}{1-t}+i0\right) - 2\ln\left(\sqrt{-\frac{t}{1-t}+i0}+1\right)\right]dt =$$

$$= \int_0^1 i\sqrt{\frac{t}{1-t}}\left[\ln\left(\frac{t}{1-t}\right) + \pi i - 2\ln\sqrt{\frac{t}{1-t}+1} - 2\arg\left(i\sqrt{\frac{t}{1-t}}+1\right)\right]dt$$

$$= i\int_0^1 \sqrt{\frac{t}{1-t}}\left[\ln t + 2i\arccos\sqrt{t}\right]dt$$

- POROVNÁNÍ :

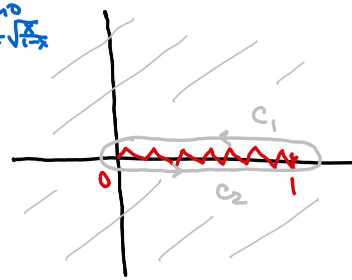
$$\pi i(1-2\ln 2) = 2i\int_0^1 \sqrt{\frac{t}{1-t}}\,\ln t\,dt$$

$$\therefore\; \boxed{I = \frac{\pi}{2}(1-2\ln 2)}$$

Př: $I = \int_0^1 \frac{\ln^2 x}{\sqrt{x}\sqrt{1-x}}\,dx$

Pozn.: J bylo nalezeno na základě $u=\sqrt{\frac{x}{1-x}}$

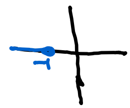
$$J = \oint_C \frac{\left(\ln\frac{z}{z-1} - 2\ln\left(\sqrt{\frac{z}{z-1}}+1\right)\right)^2}{\sqrt{z}\sqrt{z-1}}\,dz$$



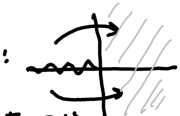
řezy: i) $\frac{z}{z-1} = -t;\ t>0 \Rightarrow z\in(0,1)$; stejně $\sqrt{\frac{z}{1-z}}$

ii) $\sqrt{\frac{z}{z-1}}+1 = -t$ nemá řez.

Důkaz: $\sqrt{\frac{z}{z-1}} = -t-1$:



ale $\sqrt{w}$:

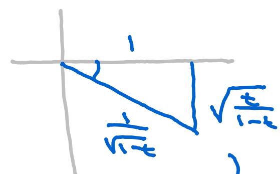


$(\arg\sqrt{w} \in (-\frac{\pi}{2},\frac{\pi}{2}))$

$\arg(-t-1) = \pm\pi$ vs.

skoky: $\left(\frac{z}{z-1}\right)\Big|_{t+i0} \overset{t\in(0,1)}{=} \frac{z}{z-1}\Big|_t + \left(\frac{z}{z-1}\right)'\Big|_t i0 = -\frac{t}{1-t} - \frac{1}{(1-t)^2}i0 = -\frac{t}{1-t} - i0$

- RESIDUOVÁ VĚTA: $f(z) \approx \frac{4\ln^2 2}{z} + O\left(\frac{1}{z^2}\right) \therefore \mathrm{Res}_\infty f(z) = -4\ln^2 2 \Rightarrow J = 8\pi i \ln^2 2$
- PARAMETRIZACE:



$\longrightarrow \ominus\ C_1: z = t+i0;\ t\in(0,1);\ dz = dt$

$$J_1 = \ominus\int_0^1 \frac{\left(\ln\left(-\frac{t}{1-t}-i0\right) - 2\ln\left(-i\sqrt{\frac{t}{1-t}}+1\right)\right)^2}{\sqrt{t}\sqrt{t+i0-1}}\,dt =$$

arcsin$(-\sqrt{t})$

$$=\ominus\int_0^1 \frac{\left(\ln\frac{t}{1-t} - \pi i - \ln\left(\frac{t}{1-t}+1\right) - 2i\arg\left(-i\sqrt{\frac{t}{1-t}}+1\right)\right)^2}{i\sqrt{t}\sqrt{1-t}}\,dt =$$

$$= i\int_0^1 \frac{(\ln t - 2i\arccos\sqrt{t})^2}{\sqrt{t}\sqrt{1-t}}\,dt = iI + 4\int_0^1\frac{\ln t\arccos\sqrt{t}}{\sqrt{t}\sqrt{1-t}}\,dt - 4i\underbrace{\int_0^1 \frac{\arccos^2\sqrt{t}}{\sqrt{t}\sqrt{1-t}}\,dt}_{\underbrace{\left[-\frac{2}{3}\arccos^3\sqrt{t}\right]_0^1}_{\frac{\pi^3}{12}}}$$

$\longrightarrow C_2: z = t-i0;\ t\in(0,1);\ dz = dt$

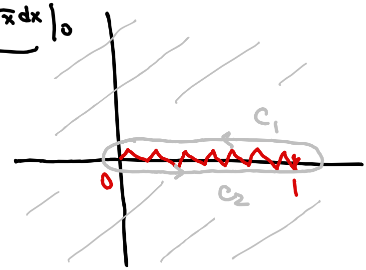
$$J_2 = \int_0^1 \frac{\left(\ln\left(-\frac{t}{1-t}+i0\right) - 2\ln\left(i\sqrt{\frac{t}{1-t}}+1\right)\right)^2}{\sqrt{t}\sqrt{t-i0-1}}\,dt =$$

$$=\int_0^1 \frac{\left(\ln\frac{t}{1-t}+\pi i - \ln\left(\frac{t}{1-t}+1\right) - 2i\arg\left(i\sqrt{\frac{t}{1-t}}+1\right)\right)^2}{-i\sqrt{t}\sqrt{1-t}}\,dt =$$

$$= i\int_0^1\frac{(\ln t + 2i\arccos\sqrt{t})^2}{\sqrt{t}\sqrt{1-t}}\,dt = iI - 4\int_0^1\frac{\ln t \arccos\sqrt{t}}{\sqrt{t}\sqrt{1-t}}\,dt - 4i\int_0^1\frac{\arccos^2\sqrt{t}}{\sqrt{t}\sqrt{1-t}}\,dt$$

- POROVNÁNÍ: $8\pi i\ln^2 2 = 2iI - \frac{2i\pi^3}{3} \therefore \boxed{I = 4\pi\ln^2 2 + \frac{\pi^3}{3}}$

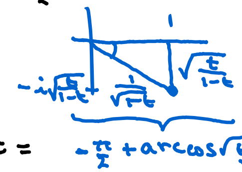
(P) $$I=\int_0^1 \ln^2\frac{x}{1-x}\,\arccos\sqrt{x}\,\underset{\in\mathbb{R}}{dx} = \frac{\partial^2}{\partial\alpha^2}\underbrace{\int_0^1\left(\frac{x}{1-x}\right)^{\alpha}\arccos\sqrt{x}\,dx}_{I(\alpha)}\Bigg|_0$$



$$\hookrightarrow J=\oint_C\left(\frac{z}{z-1}\right)^{\alpha}\left[\ln\frac{z}{z-1}-2\ln\left(\sqrt{\frac{z}{z-1}}+1\right)\right]dz$$

[Řezy a skoky viz předchozí strana]

- RESIDUOVÁ VĚTA : $f(z)\approx\left(1+\frac{\alpha}{z}\right)\left(\frac{1}{z}-2\ln 2-\frac{1}{2z}\right)\ \therefore\ \mathrm{Res}_\infty f = 2\alpha\ln 2-\frac{1}{2}\ \therefore\ J=\pi i(1-4\alpha\ln 2)$
- PARAMETRIZACE :



