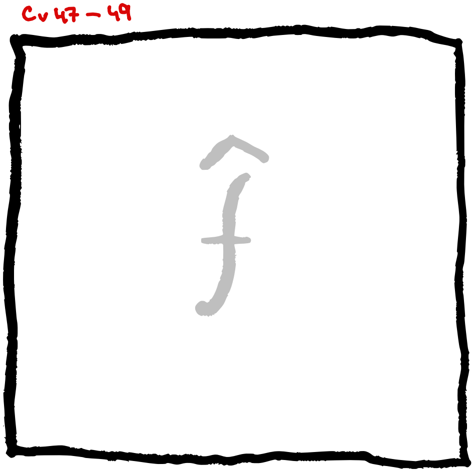
Cv 47 – 49



FOURIEROVA TRANSFORMACE

$$(1 + x^n) f \in L^1 \iff \hat{f} \in C^n$$

FOURIEROVA TRANSFORMACE

$\vec{x}, \vec{y} \in \mathbb{R}^n$

$\boxed{D}$ Konvoluce: $(f * g)(\vec{x}) := \int_{\mathbb{R}^n} f(\vec{y})\, g(\vec{x}-\vec{y})\, d\vec{y}$ ~ objemový element

$\boxed{D}$ Fourierova transformace

$$\hat{f}(\vec{k}) = \mathcal{F}(f(\vec{x})) := \int_{\mathbb{R}^n} f(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}\, d\vec{x} \overset{k_j = 2\pi \xi_j}{=} \int_{\mathbb{R}^n} f(\vec{x})\, e^{-2\pi i \vec{\xi}\cdot\vec{x}}\, d\vec{x}$$

$$f(\vec{x}) = \mathcal{F}^{-1}(\hat{f}(\vec{k})) = \check{\hat{f}}(\vec{k}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}\, d\vec{k} \overset{dk = (2\pi)^n d\xi}{=} \int_{\mathbb{R}^n} \hat{f}(\vec{\xi})\, e^{2\pi i \vec{\xi}\cdot\vec{x}}\, d\vec{\xi}$$

<u>VLASTNOSTI</u> $f, g \in \mathcal{S}(\mathbb{R}^n)$

- $\hat{f}(-k) = \overline{\hat{f}(k)}$ (jen pro reálné funkce!)
- $\check{\hat{f}}(\vec{x}) = f(\vec{x})$ [VZÁJEMNÁ INVERZE]
- $\check{f}(\vec{x}) = \frac{1}{(2\pi)^n}\widehat{f(-\vec{x})}$ [TENTÝŽ TVAR FORMULÍ]
- $\widehat{f(A\cdot\vec{x}+\vec{b})} = \frac{1}{|\det A|} e^{i\vec{k}\cdot A^{-1}\cdot\vec{b}}\, \hat{f}(\vec{k}\cdot A^{-1})$ [OBECNÁ LINEÁRNÍ TRANSFORMACE]

  speciálně $x \in \mathbb{R}$: $\widehat{f(ax+b)} = \frac{1}{|a|} e^{ik\frac{b}{a}}\, \hat{f}\left(\frac{k}{a}\right)$

- $\widehat{f^{(\alpha)}} = (ik)^\alpha \hat{f}$ [PER PARTES] (α význam multi derivace ve více dimenzích)
- $\widehat{x^\alpha f} = (i\partial_k)^\alpha \hat{f}$ [DERIVACE DLE PARAMETRU]
- $\widehat{f*g} = \hat{f}\,\hat{g}$; $\widehat{fg} = \frac{1}{(2\pi)^n}\hat{f} * \hat{g}$ [KONVOLUČNÍ ZÁKONY]
- $\langle f, g\rangle = \frac{1}{(2\pi)^n}\langle \hat{f}, \hat{g}\rangle$ čili $\int_{\mathbb{R}^n} f\bar{g}\, d\vec{x} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}\bar{\hat{g}}\, d\vec{k}$ [PARSEVALOVA ROVNOST]
- $\|f\|^2 = \frac{1}{(2\pi)^n}\|\hat{f}\|^2$ čili $\int_{\mathbb{R}^n} |f|^2 d\vec{x} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\hat{f}|^2 d\vec{k}$ [PLANCHERELOVA ROVNOST]
- $\langle f, \hat{g}\rangle = \langle \hat{f}, g\rangle$ čili $\int_{\mathbb{R}^n} f(\vec{k})\hat{g}(\vec{k})\, d\vec{k} = \int_{\mathbb{R}^n} \hat{f}(\vec{k})\, g(\vec{k})\, d\vec{k}$

Poznámky k vlastnostem:

- VZÁJEMNÁ INVERZE: $x, y \in \mathbb{R}$ (pro $\vec{x}, \vec{y} \in \mathbb{R}$ po složkách)

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \qquad \text{funguje to zpátky ? } \Leftrightarrow$$

$$f(x) \overset{?}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)\, e^{-iky} dy \right) e^{ikx}\, dk$$

$$= \int_{-\infty}^{\infty} f(y) \underbrace{\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)}\, dk \right)}_{?\ \delta(x-y)} dy \overset{?}{=} f(x)$$

