G51–52

$$\boxed{\hat{T}}$$

FOURIEROVA TRANSFORMACE DISTRIBUCÍ

FOURIEROVA TRANSFORMACE DISTRIBUCÍ

Pro $f, \varphi \in \mathcal{S}(\mathbb{R}^n)$: $\langle \hat{f}, \varphi \rangle = \int_{\mathbb{R}^n} \hat{f}(\vec{k}) \varphi(\vec{k})\, d\vec{k} = \int_{\mathbb{R}^n} f(\vec{k}) \hat{\varphi}(\vec{k})\, dk = \langle f, \hat{\varphi} \rangle$

zobecnění

$\boxed{D}$

$$\boxed{\langle \hat{T}, \varphi \rangle := \langle T, \hat{\varphi} \rangle; \quad T \in \mathcal{S}'}$$

VLASTNOSTI:

$n=0$

$\uparrow$

$$\frac{-i}{(k - i0)} = -i\left(T_{\mathrm{p.v.}\frac{1}{k}} + \pi i \delta(k) \right)$$

- $\check{\hat{T}} = T$
- $\widehat{T(\vec{x})} = (2\pi)^n \, \widecheck{T(-\vec{x})}$
- $\widehat{T(\vec{x}) e^{i\vec{b}\cdot\vec{x}}} = \hat{T}(\vec{k} - \vec{\beta})$
- $\widehat{T(A\vec{x} + \vec{b})} = \frac{1}{|\det A|} \hat{T}(\vec{k} A^{-1}) \, e^{i\vec{k}\cdot\vec{x}}$
- $\widehat{T^{(\alpha)}} = (ik)^\alpha \hat{T}$
- $\widehat{x^\alpha T} = (i\partial_k)^\alpha \hat{T}$
- $\widehat{f * T} = \hat{f} \hat{T}$

TABULKA FOURIEROVÝCH TRANSFORMACÍ

$x, k \in \mathbb{R}$; $\alpha > 0$; $\beta \in \mathbb{R}$; $n = 0, 1, 2, \dots$

$f(x)$	$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$
$\delta^{(n)}(x)$	$i^n k^n$
x^n	$2\pi i^n \delta^{(n)}(k)$
$x^n \operatorname{sgn} x$	$2(-i)^{n+1} n! \, T_{f.v. \frac{1}{k^{n+1}}}$
$\theta(x)$	$-i T_{p.v. \frac{1}{k}} + \pi \delta(k)$
$x_\pm^n$	$\frac{n! \, (\mp i)^{n+1}}{(k \mp i0)^{n+1}} = n! (\mp i)^{n+1} T_{f.v. \frac{1}{k^{n+1}}} + \pi (\pm i)^n \delta^{(n)}(k)$
$T_{f.v. \frac{1}{x^{n+1}}}$	$\pi \frac{(-i)^{n+1}}{n!} k^n \operatorname{sgn} k$
$\frac{1}{(x \pm i0)^{n+1}} = T_{f.v. \frac{1}{x^{n+1}}} \mp \frac{(-1)^n \pi i \delta^{(n)}(x)}{n!}$	$2\pi \frac{(\mp i)^{n+1}}{n!} k_\pm^n$
$x_\pm^\lambda$	$e^{\mp \frac{\pi}{2} i (\lambda + 1)} \Gamma(\lambda + 1) (k \mp i0)^{-\lambda - 1}$
$\lvert x \rvert^\lambda$	$\frac{\pi \lvert k \rvert^{-\lambda - 1}}{\Gamma(-\lambda) \cos \frac{\pi \lambda}{2}}$
$\lvert x \rvert^\lambda \operatorname{sgn} x$	$\frac{\pi i \lvert k \rvert^{-\lambda - 1} \operatorname{sgn} k}{\Gamma(-\lambda) \sin \frac{\pi \lambda}{2}}$
$(x \pm i0)^\lambda$	$e^{\pm \frac{\pi \lambda i}{2}} \frac{2\pi k_\pm^{-\lambda - 1}}{\Gamma(-\lambda)}$

OBECNÝ POČET DIMENZÍ

$\vec{x}, \vec{k} \in \mathbb{R}^n$; $r = \lVert \vec{x} \rVert$; $k = \lVert \vec{k} \rVert$; $l = 0, 1, \dots$

$f(\vec{x})$	$\hat{f}(\vec{k}) = \int_{-\infty}^{\infty} f(\vec{x}) e^{-i \vec{k} \cdot \vec{x}} d\vec{x}$
$\Delta^l \delta(\vec{x})$	$(-1)^l k^{2l}$
r^{2l}	$(2\pi)^n (-1)^l \Delta^l \delta(\vec{k})$
r^λ	$\frac{\pi^{\frac{n}{2}} 2^{\lambda + n} \Gamma\left(\frac{\lambda + n}{2}\right)}{k^{\lambda + n} \Gamma\left(-\frac{\lambda}{2}\right)}$ (RIESZ)

FT DIRAKOVY DISTRIBUCE

Př: $f(x) = \delta(x-\alpha) \;;\; x \in \mathbb{R}$

$$\hat{f}(x) = \int_{-\infty}^{\infty} \delta(x-\alpha)\, e^{-ikx}\, dx = e^{-ik\alpha}$$

inverze: $$\delta(x-\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\delta(x-\alpha)}\, e^{ikx}\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\alpha} e^{-ikx}\, dk$$