$\longrightarrow \ominus C_1$: $z=t+i0;\ t\in(0,1);\ dz=dt$

$$J_1=\ominus\int_0^1\left(-\frac{t}{1-t}-i0\right)^{\alpha}\left[\ln\left(-\frac{t}{1-t}-i0\right)-2\ln\left(\sqrt{-\frac{t}{1-t}-i0}+1\right)\right]dt=$$

$$=\ominus\int_0^1 e^{-\pi\alpha i}\left(\frac{t}{1-t}\right)^{\alpha}\left[\ln\frac{t}{1-t}-\pi i-2\ln\left(-i\sqrt{\frac{t}{1-t}}+1\right)\right]dt \overset{\cdots}{=} -e^{-\pi\alpha i}\int_0^1\left(\frac{t}{1-t}\right)^{\alpha}\left(\ln t-2i\arccos\sqrt{t}\right)dt$$

$$=-e^{-\pi\alpha i}\underbrace{\int_0^1\left(\frac{t}{1-t}\right)^{\alpha}\ln t\,dt}_{I_0(\alpha)\in\mathbb{R}}+2ie^{-\pi\alpha i}\underbrace{\int_0^1\left(\frac{t}{1-t}\right)^{\alpha}\arccos\sqrt{t}\,dt}_{I(\alpha)}=e^{-\pi\alpha i}\left(-I_0(\alpha)+2iI(\alpha)\right)$$

$\longrightarrow C_2$: $z=t-i0;\ t\in(0,1);\ dz=dt$

$$J_2=\int_0^1\left(-\frac{t}{1-t}+i0\right)^{\alpha}\left[\ln\left(-\frac{t}{1-t}+i0\right)-2\ln\left(\sqrt{-\frac{t}{1-t}+i0}+1\right)\right]dt=$$

$$=\int_0^1 e^{\pi\alpha i}\left(\frac{t}{1-t}\right)^{\alpha}\left[\ln\frac{t}{1-t}+\pi i-2\ln\left(i\sqrt{\frac{t}{1-t}}+1\right)\right]dt\overset{\cdots}{=}e^{\pi\alpha i}\int_0^1\left(\frac{t}{1-t}\right)^{\alpha}\left(\ln t+2i\arccos\sqrt{t}\right)dt$$

$$=e^{\pi\alpha i}\left(I_0(\alpha)+2iI(\alpha)\right)$$

- POROVNÁNÍ: $\pi i(1-4\alpha\ln 2)=I_0(\alpha)\big(\overbrace{e^{\pi\alpha i}-e^{-\pi\alpha i}}^{2i\sin\pi\alpha}\big)+2iI(\alpha)\big(\overbrace{e^{\pi\alpha i}+e^{-\pi\alpha i}}^{2\cos\pi\alpha}\big)$

$$\therefore I(\alpha)=\frac{\pi}{4}(1-4\alpha\ln 2)\sec\pi\alpha-\frac{1}{2}I_0(\alpha)\,\mathrm{tg}\,\pi\alpha=\frac{\pi}{4}(1-4\alpha\ln 2)\left(1+\frac{1}{2}(\pi\alpha)^2\right)-\frac{1}{2}\left(I_0(0)+I_0'(0)\alpha+O(\alpha^2)\right)\left(\pi\alpha+O(\alpha^3)\right)$$

$$=\frac{\pi}{4}-\frac{\pi\alpha}{2}I_0(0)-\pi\alpha\ln 2+\frac{\pi^3\alpha^2}{8}-\frac{\pi\alpha^2}{2}I_0'(0)+O(\alpha^3)\quad\Big/\frac{\partial^2}{\partial\alpha^2}$$

$$\therefore I=\frac{\partial^2 I(\alpha)}{\partial\alpha^2}\Big|_0=\frac{\pi^3}{4}-\pi I_0'(0)=\frac{\pi^3}{4}-\pi\int_0^1\ln\frac{t}{1-t}\ln t\,dt=\frac{\pi^3}{4}-\pi\left(\int_0^1\ln^2 t\,dt-\int_0^1\ln t\ln(1-t)\,dt\right)=$$

$$=\frac{\pi^3}{4}-\pi\left(2-\left(2-\frac{\pi^2}{6}\right)\right)=\frac{\pi^3}{4}-\frac{\pi^3}{6}=\frac{\pi^3}{12}$$

$$\boxed{I=\frac{\pi^3}{12}}$$

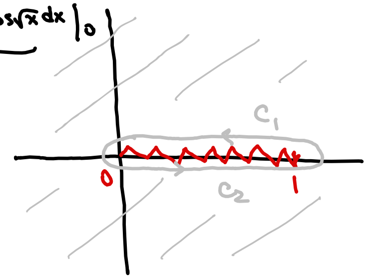
BONUS: $-4\pi i\ln 2=2iI_0'\left(\tfrac{1}{2}\right)-4\pi i\,I\left(\tfrac{1}{2}\right)$

$$\therefore I_0'\left(\tfrac{1}{2}\right)=\int_0^1\sqrt{\frac{t}{1-t}}\ln\frac{t}{1-t}\ln t\,dt=-2\pi\ln 2+2\pi I\left(\tfrac{1}{2}\right)=-2\pi\ln 2+2\pi\int_0^1\sqrt{\frac{x}{1-x}}\arccos\sqrt{x}\,dx$$

$$=\left|\,x=\cos^2\frac{u}{2}\,\right|=-2\pi\ln 2+\frac{\pi}{2}\int_0^{\pi}(1+\cos u)\,u\,du=-2\pi\ln 2+\frac{\pi^3}{4}-\pi$$

(P_{V}^{-}) $$I = \int_0^1 \ln\frac{x}{1-x}\ln x \arccos\sqrt{x}\,dx = \frac{\partial}{\partial\alpha}\underbrace{\int_0^1\left(\frac{x}{1-x}\right)^{\alpha}\ln x \arccos\sqrt{x}\,dx}_{I(\alpha)}\Big|_0 \quad \alpha \in \mathbb{R}$$

$$J = \oint_C \left(\frac{z}{z-1}\right)^{\alpha}\left[\ln\frac{z}{z-1} - 2\ln\left(\sqrt{\frac{z}{z-1}}+1\right)\right]^2 dz$$

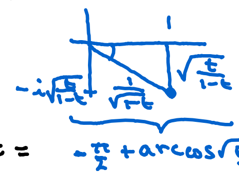


[řezy + skoky vizte předchozí stránka]

- **RESIDUOVÁ VĚTA**:

$$z\to\infty:\ f(z) \simeq \left(1+\tfrac{1}{z}\right)^{\alpha}\left[\ln\left(1+\tfrac{1}{z}\right) - 2\ln\left(1+\tfrac{1}{2z}+1\right)\right]^2 = \left(1+\tfrac{\alpha}{z}\right)\left[\tfrac{1}{z}-2\ln 2-\tfrac{1}{2z}\right]^2 + O\left(\tfrac{1}{z^2}\right)$$
$$= \left(1+\tfrac{\alpha}{z}\right)\left(4\ln^2 2 - \tfrac{2\ln 2}{z}\right) + O\left(\tfrac{1}{z^2}\right) = 4\ln^2 2 + \frac{4\alpha\ln^2 2 - 2\ln 2}{z} + O\left(\tfrac{1}{z^2}\right)$$
$$\therefore\ \operatorname{Res}_\infty f = 2\ln 2\,(1-2\alpha\ln 2) \implies J = 4\pi i\ln 2\,(2\alpha\ln 2 - 1)$$

- **PARAMETRIZACE**:



$\longrightarrow \ominus C_1:\ z = t+i0;\ t\in(0,1);\ dz = dt$

$$J_1 = \ominus\int_0^1\left(-\tfrac{t}{1-t}-i0\right)^{\alpha}\left[\ln\left(-\tfrac{t}{1-t}-i0\right) - 2\ln\left(\sqrt{-\tfrac{t}{1-t}-i0}+1\right)\right]^2 dt =$$
$$= \ominus\int_0^1 e^{-\pi\alpha i}\left(\tfrac{t}{1-t}\right)^{\alpha}\left[\ln\tfrac{t}{1-t} - \pi i - 2\ln\left(-i\sqrt{\tfrac{t}{1-t}}+1\right)\right]^2 dt \overset{\because}{=} -e^{-\pi\alpha i}\int_0^1\left(\tfrac{t}{1-t}\right)^{\alpha}\left(\ln t - 2i\arccos\sqrt{t}\right)^2 dt$$
$$= -e^{-\pi\alpha i}\underbrace{\int_0^1\left(\tfrac{t}{1-t}\right)^{\alpha}\ln^2 t\,dt}_{I_0(\alpha)\in\mathbb{R}} + 4ie^{-\pi\alpha i}\underbrace{\int_0^1\left(\tfrac{t}{1-t}\right)^{\alpha}\ln t\arccos\sqrt{t}\,dt}_{I(\alpha)} + 4e^{-\pi\alpha i}\underbrace{\int_0^1\left(\tfrac{t}{1-t}\right)^{\alpha}\arccos^2\sqrt{t}\,dt}_{I_1(\alpha)\in\mathbb{R}}$$

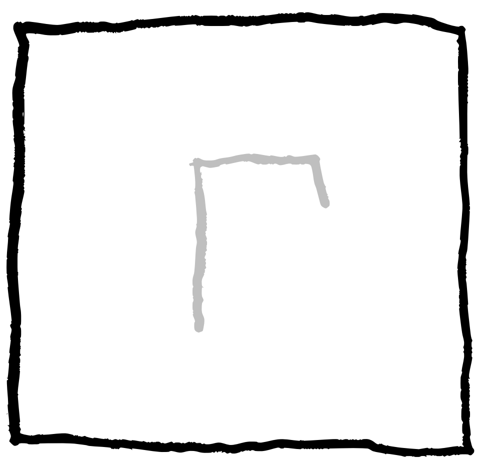
$\longrightarrow C_2:\ z = t-i0;\ t\in(0,1);\ dz = dt$

$$J_2 = \int_0^1\left(-\tfrac{t}{1-t}+i0\right)^{\alpha}\left[\ln\left(-\tfrac{t}{1-t}+i0\right) - 2\ln\left(\sqrt{-\tfrac{t}{1-t}+i0}+1\right)\right]^2 dt =$$
$$= \int_0^1 e^{\pi\alpha i}\left(\tfrac{t}{1-t}\right)^{\alpha}\left[\ln\tfrac{t}{1-t} + \pi i - 2\ln\left(i\sqrt{\tfrac{t}{1-t}}+1\right)\right]^2 dt \overset{\because}{=} e^{\pi\alpha i}\int_0^1\left(\tfrac{t}{1-t}\right)^{\alpha}\left(\ln t + 2i\arccos\sqrt{t}\right)^2 dt$$
$$= e^{\pi\alpha i}\left(I_0(\alpha) + 4iI(\alpha) - 4I_1(\alpha)\right)$$

- **POROVNÁNÍ**:

$$4\pi i\ln 2\,(2\alpha\ln 2-1) = I_0(\alpha)\overbrace{\left(e^{\pi\alpha i}-e^{-\pi\alpha i}\right)}^{2i\sin\pi\alpha} + 4iI(\alpha)\overbrace{\left(e^{\pi\alpha i}+e^{-\pi\alpha i}\right)}^{2\cos\pi\alpha} - 4J_1(\alpha)\overbrace{\left(e^{\pi\alpha i}-e^{-\pi\alpha i}\right)}^{2i\sin\pi\alpha}$$
$$\therefore\ I(\alpha) = \tfrac{\pi}{2}\ln 2\,(2\alpha\ln 2-1)\underbrace{\sec\pi\alpha}_{1+O(\alpha^2)} + \left(I_1(\alpha)-\tfrac{1}{4}I_0(\alpha)\right)\underbrace{\operatorname{tg}\pi\alpha}_{\pi\alpha+O(\alpha^3)} = \tfrac{\pi}{2}\ln 2(2\alpha\ln 2-1) + \pi\alpha\left(I_1(0)-\tfrac{1}{4}I_0(0)\right) + O(\alpha^2)$$
$$\therefore\ I = \frac{\partial I(\alpha)}{\partial\alpha}\Big|_0 = \pi\ln^2 2 + \pi I_1(0) - \tfrac{\pi}{4}I_0(0) = \pi\ln^2 2 + \pi\underbrace{\int_0^1\arccos^2\sqrt{t}\,dt}_{\frac{\pi^2}{8}-\frac{1}{2}} - \tfrac{\pi}{4}\underbrace{\int_0^1\ln^2 t\,dt}_{2!}$$
$$\therefore\ \boxed{I = \pi\ln^2 2 - \pi + \frac{\pi^3}{8}}$$

$$\left\langle \int_0^1\arccos^2\sqrt{t}\,dt = \left|\begin{matrix} t=\cos^2 u \\ dt = -2\cos u\sin u\,du\end{matrix}\right| = 2\int_0^{\pi/2}u^2\sin u\cos u\,du = \left|\begin{matrix} u=\theta/2 \\ du=\frac{1}{2}d\theta\end{matrix}\right| = \right.$$
$$\left. = \frac{1}{8}\int_0^{\pi}\theta^2\sin\theta\,d\theta \overset{PP}{=} \left|\begin{matrix} \theta^2 & + & \sin\theta \\ 2\theta & - & -\cos\theta \\ 2 & + & -\sin\theta \\ 0 & & \cos\theta\end{matrix}\right| = \frac{1}{8}\left(-\theta^2\cos\theta + 2\cos\theta\right)\Big|_0^{\pi} = \frac{\pi^2}{8} - \frac{1}{2}\right\rangle$$



$\boxed{D}$ (GAMMA FCE) : $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\, dt$; $\operatorname{Re} z > 0$

$\boxed{V}$ (VLASTNOSTI Γ FCE)

i) Speciální hodnoty : $\Gamma(n) = (n-1)!$; $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$; $n = 1, 2, 3, \dots$

ii) Rekurence : $\Gamma(z+1) = z\,\Gamma(z)$ (dodefinovává Γ pro $\forall z \in \mathbb{C}$)

iii) Reflekční forule : $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$; $\forall z \in \mathbb{C} / \mathbb{Z}$

iv) Duplicitní formule : $\Gamma(2z) = \dfrac{1}{\sqrt{\pi}}\, 2^{2z-1}\, \Gamma(z)\Gamma(z+\tfrac{1}{2})$

v) Weierstrassův součin : $\Gamma(z) = \dfrac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \dfrac{e^{\frac{z}{n}}}{1+\frac{z}{n}}$

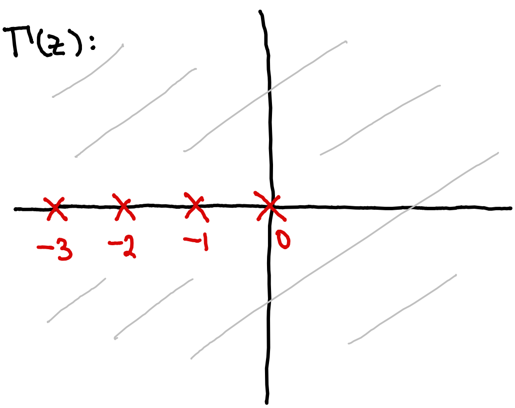
$$\Big[\, \gamma := \lim_{n\to\infty} \Big(\ln n - \underbrace{\sum_{k=1}^{n} \frac{1}{k}}_{H_n} \Big) \quad \text{Euler-Mascheroni} \,\Big]$$

vi) Póly : $\lim_{z \to -n} (z+n)\,\Gamma(z) = \dfrac{(-1)^n}{n!}$; $n = 0, 1, 2, \dots$

vii) Asymptotika v ∞ :