TRIK:

$$f(x) \overset{?}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx} e^{-\varepsilon k^2} dk =$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)\, e^{-iky}\, dy \right) e^{ikx} e^{-\varepsilon k^2} dk =$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \left(\int_{-\infty}^{\infty} e^{ik(x-y) - \varepsilon k^2} dk \right) dy = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \sqrt{\frac{\pi}{\varepsilon}} \int_{-\infty}^{\infty} f(y)\, e^{-\frac{(x-y)^2}{4\varepsilon}} dy =$$

$$\left| \begin{array}{l} y = x + 2t\sqrt{\varepsilon} \\ dy = 2\sqrt{\varepsilon}\, dt \end{array} \right| = \lim_{\varepsilon \to 0^+} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2t\sqrt{\varepsilon})\, e^{-t^2} dt = f(x) \underbrace{\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt}_{1}$$

- OBECNÁ LINEÁRNÍ TRANSFORMACE $\vec{x}, \vec{y} \in \mathbb{R}^n$; A matice $n \times n$

$$\widehat{f(A\cdot\vec{x}+\vec{b})} = \int_{\mathbb{R}^n} f(A\cdot\vec{x} + \vec{b})\, e^{-i\vec{k}\cdot\vec{x}}\, d\vec{x} = \frac{1}{|\det A|}\, e^{i\vec{k}\cdot A^{-1}\cdot\vec{b}}\, \hat{f}(\vec{k}\cdot A^{-1})$$

$\to$ Substituce $\vec{y} = A\cdot\vec{x} + \vec{b}$ čili ve složkách $y_i = A_{ij} x_j + b_i$

potřebujeme $d\vec{x} = \left|\frac{\partial x}{\partial y}\right| d\vec{y}$; $J = \left|\frac{\partial x}{\partial y}\right|$ jakobián.

platí $J^{-1} = \left|\frac{\partial y}{\partial x}\right| = \left|\det \frac{\partial y_i}{\partial x_j}\right| = |\det A_{ij}| = |\det A|$

navíc $\vec{y} = A\cdot\vec{x} + \vec{b} \Rightarrow \vec{x} = A^{-1}\cdot(\vec{y} - \vec{b})$

$$\therefore \widehat{f(A\vec{x}+\vec{b})} = \int_{\mathbb{R}^n} f(\vec{y})\, e^{-i\vec{k}\cdot A^{-1}\cdot(\vec{y}-\vec{b})} \frac{d\vec{y}}{|\det A|}$$

TABULKA FOURIEROVÝCH TRANSFORMACÍ $[\alpha > 0,\ \beta \in \mathbb{R}]$

$f(x)$	$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$
$e^{-\alpha x^2}$	$\sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{k^2}{4\alpha}}$
$\chi_{[-\alpha,\alpha]}(x)$	$2\,\frac{\sin \alpha k}{k}$
$\frac{\sin \alpha x}{x}$	$\chi_{[-\alpha,\alpha]}(k)$
$\frac{1}{\alpha^2 + x^2}$	$\frac{\pi}{\alpha}\, e^{-\alpha\lvert k\rvert}$
$\frac{x}{\alpha^2 + x^2}$	$-i\pi\, e^{-\alpha\lvert k\rvert}\, \operatorname{sgn} k$
$e^{-\alpha\lvert x\rvert}$	$\frac{2\alpha}{\alpha^2 + k^2}$
$\frac{\chi_{[-1,1]}(x)}{\sqrt{1-x^2}}$	$\pi J_0(k)$
$J_n(x)$	$(-i)^n\, \frac{2\,T_n(k)}{\sqrt{1-k^2}}\, \chi_{[-1,1]}(k)$

OBECNÝ POČET DIMENZIÍ $\vec{x}, \vec{k} \in \mathbb{R}^n$; $r = \lVert \vec{x} \rVert$; $k = \lVert \vec{k} \rVert$

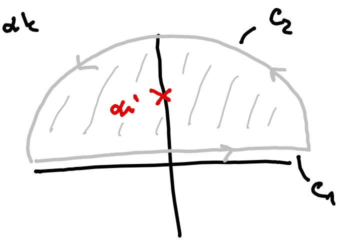
$f(\vec{x})$	$\hat{f}(\vec{k}) = \int_{-\infty}^{\infty} f(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}\, d\vec{x}$	
$e^{-\alpha r^2}$	$\left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} e^{-\frac{k^2}{4\alpha}}$	
$\frac{e^{-\alpha r}}{r}$	$\frac{4\pi}{\alpha^2 + k^2}$	$(n = 3)$
$\frac{1}{\alpha^2 + r^2}$	$\frac{2\pi^2}{k}\, e^{-k\alpha}$	$(n = 3)$
r^λ	$\frac{\pi^{\frac{n}{2}}\, 2^{\lambda+n}\, \Gamma\left(\frac{\lambda+n}{2}\right)}{k^{\lambda+n}\, \Gamma\left(-\frac{\lambda}{2}\right)}$	(RIESZ)
$e^{-\alpha r}$	$\frac{2^n \pi^{\frac{n-1}{2}} \alpha\, \Gamma\left(\frac{n+1}{2}\right)}{(\alpha^2 + k^2)^{\frac{n+1}{2}}}$	