Př: $f(x) = e^{i\alpha x} \;;\; x \in \mathbb{R}$

$$\widehat{e^{ik\alpha}} = \int_{-\infty}^{\infty} e^{i\alpha x} e^{-ikx}\, dx = 2\pi\, \delta(k-\alpha)$$

porovnejte si

NEBOLI $$\boxed{\int_{-\infty}^{\infty} e^{i\beta x}\, dx = 2\pi\, \delta(\beta)}$$ ve smyslu distribucí

Př: Najděte FT fce $f(\vec{x}) = \delta(\vec{x}-\vec{a})$ a $g(\vec{x}) = e^{i\vec{x}\cdot\vec{a}}$

$$\widehat{\delta(\vec{x}-\vec{a})} = \int_{\mathbb{R}} \delta(\vec{x}-\vec{a})\, e^{-i\vec{k}\cdot\vec{x}}\, d\vec{x} = e^{-i\vec{k}\cdot\vec{a}}$$

$$\therefore \quad \widehat{e^{i\vec{x}\cdot\vec{a}}} = (2\pi)^n\, \widecheck{e^{-i\vec{x}\cdot\vec{a}}} = (2\pi)^n\, \delta(\vec{k}-\vec{a})$$

Př* Najděte FT Besselových funkcí $J_n(x) \;;\; n \in \mathbb{N}_0$:

$$J_n(x) = \frac{(-i)^n}{2\pi}\int_0^{2\pi} e^{in\theta + ix\cos\theta}\, d\theta \;;$$ řešení:

$$\therefore \quad \widehat{J_n(x)} = \int_{-\infty}^{\infty} J_n(x)\, e^{-ikx}\, dx = \frac{(-i)^n}{2\pi}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} e^{in\theta + ix\cos\theta - ikx}\, d\theta\, dx$$

$$= (-i)^n \int_{-\pi}^{\pi} e^{in\theta}\, \delta(\cos\theta - k)\, d\theta = (-i)^n\, \frac{2\,T_n(k)}{\sqrt{1-k^2}}\, \chi_{[-1,1]}(k)$$

Speciálně $$\widehat{J_0(x)} = \frac{2\chi_{[-1,1]}(k)}{\sqrt{1-k^2}}$$

DŮSLEDEK : $$\widehat{\frac{\chi_{[-1,1]}(x)}{\sqrt{1-x^2}}} = 2\pi\, \widecheck{\;-''-\;} = \pi J_0(k)$$ (toto už víme)

Př: Řešte PDR: $u_{,t} + \vec{b}\cdot\nabla u = 0$; $u(\vec{x},t)$; $\vec{b} \in \mathbb{R}^n$ konst.; $\vec{u}(\vec{x},0) = \vec{u}_0(\vec{x})$

<u>Řešení</u>: $u(\vec{x},t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{u}(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}\, d\vec{k}$

$\downarrow$

$$\nabla u = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{u}(\vec{k},t)\, \underbrace{\nabla(i\vec{k}\cdot\vec{x})}_{i\vec{k}}\, e^{i\vec{k}\cdot\vec{x}}\, d\vec{k}$$

$$\therefore \quad u_{,t} + \vec{b}\cdot\nabla u = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (\hat{u}_{,t} + i\vec{b}\cdot\vec{k}\, u)\, e^{i\vec{k}\cdot\vec{x}}\, d\vec{k} \equiv 0$$

čili $\hat{u}_{,t} + i\vec{b}\cdot\vec{k}\, u = 0 \Rightarrow \hat{u} = \hat{u}_0\, e^{-i(\vec{b}\cdot\vec{k})t} = \hat{u}_0\, \widehat{\delta(\vec{x}-t\vec{b})}$

$$\Rightarrow u(\vec{x},t) = u_0 * \delta(\vec{x}-t\vec{b}) = \int_{\mathbb{R}^n} u_0(\vec{y})\, \delta(\vec{x}-\vec{y}-t\vec{b})\, d\vec{y} = u_0(\vec{x}-t\vec{b})$$

$\boxed{\text{FT DERIVACÍ } \delta \text{ A MOCNIN } x}$

Př: $f(x) = \delta^{(n)}(x-a) \; ; \; x \in \mathbb{R}$

Z pravidel pro FT: $\widehat{\delta^{(n)}(x-a)} = (ik)^n \, \widehat{\delta(x-a)} = \underline{(ik)^n e^{-ika}}$

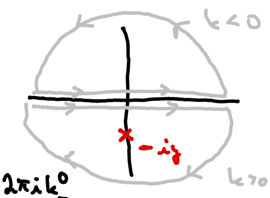
Př: $f(x) = x^n \; ; \; x \in \mathbb{R} \; ; \; n = 0, 1, 2, \dots$

Z pravidel pro FT: $\widehat{x^n} = \widehat{x^n \cdot 1} = (i\partial_k)^n \, \hat{1} = (i\partial_k)^n \delta(k) = 2\pi i^n \delta^{(n)}(k)$

NEBO: $\widehat{x^n} = 2\pi \, \widecheck{(-x)^n} = 2\pi i^n \, \widecheck{\underline{i^n x^n e^{-ix \cdot 0}}} = 2\pi i^n \delta^{(n)}(x)$