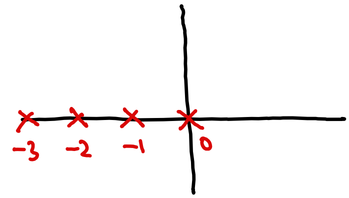
$$\Gamma(z) \cong \sqrt{\frac{2\pi}{z}}\, z^z e^{-z}$$

$\Gamma(z)$:



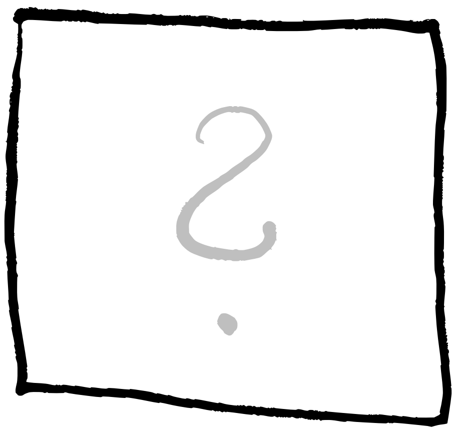
Př $\sum_{n=0}^{\infty} \frac{g(n)}{n!}(-1)^n$

$\hookrightarrow$ $J = \oint_C g(-z)\Gamma(z)\,dz$



- <u>RESIDUOVÁ VĚTA</u>

$\mathrm{Res}_{-n} f(z) = \lim_{z \to -n} (z+n) g(-z)\Gamma(z) = g(n)\frac{(-1)^n}{n!}$



MISCELLANEOUS

Př. $I = \int_0^1 \frac{\ln^2 x}{\sqrt{x(1-x)}}\,dx \in \mathbb{R}$

užijeme substituce $u = \sqrt{\frac{x}{1-x}}$; čili $x = \frac{u^2}{1+u^2}$; $dx = \frac{2u}{(1+u^2)^2}\,du$

$$\therefore I = \int_0^1 \frac{\ln^2 x}{x}\sqrt{\frac{x}{1-x}}\,dx = \int_0^\infty \frac{\ln^2\left(\frac{u^2}{1+u^2}\right)}{\frac{u^2}{1+u^2}}\, u\,\frac{2u}{(1+u^2)}\,du =$$

$$= 2\int_0^\infty \frac{\ln^2\frac{u^2}{1+u^2}}{1+u^2}\,du = 2\int_0^\infty \frac{(2\ln u - \ln(1+u^2))^2}{1+u^2}\,du =$$

$$= 8\int_0^\infty \frac{\ln^2 u}{1+u^2}\,du - 8\int_0^\infty \frac{\ln u\,\ln(1+u^2)}{1+u^2}\,du + 2\int_0^\infty \frac{\ln^2(1+u^2)}{1+u^2}\,du$$

$$= 8\left(\frac{\pi^3}{8}\right) - 8\left(\frac{\pi^3}{8}\right) + 2\left(2\pi\ln^2 2 + \frac{\pi^3}{6}\right) = 4\pi\ln^2 2 + \frac{\pi^3}{3}$$

Př. $I = \int_0^1 \frac{\ln(1-x+x^2)}{x-x^2}\,dx \in \mathbb{R}$ $\left\langle \text{PZ}: \frac{1}{x-x^2} = \frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}\right\rangle$;

$$\text{PP}: I = \cancelto{0}{\ln(1-x+x^2)(\ln x - \ln(1-x))\Big|_0^1} - \int_0^1 \frac{2x-1}{1-x+x^2}\ln\left(\frac{x}{1-x}\right)dx = -\frac{\pi^2}{9}$$

Př. $I = \int_0^\infty \frac{x\ln^2 x}{x^4+x^2+1}\,dx = \left|\begin{matrix} x = \sqrt{u} \\ dx = \frac{1}{2\sqrt{u}}du \end{matrix}\right| = \frac{1}{8}\int_0^\infty \frac{\ln^2 u}{u^2+u+1}\,du =$

Př. $I = \int_0^\infty \frac{\operatorname{arctg}^3 x}{x^3}\,dx \overset{PP}{=} -\cancelto{0}{\frac{\operatorname{arctg}^3 x}{2x^2}}\Big|_0^\infty + \frac{3}{2}\int_0^\infty \frac{\operatorname{arctg}^2 x}{x^2(1+x^2)}\,dx \overset{PZ}{=}$

$$= \frac{3}{2}\int_0^\infty \frac{\operatorname{arctg}^2 x}{x^2}\,dx - \frac{3}{2}\int_0^\infty \frac{\operatorname{arctg}^2 x}{1+x^2}\,dx \overset{PP}{=}$$

$$= -\frac{3}{2}\left[\frac{1}{x}\operatorname{arctg}^2 x\right]_0^\infty + 3\underbrace{\int_0^\infty \frac{\operatorname{arctg} x}{x(1+x^2)}\,dx} - \frac{1}{2}\left[\operatorname{arctg}^3 x\right]_0^\infty = \frac{3\pi}{2}\ln 2 - \frac{\pi^3}{16}$$

Př. $\int_0^1 \ln x\,\ln(1-x)\,dx = \frac{1}{2}\int_0^1 \ln^2 x + \ln^2(1-x)\,dx - \frac{1}{2}\int_0^1 (\ln x - \ln(1-x))^2 =$

$$= \frac{1}{2}\left[2! + 2! - \int_0^1 \ln^2\frac{x}{1-x}\,dx\right] = \frac{1}{2}\left[4 - \frac{\pi^2}{3}\right] = 2 - \frac{\pi^2}{6}$$

Př. $I = \int_0^1 (\ln^2 x - \ln^2(1-x))\arccos\sqrt{x}\,dx$

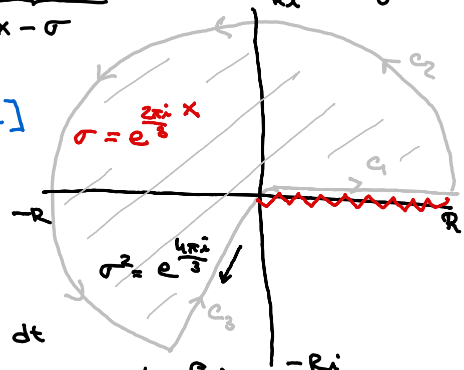
trik: $\ln^2 x - \ln^2(1-x) = (\ln x - \ln(1-x))(\underbrace{\ln x + \ln(1-x)}_{2\ln x - (\ln x - \ln(1-x))}) = \ln\frac{x}{1-x}\left(2\ln x - \ln\frac{x}{1-x}\right)$

$$\therefore I = 2\underbrace{\int_0^1 \ln\frac{x}{1-x}\ln x\,\arccos\sqrt{x}\,dx}_{\pi\ln^2 2 - \pi + \frac{\pi^3}{8}} - \underbrace{\int_0^1 \ln^2\frac{x}{1-x}\arccos\sqrt{x}}_{\pi^3/12} = 2\pi\ln^2 2 - 2\pi + \frac{\pi^3}{6}$$

Př $I = \int_0^\infty \frac{\ln^2 x}{x^2+x+1}\,dx \in \mathbb{R}$ [Nevhodná zdlouhavá metoda]

trik : $\frac{1}{x^2+x+1} = \frac{1}{(x+\frac12)^2+\frac34} = \frac{2}{\sqrt3}\operatorname{Im}\underbrace{\frac{1}{x+\frac12-\frac{\sqrt3}{2}i}}_{x-\sigma} = \frac{2}{\sqrt3}\operatorname{Im}\frac{1}{x-\underbrace{e^{2\pi i/3}}_{\sigma}}$

