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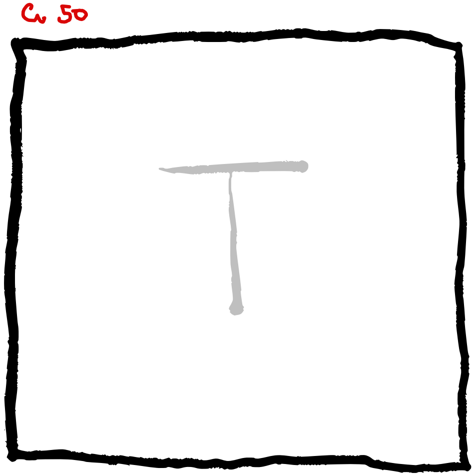


DISTRIBUCE

DISTRIBUCE

D $T : \mathcal{S}(\mathbb{R}^n) \xrightarrow[\text{spojitý}]{\text{lineární}} \mathbb{R}$; $\varphi \in \mathcal{S}(\mathbb{R}^n)$; píšu $\langle T, \varphi \rangle$; $T \in \mathcal{S}'(\mathbb{R}^n)$

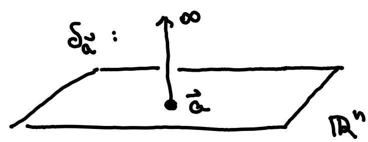
- (RD) Regulární distribuce : $f \in L^1_{loc}(\mathbb{R}^n)$

$$\boxed{\langle T_f , \varphi \rangle := \int_{\mathbb{R}^n} f(\vec{x})\, \varphi(\vec{x})\, d\vec{x}}$$ zkráceně $\langle f, \varphi \rangle$

- Diracova distribuce

$\langle \delta_{\vec{a}} , \varphi \rangle := \varphi(\vec{a})$ fyzikálně

"$\int_{\mathbb{R}^n} \delta_{\vec{a}}(\vec{x})\, \varphi(\vec{x})\, d\vec{x} = \varphi(\vec{a})$"



D TRANSFORMACE DISTRIBUCÍ (podle AD)

$$\langle f(A\cdot\vec{x}+\vec{b}) , \varphi(\vec{x})\rangle = \int_{\mathbb{R}^n} f(A\vec{x}+\vec{b})\, \varphi(\vec{x})\, d\vec{x}$$

$$= \frac{1}{|\det A|} \int_{\mathbb{R}^n} f(\vec{y})\, \varphi(A^{-1}(\vec{y}-\vec{b}))\, d\vec{y} = \frac{1}{|\det A|} \langle f(\vec{y}), \varphi(A^{-1}(\vec{y}-\vec{b})) \rangle$$

Př: $\langle \delta(\vec{x}-\vec{a}), \varphi(\vec{x}) \rangle = \langle \delta(\vec{x}), \varphi(\vec{x}+\vec{a}) \rangle = \varphi(\vec{0}+\vec{a}) = \langle \delta_a(\vec{x}), \varphi(\vec{x}) \rangle$

$\therefore$ platí $\delta_{\vec{a}}(\vec{x}) = \delta_{\vec{0}}(\vec{x}-\vec{a}) = \delta(\vec{x}-\vec{a})$

Př: $\langle \delta(A\cdot\vec{x}), \varphi(\vec{x}) \rangle = \frac{1}{|\det A|} \langle \delta(\vec{y}), \varphi(A^{-1}\cdot\vec{y}) \rangle = \frac{1}{|\det A|} \varphi(\vec{0}) = \frac{\langle \delta, \varphi \rangle}{|\det A|}$

$\therefore$ platí $\delta(A\cdot\vec{x}) = \frac{1}{|\det A|} \delta(\vec{x})$ speciálně $\delta(ax) = \frac{1}{|a|}\delta(x)$

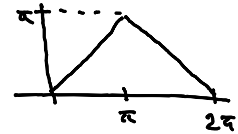
Př: $\delta(\vec{x}) = \delta(x_1)\delta(x_2) \dots \delta(x_n)$; $\vec{x} \in \mathbb{R}^n$

neboť $\int_{\mathbb{R}^n} \delta(\vec{x})\, \varphi(\vec{x})\, d\vec{x} = \int_{-\infty}^{\infty} \delta(x_1) \int_{-\infty}^{\infty} \delta(x_2) \int_{-\infty}^{\infty} \dots \varphi(x_1 \dots x_n)\, dx_1 \dots dx_n$

(zevnitř dosazuji nuly!) $= \varphi(0, 0, \dots, 0)$

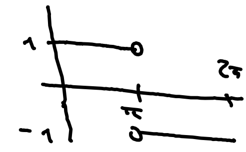
POUŽITÍ DIRAKOVY DISTRIBUCE

Př $$I = \int_0^{2\pi} f(x)\cos nx \, dx \qquad ; \quad f(x):$$



$$PP: \; I = \left| \begin{array}{ll} f & \cos nx \\ f' & \frac{1}{n}\sin nx \\ f'' & -\frac{1}{n^2}\cos nx \end{array} \right| =$$

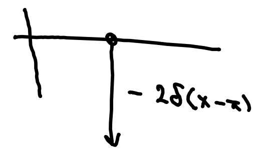
$f'(x) =$



$$= \frac{f}{n}\sin nx \Big|_0^{2\pi} + \frac{f'}{n}\cos nx \Big|_0^{2\pi} - \int_0^{2\pi} \frac{-2\delta(x-\pi)}{n^2}\cos nx \, dx$$

$$= 0 + \frac{(-1)}{n} - \frac{1}{n} + \frac{2}{n^2}\cos n\pi = -\frac{2}{n} + \frac{2}{n^2}(-1)^n$$

$f''(x):$



$\boxed{V}$ Buď $a < x_1 < \dots < x_n < b$; $g(x_j) = 0$, $g(x)$ pouze jednoduché kořeny

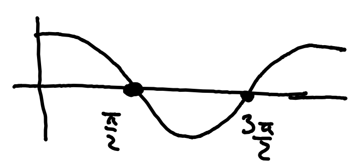
pak
$$\boxed{\int_a^b f(x)\,\delta(g(x))\,dx = \sum_{j=1}^{n} \frac{f(x_j)}{|g'(x_j)|}}$$