Př: Spočtěte FT funkce $f(x) = \frac{1}{\alpha^2+x^2}$; $x \in \mathbb{R}$; BÚNO $\alpha > 0$

$$\hat{f}(k) = \int_{-\infty}^{\infty} \frac{1}{\alpha^2+x^2} e^{-ikx} dx = \left(\int_{-\infty}^{\infty} \frac{1}{\alpha^2+x^2} \cos kx - \underbrace{\cancel{i \sin kx}}_{\text{liché}} dx \right)$$

vidíme, že $\hat{f}(k)$ je sudá v k, čili stačí $k > 0$.

$$J_1 + J_2 = 2\pi i \operatorname{Res}_{\alpha i} \frac{e^{ikz}}{\alpha^2+z^2} = 2\pi i \frac{e^{ik(\alpha i)}}{2(\alpha i)} = \frac{\pi}{\alpha} e^{-\alpha k}$$



pro $z = x+iy$ je $|e^{ikz}| = |e^{ikx-ky}| = e^{-ky} \le 1$

$$\therefore \quad |J_2| \le \frac{1}{R^2-\alpha^2} 2\pi R \longrightarrow R$$

Zároveň $J_1 \longrightarrow \hat{f}(k)$, čili $\hat{f}(k) = \frac{\pi}{\alpha} e^{-\alpha k}$; $k > 0$

obecně tedy $\boxed{\hat{f}(k) = \frac{\pi}{\alpha} e^{-\alpha|k|}}$; $k \in \mathbb{R}$

Př: Spočtěte FT fce $f(x) = \frac{x}{\alpha^2+x^2}$

$$\longrightarrow \widehat{\frac{x}{\alpha^2+x^2}} = \widehat{x \frac{1}{\alpha^2+x^2}} = i \frac{\partial}{\partial k} \widehat{\frac{1}{\alpha^2+x^2}} = i \frac{\pi}{\alpha} \left(e^{-\alpha|k|}\right)' = -i\pi e^{-\alpha|k|} \operatorname{sgn} k$$

WILD

$$\longrightarrow \widehat{\frac{x}{\alpha^2+x^2}} = \widehat{x \frac{1}{\alpha^2+x^2}} = \frac{1}{2\pi} \hat{x} * \widehat{\frac{1}{\alpha^2+x^2}} =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi i \delta'(k-l) \frac{\pi}{\alpha} e^{-\alpha|l|} dl \overset{PP}{=} -i\frac{\pi}{\alpha} \int_{-\infty}^{\infty} (-\delta(k-l))(-\alpha e^{-|l|} \operatorname{sgn} l) dl$$

$$= -\frac{i\pi}{\alpha} \alpha e^{-|k|} \operatorname{sgn} k \quad \checkmark$$

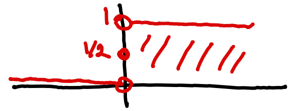
Př: Spočtěte FT $f(x) = \frac{1}{(\alpha^2+x^2)^2}$

Trik: Derivace dle parametru: $\widehat{\frac{1}{\alpha^2+x^2}} = \frac{\pi}{\alpha} e^{-\alpha|k|} \quad / \frac{\partial}{\partial\alpha}$

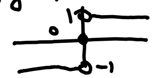
$$-\widehat{\frac{2\alpha}{(\alpha^2+x^2)^2}} = -\frac{\pi}{\alpha^2} e^{-\alpha|k|} - \frac{\pi|k|}{\alpha} e^{-\alpha|k|} \quad \therefore \quad \widehat{\frac{1}{(\alpha^2+x^2)^2}} = \frac{\pi(1+\alpha|k|)}{2\alpha^3} e^{-\alpha|k|}$$

D Heavisideova funkce $\Theta(x)$; $x \in \mathbb{R}$

$$\Theta(x) := \begin{cases} 1 & ; x > 0 \\ 1/2 & ; x = 0 \\ 0 & ; x < 0 \end{cases}$$



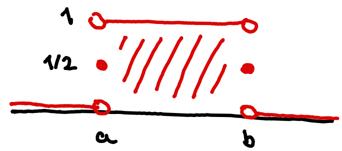
$\operatorname{sgn} x$



$$\operatorname{sgn} x = 2\,\Theta(x) - 1$$

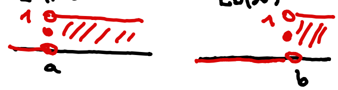
D Charakteristická funkce $\chi_M(\vec{x})$; $\vec{x} \in \mathbb{R}^n$; $M \subset \mathbb{R}^n$ oblast

$$\chi_M(\vec{x}) := \begin{cases} 1 & ; \vec{x} \in \operatorname{Int} M \\ 1/2 & ; \vec{x} \in \partial M \\ 0 & ; \vec{x} \in \operatorname{Ext} M \end{cases}$$



Př. Box-funkce $\chi_{[a,b]}(x)$; $a < b$

Platí : $\chi_{[a,b]}(x) = \chi_{[a,\infty)}(x) - \chi_{[b,\infty)}(x) = \Theta(x-a) - \Theta(x-b)$