$J = \oint_C \frac{\ln^2 z}{z-}\,dz$; C :



- RESIDUOVÁ VĚTA : $[\sigma = e^{\frac{2\pi i}{3}} = -\frac12+\frac{\sqrt3}{2}i]$

$\longrightarrow$ $\operatorname{Res}_\sigma f(z) = \ln^2 z\big|_\sigma = \ln^2 e^{\frac{2\pi i}{3}} = -\frac{4\pi^2}{9}$

$\therefore J = 2\pi i \sum_{\sigma \in \mathrm{Int}\,c} \operatorname{Res}_\sigma f(z) = -\frac{8\pi^3 i}{9}$

- PARAMETRIZACE :

$\longrightarrow$ $c_1 : z = t + i0$; $t \in (0,R)$; $dz = dt$

$J_1 = \int_0^R \frac{\ln^2(t+i0)}{t-\sigma}\,dt = \int_0^R \frac{\ln^2 t\,(t+\frac12+\frac{\sqrt3}{2}i)}{t^2+t+1}\,dt =$

$= \int_0^R \frac{(t+\frac12)\ln^2 t}{t^2+t+1}\,dt + \frac{\sqrt3}{2}i\int_0^R \frac{\ln^2 t}{t^2+t+1}\,dt = I_0(R) + \frac{\sqrt3}{2}i\,I + O\left(\frac{\ln^2 R}{R}\right)$ lim

$\longrightarrow$ $c_2 : z = Re^{it}$; $t \in (0, \frac{4\pi}{3})$; $dz = Ri e^{it}\,dt$

$J_2 = \int_0^{\frac{4\pi}{3}} \frac{\ln^2(Re^{it})}{Re^{it}-\sigma} Rie^{it}\,dt = \int_0^{\frac{4\pi}{3}} (\ln R + it)^2 \left(1-\frac{\sigma}{R}e^{-it}\right)^{-1} i\,dt =$

$= i\int_0^{\frac{4\pi}{3}} (\ln^2 R + 2it\ln R - t^2)(1+O(\tfrac1R))\,dt = \frac{4\pi i}{3}\ln^2 R - \frac{16\pi^2}{9}\ln R - \frac{64\pi^3 i}{81} + O\left(\frac{\ln^2 R}{R}\right)$

$\longrightarrow$ $\ominus\, c_3 : z = te^{\frac{4\pi i}{3}}$; $t \in (0,R)$; $dz = e^{\frac{4\pi i}{3}}\,dt$

$J_3 = \ominus\int_0^R \frac{\ln^2(te^{\frac{4\pi i}{3}})}{te^{\frac{4\pi i}{3}} - e^{\frac{2\pi i}{3}}} e^{\frac{4\pi i}{3}}\,dt = -\int_0^R \frac{(\ln t + \frac{4\pi i}{3})^2}{t+\frac12+\frac{\sqrt3}{2}i}\,dt =$

$= -\underbrace{\int_0^R \frac{(t+\frac12)\ln^2 t}{t^2+t+1}\,dt}_{I_0(R)\in\mathbb{R},\ \text{diverguje}} + \frac{\sqrt3}{2}i\int_0^R \frac{\ln^2 t}{t^2+t+1}\,dt - \frac{8\pi i}{3}\underbrace{\int_0^R \frac{(t+\frac12)\ln t}{t^2+t+1}\,dt}_{I_1(R)\in\mathbb{R}\ \text{ale diverguje}}$

$- \frac{4\pi}{\sqrt3}\underbrace{\int_0^R \frac{\ln t}{t^2+t+1}\,dt}_{\xrightarrow{R\to\infty} I_2 \in \mathbb{R}} + \frac{16\pi^2}{9}\underbrace{\int_0^R \frac{t+\frac12}{t^2+t+1}\,dt}_{\ln(R^2+R+1)} - \frac{8\pi^2 i}{3\sqrt3}\underbrace{\int_0^R \frac{dt}{t^2+t+1}}_{\xrightarrow{R\to\infty} I_3 \in \mathbb{R}}$

- POROVNÁNÍ : $-\frac{8\pi^3}{9}i = \sqrt3 i\,I + \frac{4\pi i}{3}\ln^2 R - \frac{16\pi^2}{9}\ln R - \frac{64\pi^3 i}{81} - \frac{8\pi i}{3}I_1(R) - \frac{4\pi}{\sqrt3}I_2 +$

$+ \frac{16\pi^2}{9}\ln(R^2+R+1) - \frac{8\pi^2 i}{3\sqrt3}I_3 + O\left(\frac{\ln^2 R}{R}\right)$ // $\boxed{\operatorname{Im}:}$

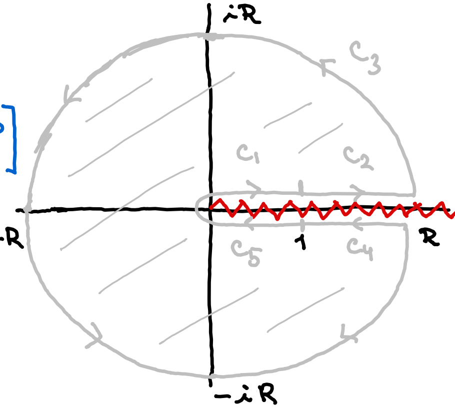
$I = \frac{1}{\sqrt3}\lim_{R\to\infty}\left[-\frac{8\pi^3}{9} - \frac{4\pi}{3}\ln^2 R + \frac{64\pi^3}{81} + \frac{8\pi}{3}I_1(R) + \frac{8\pi^2}{3\sqrt3}I_3\right]$

asi by to šlo
... ale potřebuji $I_1(R)$ a I_3

Př: $I = \int_0^1 \frac{\ln^2 x}{\sqrt{x(1-x)}}\,dx \in \mathbb{R}$ $\boxed{?!}$

$$J := \oint_C \frac{\ln^2 z}{\sqrt{z}\sqrt{z-1}}\,dz\ ; \quad C:$$

[vyřešeno jinak]



- CAUCHYHO VĚTA: $J = 0$
- PARAMETRIZACE:

$\longrightarrow$ C_1: $z = t+i0$; $t\in(0,1)$; $dz = dt$

$$J_1 = \int_0^1 \frac{\ln^2(t+i0)}{\sqrt{t+i0}\,\underbrace{\sqrt{t-1}}_{i\sqrt{1-t}}}\,dt = \frac{1}{i}\int_0^1 \frac{\ln^2 t}{\sqrt{t}\sqrt{1-t}}\,dt = -iI$$

$\longrightarrow$ C_2: $z = t+i0$; $t\in(1,R)$; $dz = dt$

$$J_2 = \int_1^R \frac{\ln^2(t+i0)}{\sqrt{t+i0}\sqrt{t-1+i0}}\,dt = \int_1^R \frac{\ln^2 t}{\sqrt{t}\sqrt{t-1}}\,dt \quad (\text{diverguje pro } R\to\infty)$$

$\longrightarrow$ C_3: $z = Re^{it}$; $t\in(0,2\pi)$; $dz = Rie^{it}dt$

$$J_3 = \int_0^{2\pi} \frac{\ln^2(Re^{it})}{\sqrt{Re^{it}}\sqrt{Re^{it}-1}}\,Rie^{it}dt = i\int_0^{2\pi}(\ln R + it)^2\left(1-\tfrac{1}{R}e^{-it}\right)^{-\frac12}dt =$$

$$= i\int_0^{2\pi}(\ln R+it)^2\left(1+O\left(\tfrac{1}{R}\right)\right)dt = 2\pi i\ln^2 R - 2\pi^2\ln R - \frac{8\pi^3 i}{3} + O\left(\frac{\ln^2 R}{R}\right)$$