„Dk" volme ε malé; aby na $(x_j - \varepsilon, x_j + \varepsilon)$ $\forall j$. právě 1 kořen

$$\int_a^b f(x)\,\delta(g(x))\,dx = \sum_{j=1}^{n} \int_{x_j-\varepsilon}^{x_j+\varepsilon} f(x)\,\delta(g(x))\,dx =$$

$$\approx \sum_{j=1}^{n} \int_{x_j-\varepsilon}^{x_j+\varepsilon} f(x)\,\delta(g'(x_j)(x-x_j))\,dx = \sum_{j=1}^{n} \frac{1}{|g'(x_j)|} \int_{x_j-\varepsilon}^{x_j+\varepsilon} f(x)\,\delta(x-x_j)\,dx$$

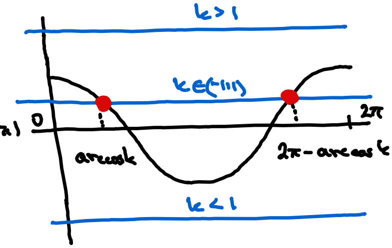
Př $$I = \int_0^{2\pi} \theta \, \delta(\cos\theta)\, d\theta$$



$$(\cos\theta)' = -\sin\theta = \begin{cases} -\sin\frac{\pi}{2} = -1 \; ; \; \theta = \frac{\pi}{2} \\ -\sin\frac{3\pi}{2} = 1 \; ; \; \theta = \frac{3\pi}{2} \end{cases}$$

$$\therefore \; I = \frac{1}{|-1|}\frac{\pi}{2} + \frac{1}{|1|}\frac{3\pi}{2} = 2\pi$$

Př $I(k) = \int_0^{2\pi} e^{in\theta}\,\delta(\cos\theta - k)\,d\theta$; $n\in\mathbb{N}_0,\ k\in\mathbb{R}$



máme 2 nulové body $\theta_1 = \arccos k \in (0,\pi)$

$\theta_2 = 2\pi - \arccos k \in (\pi, 2\pi)$

$$(\cos\theta - k)' = -\sin\theta = \begin{cases} -\sqrt{1-\cos^2\theta} & ; \theta\in(0,\pi) \\ \sqrt{1-\cos^2\theta} & ; \theta\in(\pi,2\pi)\end{cases}$$

$$\therefore\ I(k) \overset{k\in(-1,1)}{=} \frac{e^{in\arccos k}}{\left|-\sqrt{1-k^2}\right|} + \frac{e^{in(2\pi-\arccos k)}}{\left|\sqrt{1-k^2}\right|} =$$

$$= \frac{e^{in\arccos k}+e^{-in\arccos k}}{\sqrt{1-k^2}} = \frac{2\cos(n\arccos k)}{\sqrt{1-k^2}} = \frac{2T_n(k)}{\sqrt{1-k^2}}$$

Čebyševovy polynomy

Pro $k\in\mathbb{R}$: $\boxed{I(k) = \frac{2T_n(k)}{\sqrt{1-k^2}}\,\chi_{[-1,1]}(k)}$

SLABÁ LIMITA

D $T_n \overset{\mathcal{S}'(\mathbb{R})}{\longrightarrow}{}^{*}\ T \iff \forall\varphi\in\mathcal{S}(\mathbb{R}) : \langle T_n,\varphi\rangle \longrightarrow \langle T,\varphi\rangle$

Př Spočtěte $\lim_{n\to\infty} T_{\frac{n}{1+n^2x^2}}$; $x\in\mathbb{R}$

$$\left\langle T_{\frac{n}{x^2+n^2}},\varphi\right\rangle = \int_{-\infty}^{\infty} \frac{n}{1+n^2x^2}\varphi(x)\,dx = \left| \begin{array}{ll} x=\frac{u}{n} & \vdots\ \infty\to\infty \\ dx = \frac{1}{n}du & \vdots\ -\infty\to-\infty\end{array}\right|$$

$$= \int_{-\infty}^{\infty} \frac{\varphi\left(\frac{u}{n}\right)}{1+u^2}\,du \xrightarrow[\text{major.}]{n\to\infty} \int_{-\infty}^{\infty} \frac{\varphi(0)}{1+u^2}\,du = \pi\varphi(0) = \pi\langle\delta,\varphi\rangle$$

$$\therefore\ \lim_{n\to\infty} T_{\frac{n}{1+n^2x^2}} = \pi\delta$$

$\boxed{D}$ SOUČIN DISTRIBUCÍ $\langle f T, \varphi \rangle := \langle T, f\varphi \rangle$; f spoj. p.r. ; $T \in S'(\mathbb{R})$

Př: Zjednodušte a) $x^2 \delta_a$ ($= x^2 \delta(x-a)$)

Sol $\langle x^2 \delta_a, \varphi \rangle \overset{def.}{=} \langle \delta_a, x^2 \varphi \rangle \overset{def.}{=} a^2 \varphi(a) = \langle a^2 \delta_a, \varphi \rangle$ ∴ $x^2 \delta_a = a^2 \delta_a$

OBECNĚ $\boxed{f(x)\delta_a = f(a)\delta_a}$

$\boxed{D}$ DERIVACE DISTRIBUCÍ (podle RD) ; $\varphi(x) \in S(\mathbb{R})$

$$\langle f', \varphi \rangle = \int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx \overset{PP}{=} f(x)\varphi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx = -\langle f, \varphi' \rangle$$

tuto definici opět rozšíříme na libov. T ; $\boxed{\langle T^{(k)}, \varphi \rangle = (-1)^k \langle T, \varphi^{(k)} \rangle}$

Obecně pro $\varphi \in S(\mathbb{R}^n)$: $\langle \nabla f, \varphi \rangle = -\langle f, \nabla \varphi \rangle$