Př. $f(x) = \chi_{[-1,1]}(x) \Rightarrow \hat{f}(k) = \int_{-\infty}^{\infty} \chi_{[-1,1]}(x)\, e^{-ikx} dx = \int_{-1}^{1} e^{-ikx} dx = \dfrac{e^{-ik} - e^{ik}}{-ik} = 2\,\dfrac{\sin k}{k}$

Intermezzo :

$I(\alpha) = \int_{-\infty}^{\infty} \dfrac{\sin \alpha x}{x} dx$; evidentně $I(-\alpha) = -I(\alpha)$ lichá fce ; $I(0) = 0$

$\therefore$ BÚNO $\alpha > 0$; $I(\alpha) \overset{\alpha>0}{=} \left| \begin{matrix} x = \frac{y}{\alpha} \\ dx = \frac{1}{\alpha} dy \end{matrix} \;\middle|\; \begin{matrix} \infty \to \infty \\ -\infty \to -\infty \end{matrix} \right| = \int_{-\infty}^{\infty} \dfrac{\sin y}{\frac{y}{\alpha}} \frac{1}{\alpha} dy =$

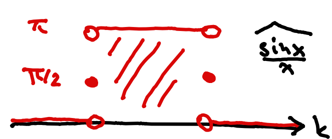
$= \int_{-\infty}^{\infty} \dfrac{\sin y}{y} dy = \pi$ [DIRICHLETŮV INTEGRÁL]

$\therefore \quad I(\alpha) = \begin{cases} \pi & ; \alpha > 0 \\ 0 & ; \alpha = 0 \\ -\pi & ; \alpha < 0 \end{cases}$ čili $\boxed{I(\alpha) = \pi \operatorname{sgn} \alpha}$

Př. $f(x) = \dfrac{\sin x}{x} \Rightarrow \hat{f}(k) = \int_{-\infty}^{\infty} \dfrac{\sin x}{x} e^{-ikx} dx = \int_{-\infty}^{\infty} \dfrac{\sin x}{x} \cos kx \, dx =$

$= \frac{1}{2} \int_{-\infty}^{\infty} \dfrac{\sin(kx+x) - \sin(kx-x)}{x} dx = \frac{\pi}{2} \left(\operatorname{sgn}(k+1) - \operatorname{sgn}(k-1) \right) =$

$= \pi \left(\Theta(k+1) - \Theta(k-1) \right) = \pi\, \chi_{[-1,1]}(k)$



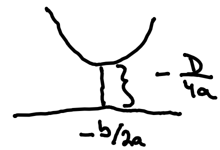
Důsledek: $\dfrac{\sin x}{x} = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} \pi \chi_{[-1,1]}(k)\, e^{ikx} dk$

$\therefore \quad \hat{\chi}_{[-1,1]}(k) = 2\, \dfrac{\sin k}{k}$

Př: 1D - GAUSS: $x \in \mathbb{R}$

$$f(x) = e^{-\alpha x^2} \; ; \; \alpha > 0 \Rightarrow \qquad \hat{f}(k) = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{k^2}{4\alpha}}$$

Intermezzo s kvadratickou funkcí:



$$Q = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = a\left(x + \frac{b}{2a}\right) - \frac{D}{4a}$$

$$\therefore \; -\alpha x^2 - ikx = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} = -\alpha\left(x + \frac{-ik}{-2\alpha}\right)^2 - \frac{(-ik)^2}{-4\alpha} = -\alpha\left(x + \frac{ik}{2\alpha}\right)^2 - \frac{k^2}{4\alpha}$$

čili $$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{-ikx}\, dx = \int_{-\infty}^{\infty} e^{-\alpha\left(x + \frac{ik}{2\alpha}\right)^2 - \frac{k^2}{4\alpha}}\, dx \overset{\text{posun kontury}}{=} e^{-\frac{k^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx =$$

$$= \left| \begin{matrix} x = \frac{y}{\sqrt{\alpha}} \\ dx = \frac{1}{\sqrt{\alpha}}\, dy \end{matrix} \right| = \frac{1}{\sqrt{\alpha}}\, e^{-\frac{k^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{k^2}{4\alpha}}$$

Př: OBECNÝ GAUSS $\vec{x} \in \mathbb{R}^n$

$$f(\vec{x}) = e^{-\alpha r^2} \; ; \; r^2 = x^2 = x_1^2 + \dots + x_n^2 \; ; \; \alpha > 0 \Rightarrow \qquad \hat{f}(\vec{k}) = \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} e^{-\frac{k^2}{4\alpha}}$$

$$\hat{f}(\vec{k}) = \int_{\mathbb{R}^n} e^{-\alpha r^2} e^{-i\vec{k}\cdot\vec{x}}\, d\vec{x} = \int_{\mathbb{R}^N} \prod_{j=1}^{n} e^{-\alpha x_j^2 - ik_j x_j}\, d\vec{x} =$$

$$= \prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{-\alpha x_j^2 - ik_j x_j}\, dx_j = \prod_{j=1}^{n} \hat{f}_j(k_j) \overset{f_j(x_j) = e^{-\alpha x_j^2}}{=} \prod_{j=1}^{n} \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{k_j^2}{4\alpha}}$$