$\longrightarrow$ $\ominus C_4$: $z = t-i0$; $t\in(1,R)$; $dz = dt$

$$J_4 = \ominus\int_1^R \frac{\ln^2(t-i0)\,dt}{\underbrace{\sqrt{t-i0}}_{-\sqrt{t}}\,\underbrace{\sqrt{t-1-i0}}_{-\sqrt{t-1}}} = -\int_1^R \frac{(\ln t+2\pi i)^2}{\sqrt{t}\sqrt{t-1}}\,dt = -\int_1^R \frac{\ln^2 t}{\sqrt{t}\sqrt{t-1}}\,dt - 4\pi i\underbrace{\int_1^R\frac{\ln t\,dt}{\sqrt{t}\sqrt{t-1}}}_{I_2(R)\in\mathbb{R}} + 4\pi^2\underbrace{\int_1^R\frac{dt}{\sqrt{t}\sqrt{t-1}}}_{I_1(R)\in\mathbb{R}}$$

$\longrightarrow$ $\ominus C_5$: $z = t-i0$; $t\in(0,1)$; $dz = dt$

$$J_5 = \ominus\int_0^1 \frac{\ln(t-i0)}{\underbrace{\sqrt{t-i0}}_{-\sqrt{t}}\,\underbrace{\sqrt{t-1}}_{i\sqrt{1-t}}}\,dt = \frac{1}{i}\int_0^1\frac{(\ln t+2\pi i)^2}{\sqrt{t}\sqrt{1-t}}\,dt = -iI + 4\pi\underbrace{\int_0^1\frac{\ln t\,dt}{\sqrt{t(1-t)}}}_{-2\pi\ln 2} + 4\pi^2 i\underbrace{\int_0^1\frac{dt}{\sqrt{t}\sqrt{1-t}}}_{\pi}$$

- POROVNÁNÍ:

$$0 = -2iI + 2\pi i\ln^2 R - 2\pi^2\ln R - \frac{8\pi^3 i}{3} - 4\pi i I_2(R) + 4\pi^2 I_1(R) - 8\pi^2\ln 2 + \frac{4\pi^3 i}{3} + O\left(\frac{\ln^2 R}{R}\right)$$

$\boxed{\mathrm{Im}:}$ $$I = \lim_{R\to\infty} \pi\ln^2 R - \frac{2\pi^3}{3} - 2\pi\underbrace{\int_1^R \frac{\ln t}{\sqrt{t}\sqrt{t-1}}\,dt}_{\text{asymptotika integrálu?}}$$

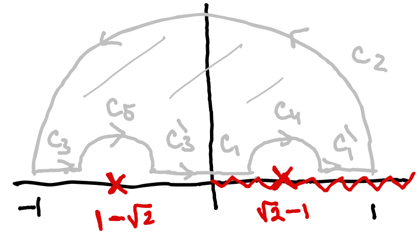
Př $I = \underbrace{\int_0^\pi \frac{x\cos x}{1+\sin^2 x}\,dx}_{\in \mathbb{R}} = \mathrm{Re}\int_0^\pi \frac{x e^{ix}}{1+\sin^2 x}\,dx$ $\boxed{?}$

$$J = \oint \frac{\ln z}{1+\left(\frac{z-\frac{1}{z}}{2i}\right)^2}\,dz = \oint \frac{4z^2\ln z\,dz}{\underset{6z^2-z^4-1}{4z^2-(z^2-1)^2}} = \oint \frac{4z^2 \ln z\,dz}{(2z-z^2+1)(2z+z^2-1)}$$

$$= -(z-1-\sqrt{2})(z-1+\sqrt{2})(z+1-\sqrt{2})(z+1+\sqrt{2})$$

singularity: $\sigma = \pm 1 \pm \sqrt{2}$

- REZIDUOVÁ VĚTA : [CAUCHY : $J=0$]



$\longrightarrow$ $\mathrm{Res}_{\sqrt{2}-1} f(z) = \dfrac{4(\sqrt{2}-1)^2\ln(\sqrt{2}-1)}{-(-2)\,2(\sqrt{2}-1)\,2\sqrt{2}}$

$\longrightarrow$ $\mathrm{Res}_{1-\sqrt{2}} f(z) = \dfrac{4(\sqrt{2}-1)^2(\ln(\sqrt{2}-1)+\pi i)}{-(-2\sqrt{2})(-2)(\sqrt{2}-1)\cdot 2}$

- PARAMETRIZACE :

$\longrightarrow$ $C_1: z = t + i0;\ t \in (0, \sqrt{2}-1-\varepsilon) \cup (\sqrt{2}-1+\varepsilon, 1)$

$$J_1 = -\!\!\!\!\!\!\int_0^1 \frac{4t^2\ln t}{6t^2-t^4-1} \in \mathbb{R}$$

$\longrightarrow$ $C_2: z = e^{it};\ t\in(0,\pi);\ dz = i e^{it}dt$

$$J_2 = \int_0^\pi \frac{\ln(e^{it})}{1+\sin^2 t}\, i e^{it}\,dt = -\int_0^\pi \frac{t e^{it}}{1+\sin^2 t}\,dt =$$

$$= -\underbrace{\int_0^\pi \frac{t\cos t}{1+\sin^2 t}\,dt}_{I} - i\underbrace{\int_0^\pi \frac{t\sin t}{1+\sin^2 t}\,dt}_{I_0 \in \mathbb{R}}$$

$\longrightarrow$ $\ominus C_3: z = -t;\ t \in (0,\sqrt{2}-1-\varepsilon)\cup(\sqrt{2}-1+\varepsilon,1);\ dz = -dt$

$$J_3 = \ominus -\!\!\!\!\!\!\int_0^1 \frac{4t^2(\ln t+\pi i)}{6t^2-t^4-1}(-dt) = J_1 + 4\pi i \underbrace{-\!\!\!\!\!\!\int_0^1 \frac{t^2\,dt}{6t^2-t^4-1}}_{I_1 \in \mathbb{R}}$$

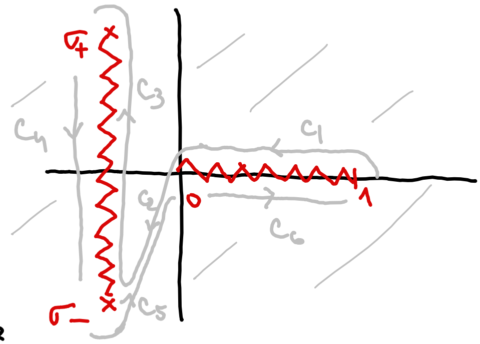
$\longrightarrow$ $C_4: J_4 \xrightarrow{\varepsilon\to 0^+} -\pi i\,\mathrm{Res}_{\sqrt{2}-1} = -\dfrac{\pi i}{2\sqrt{2}}(\sqrt{2}-1)\ln(\sqrt{2}-1)$

$\longrightarrow$ $C_5: J_5 \xrightarrow{\varepsilon\to 0^+} -\pi i\,\mathrm{Res}_{1-\sqrt{2}} = \dfrac{\pi i}{2\sqrt{2}}(\sqrt{2}-1)(\ln(\sqrt{2}-1)+\pi i)$

- POROVNÁNÍ : $0 = -\dfrac{\pi^2}{2\sqrt{2}}(\sqrt{2}-1) + 2J_1 - I - iI_0 + 4\pi i I_1$