Př $\langle \delta', \varphi \rangle = -\langle \delta, \varphi' \rangle = -\varphi'(0)$ (resp. $-\nabla\varphi(\vec{0})$)

fyzikálně $\int_{-\infty}^{\infty} \delta'(\vec{x})\varphi(\vec{x})\,d\vec{x} \overset{PP}{=} -\int_{-\infty}^{\infty} \delta(\vec{x})\varphi'(\vec{x})\,d\vec{x} = \varphi(\vec{0})$

Př Zjednodušte $x e^{-x} \delta''(x)$; $x \in \mathbb{R}$

Řešení: $\langle x e^{-x} \delta'', \varphi \rangle = \langle \delta'', x e^{-x} \varphi \rangle \overset{def}{=} (-1)^2 \langle \delta, (x e^{-x} \varphi)'' \rangle =$

$= \langle \delta, \overset{0}{\cancel{x''(e^{-x}\varphi)}} + 2x'(e^{-x}\varphi)' + x(e^{-x}\varphi)'' \rangle =$

$= 2(e^{-x}\varphi)'\big|_0 + 0 = -2\varphi(0) + 2\varphi'(0) = -2\langle \delta, \varphi \rangle + 2\langle \delta, \varphi' \rangle$

$= \langle -2\delta - 2\delta', \varphi \rangle$ ∴ $\boxed{x e^{-x} \delta'' = -2\delta - 2\delta'}$

Př Zjednodušte $e^{-\alpha r^2} \Delta \delta(\vec{x})$;

Řešení: $\langle e^{-\alpha r^2} \Delta \delta, \varphi \rangle = \langle \Delta \delta, e^{-\alpha r^2} \varphi \rangle \overset{def}{=} \langle \delta, \Delta(e^{-\alpha r^2} \varphi) \rangle =$

$= \langle \delta, \Delta(e^{-\alpha r^2})\varphi + 2\nabla(e^{-\alpha r^2}) \cdot \nabla\varphi + e^{-\alpha r^2} \Delta \varphi \rangle =$

$= \langle \delta, -2\alpha(3-\alpha r^2) e^{-\alpha r^2} \varphi - 4\alpha e^{-\alpha r^2} \vec{r} \cdot \nabla \varphi + e^{-\alpha r^2} \Delta \varphi \rangle = -6\alpha \varphi(\vec{0}) + \Delta \varphi(\vec{0})$

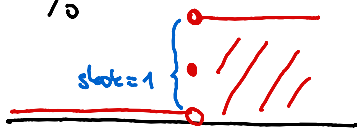
$= \langle \delta, \Delta\varphi \rangle - 6\alpha \langle \delta, \varphi \rangle = \langle \Delta \delta, \varphi \rangle - 6\alpha \langle \delta, \varphi \rangle$ ∴ $\boxed{e^{-\alpha r^2} \Delta \delta = \Delta \delta - 6\alpha \delta}$

(Př) Ukažme $\theta' = \delta$ (kde θ je Heavisideova fce) v 1D

$$\langle \theta', \varphi \rangle \overset{def}{=} - \langle \theta, \varphi' \rangle = - \int_{-\infty}^{\infty} \theta(x)\, \varphi'(x)\, dx =$$

$$= - \int_0^{\infty} \varphi'(x)\, dx = - \varphi(x) \Big|_0^{\infty} = \varphi(0) = \langle \delta, \varphi \rangle$$

fyzikálně $\theta'(x) = \delta(x)$



$\Rightarrow 1 \cdot \delta(\vec{x})$

V DERIVOVÁNÍ SOUČINU DISTRIBUCÍ $\boxed{(fT)' = f'T + fT'}$

Dk: $\langle (fT)', \varphi \rangle \overset{def}{=} - \langle fT, \varphi' \rangle = - \langle T, f\varphi' \rangle =$

$$= - \langle T, (f\varphi)' - f'\varphi \rangle = \langle T, f'\varphi \rangle - \langle T, (f\varphi)' \rangle$$

$$= \langle f'T, \varphi \rangle + \langle T', f\varphi \rangle = \langle f'T + fT', \varphi \rangle$$

(Př) Zjednodušte ve smyslu distribucí

a) $x e^{-x} \delta''(x) \Rightarrow$ trik: $fT'' = (fT)'' - 2(f'T)' + f''T$

$$\left(\because (fT)'' = f''T + 2f'T' + fT'' = f''T + 2\left((f'T)' - f''T\right) + fT'' \right)$$

$$\Rightarrow x e^{-x} \delta'' = \underbrace{(x e^{-x} \delta)''}_{0 \cdot \delta} - 2\big(\underbrace{(x e^{-x})' \delta}_{1 \cdot \delta}\big)' + \underbrace{(x e^{-x})'' \delta}_{-2 \cdot \delta} = \underline{-2\delta' - 2\delta} \checkmark$$

b) $x^2 g_n''(x) - n(n+1) g_n(x)$; kde $g_n(x) = \begin{cases} \left(\frac{a}{x}\right)^n & x > a \\ \left(\frac{x}{a}\right)^{n+1}; & x < a \end{cases}$; $a > 0$; $n = 1, 2, \ldots$

Trik: $g_n(x) = \left(\frac{x}{a}\right)^{n+1} + \theta(x-a)\left(\left(\frac{a}{x}\right)^n - \left(\frac{x}{a}\right)^{n+1}\right)$

$$g_n'(x) = \frac{(n+1)}{a}\left(\frac{x}{a}\right)^n + \cancel{\delta(x-a)\left(\left(\frac{a}{x}\right)^n - \left(\frac{x}{a}\right)^{n+1}\right)} \to 0 + \theta(x-a)\left(-\frac{n}{a}\left(\frac{a}{x}\right)^{n+1} - \frac{n+1}{a}\left(\frac{x}{a}\right)^n\right)$$

$$g_n''(x) = \frac{n(n+1)}{a^2}\left(\frac{x}{a}\right)^{n-1} + \delta(x-a)\left(-\frac{2n+1}{a}\right) + \theta(x-a)\left(\frac{n(n+1)}{a^2}\left(\frac{a}{x}\right)^{n+2} - \frac{n(n+1)}{a^2}\left(\frac{x}{a}\right)^{n-1}\right)$$

$$\Rightarrow x^2 g_n''(x) - n(n+1) g_n(x) = -(2n+1)\, a\, \delta(x-a)$$

c) $\Delta \dfrac{e^{-\alpha r}}{r}$; $r = \|\vec{x}\|$; $\vec{x} \in \mathbb{R}^3$; Hint $\Delta \frac{1}{r} = -4\pi\delta(\vec{x})$