Př: NAJDĚTE FT fce $f(x) = x e^{-\alpha x^2}$; $x \in \mathbb{R}$; $\alpha > 0$

jest $$\widehat{x e^{-\alpha x^2}} = i \frac{\partial}{\partial k} \widehat{e^{-\alpha x^2}} = i \frac{\partial}{\partial k} \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{k^2}{4\alpha}} = -\frac{ik}{2\alpha} \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{k^2}{4\alpha}}$$

NEBO: $$\widehat{x e^{-\alpha x^2}} = -\frac{1}{2\alpha} \widehat{\left(e^{-\alpha x^2}\right)'} = -\frac{1}{2\alpha}\, ik\, \widehat{e^{-\alpha x^2}} = -\frac{1}{2\alpha}\, ik \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{k^2}{4\alpha}}$$

(Př) Řešte (PDR) $u_{,t} - \Delta u = 0$; $u(\vec{x},t)$; $u(\vec{x},0) = u_0(\vec{x}) \in \mathcal{S}(\mathbb{R}^n)$ BC.
TEPELNOU

<u>Řešení</u> : Pišme $u(\vec{x},t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}(\vec{k},t) e^{i\vec{k}\cdot\vec{x}} d\vec{k}$ nějaké $\hat{u}(\vec{k},t)$

BC : $u_0(\vec{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}(\vec{k},0) e^{i\vec{k}\cdot\vec{x}} d\vec{k}$ čili $\hat{u}(k,0) = \hat{u}_0$

jest $\Delta u(\vec{x},t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}(\vec{k},t) \Delta e^{i\vec{k}\cdot\vec{x}} d\vec{k} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}(\vec{k},t) (-k^2 e^{i\vec{k}\cdot\vec{x}}) d\vec{k}$

$\therefore \quad u_{,t} - \Delta u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\hat{u}_{,t} - k^2 \hat{u}) e^{i\vec{k}\cdot\vec{x}} d\vec{k} \equiv 0$

$\Rightarrow \quad \hat{u}_{,t} - k^2 \hat{u} = 0 \qquad (\text{ODR}) \quad \Rightarrow \quad \hat{u} = C(\vec{k}) e^{-k^2 t}$

BC: $t = 0$: $\hat{u}_0 = C(\vec{k}) \quad \therefore \quad \hat{u} = \hat{u}_0 e^{-k^2 t} = \hat{u}_0 \hat{f}$

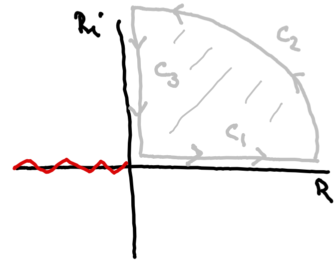
DLE ZÁKONA O KONVOLUCI : $\mathcal{F}(e^{-\alpha x^2}) = \left(\frac{\pi}{\alpha}\right)^n e^{-\frac{k^2}{4\alpha}}$

$u = u_0 * \mathcal{F}^{-1}(e^{-k^2 t}) = u_0 * \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha x^2}$; $\alpha = \frac{1}{4t}$

čili $\boxed{u(\vec{x},t) = \frac{1}{(4t\pi)^n} \int_{\mathbb{R}^n} u_0(\vec{x}-\vec{y}) e^{-\frac{y^2}{4t}} dy}$

(4*) Spočtěte FT funkce $f(x) = \frac{\theta(x)}{\sqrt{x}} = x_+^{-1/2}$

<u>Sol</u>: $\hat{f}(k) = \widehat{\frac{\theta(x)}{\sqrt{x}}} = \int_{-\infty}^{\infty} \frac{\theta(x)}{\sqrt{x}} e^{-ikx}\,dx = \int_0^{\infty} \frac{1}{\sqrt{x}} e^{-ikx}\,dx$



1) $k < 0$: $\quad J = \oint \frac{1}{\sqrt{z}} e^{-ikz}\,dz$

- <u>Cauchy</u> $\quad J = 0$
- <u>Parametrizace</u>

$\longrightarrow \quad J_1 = \int_0^R \frac{1}{\sqrt{x}} e^{-ikx}\,dx \xrightarrow{R\to\infty} \hat{f}(k)$

$\longrightarrow \quad C_2: \; z = Re^{it}; \; t \in (0, \frac{\pi}{2})$

$$J_2 = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{R}\, e^{i\frac{t}{2}}} e^{-ikRe^{it}} Rie^{it}\,dt$$

$$|J_2| \le \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{R}} e^{Rk\sin t} R\,dt \overset{k<0}{\le} \sqrt{R}\int_0^{\frac{\pi}{2}} e^{Rk\frac{2t}{\pi}}\,dt = \frac{\pi}{2k\sqrt{R}}\left(e^{Rk}-1\right) \xrightarrow{R\to\infty} 0$$

$\longrightarrow \quad (-)\,C_3: \; z = it; \; dz = i\,dt; \; t \in (0, \infty)$

$$J_3 = (-)\int_0^R \frac{1}{\sqrt{\underbrace{it}_{e^{\frac{\pi}{2}i}t}}} e^{kt}\, i\,dt \xrightarrow{R\to\infty} -ie^{-\frac{\pi}{4}i}\int_0^{\infty} \frac{1}{\sqrt{t}} e^{kt}\,dt$$