$\boxed{\text{FT SOCHOTSKÝ–PLEMELJOVÝCH DISTRIBUCÍ}}$

$$\left\langle T_{\frac{1}{(x \pm i0)^n}} ; \varphi \right\rangle := \lim_{y \to 0^+} \left\langle T_{\frac{1}{(x \pm iy)^n}} ; \varphi \right\rangle \; ; \; n = 1, 2, 3, \dots$$



Nejdřív $\hat{T}_{\frac{1}{x+i0}} = \lim_{y \to 0^+} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x+iy} \, dx = \theta(k)(-2\pi i) \lim_{y \to 0^+} \operatorname{Res}_{-iy} \frac{e^{-ikz}}{z+iy}$

$= -2\pi i \theta(k) = -2\pi k_+^0$; Analogicky $\hat{T}_{\frac{1}{x-i0}} = 2\pi i \theta(-k) = 2\pi i k_-^0$

Víme: $T_{\frac{1}{(x \pm i0)^{n+1}}} = \frac{(-1)^n}{n!} T^{(n)}_{\frac{1}{x \pm i0}}$; $n = 0, 1, 2, \dots$

$\therefore \; \hat{T}_{\frac{1}{(x \pm i0)^{n+1}}} = \frac{(-1)^n}{n!} (ik)^n \, \hat{T}_{\frac{1}{x \pm i0}} = \mp \frac{(-1)^n}{n!} (ik)^n \, 2\pi i \, k_\pm^0$

Nebo také $\hat{T}_{\frac{1}{(x+i0)^{n+1}}} = 2\pi \frac{(-i)^{n+1}}{n!} k_+^n$; $\hat{T}_{\frac{1}{(x-i0)^{n+1}}} = 2\pi \frac{i^{n+1}}{n!} k_-^n$

$\boxed{\text{FT } x_+^n, \, x_-^n}$

- Inverzí předchozího výsledku: [NEBO $x_+^n = x^n \theta(x)$ ...]

$\widehat{x_+^n} = 2\pi \, \widecheck{(-x)_+^n} = 2\pi \, \widecheck{x_-^n} = \frac{n! (-i)^{n+1}}{(k-i0)^{n+1}}$; Analogicky $\widehat{x_-^n} = \frac{n! \, i^{n+1}}{(k+i0)^{n+1}}$

$\boxed{\text{FT HADAMARDOVÝCH REG. } \int}$

$$\langle T_{\text{p.v.}\frac{1}{x}}, \varphi\rangle := \fint_{-\infty}^{\infty} \frac{\varphi(x)}{x}\,dx = \lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\frac{\varphi(x)}{x}\,dx$$

<u>Trik</u> : $T_{\frac{1}{x\pm i0}} = T_{\text{p.v.}\frac{1}{x}} \mp \pi i \delta$ (Sochocký-Plemelj)

$$\therefore\ \widehat{T_{\text{p.v.}\frac{1}{x}}} = \widehat{T}_{\frac{1}{x+i0}} + \pi i \hat{\delta} = -2\pi i\,\Theta(k) + \pi i = -\pi i \operatorname{sgn} k$$

Víme, že $T'_{\text{p.v.}\frac{1}{x}} = -T_{\text{f.v.}\frac{1}{x^2}}$ obecně $\boxed{T^{(n)}_{\text{p.v.}\frac{1}{x}} = (-1)^n n!\, T_{\text{f.v.}\frac{1}{x^{n+1}}}}$

$$\therefore\ \widehat{T_{\text{f.v.}\frac{1}{x^{n+1}}}} = \frac{(-1)^n}{n!}\widehat{T^{(n)}_{\text{p.v.}\frac{1}{x}}} = \frac{(-1)^n}{n!}(ik)^n(-\pi i \operatorname{sgn} k) = \pi\frac{(-i)^{n+1}}{n!}k^n \operatorname{sgn} k$$

$\boxed{\text{FT } x^n \operatorname{sgn} x}$

- Inverzí předchozího výsledku :

$$\widehat{x^n \operatorname{sgn} x} = 2\pi\,\widecheck{(-x)^n \operatorname{sgn}(-x)} = (-1)^{n+1}2\pi\,\widecheck{x^n \operatorname{sgn} x} = 2(-i)^{n+1} n!\, T_{\text{f.v.}\frac{1}{k^{n+1}}}$$

FOURIEROVA TRANSFORMACE HOMOGENNÍCH DISTRIBUCÍ

→ Jiný způsob jak získat $\widehat{T_{\text{f.v.}\frac{1}{x^n}}}$ aj.