Sol: Pro regulární radiální fci $\Delta f(r) = \frac{1}{r}(fr)'' = \frac{1}{r^2}(f' r^2)'$

$$\Delta\left(\frac{e^{-\alpha r}}{r}\right) = \underbrace{(\Delta e^{-\alpha r})\frac{1}{r} + 2\nabla(e^{-\alpha r}) \cdot \nabla\left(\frac{1}{r}\right)}_{\Delta\left(\frac{e^{-\alpha r}}{r}\right) \text{ regulární}} + e^{-\alpha r}\Delta\frac{1}{r} =$$

$$= \frac{1}{r}(e^{-\alpha r})'' + e^{-\alpha r}(-4\pi\delta(\vec{x})) = \frac{\alpha^2}{r}e^{-\alpha r} - 4\pi\delta(\vec{x})$$

$\boxed{\text{HADAMARDOVA REGULARIZACE}}$ $T_{f.p.\,x^{-m}}$ (finite part) ; $m = 1,2,3,\dots$

zobecnění

CAUCHY: $$\fint_{-\infty}^{\infty} \frac{\varphi(x)}{x}\,dx := \lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\right)\frac{\varphi(x)}{x}\,dx := \left\langle T_{p.v.\frac{1}{x}},\varphi\right\rangle$$

zobecnění

HADAMARD: $$\fint_{-\infty}^{\infty} \frac{\varphi(x)}{x^2}\,dx := \fint_{-\infty}^{\infty}\frac{\varphi(x)-\varphi(0)}{x^2}\,dx := \left\langle T_{f.p.\frac{1}{x^2}},\varphi\right\rangle$$

$\boxed{D}$ $n=1,2,\dots$

$$\boxed{\fint_{-\infty}^{\infty}\frac{\varphi(x)}{x^{n+1}}\,dx := \fint_{-\infty}^{\infty}\frac{\varphi(x)-\varphi(0)-x\varphi'(0)-\frac{x^2}{2!}\varphi''(0)-\dots-\frac{x^{n-1}}{(n-1)!}\varphi^{(n-1)}(0)}{x^{n+1}}\,dx := \left\langle T_{f.p.\frac{1}{x^{n+1}}};\varphi\right\rangle}$$

Př $$T_{f.p.\frac{1}{x^2}} = -T''_{\ln|x|}$$ (analogie $(\ln|x|)'' = \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$)

Dk $$\left\langle T''_{\ln|x|},\varphi\right\rangle \overset{\text{def}}{=} (-1)^2\left\langle T_{\ln|x|},\varphi''\right\rangle \overset{\text{ID}}{=} \int_{-\infty}^{\infty}\ln|x|\,\varphi''(x)\,dx$$

$$= \lim_{\varepsilon\to0^+}\left(\int_{-\infty}^{-\varepsilon}\ln|x|\,\varphi''(x)\,dx + \int_{\varepsilon}^{\infty}\ln|x|\,\varphi''(x)\,dx\right) \overset{\text{PP}}{=} \left|\begin{array}{ccc} \searrow & & \int \\ \ln|x| & + & \varphi''(x) \\ \frac{1}{x} & - & \varphi'(x) \\ -\frac{1}{x^2} & +\int & \varphi(x)\end{array}\right|$$

$$= \lim_{\varepsilon\to0^+}\left(\left[\ln|x|\,\varphi'(x)-\frac{1}{x}\varphi(x)\right]_{-\infty}^{-\varepsilon} + \left[\ln|x|\,\varphi'(x)-\frac{1}{x}\varphi(x)\right]_{\varepsilon}^{\infty} - \underbrace{\left(\int_{\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\right)\frac{\varphi(x)}{x^2}\,dx}_{\text{nelze provést limitu } \varepsilon\to0^+ \,!}\right)$$

$$= \lim_{\varepsilon\to0^+}\left((\varphi'(-\varepsilon)-\varphi'(\varepsilon))\ln\varepsilon + \frac{\varphi(-\varepsilon)+\varphi(\varepsilon)}{\varepsilon} - \text{——}\,''\,\text{——}\right) \quad // \ \varphi(\varepsilon) = \varphi(0)+\varepsilon\varphi'(0)+O(\varepsilon^2)$$

$$= \lim_{\varepsilon\to0^+}\left(\frac{2\varphi(0)}{\varepsilon} - \left(\int_{\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\right)\frac{\varphi(x)}{x^2}\,dx\right) \quad // \ \text{Trik: } \frac{2}{\varepsilon} = \left(\int_{\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\right)\frac{dx}{x^2}$$

$$= -\lim_{\varepsilon\to0^+}\left(\int_{\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\right)\frac{\varphi(x)-\varphi(0)}{x^2}\,dx = -\fint_{-\infty}^{\infty}\frac{\varphi(x)-\varphi(0)}{x^2}\,dx = -\fint_{-\infty}^{\infty}\frac{\varphi(x)}{x^2}\,dx = \left\langle -T_{f.p.\frac{1}{x^2}},\varphi\right\rangle$$

OBECNĚ $$T'_{f.v.\frac{1}{x^m}} = -m\,T_{f.v.\frac{1}{x^{m+1}}} \quad ; \ m = 1,2,3,\dots$$

SOCHOCKÝ – PLEMEJL :

obecně $\boxed{T_{(x\pm i0)^{-n}} := \lim_{y\to 0^+} T_{(x\pm iy)^{-n}} \quad ; \quad n = 1, 2, 3, \dots}$

$\longrightarrow$ čemu se rovnají ??