- <u>Porovnání</u>: $\hat{f}(k) = ie^{-\frac{\pi}{4}i}\int_0^{\infty} \frac{1}{\sqrt{t}} e^{-|k|t}\,dt = \left| \begin{array}{l|l} t = \frac{u}{|k|} & 0 \to 0 \\ dt = \frac{1}{|k|}du & \infty \to \infty \end{array} \right| =$

$$= \frac{ie^{-\frac{\pi}{4}i}}{\sqrt{|k|}} \int_0^{\infty} \frac{1}{\sqrt{u}} e^{-u}\,du = \frac{e^{\frac{\pi}{4}i}}{\sqrt{|k|}}\Gamma\left(\tfrac{1}{2}\right) = e^{\frac{\pi}{4}i}\sqrt{\frac{\pi}{|k|}}$$

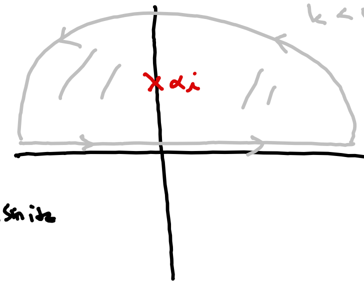
2) $k > 0$: $\quad \overline{\hat{f}(k)} = \overline{\int_{-\infty}^{\infty} f(x) e^{-ikx}\,dx} = \int_{-\infty}^{\infty} f(x) e^{ikx}\,dx = \hat{f}(-k)$

$\therefore \; \hat{f}(k) = \overline{\hat{f}(-k)} = e^{-\frac{\pi}{4}i}\sqrt{\frac{\pi}{|k|}}; \quad k > 0$

<u>Celkově</u>: $\boxed{\widehat{\frac{\theta(x)}{\sqrt{x}}} = e^{-\frac{\pi}{4}i\,\mathrm{sgn}\,k}\sqrt{\frac{\pi}{|k|}}} \qquad k \in \mathbb{R}/\{0\}$

(Př)* Spočtěte FT $f(x) = \frac{1}{(\alpha^2+x^2)^{n+1}}$; $n = 0,1,2,3,\dots$; $x\in\mathbb{R}$

Sol : $\hat f(k) = \int_{-\infty}^{\infty} \frac{1}{(\alpha^2+x^2)^{n+1}}\, e^{-ikx}\, dx$



BÚNO k<0

$$= 2\pi i \,\mathrm{Res}_{\alpha i}\, \frac{1}{(\alpha^2+z^2)^{n+1}}\, e^{i|k|z} =$$

$$= 2\pi i\, \frac{1}{n!}\left[\frac{1}{(z+\alpha i)^{n+1}}\, e^{i|k|z}\right]^{(n)}\Bigg|_{\alpha i} \overset{\text{Leibnitz}}{=}$$

$$= \frac{2\pi i}{n!}\sum_{j=0}^{n} \binom{n}{j} \underbrace{\left(\frac{1}{(z+\alpha i)^{n+1}}\right)^{(j)}}_{(-1)^j \frac{(n+j)!}{n!}\frac{1}{(z+\alpha i)^{n+j+1}}} \underbrace{\left(e^{i|k|z}\right)^{(n-j)}}_{(i|k|)^{n-j}\, e^{i|k|z}} \Bigg|_{\alpha i} =$$

$$= \frac{2\pi i}{n!}\sum_{j=0}^{n} \frac{\cancel{n!}}{j!\,(n-j)!}\, (-1)^j\, \frac{(n+j)!}{\cancel{n!}}\, \frac{1}{(2\alpha i)^{n+j+1}}\, (i|k|)^{n-j}\, e^{-\alpha|k|}$$

$$= \frac{\pi}{\alpha}\, e^{-\alpha|k|} \sum_{j=0}^{n} \frac{(n+j)!}{j!\,(n-j)!}\, \frac{|k|^{n-j}}{(2\alpha)^{n+j}}$$

(Př)* Najděte FT $f(x) = \frac{1}{\sqrt{\alpha^2+x^2}}$;

$$\hat f(k) = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{\sqrt{\alpha^2+x^2}}\,dx \overset{Re}{=} \int_{-\infty}^{\infty} \frac{\cos(kx)}{\sqrt{\alpha^2+x^2}}\,dx \overset{sym.}{=} 2\int_0^{\infty} \frac{\cos(kx)}{\sqrt{\alpha^2+x^2}}\,dx$$

$$\overset{x=\alpha t}{=} 2\int_0^{\infty} \frac{\cos(\alpha k t)}{\sqrt{1+t^2}}\,dt = 2K_0(\alpha k)$$ [Modif. Bess. fce II. dr.]