Je totiž $\lim\limits_{\lambda\to -2m} H_{|x|^\lambda} = T_{\text{f.v.}x^{-2m}}$; $\lim\limits_{\lambda\to-2m+1} H_{|x|^\lambda \operatorname{sgn} x} = T_{\text{f.v.}x^{-2m+1}}$; $m=1,2,3,\dots$

- Metoda: Počítejme $\widehat{x_\pm^\lambda}$; $\operatorname{Re}\lambda\in(-1,0)$ → komplexní rozšíření na $\hat{H}_{x_\pm^\lambda}$:

Jest $\widehat{x_+^\lambda} = \int_{-\infty}^\infty x_+^\lambda e^{-ikx}\,dx = \int_0^\infty x^\lambda e^{-ikx}\,dx =$

$k<0$: $$= \int_0^\infty (it)^\lambda e^{-|k|t}\, i\,dt = i e^{\frac{\pi}{2}i\lambda}\int_0^\infty t^\lambda e^{-|k|t}\,dt = \frac{e^{\frac{\pi}{2}i(\lambda+1)}}{|k|^{\lambda+1}}\Gamma(\lambda+1)$$

$k>0$: $$= \int_0^\infty (-it)^\lambda e^{-kt}(-i)\,dt = e^{-\frac{\pi}{2}i(\lambda+1)}\frac{\Gamma(\lambda+1)}{k^{\lambda+1}}$$

Celkově $\widehat{x_+^\lambda} = e^{-\frac{\pi}{2}i(\lambda+1)\operatorname{sgn}k}\,\Gamma(\lambda+1)|k|^{-\lambda-1} =$

$$e^{-\frac{\pi}{2}i(\lambda+1)}\Gamma(\lambda+1)\left(k_+^{-\lambda-1} + e^{-\pi i(-\lambda-1)}k_-^{-\lambda-1}\right) = e^{-\frac{\pi}{2}i(\lambda+1)}\Gamma(\lambda+1)(k-i0)^{-\lambda-1}$$

- Jest $x_-^\lambda = (-x)_+^\lambda \Rightarrow \widehat{x_-^\lambda} = \widehat{(-x)_+^\lambda} = \widehat{x_+^\lambda}(-k) = e^{\frac{\pi}{2}i(\lambda+1)\operatorname{sgn}k}\,\Gamma(\lambda+1)|k|^{-\lambda-1}$

- $$\widehat{|x|^\lambda} = \widehat{x_+^\lambda}+\widehat{x_-^\lambda} = \Big(\underbrace{e^{-\frac{\pi}{2}i(\lambda+1)\operatorname{sgn}k}+e^{\frac{\pi}{2}i(\lambda+1)\operatorname{sgn}k}}_{2\cos\left(\frac{\pi}{2}(\lambda+1)\operatorname{sgn}k\right)}\Big)|k|^{-\lambda-1}\Gamma(\lambda+1) = -2\sin\left(\frac{\pi\lambda}{2}\right)\Gamma(\lambda+1)|k|^{-\lambda-1}$$

↳ BONUS: z inverze $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin\pi z}$

úprava: $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ // $z=-\lambda$ $\Rightarrow$ $\widehat{|x|^\lambda} = \dfrac{\pi|k|^{-\lambda-1}}{\Gamma(-\lambda)\cos\frac{\pi\lambda}{2}}$

- $$\widehat{|x|^\lambda\operatorname{sgn}x} = \widehat{x_+^\lambda}-\widehat{x_-^\lambda} = \Big(\underbrace{e^{-\frac{\pi}{2}i(\lambda+1)\operatorname{sgn}k}-e^{\frac{\pi}{2}i(\lambda+1)\operatorname{sgn}k}}_{-2i\sin\left(\frac{\pi}{2}(\lambda+1)\operatorname{sgn}k\right)}\Big)|k|^{-\lambda-1}\Gamma(\lambda+1) = \underbrace{-2i\cos\left(\frac{\pi\lambda}{2}\right)|k|^{-\lambda-1}\operatorname{sgn}k\,\Gamma(\lambda+1)}_{\frac{\pi i|k|^{-\lambda-1}\operatorname{sgn}k}{\Gamma(-\lambda)\sin\frac{\pi\lambda}{2}}}$$

- Jest $(x\pm i0)^\lambda = e^{\pm\frac{\pi\lambda i}{2}}\left(|x|^\lambda\cos\frac{\pi\lambda}{2} \mp i|x|^\lambda\operatorname{sgn}x\,\sin\frac{\pi\lambda}{2}\right)$ (viz Distribuce)

$$\therefore\ \widehat{(x\pm i0)^\lambda} = e^{\pm\frac{\pi\lambda i}{2}}\left(\widehat{|x|^\lambda}\cos\frac{\pi\lambda}{2}\mp i\,\widehat{|x|^\lambda\operatorname{sgn}x}\,\sin\frac{\pi\lambda}{2}\right)$$

$$= e^{\pm\frac{\pi\lambda i}{2}}\frac{\pi|k|^{-\lambda-1}}{\Gamma(-\lambda)}\left(\frac{\cos\frac{\pi\lambda}{2}}{\cos\frac{\pi\lambda}{2}} \mp i\,\frac{i\operatorname{sgn}k\,\sin\frac{\pi\lambda}{2}}{\sin\frac{\pi\lambda}{2}}\right) =$$

$$= e^{\pm\frac{\pi\lambda i}{2}}\frac{\pi|k|^{-\lambda-1}}{\Gamma(-\lambda)}(1\pm\operatorname{sgn}k) = e^{\pm\frac{\pi\lambda i}{2}}\frac{2\pi k_\pm^{-\lambda-1}}{\Gamma(-\lambda)}$$