$$\ln'(x\pm i0) = \ln|x| \pm \pi i\, \overbrace{\theta(-x)}^{\theta(-x)=1-\theta(x)} = \ln|x| \mp \pi i\, \theta(x) \pm \pi i$$

derivace ve smyslu distribucí

$$\frac{1}{x\pm i0} = \text{p.v.}\,\frac{1}{x} \mp \pi i\, \delta(x) \qquad \text{neboť} \quad T'_{\ln|x|} = T_{\text{p.v.}\frac{1}{x}}$$

$$-\frac{1}{(x\pm i0)^2} = -\,\text{f.v.}\,\frac{1}{x^2} \mp \pi i\, \delta'(x)$$

obecně $$\frac{1}{(x\pm i0)^{n+1}} = \text{f.v.}\,\frac{1}{x^{n+1}} \mp \frac{(-1)^n \pi i\, \delta^{(n)}(x)}{n!}$$

čili např. $$\lim_{y\to 0^+} \int_{-\infty}^{\infty} \frac{\varphi(x)}{(x+iy)^2}\,dx = \fint_{-\infty}^{\infty} \frac{\varphi(x)}{x^2}\,dx - (-1)^1 \pi i \overbrace{\int_{-\infty}^{\infty} \varphi(x)\, \delta'(x)\,dx}^{-\int_{-\infty}^{\infty} \varphi'(x)\delta(x)\,dx} =$$

$$= \fint_{-\infty}^{\infty} \frac{\varphi(x) - \varphi(0)}{x^2}\,dx - \pi i\, \varphi'(0)$$

Obecně $$\lim_{y\to 0^+} \int_{-\infty}^{\infty} \frac{\varphi(x)}{(x\pm iy)^n}\,dx = \fint_{-\infty}^{\infty} \frac{\varphi(x)}{x^{n+1}}\,dx \mp \frac{\pi i}{n!}\, \varphi^{(n)}(0)$$

HOMOGENNÍ DISTRIBUCE $\quad H_{|x|^\lambda}\,;\, H_{|x|^\lambda \operatorname{sgn} x}\,;\, H_{x_+^\lambda}\,;\, H_{x_-^\lambda}$

→ jiný způsob regularizace

D $\quad \langle T_{x_+^\lambda}, \varphi \rangle := \int_{-\infty}^{\infty} x_+^\lambda \varphi(x)\, dx = \int_0^\infty x^\lambda \varphi(x)\, dx \quad ;$

nekonverguje $\operatorname{Re}\lambda \leq -1$ např. $\int_0^\infty \frac{1}{x^{3/2}} e^{-x^2} dx \Rightarrow$ regularizace?

$(x_+^\lambda)' = \lambda x_+^{\lambda-1}$ čili třeba $(x_+^{-1/2})' = -\frac{1}{2} x_+^{-3/2}$

$\therefore \quad H_{x_+^{-3/2}} := -2 H'_{x_+^{-1/2}} = -2 \underbrace{T'_{x_+^{-1/2}}}$ $\Leftrightarrow \langle H_{x_+^{-3/2}}, \varphi \rangle = 2 \langle T_{x_+^{-1/2}}, \varphi' \rangle$

toto už je regulární distribuce

(OBECNĚ)

$$\boxed{H_{x_\pm^\lambda} := \frac{\pm 1}{\lambda+1} H'_{x_\pm^{\lambda+1}}\ ;\ \lambda \neq -1 \quad \& \quad H_{x_\pm^\lambda} := T_{x_\pm^\lambda}\ ;\ \operatorname{Re}\lambda > -1}$$

(dobře definované pro $\lambda \in \mathbb{C} / \{-1,-2,-3,\dots\}$)

dále $H_{|x|^\lambda} := H_{x_+^\lambda} + H_{x_-^\lambda}\,;\, H_{|x|^\lambda \operatorname{sgn} x} := H_{x_+^\lambda} - H_{x_-^\lambda}\,;\, H_{(x\pm i0)^\lambda} := H_{x_+^\lambda} + e^{\pm\lambda\pi i} H_{x_-^\lambda}$

Př Ukažte $H_{(x\pm i0)^\lambda} = H_{|x|^\lambda} e^{\pm \pi i \lambda \theta(-x)}$

Př $\quad H_{|x|^\lambda} = \frac{1}{\lambda+1} H'_{|x|^{\lambda+1} \operatorname{sgn} x} \quad ; \quad H_{|x|^\lambda \operatorname{sgn} x} = \frac{1}{\lambda+1} H'_{|x|^{\lambda+1}}$

Dk: $\quad H_{|x|^\lambda} = H_{x_+^\lambda} + H_{x_-^\lambda} = \frac{1}{\lambda+1}\left(H'_{x_+^{\lambda+1}} - H'_{x_-^{\lambda+1}}\right) = \frac{1}{\lambda+1} H_{|x|^{\lambda+1} \operatorname{sgn} x}$

& $\quad H_{|x|^\lambda \operatorname{sgn} x} = H_{x_+^\lambda} - H_{x_-^\lambda} = \frac{1}{\lambda+1}\left(H'_{x_+^{\lambda+1}} + H'_{x_-^{\lambda+1}}\right) = \frac{1}{\lambda+1} H_{|x|^{\lambda+1}}$

Př $\quad (x \pm i0)^\lambda = e^{\pm \frac{\pi\lambda i}{2}} \left(|x|^\lambda \cos\frac{\pi\lambda}{2} \mp i\, |x|^\lambda \operatorname{sgn} x \sin\frac{\pi\lambda}{2} \right)$

Dk. Jest $(x\pm i0)^\lambda = x_+^\lambda + e^{\pm\lambda\pi i} x_-^\lambda = \frac{1}{2}\left[|x|^\lambda + |x|^\lambda \operatorname{sgn} x + e^{\pm\lambda\pi i}\left(|x|^\lambda - |x|^\lambda \operatorname{sgn} x\right)\right]$

$= \frac{1}{2}|x|^\lambda (1 + e^{\pm\lambda\pi i}) + \frac{1}{2}|x|^\lambda \operatorname{sgn} x (1 - e^{\pm\lambda\pi i}) =$

$= \frac{1}{2}|x|^\lambda e^{\pm\frac{\pi}{2}\lambda i}\left(e^{\mp\frac{\pi}{2}\lambda i} + e^{\pm\frac{\pi}{2}\lambda i}\right) + \frac{1}{2}|x|^\lambda \operatorname{sgn} x\, e^{\pm\frac{\pi}{2}\lambda i}\left(e^{\mp\frac{\pi}{2}\lambda i} - e^{\pm\frac{\pi}{2}\lambda i}\right)$