RADIÁLNÍ FUNKCE

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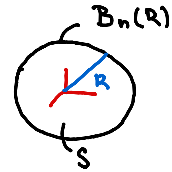
$r = \|\vec{x}\|$; $r^2 = x^2 = \sum_{j=1}^{n} x_j^2$

D $f: \mathbb{R}^n \to \mathbb{R}$ je radiální pokud $f(\vec{x}) = g(r)$ pro nějakou $g: \mathbb{R} \to \mathbb{R}$

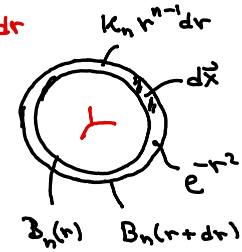


D $K_n :=$ Povrch jednotkové n-koule v $\mathbb{R}^n$ $(B_n(1))$

Platí $S_n(R) = K_n R^{n-1}$ povrch koule o poloměru $r = R$

A jelikož $d\vec{x} = S\, dr = K_n r^{n-1}\, dr$; $V_n(R) = \int dx = \int_0^R K_n r^{n-1}\, dr = \frac{K_n}{n} R^n$

Pro libov f: $\int_{\mathbb{R}^n} f(\vec{x})\, d\vec{x} = \int_0^\infty g(r)\, S_n(r)\, dr = K_n \int_0^\infty g(r)\, r^{n-1}\, dr$



V $\boxed{K_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}}$

<u>Dk</u>: Trik výpočet $\int_{\mathbb{R}^n} e^{-x^2}\, d\vec{x}$ dvěma způsoby

I. $\int_{\mathbb{R}^n} e^{-x^2}\, d\vec{x} = \int_{\mathbb{R}} e^{-(x_1^2 + \dots + x_n^2)}\, dx = \left(\int_{-\infty}^{\infty} e^{-x_1^2}\, dx_1 \right) = (\sqrt{\pi})^n$

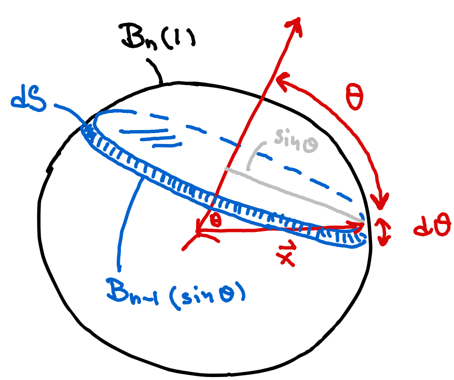
II. $\int_{\mathbb{R}^n} e^{-x^2}\, d\vec{x} = \int_0^\infty e^{-r^2} \underbrace{K_n r^{n-1}\, dr}_{dx \text{ slupka}} =$

$= \left| \begin{array}{l} r = u^{1/2} \\ dr = \frac{1}{2} u^{-1/2} \end{array} \right| = \frac{K_n}{2} \int_0^\infty u^{\frac{n}{2}-1} e^{-u}\, du = \frac{K_n}{2} \Gamma(\tfrac{n}{2})$

<u>Dk2</u>: Rekurentní formule

$K_n = S_n(1) = \int dS = \int_0^\pi S_{n-1}(\sin\theta)\, d\theta$

$= K_{n-1} \int_0^\pi \sin^{n-2}\theta\, d\theta$



NEBO $\frac{K_n}{n} = V_n(1) = \int dV =$

$= \int_0^\pi V_{n-1}(\sin\theta) \underbrace{dz}_{\sin\theta\, d\theta} = \frac{K_{n-1}}{n-1} \int_0^\pi \sin^n\theta\, d\theta$

FOURIEROVA TRANSFORMACE RADIÁLNÍCH FUNKCÍ

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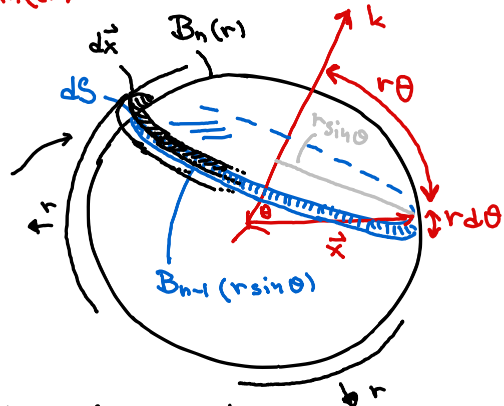
$\boxed{V}$ $\hat f(\vec k) = \frac{(2\pi)^{\frac n2}}{k^{\frac n2-1}} \int_0^\infty r^{\frac n2} g(r)\, J_{\frac n2-1}(kr)\, dr$; $n = 1,2,3,\dots$

speciálně $n=3$: $\hat f(\vec k) = \frac{4\pi}{k} \int_0^\infty r\, g(r) \sin(kr)\, dr$



Dk : $\hat f(\vec k) = \int_{\mathbb{R}^n} f(\vec x)\, e^{-i\vec k\cdot\vec x}\, d\vec x$

Víme $\vec k\cdot\vec x = kr\cos\theta$ = konst. na $d\vec x$

$\rightarrow \underbrace{d\vec x}_{\text{objem kroužku}} = \underbrace{S_{n-1}(r\sin\theta)}_{\text{„délka"}}\ \underbrace{r\,d\theta}_{\text{tloušťka}}\ \underbrace{dr}_{\text{šířka}}$