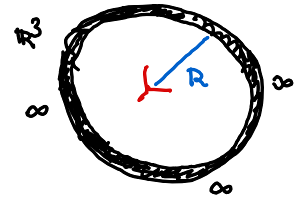
FOURIEROVA TRASFORMACE RADIÁLNÍCH DISTRIBUCÍ

Připomenutí: Pro $f(\vec{x}) = g(r)$ radiální ($f \in \mathcal{S}(\mathbb{R}^n)$; $\vec{x} \in \mathbb{R}^n$; $r = \|\vec{x}\|$)

$$\hat{f}(\vec{k}) = k_{n-1} \int_0^\infty \int_0^\pi g(r)\, e^{-ikr\cos\theta}\, r^{n-1} \sin^{n-2}\theta \; d\theta \, dr$$

Speciálně $n=3$: $\hat{f}(\vec{k}) = \frac{4\pi}{k} \int_0^\infty r\, g(r) \sin(kr)\, dr$; nebo obecně

$$\hat{f}(\vec{k}) = \frac{(2\pi)^{\frac{n}{2}}}{k^{\frac{n}{2}-1}} \int_0^\infty r^{\frac{n}{2}} g(r)\, J_{\frac{n}{2}-1}(kr)\, dr$$



Př: $f(\vec{x}) = \nu_R = \delta(r-R)$; $R>0$; $\vec{x} \in \mathbb{R}^3$

$$\rightarrow \widehat{\delta(r-R)} = \frac{(2\pi)^{\frac{n}{2}}}{k^{\frac{n}{2}-1}} \int_0^\infty r^{\frac{n}{2}} \delta(r-R)\, J_{\frac{n}{2}-1}(kr)\, dr = \frac{(2\pi)^{\frac{n}{2}}}{k^{\frac{n}{2}-1}} R^{\frac{n}{2}} J_{\frac{n}{2}-1}(kR)$$

Speciálně pro $n=3$: $\widehat{\delta(r-R)} = \frac{4\pi}{k} \int_0^\infty r\, \delta(r-R) \sin(kr)\, dr = \frac{4\pi R}{k} \sin(kR)$

HOMOGENNÍ RADIÁLNÍ DISTRIBUCE (RIESZOVA)

odvození ve Fourierových tr. fcí

$$\boxed{\therefore \; \widehat{r^\lambda} = \frac{\pi^{\frac{n}{2}}\, 2^{\lambda+n}\, \Gamma\left(\frac{\lambda+n}{2}\right)}{k^{\lambda+n}\, \Gamma\left(-\frac{\lambda}{2}\right)}}$$

$\Rightarrow$ Rozšíření na H_{r^λ} ; $\lambda \in \mathbb{C}$

$\lambda \neq \dots, -4-n, -2-n, -n, 0, 2, 4, \dots$

FT DERIVACÍ DIRAKA

Nejdřív vím $\widehat{\delta(\vec{x})} = 1 \Rightarrow \widehat{\nabla\delta(\vec{x})} = i\vec{k}\, \widehat{\delta(\vec{x})} = i\vec{k}$;

$\widehat{\Delta\delta(\vec{x})} = (i\vec{k} \cdot i\vec{k})\, \widehat{\delta(\vec{x})} = -k^2$; $k = \|\vec{k}\|$ (dle pravidel FT)

$$\therefore \; \widehat{\Delta^\ell \delta(\vec{x})} = (-k^2)^\ell = (-1)^\ell k^{2\ell}$$

FT RADIÁLNÍCH SUDÝCH MOCNIN

(liché viz vzorec $\widehat{r^\lambda}$ výše)

Inverzí získám: $\widehat{r^{2\ell}} = (2\pi)^n \, \check{r^{2\ell}} = (2\pi)^n (-1)^\ell \Delta^\ell \delta(\vec{k})$

(Př) Řešte vlnovou PDR ve 3D : $\vec{x} \in \mathbb{R}^3$

$$u_{,tt} - \Delta u = 0 \ ; \quad u(\vec{x},t) \ ; \quad BC: \left.\begin{array}{l} u(\vec{x},0) = u_0(\vec{x}) \\ \frac{\partial u}{\partial t}(\vec{x},0) = v_0(\vec{x}) \end{array}\right\} \in \mathcal{S}(\mathbb{R}^3)$$

Řešení :

$$u(\vec{x},t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}\, dk$$

$$\therefore \quad u_{,tt} - \Delta u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\hat{u}_{,tt} + k^2 \hat{u})\, e^{i\vec{k}\cdot\vec{x}}\, dk \ ; \quad \text{čili}$$

$$\hat{u}_{,tt} + k^2 \hat{u} = 0 \ ; \qquad BC: \ \hat{u}(\vec{k},0) = \hat{u}_0 \ ; \ \frac{\partial \hat{u}}{\partial t}(\vec{k},0) = \hat{v}_0$$

$$\Rightarrow \quad \hat{u} = C(\vec{k}) \cos(kt) + D(\vec{k}) \sin(kt) =$$

$$\overset{BC}{=} \hat{u}_0(\vec{k}) \cos(kt) + \hat{v}_0(\vec{k}) \frac{\sin(kt)}{k} =$$

$$= \left(\hat{u}_0 \frac{\sin(kt)}{k}\right)_{,t} + \hat{v}_0 \frac{\sin(kt)}{k}$$

$$\therefore \quad \text{Formálně} \quad u = \left(u_0 * \widecheck{\frac{\sin(kt)}{k}}\right)_{,t} + v_0 * \widecheck{\frac{\sin(kt)}{k}} \quad \text{je řešením}$$