$= \frac{1}{2}|x|^\lambda e^{\pm\frac{\pi}{2}\lambda i}\, 2\cos\left(\pm\frac{\pi}{2}\lambda\right) + \frac{1}{2}|x|^\lambda \operatorname{sgn} x\, 2i \sin\left(\mp\frac{\pi}{2}\lambda\right)$

(Př) $\lim\limits_{\lambda\to -1} H_{|x|^{\lambda}\operatorname{sgn} x} := H_{x^{-1}} = T_{p.v.\frac{1}{x}}$

Sol $\langle H_{x^{-1}}, \varphi\rangle = \lim\limits_{\lambda\to-1} \langle H_{|x|^{\lambda}\operatorname{sgn}x}, \varphi\rangle = \lim\limits_{\lambda\to-1} \frac{1}{\lambda+1}\langle H'_{|x|^{\lambda+1}}, \varphi\rangle$

$$= \lim_{\lambda\to-1} \frac{-1}{\lambda+1}\langle H_{|x|^{\lambda+1}}, \varphi'\rangle \overset{def.}{=} \lim_{\lambda\to-1} \frac{-1}{\lambda+1}\langle T_{|x|^{\lambda+1}}, \varphi'\rangle =$$

$$= \lim_{\lambda\to-1} \frac{-1}{\lambda+1}\int_{-\infty}^{\infty} |x|^{\lambda+1}\varphi'\,dx \overset{Tayl.}{=} \lim_{\lambda\to-1} -\frac{1}{\lambda+1}\int_{-\infty}^{\infty} \varphi' + (\lambda+1)\ln|x|\,\varphi' + O((\lambda+2)^2)\,dx$$

$$= \lim_{\lambda\to-1} -\frac{1}{\lambda+1}\left[\varphi(x)\Big|_{-\infty}^{\infty} + (\lambda+1)\int_{-\infty}^{\infty} \ln|x|\,\varphi'\,dx + O((\lambda+2)^2)\right] =$$

$$= -\int_{-\infty}^{\infty} \ln|x|\,\varphi'\,dx = -\langle T_{\ln|x|}, \varphi'\rangle = \langle T'_{\ln|x|}, \varphi\rangle = \langle T_{p.v.}, \varphi\rangle$$

(Př) Ukažte

(Př) $\lim\limits_{\lambda\to -2} H_{|x|^{\lambda}} := H_{x^{-2}} = T_{f.p.\frac{1}{x^2}}$

Dk: $\langle H_{|x|^{\lambda}}, \varphi\rangle = \frac{1}{\lambda+1}\langle H'_{|x|^{\lambda+1}\operatorname{sgn}x}, \varphi\rangle =$

$$= \frac{1}{(\lambda+1)(\lambda+2)}\langle H''_{|x|^{\lambda+2}}, \varphi\rangle \overset{\lambda>-3}{=} \frac{1}{(\lambda+1)(\lambda+2)}\langle T''_{|x|^{\lambda+2}}, \varphi\rangle$$

$$\overset{def}{=} \frac{1}{(\lambda+1)(\lambda+2)}(-1)^2\langle T_{|x|^{\lambda+2}}, \varphi''\rangle = \frac{1}{(\lambda+1)(\lambda+2)}\int_{-\infty}^{\infty} |x|^{\lambda+2}\varphi''(x)\,dx$$

Taylorova řada $|x|^{\lambda+2}$; $\lambda\to-2$: $|x|^{\lambda+2} = 1 + \ln|x|(\lambda+2) + \underbrace{\frac{1}{2!}\ln^2|x|(\lambda+2)^2}_{O((\lambda+2)^2)}$

$$\langle H_{|x|^{\lambda}}, \varphi\rangle = \frac{1}{(\lambda+1)(\lambda+2)}\underbrace{\int_{-\infty}^{\infty}\varphi''(x)\,dx}_{0} + \frac{1}{\lambda+1}\int_{-\infty}^{\infty}\ln|x|\,\varphi''(x)\,dx + O(\lambda+2)$$

$$\therefore \lim_{\lambda\to-2}\langle H_{|x|^{\lambda}}, \varphi\rangle = -\int_{-\infty}^{\infty}\ln|x|\,\varphi''(x)\,dx = -\langle T_{\ln|x|}, \varphi''\rangle = -\langle T''_{\ln|x|}, \varphi\rangle = \langle T_{f.p.\frac{1}{x^2}}, \varphi\rangle$$

OBECNĚ: $\lim\limits_{\lambda\to-2m} H_{|x|^{\lambda}} = T_{f.v.\frac{1}{x^{2m}}}$; $\lim\limits_{\lambda\to-2m+1} H_{|x|^{\lambda}\operatorname{sgn}x} = T_{f.v.\frac{1}{x^{2m-1}}}$; $\lim\limits_{\lambda\to-n} H_{(x\pm i0)^{\lambda}} = T_{(x\pm i0)^{-n}}$

REZIDUA $H_{x_\pm^\lambda}$ v $\lambda = -n$; $n = 1, 2, 3, \dots$

$\boxed{D}$ $\mathrm{Res}_{\lambda=\sigma} T_\lambda$ je distribuce definovaná pro parametrický systém T_λ

$$\langle \mathrm{Res}_\sigma T_\lambda , \varphi \rangle = \mathrm{Res}_\sigma \langle T_\lambda , \varphi \rangle$$

Př. $\mathrm{Res}_{-n} H_{x_\pm^\lambda} = \mathrm{Res}_{-n} \dfrac{H^{(n)}_{x_\pm^{\lambda+n}}}{(\lambda+n)(\lambda+n-1)\cdots(\lambda+1)}$ $\overset{\text{jednoduchý pól}}{=}$

$$= \frac{H^{(n)}_{x_\pm^0}}{(-1)(-2)\cdots(-n+1)} = \frac{(-1)^{n-1}}{(n-1)!} (\pm\, \delta^{(n)}(x))$$