$\boxed{\mathrm{Re}}$: $I = -\dfrac{\pi^2}{2\sqrt{2}}(\sqrt{2}-1) + 8 -\!\!\!\!\!\!\int_0^1 \dfrac{t^2\ln t}{6t^2-t^4-1}\,dt \overset{t=\sqrt{u}}{=} \dfrac{1}{4} -\!\!\!\!\!\!\int_0^1 \dfrac{\sqrt{u}\ln u}{6u-u^2-1}\,du$

(Př) $I := \int_0^1 \ln\frac{x}{1-x}\,\frac{dx}{\sqrt{1+x+x^2}}$ $\boxed{?}$

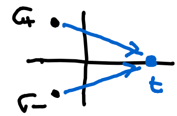


$\downarrow$ $J := \oint_C \ln^2\frac{z}{1-z}\,\frac{dz}{\sqrt{z-\sigma_+}\sqrt{z-\sigma_-}}$

řezy:

$$\frac{z}{1-z}\Big|_{t+i0} = \frac{z}{1-z}\Big|_t + \left(\frac{z}{1-z}\right)'\Big|_t i0 = \frac{t}{1-t} + \underbrace{\frac{i0}{(1-t)^2}}_{i0}$$

$$\sqrt{z-\sigma_+}\sqrt{z-\sigma_-} = \sqrt{|z-\sigma_+|}\sqrt{|z-\sigma_-|}\,e^{i\arg(z-\sigma_+)+i\arg(z-\sigma_+)}$$; arg w:



• RESIDUOVÁ VĚTA

Res_∞: $f(t) = \ln^2\frac{t}{1-t}\,\frac{1}{\sqrt{|t-\sigma_+||t-\sigma_-|}} \xrightarrow{t\to\infty} \frac{(\pi i)^2}{t}$

$t>0$: $\arg(t-\sigma_+) + \arg(t-\sigma_-) = 0$

$\therefore$ $\mathrm{Res}_\infty f(z) = \pi^2 \Rightarrow J = -2\pi i(\pi^2) = -2\pi^3 i$

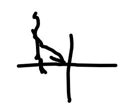
• PARAMETRIZACE :

$(x+\frac12)^2+\frac34$, $\operatorname{arcsinh}\frac{x+\frac12}{\frac{\sqrt3}{2}}$

$\frac{2x+1}{\sqrt3}$

$\ln(x+\sqrt{1+x^2})$

$x=-\frac12+\frac{\sqrt3}{2}\sinh u$



$\longrightarrow$ C_1: $z = t+i0$; $t\in(0,1)$; $dz = dt$

Předělat: $\ln\frac{z}{1-z}$ na $\ln z - \ln(1-z)$

C_3: $z = -\frac12 + it \pm 0$; $t\in\left(-\frac{\sqrt3}{2},\frac{\sqrt3}{2}\right)$

jest. $\ln z - \ln(1-z) = \ln\left(-\frac12+it\right) - \ln\left(\frac12 - it\right) =$

$= (\pm)\ 2\arctan 2t \ \dots$

$\sqrt{z-\sigma_+}\sqrt{z-\sigma_-} \overset{?}{=} \sqrt{\left(t-\frac{\sqrt3}{2}\right)\left(t+\frac{\sqrt3}{2}\right)} = \sqrt{t^2-\frac34}$

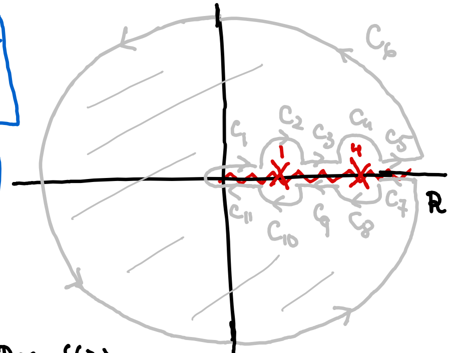
$-\frac12+\frac12+it-\frac{\sqrt3}{2}$

$\therefore$ $\rightsquigarrow \int_{C_3} \int_{-\frac{\sqrt3}{2}}^{\frac{\sqrt3}{2}} \frac{\arctan 2t}{\sqrt{t^2-\frac34}}\,dt$

Př: $I := \int_0^\infty \frac{\ln^2 x}{(x-1)^2(x-4)\sqrt{x}}\,dx$ $\boxed{\text{nevhodná metoda}}$

$\downarrow$

$J := \oint_C \frac{\ln^2 z}{(z-1)^2(z-4)\sqrt{z}}\,dz$ (vyřešeno jinak)



- RESIDUOVÁ VĚTA

$\longrightarrow C_4:\ \mathrm{Res}_4 f(z) = \frac{\ln^2 4}{(4-1)^2\sqrt{4}} = \frac{2\ln^2 2}{9}$

$\longrightarrow C_2:$ $z=1$ odstranitelná singularita : $\mathrm{Res}_1 f(z)$ 0

$\longrightarrow C_8:\ \mathrm{Res}_4 f = \left(\frac{\ln^2 z}{\sqrt{z}}\right)'\Big|_{z=4-i0} = \frac{2\ln z}{\sqrt{z}} - \frac{\ln^2 z}{2z^{3/2}}\Big|_{4-i0} = \frac{\ln^2 2}{2} - 2\ln 2 - \frac{\pi^2}{4} - 2\pi i + \frac{\pi i \ln 2}{2}$

$\longrightarrow C_{10}:\ f(z) = -\frac{4\pi^3}{3}\frac{1}{(z-1)^2} + \left(\frac{2}{9}\pi^2 + \frac{4\pi i}{3}\right)\frac{1}{z-1} + O(1)$

- PARAMETRIZACE

$\longrightarrow C_6:\ |J_6| \le \frac{(\ln R + 2\pi)^2}{(R-1)^2(R-4)\sqrt{R}} 2\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow C_1 + C_3 + C_5:\ z = t + i0\ ;\ t \in (0, 1-\varepsilon) \cup (1+\varepsilon, 4-\varepsilon) \cup (4+\varepsilon, R)$

$$J_1 + J_3 + J_5 \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{\ln^2 t}{(t-1)^2(t-4)\sqrt{t}}\,dt = I$$

$\longrightarrow \ominus(C_7 + C_9 + C_{11}):\ z = t - i0\ ;\ t \in (0, 1-\varepsilon) \cup (1+\varepsilon, 4-\varepsilon) \cup (4+\varepsilon, R)$

$$J_7 + J_9 + J_{11} = \ominus\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{4-\varepsilon} + \int_{4+\varepsilon}^{R}\right)\frac{\ln^2(\overbrace{t-i0}^{\ln t + 2\pi i})\,dt}{(t-1)^2(t-4)\underbrace{\sqrt{t-i0}}_{-\sqrt{t}}} =$$

$$= \ominus\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{4-\varepsilon} + \int_{4+\varepsilon}^{R}\right)\frac{\ln^2 t + 4\pi i \ln t - 4\pi^2}{(t-1)^2(t-4)(-\sqrt{t})}\,dt =$$

$$= \int_0^\infty \frac{\ln^2 t\,dt}{(t-1)^2(t-4)\sqrt{t}} + 4\pi i\int_0^\infty \frac{\ln t\,dt}{(t-1)^2(t-4)\sqrt{t}} - 4\pi^2 \underbrace{\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{\infty}\right)\frac{dt}{(t-1)^2(t-4)\sqrt{t}}}_{I_0(\varepsilon)}$$

$I_0(\varepsilon) \xrightarrow{\varepsilon\to 0^+}$ diverguje

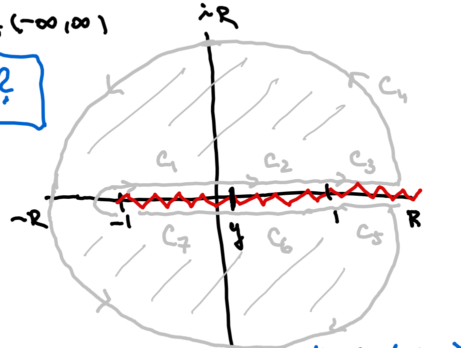
$\longrightarrow C_4:\ J_4 \xrightarrow{\varepsilon\to 0^+} -\pi i\,\mathrm{Res}_4^{C_4} f(z) = -\frac{2\pi i \ln^2 2}{9}$