NYNÍ NAJDEME INVERZNÍ TRANSFORMACE : $(k \leftrightarrow r)$

- $$\widehat{\frac{\sin(rt)}{r}} \overset{\text{radiální}}{=} \frac{4\pi}{k} \int_0^\infty r \frac{\sin(rt)}{r} \sin(kr)\, dr =$$

$$\overset{\text{sym.}}{=} \frac{2\pi}{k} \int_{-\infty}^{\infty} \sin(rt) \sin(kr)\, dr = \frac{\pi}{k} \int_{-\infty}^{\infty} \cos(rt - kr) - \cos(rt + kr)\, dr$$

$$= \frac{\pi}{k} \operatorname{Re} \int_{-\infty}^{\infty} e^{-i(rt-kr)} - e^{-i(rt+kr)}\, dr = \frac{2\pi^2}{k} (\delta(t-k) - \delta(t+k))$$

$$\therefore \quad \widecheck{\frac{\sin(kt)}{k}} \overset{\vec{k} \to -\vec{k}}{=} \frac{1}{(2\pi)^3} \widehat{\frac{\sin(rt)}{r}} = \frac{1}{4\pi r} (\delta(t-r) - \delta(t+r))$$

člen $\delta(t+y)$ tam není protože $y = |\vec{y}| > 0$

$$\Rightarrow \quad v_0 * \widecheck{\frac{\sin(kt)}{k}} = \int_{\mathbb{R}^3} v_0(\vec{x} - \vec{y}) \frac{1}{4\pi y} \delta(t-y)\, d\vec{y}$$

$$\overset{\text{radiální}}{=} \int_0^\infty \oint_\Omega v_0(\vec{x} - r\hat{r}) \frac{1}{4\pi r} \delta(t-r)\, r^2\, d\Omega\, dr = \frac{t}{4\pi} \oint_\Omega v_0(\vec{x} - t\hat{r})\, d\Omega$$

$$\left[\text{OBDOBNĚ} \quad u_0 * \widecheck{\cos kt} = \frac{\partial}{\partial t} \frac{t}{4\pi} \oint_\Omega u_0(\vec{x} - t\hat{r})\, d\Omega \right]$$

KONVOLUCE DISTRIBUCÍ

Př Jaký význam má $\frac{1}{x_+^{3/2}} * e^{-|x|} = \int_{-\infty}^{\infty} \frac{1}{y_+^{3/2}} e^{-|x-y|}\, dy$ <u>nekonverguje</u>

<u>ALE</u> my můžeme vnímat $\frac{1}{x^{3/2}}$ jako $H_{x^{-3/2}}$ =

Pro funkce $f, g, h \in \mathcal{S}(\mathbb{R}^n)$ platí asociativita:

$\tilde{g}(\vec{x}) := g(-\vec{x})$

$$\int_{\mathbb{R}^n} (f * g)(\vec{x})\, h(\vec{x})\, d\vec{x} = \int_{\mathbb{R}^n} f(\vec{x})\, (\tilde{g} * h)(\vec{x})\, d\vec{x}$$

zobecnění

D $\langle f * T, \varphi \rangle := \langle T, \tilde{f} * \varphi \rangle$; $f \in L^1_{loc}(\mathbb{R}^n)$, $T \in \mathcal{S}'(\mathbb{R}^n)$

! jen T může být obecná distribuce, f musí být regulární

platí $\langle f * T', \varphi \rangle = \langle T', \tilde{f} * \varphi \rangle =$

$$- \langle T, \tilde{f} * \varphi' \rangle = - \langle f * T, \varphi' \rangle$$

$$\therefore \quad (f * T)' = f * T'$$

V $\widehat{f * T} = \hat{f}\, \hat{T}$ ve smyslu distribucí ;

(Př) Spočtěte $T_{x^2} * T_{e^{-\alpha|x|}} =$

Metoda I : Přímo výpočtem (vše regulární)

$$T_{x^2} * T_{e^{-\alpha|x|}} = \int_{-\infty}^{\infty} e^{-\alpha|y|}(x-y)^2\,dy = \int_{-\infty}^{\infty} e^{-\alpha|y|}(x^2-2xy+y^2)\,dy =$$

$$\overset{\text{sym.}}{=} x^2 \underbrace{\int_{-\infty}^{\infty} e^{-\alpha|y|}\,dy}_{\frac{2}{\alpha}} + \underbrace{\int_{-\infty}^{\infty} y^2 e^{-\alpha|y|}\,dy}_{\partial_\alpha^2 \int_{-\infty}^{\infty} e^{-\alpha|y|}dy} = \frac{2x^2}{\alpha} + \frac{4}{\alpha^3}$$