RADIÁLNÍ DISTRIBUCE

$\vec{x} \in \mathbb{R}^n;\ r = \|\vec{x}\|;\ n = 1,2,3,\dots$

$\boxed{D}$ $$\langle T_{r^\lambda}; \varphi \rangle := \int_{\mathbb{R}^n} r^\lambda \varphi(\vec x)\, d\vec x = K_n \int_0^\infty \oint_{B_n(1)} r^{\lambda+n-1}\varphi(\vec x)\, d\Omega\, dr;\ \mathrm{Re}\,\lambda > -n$$

Dú $$\Delta T_{r^{2-n}} = (2-n) K_n \delta(\vec x);\quad n \in \mathbb{N}/\{2\}$$

Dk: $$\langle \Delta T_{r^{2-n}}, \varphi\rangle \overset{\text{def.}}{=} \langle T_{r^{2-n}}, \Delta\varphi\rangle = \int_{\mathbb{R}^n} r^{2-n}\,\Delta\varphi\, d\vec x =$$

$$\left\langle \text{Green II}: \int_V \psi\Delta\varphi - \varphi\Delta\psi\, dV = \oint_{\partial V} (\psi\nabla\varphi - \varphi\nabla\psi)\cdot d\vec S \right\rangle$$

$$= \lim_{\varepsilon\to0^+} \int_{\mathbb{R}^n/B_n(\varepsilon)} r^{2-n}\Delta\varphi\, d\vec x = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n/B_n(\varepsilon)} \varphi\, \cancelto{0}{\Delta r^{2-n}}\, d\vec x - \oint_{B_n(\varepsilon)} (r^{2-n}\nabla\varphi - \varphi\nabla r^{2-n})\cdot d\vec S$$

$$= -\lim_{\varepsilon\to0^+} \oint_{B_n(\varepsilon)} (\varepsilon^{2-n}\nabla\varphi - (2-n)\varphi\,\hat r\, \varepsilon^{1-n})\cdot \hat r\, dS \quad // [dS = \varepsilon^{n-1} d\Omega;\ \Omega\in B_n(1)]$$

$$= -\lim_{\varepsilon\to0^+} \oint \varepsilon\, \hat r\cdot\nabla\varphi - (2-n)\varphi\, d\Omega = (2-n)\,\varphi(0)\underbrace{\oint d\Omega}_{K_n}$$

Př Pro $n=2$, ověřte $\Delta \ln r = -2\pi\,\delta(\vec x)$

$\boxed{\text{HOMOGENNÍ RADIÁLNÍ DISTRIBUCE}}$ ($\lambda \neq \dots -4-n, -2-n, -n$)

Neceločíselné λ budou odvislé od identity:

$$\Delta r^\lambda = \nabla\cdot\nabla r^\lambda = \lambda\nabla\cdot(\hat r\, r^{\lambda-1}) = \lambda\nabla\cdot(\vec x\, r^{\lambda-2}) = \lambda\Big[\underbrace{(\nabla\cdot\vec x)}_{n}\, r^{\lambda-2} + \vec x\cdot\nabla r^{\lambda-2}\Big]$$

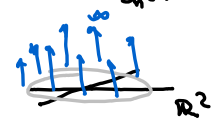
$$= \lambda\left[n\, r^{\lambda-2} + \vec x\cdot(\lambda-2)\,\hat r\, r^{\lambda-3}\right] = \lambda(\lambda+n-2)\, r^{\lambda-2}$$

čili definujme $$\boxed{H_{r^\lambda} := \frac{\Delta H_{r^{\lambda+2}}}{(\lambda+2)(\lambda+n)} \quad \& \quad H_{r^\lambda} := T_{r^\lambda};\ \mathrm{Re}\,\lambda > -n}$$

$\boxed{\text{PLOŠNÁ MÍRA}}$ $\vec{x} \in \mathbb{R}^n$; $r = \|\vec{x}\|$; $R > 0$ fixní

$\boxed{D}$ $\langle \nu_R, \varphi \rangle := \oint_{r=R} \varphi(\vec{x})\, dS$ (přes povrch $B_n(R)$) $= R^{n-1} \oint_{B_n(1)} \varphi(\vec{x})\Big|_{r=R} d\Omega$

$\boxed{V}$ $\boxed{\nu_R = \delta(r - R)}$



Dk: $\int_{\mathbb{R}^n} \delta(r-R)\varphi(\vec{x})\, d\vec{x} = \oint_{r=R} \int_0^\infty \delta(r-R)\, \varphi(\vec{x})\, dr\, dS = \oint_{r=R} \varphi(\vec{x})\, dS = \langle \nu_R, \varphi \rangle$

Př: Spočtěte $\Delta \chi_{[0,R]}(\vec{x})$; $R > 0$.

ve smyslu distribucí (tj $\Delta T_{\chi_{[0,R]}}$)

Sol: $\langle \Delta T_{\chi_{[0,R]}}, \varphi \rangle \overset{\text{Def}}{=} \langle T_{\chi_{[0,R]}}, \Delta\varphi \rangle =$

$= \int_{\mathbb{R}^3} \chi_{[0,R]}(\vec{x})\, \Delta\varphi\, d = \int_{r \le R} \Delta\varphi\, dV = \int_{r \le R} \nabla \cdot \nabla\varphi\, dV =$

$\overset{\text{Gauss}}{=} \oint_{r=R} \hat{r} \cdot \nabla\varphi\, dS = \oint_{r=R} \partial_r \varphi\, dS = \langle \nu_R, \partial_r \varphi \rangle = -\langle \partial_r \nu_R, \varphi \rangle$

$\therefore \Delta \chi_{[0,R]} = -\partial_r \nu_R$

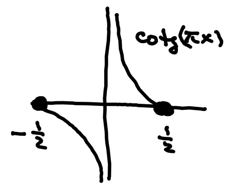
FOURIEROVY ŘADY DISTRIBUCÍ

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi n x}{L} + b_n \sin\frac{2\pi n x}{L}$$

$$a_n = \frac{2}{L}\int_L f(x)\cos\frac{2\pi n x}{L}\,dx \overset{\text{zobec.}}{=} \frac{2}{L}\left\langle T_f, \cos\frac{2\pi n x}{L}\right\rangle_L$$

$$b_n = \frac{2}{L}\int_L f(x)\sin\frac{2\pi n x}{L}\,dx = \frac{2}{L}\left\langle T_f, \sin\frac{2\pi n x}{L}\right\rangle_L$$