$\longrightarrow C_8:\ J_8 \xrightarrow{\varepsilon\to 0^+} -\pi i\,\mathrm{Res}_4^{C_8} f(z) = -\frac{\pi i \ln^2 2}{2} + 2\pi i \ln 2 + \frac{\pi^3 i}{4} - 2\pi^2 - \frac{\pi^2 \ln 2}{2}$

$\longrightarrow C_{10}:$

$\int_0^\infty$

(Př) $I(y) = \int_{-1}^{1} \frac{|x-y|}{\sqrt{1-x^2}}\, dx \in \mathbb{R} \quad \forall y \in (-\infty,\infty)$ $\boxed{?}$



$y \in (-1,1)$

$$\downarrow \quad J(y,\alpha) = \oint_C \frac{(z-y)^\alpha}{\sqrt{z+1}\,\sqrt{z-1}}\, dz$$

- **REZIDUOVÁ VĚTA**: $J = 0$ [CAUCHY]
- **PARAMETRIZACE**:

→ C_1: $z = t+i0$; $t \in (-1,y)$; $dz = dt$

$$J_1 = \int_{-1}^{y} \frac{(t-y)^\alpha}{\sqrt{t+i0+1}\,\sqrt{t-1}}\, dt = \int_{-1}^{y} \frac{e^{\pi i\alpha}(y-t)^\alpha}{\sqrt{t+1}\; i\sqrt{1-t}}\, dt = -ie^{\pi i\alpha} \underbrace{\int_{-1}^{y} \frac{|t-y|^\alpha}{\sqrt{1-t^2}}\, dt}_{I_0 \in \mathbb{R}}$$

→ C_2: $z = t+i0$; $t \in (y,1)$; $dz = dt$

$$J_2 = \int_{y}^{1} \frac{(t-y+i0)^\alpha}{\sqrt{t+i0+1}\,\sqrt{t-1}}\, dt = \int_{y}^{1} \frac{(t-y)^\alpha}{\sqrt{t+1}\; i\sqrt{1-t}}\, dt = -i \underbrace{\int_{y}^{1} \frac{|t-y|^\alpha}{\sqrt{1-t^2}}\, dt}_{I_1 \in \mathbb{R}}$$

→ C_3: $z = t+i0$; $t \in (1,R)$; $dz = dt$

$$J_3 = \int_{1}^{R} \frac{(t-y+i0)^\alpha\, dt}{\sqrt{t+i0+1}\,\sqrt{t+i0-1}} = \int_{1}^{R} \frac{(t-y)^\alpha\, dt}{\sqrt{t+1}\,\sqrt{t-1}} = \underbrace{\int_{1}^{R} \frac{(t-y)^\alpha\, dt}{\sqrt{t^2-1}}}_{I_2(R) \in \mathbb{R} \text{ ale diverg.}}$$

→ C_4: $z = Re^{it}$; $t \in (0,2\pi]$; $dz = Rie^{it}\, dt$

$$J_4 = \int_{0}^{2\pi} \frac{(Re^{it}-y)^\alpha}{\sqrt{Re^{it}+1}\,\sqrt{Re^{it}-1}}\, Rie^{it}\, dt = \int_0^{2\pi} \frac{R^\alpha e^{\alpha it}\left(1 - \frac{y\alpha}{R}e^{-it} + O(\frac{1}{R^2})\right) Rie^{it}\, dt}{\sqrt{R}\, e^{\frac{it}{2}}\left(1+\frac{e^{-it}}{R}\right)^{1/2} \sqrt{R}\, e^{\frac{it}{2}}\left(1-\frac{e^{-it}}{R}\right)^{1/2}} =$$

$$= iR^\alpha \int_0^{2\pi} e^{\alpha it}\left(1 - \frac{y\alpha}{R}e^{-it} + O(\tfrac{1}{R^2})\right) dt = R^\alpha\left(e^{2\pi\alpha i}-1\right)\left(\frac{1}{\alpha} - \frac{y\alpha}{R(\alpha-1)}\right) + O(R^{\alpha-2})$$

→ ⊖ C_5: $z = t-i0$; $t \in (1,R)$; $dz = dt$

$$J_5 = ⊖\int_{1}^{R} \frac{(t-y-i0)^\alpha}{\sqrt{t-i0+1}\,\sqrt{t-i0-1}}\, dt = ⊖\int_1^R \frac{e^{2\pi i\alpha}(t-y)^\alpha}{(-\sqrt{t+1})(-\sqrt{t-1})}\, dt = -e^{2\pi i\alpha} I_2(R)$$

→ ⊖ C_6: $z = t-i0$; $t \in (y,1)$; $dz = dt$

$$J_6 = ⊖\int_{y}^{1} \frac{(t-y-i0)^\alpha}{\sqrt{t-i0+1}\,\sqrt{t-1}}\, dt = ⊖\int_y^1 \frac{e^{2\pi i\alpha}(t-y)^\alpha}{(-\sqrt{t+1})\; i\sqrt{1-t}}\, dt = -ie^{2\pi i\alpha} I_1$$

→ ⊖ C_7: $z = t-i0$; $t \in (-1,y)$; $dz = dt$

$$J_7 = ⊖\int_{-1}^{y} \frac{(t-y)^\alpha}{\sqrt{t-i0+1}\,\sqrt{t-1}}\, dt = ⊖\int_{-1}^{y} \frac{e^{\pi i\alpha}(y-t)^\alpha}{(-\sqrt{t+1})(i\sqrt{1-t})}\, dt = -ie^{\pi\alpha i} I_0$$

- **POROVNÁNÍ**: $0 = R\left(e^{2\pi\alpha i}-1\right)\left(\frac{1}{\alpha} - \frac{y\alpha}{R(\alpha-1)}\right) - 2ie^{\pi\alpha i} I_0 - iI_1 - ie^{2\pi i\alpha} I_1 + \left(1-e^{2\pi i\alpha}\right) I_2(R)$

→ NO FURTHER INFO ?

7 $\int_0^{\pi} \frac{x^2\,dx}{\sqrt{5} - 2\cos x} \in \mathbb{R}$ (Li)

$\boxed{?}$

Pr $I = \int_0^{\pi} \frac{x\cos x}{1+\sin^2 x}\,dx$ (Li) $\boxed{?}$

Př. $I = \int_0^1 \frac{x \ln x \ln(1-x)}{1+x^2}\,dx$ $\boxed{?}$

$J := \oint z$

$$\int_0^1 \frac{x}{1+x^2}\ln^2\left(\frac{x}{1-x}\right) dx$$

$$\int_0^1 \frac{x \ln^2 x}{1+x^2}\,dx + \int_0^1 x\,\frac{\ln^2(1-x)}{1+x^2}\,dx$$

$$\hookrightarrow \int_1^\infty \frac{\ln^2 x}{x(1+x^2)}\,dx \qquad \frac{1}{x}\frac{1}{1+x^2} = \frac{1}{x} - \frac{x}{1+x^2}$$

$$\parallel$$

$$\int_1^\infty \frac{\ln^2 x}{x}\,dx - \int_1^\infty \frac{x \ln^2 x}{1+x^2}\,dx$$

$$\rightarrow \int_0^\infty \frac{x \ln^2 x}{1+x^2}\,dx = \left.\frac{\ln^3 x}{3}\right|_1^\infty$$

Když ošetříš ∞, je to zajímavá úloha na

symetrii