Metoda II : Pomocí Fourierovy transformace

$$\widehat{T_{x^2} * T_{e^{-\alpha|x|}}} = \widehat{T}_{x^2}\,\widehat{T}_{e^{-|x|}} = 2\pi\,(-\delta^{(2)}(k))\,\frac{2\alpha}{\alpha^2+k^2} =$$

$$= -4\pi\alpha\,\frac{1}{\alpha^2+k^2}\,\delta^{(2)}(k) = -4\pi\alpha\left[\left(\frac{1}{\alpha^2+k^2}\,\delta(k)\right)'' - 2\left(\left(\frac{1}{\alpha^2+k^2}\right)'\delta(k)\right)' + \left(\frac{1}{\alpha^2+k^2}\right)''\delta(k)\right]$$

$$= -4\pi\alpha\left[\left(\frac{1}{\alpha^2+k^2}\Big|_0\,\delta(k)\right)'' - 2\left(\left(\frac{1}{\alpha^2+k^2}\right)'\Big|_0\,\delta(k)\right)' + \left(\frac{1}{\alpha^2+k^2}\right)''\Big|_0\,\delta(k)\right] =$$

$$= -4\pi\alpha\left[\frac{1}{\alpha^2}\,\delta''(k) - \frac{2}{\alpha^4}\,\delta(k)\right]$$

$$\therefore \text{ Inverze : } T_{x^2} * T_{e^{-\alpha|x|}} = -4\pi\alpha\left[-\frac{1}{\alpha^2}\frac{x^2}{2\pi} - \frac{2}{\alpha^4}\frac{1}{2\pi}\right] = \frac{2x^2}{\alpha} + \frac{4}{\alpha^3}$$

(Př) Spočtěte $H_{x_+^{-3/2}} * T_{x_+^{-1/2}}$

Metoda I : Nelze přímo $\because$ $\int_{-\infty}^{\infty} y_+^{-3/2}(x-y)_+^{-1/2}\,dy$ nekonverguje

ale $H_{x_+^{-3/2}} = -2\,T'_{x_+^{-1/2}} \Rightarrow H_{x_+^{-3/2}} * T_{x_+^{-1/2}} = -2\left(T_{x_+^{-1/2}} * T_{x_+^{-1/2}}\right)'$

jest $T_{x_+^{-1/2}} * T_{x_+^{-1/2}} = \int_{-\infty}^{\infty} y_+^{-1/2}(x-y)_+^{-1/2}\,dy \overset{x>0}{=} \int_0^x \frac{1}{\sqrt{y}}\frac{1}{\sqrt{x-y}}\,dy =$

$$\overset{|y = x\sin^2 t|}{=} \int_0^{\frac{\pi}{2}} \frac{2\sin t\cos t}{\sin t\cos t}\,dt = \pi \text{ ; jinak } 0$$

$$\therefore T_{x_+^{-1/2}} * T_{x_+^{-1/2}} = \pi\,\theta(x) \Rightarrow H_{x_+^{-3/2}} * T_{x_+^{-1/2}} = -2\pi\,\delta(x)$$

Metoda II : $\widehat{H_{x_+^{-3/2}} * T_{x_+^{-1/2}}} = \widehat{H}_{x_+^{-3/2}}\,\widehat{T}_{x_+^{-1/2}} =$

$$\Gamma(-\tfrac{1}{2})\,|k|^{1/2}\,e^{-\frac{\pi}{2}i(-\frac{1}{2})\operatorname{sgn}k}\,\Gamma(\tfrac{1}{2})\,|k|^{-1/2}\,e^{-\frac{\pi}{2}i\frac{1}{2}\operatorname{sgn}k} = \Gamma(-\tfrac{1}{2})\Gamma(\tfrac{1}{2}) = -2\sqrt{\pi}\sqrt{\pi} = -2\pi$$

$$\therefore H_{x_+^{-3/2}} * T_{x_+^{-1/2}} = -2\pi\,\delta(x)$$

KONVOLUČNÍ ROVNICE

Př: Najděte řešení $f \in \mathcal{S}'$ Sol:

a) $$e^{-\beta|x-x_0|} = \int_{-\infty}^{\infty} f(y)\, e^{-\alpha|x-y|}\, dy \quad ; \alpha, \beta > 0 \; ; x_0 \in \mathbb{R}$$

b) $$e^{-\beta\|\vec{x}-\vec{x}_0\|} = \int_{\mathbb{R}^3} f(\vec{y})\, e^{-\alpha\|\vec{x}-\vec{y}\|}\, d\vec{y} \quad ; \alpha, \beta > 0 \; ; \vec{x}_0 \in \mathbb{R}^3$$

a) Vezmeme FT ; $\widehat{e^{-\beta|x-x_0|}} = \widehat{f(x) * e^{-\alpha|x|}} = \hat{f}\, \widehat{e^{-\alpha|x|}}$

jest $$\widehat{e^{-\alpha|x|}} = 2\,\mathrm{Re} \int_0^{\infty} e^{-\alpha x - ikx}\, dx = 2\,\mathrm{Re} \frac{1}{\alpha + ik} = \frac{2\alpha}{\alpha^2 + k^2}$$

obdobně $$\widehat{e^{-\beta|x-x_0|}} = \int_{-\infty} e^{-\beta|x-x_0|} e^{-ikx}\, dx = \left| x = u + x_0 \right| = e^{-ikx_0}\, \widehat{e^{-\beta|x|}} = \frac{2\beta e^{-ikx_0}}{\beta^2 + k^2}$$

$$\therefore \quad \frac{2\beta e^{-ikx_0}}{\beta^2 + k^2} = \hat{f}\, \frac{2\alpha}{\alpha^2 + k^2} \Rightarrow \hat{f} = \frac{\beta}{\alpha} \frac{\alpha^2 + k^2}{\beta^2 + k^2} e^{-ikx_0} =$$

$$= \frac{\beta}{\alpha} \frac{\alpha^2 - \beta^2 + \beta^2 + k^2}{\beta^2 + k^2} e^{-ikx_0} = \frac{\beta}{\alpha} e^{-ikx_0} + \frac{\beta}{\alpha} \frac{\alpha^2 - \beta^2}{\beta^2 + k^2} e^{-ikx_0}$$