Př: Najděte FŘ distribuce $T_{\text{p.v.}\,\operatorname{cotg}(\pi x)}$



$L = 1$; $\quad a_n = 2 ⨍_{-1/2}^{1/2} \operatorname{cotg}(\pi x)\cos(2\pi n x)\,dx \overset{\text{lichá}}{=} 0$

$$b_n = 2\int_{-1/2}^{1/2} \operatorname{cotg}(\pi x)\sin(2\pi n x)\,dx$$

Trik: $b_{n+1} - b_{n-1} = 2\int_{-1/2}^{1/2} \operatorname{cotg}(\pi x)\left(\sin(2\pi n x + 2\pi x) - \sin(2\pi n x - 2\pi x)\right)dx =$

$$= 4\int_{-1/2}^{1/2} \operatorname{cotg}(\pi x)\cos(2\pi n x)\underbrace{\sin(2\pi x)}_{2\sin(\pi x)\cos(\pi x)}\,dx = 8\int_{-1/2}^{1/2}\cos^2(\pi x)\cos(2\pi n x)\,dx =$$

$$= 4\int_{-1/2}^{1/2}(1+\cos(2\pi x))\cos(2\pi n x)\,dx \overset{\text{OG}}{=} 0 \quad ; \quad n \geq 2$$

čili $\quad b_{n+1} = b_{n-1} \quad$ pro $\quad n \geq 2$;

- pro $n = 1$ máme $\quad b_2 - b_0 = 4\int_{-1/2}^{1/2}(1+\cos(2\pi x))\cos(2\pi x)\,dx = 2$

  a jelikož $b_0 = 0$, je $\quad b_2 = 2 \; (= b_4 = b_6 = \cdots)$

- Spočteme separátně:

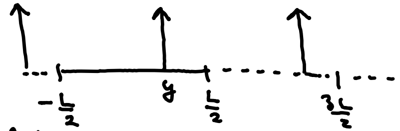
$$b_1 = 2\int_{-1/2}^{1/2}\operatorname{cotg}(\pi x)\sin(2\pi x)\,dx = 2\int_{-1/2}^{1/2} 1+\cos(2\pi x)\,dx = 2 \; (= b_3 = b_5 = \cdots)$$

Celkově: $\quad T_{\text{p.v.}\,\cot \pi x} = 2\sum_{n=1}^{\infty}\sin(2\pi n x) \; ; \; x \in (-\tfrac{1}{2}, \tfrac{1}{2}) + \pi\ell$

POISSONOVA SUMAČNÍ FORMULE

Př FŘ: $f(x) = \sum_{n\in\mathbb{Z}} c_n e^{\frac{2\pi i x n}{L}}$; $c_n = \frac{1}{L}\int_L f(x)\, e^{-\frac{2\pi i x n}{L}}\, dx$

mějme $f(x) = \delta(x-y)$; $y \in (-\frac{L}{2}, \frac{L}{2})$



$$\Rightarrow \quad c_n = \frac{1}{L}\int_L \delta(x-y)\, e^{-\frac{2\pi i x n}{L}} = \frac{1}{L} e^{-\frac{2\pi i y n}{L}}$$

$$\therefore \quad \boxed{\delta(x-y) = \frac{1}{L}\sum_{n\in\mathbb{Z}} e^{\frac{2\pi i n}{L}(x-y)}} \quad ; \ x \in (-\tfrac{L}{2}, \tfrac{L}{2})$$

přesněji

$$\delta_{L\mathbb{Z}}(z) := \sum_{n\in\mathbb{Z}} \delta(z-nL) = \frac{1}{L}\sum_{n\in\mathbb{Z}} e^{-\frac{2\pi i n}{L} z} \quad ; \ z \in \mathbb{R}/n\mathbb{Z}$$

(Dirac Comb)

Důsledek: PSF:

$$\sum_{n\in\mathbb{Z}} \varphi(n) = \int_{-\infty}^{\infty} \varphi(x) \sum_{m\in\mathbb{Z}} \delta(x-m)\, dx \overset{L=1}{=} \int_{-\infty}^{\infty} \varphi(x) \sum_{n\in\mathbb{Z}} e^{-2\pi i n x}\, dx = \sum_{n\in\mathbb{Z}} \hat{\varphi}(2\pi n)$$

Př $\sum_{n\in\mathbb{Z}} \frac{1}{n^2+\alpha^2}$; víme $\widehat{\frac{1}{x^2+\alpha^2}} = \int_{-\infty}^{\infty} \frac{1}{x^2+\alpha^2} e^{-ikx}\, dx \overset{(*)}{=} \frac{\pi}{\alpha} e^{-|k|\alpha}$; $\alpha>0$

$$\therefore \quad \sum_{n\in\mathbb{Z}} \frac{1}{n^2+\alpha^2} = \sum_{n\in\mathbb{Z}} \frac{\pi}{\alpha} e^{-2\pi|n|\alpha} = -\frac{\pi}{\alpha} + 2\sum_{n=0}^{\infty} \frac{\pi}{\alpha} e^{-2\pi n\alpha} = -\frac{\pi}{\alpha} + \frac{2\pi}{\alpha}\frac{1}{1-e^{2\pi\alpha}}$$

$$= -\frac{\pi}{\alpha} + \frac{2\pi}{\alpha}\frac{e^{\pi\alpha}}{e^{\pi\alpha}-e^{-\pi\alpha}} = \frac{\pi}{\alpha}\frac{e^{-\pi\alpha}-e^{\pi\alpha}+2e^{\pi\alpha}}{e^{\pi\alpha}-e^{-\pi\alpha}} = \frac{\pi}{\alpha}\operatorname{cotgh}(\pi\alpha)$$

Př Sečtěte $\sum_{n\in\mathbb{Z}} \frac{1}{n^4+\alpha^4}$; $\alpha > 0$