↳ inverze: $$f = \frac{\beta}{\alpha} \delta(x - x_0) + \frac{\alpha^2 - \beta^2}{2\alpha} e^{-\beta|x - x_0|}$$

b) Stejná rovnost v FT obrazech, ale

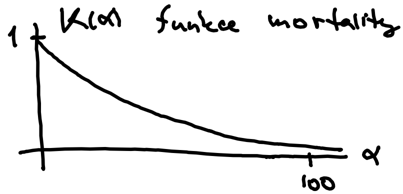
$$\widehat{\frac{e^{-\alpha r}}{r}} = \frac{4\pi}{\alpha^2 + k^2} \overset{\partial_\alpha}{\Rightarrow} \widehat{e^{-\alpha r}} = \frac{8\pi\alpha}{(\alpha^2 + k^2)^2} \quad ; \quad \widehat{e^{-\beta\|\vec{x} - \vec{x}_0\|}} = \frac{8\pi\beta\, e^{-i\vec{k}\cdot\vec{x}_0}}{(\beta^2 + k^2)^2}$$

$$\therefore \quad \hat{f} = \frac{\beta}{\alpha} \frac{(\alpha^2 + k^2)^2}{(\beta^2 + k^2)^2} e^{-i\vec{k}\cdot\vec{x}_0} = \frac{\beta}{\alpha} \frac{(\alpha^2 - \beta^2 + \beta^2 + k^2)^2}{(\beta^2 + k^2)^2} e^{-i\vec{k}\cdot\vec{x}_0} =$$

$$= \frac{\beta}{\alpha} \frac{(\alpha^2 - \beta^2)^2}{(\beta^2 + k^2)^2} e^{-i\vec{k}\cdot\vec{x}_0} + \frac{2\beta}{\alpha} \frac{(\alpha^2 - \beta^2)}{(\beta^2 + k^2)} e^{-i\vec{k}\cdot\vec{x}_0} + \frac{\beta}{\alpha} e^{-i\vec{k}\cdot\vec{x}_0}$$

$$\therefore \quad f = \frac{(\alpha^2 - \beta^2)^2}{8\pi\alpha} e^{-\beta\|\vec{x} - \vec{x}_0\|} + \frac{\alpha^2 - \beta^2}{2\pi\alpha} \frac{e^{-\beta\|\vec{x} - \vec{x}_0\|}}{\|\vec{x} - \vec{x}_0\|} + \frac{\beta}{\alpha} \delta(\vec{x} - \vec{x}_0)$$

(Př) Populační dynamika



$t = 0$ rok začátku měření

α … věk osoby $\leq t$.

N_0 … počáteční populace $\alpha = 0$.

<u>Mortalita</u> : $K(\alpha)$

<u>Natalita</u> : λ koeficient

→ model všichni lidé plodní : $N(t) \longrightarrow \lambda N(t) \Delta t$ nová populace

<u>Kohorta</u> : $\eta(t,\alpha) \Rightarrow \eta(t,0) = \lambda N(t)$

→ jako funkce α tzv. populační pyramida (nová populace jen!)

časová rovnice pro kohortu : $\eta(t,\alpha) = \underbrace{\eta(t-\alpha,0)\,K(\alpha)}_{\text{nově narození}} = \lambda N(t-\alpha) K(\alpha)$

<u>Celková populace</u> :

$$N(t) = N_0 K(t) + \int_0^t \eta(t,\alpha)\, d\alpha = N_0 K(t) + \lambda \int_0^t N(t-\alpha) K(\alpha)\, d\alpha$$

Protože $t < 0$ je $N(t) = 0$; $K(t) = 0$: $N(t) = N_0 K(t) + \lambda \int_{-\infty}^{\infty} N(t-\alpha) K(\alpha)\, d\alpha$

$$\therefore \quad \hat{N} = N_0 \hat{K} + \lambda \hat{N} \hat{K} \quad \Rightarrow \quad \hat{N} = \frac{N_0 \hat{K}}{1 - \lambda \hat{K}}$$

<u>Model</u> : $K(\alpha) = e^{-\mu\alpha}\theta(\alpha) \quad \Rightarrow \quad \hat{K} = \frac{1}{\mu + i\alpha}$

$$\therefore \quad \hat{N} = N_0 \frac{\frac{1}{\mu + i\alpha}}{1 - \lambda \frac{1}{\mu + i\alpha}} = N_0 \frac{1}{\mu - \lambda + i\alpha} \quad \Rightarrow \quad \underline{\underline{N = N_0 e^{(\lambda-\mu)t}\,\theta(t)}}$$

<u>Meze</u> : $\frac{1}{\lambda}\eta(t,0) = N_0 K(t) + \int_0^t \eta(t-\alpha,0) K(\alpha)\, d\alpha$