$\therefore\ \hat f(\vec k) \overset{\boxed{n\geq 2}}{=} K_{n-1} \int_0^\infty \int_0^\pi g(r)\, e^{-ikr\cos\theta}\, r^{n-1} \sin^{n-2}\theta\, d\theta\, dr$ (*)

intermezzo: $\int_0^\pi e^{-ikr\cos\theta} \sin^{n-2}\theta\, d\theta = \sum_{l=0}^\infty \int_0^\pi \frac{(-ikr)^l}{l!} \cos^l\theta\, \sin^{n-2}\theta\, d\theta$

$\overset{l \text{ sudé}}{=} \sum_{l=0}^\infty \int_0^\pi \frac{(-ikr)^{2l}}{(2l)!} \cos^{2l}\theta\, \sin^{n-2}\theta\, d\theta = 2 \sum_{l=0}^\infty \frac{(-1)^l (kr)^{2l}}{(2l)!} \underbrace{\int_0^{\pi/2} \cos^{2l}\theta\, \sin^{n-2}\theta\, d\theta}_{\frac12 B\left(\frac{2l+1}{2};\, \frac{n-1}{2}\right)}$

DUPLICITNÍ FORMULE $z = l + \frac12$

$= \sum_{l=0}^\infty \frac{(-1)^l (kr)^{2l}}{(2l)!} \frac{\Gamma(l+\frac12)\,\Gamma(\frac{n-1}{2})}{\Gamma(l+\frac n2)} = \Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi} \sum_{l=0}^\infty \frac{(-1)^l}{l!\,\Gamma(l+\frac n2)} \left(\frac{kr}{2}\right)^{2l} =$

$= \Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi} \left(\frac{2}{kr}\right)^{\frac n2-1} \sum_{l=0}^\infty \frac{(-1)^l}{l!\,\Gamma(l+\frac n2-1+1)} \left(\frac{kr}{2}\right)^{2l+\frac n2-1} = \Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi} \left(\frac{2}{kr}\right)^{\frac n2-1} J_{\frac n2-1}(kr)$

$\therefore\ \hat f(\vec k) = \int_0^\infty g(r)\, K_{n-1}\, r^{n-1}\, \Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi} \left(\frac{2}{kr}\right)^{\frac n2-1} J_{\frac n2-1}(kr)\, dr$

$= \frac{K_{n-1}\, \Gamma(\frac{n-1}{2})\, \sqrt{\pi}\, 2^{\frac n2-1}}{k^{\frac n2-1}} \int_0^\infty r^{\frac n2} g(r)\, J_{\frac n2-1}(kr)\, dr$ (opět radiální v $\vec k$)

$= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \frac{\Gamma(\frac{n-1}{2})\sqrt{\pi}\, 2^{\frac n2-1}}{k^{\frac n2-1}} \int_0^\infty r^{\frac n2} g(r)\, J_{\frac n2-1}(kr)\, dr$

Speciálně : $n = 3$; $J_{\frac{n}{2}-1}(x) = J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\ \sin x$

$$\therefore\ \hat{f}(\vec{r}) = \frac{(2\pi)^{\frac{3}{2}}}{\sqrt{k}}\int_0^\infty r^{\frac{3}{2}}\, g(r) \sqrt{\frac{2}{\pi k r}}\ \sin(kr)\, dr = \frac{4\pi}{k}\int_0^\infty r\, g(r) \sin(kr)\, dr$$

NEBO VÝPOČET Z MEZIVÝPOČTU:

$$\hat{f}(\vec{k}) = K_{n-1}\int_0^\infty\int_0^\pi g(r)\, e^{-ikr\cos\theta}\, r^{n-1} \sin^{n-2}\theta\ d\theta\, dr$$

$n=3$

$$\hat{f}(\vec{k}) = \int_0^\infty\int_0^\pi g(r)\, e^{-ikr\cos\theta}\, 2\pi\, r^2 \sin\theta\ d\theta =$$

$$= 2\pi\int_0^\infty r^2 g(r)\left[\frac{e^{-ikr\cos\theta}}{ikr}\right]_0^\pi dr = 2\pi\int_0^\infty r^2 g(r)\, \underbrace{\frac{e^{ikr} - e^{-ikr}}{ikr}}_{\frac{2\sin(kr)}{kr}}\, dr$$

POZNÁMKA:

Vzorec (*): $\hat{f}(\vec{k}) = K_{n-1}\int_0^\infty\int_0^\pi g(r)\, e^{-ikr\cos\theta}\, r^{n-1} \sin^{n-2}\theta\ d\theta\, dr$

nápadně připomíná integrál ve 2D ; $x = r\cos\theta$; $y = r\sin\theta$, $J = r$ $\therefore$

$$\boxed{\hat{f}(\vec{k}) \overset{n\geq 2}{=} K_{n-1}\int_0^\infty\int_{-\infty}^\infty g(\sqrt{x^2+y^2})\, e^{-ikx}\, y^{n-2}\, dx\, dy = K_{n-1}\mathcal{F}\left[\int_0^\infty g(\sqrt{x^2+y^2})\, y^{n-2}\, dy\right]}$$

jako fce $k = \|\vec{k}\|$

POZNÁMKA (*) platí i pro FT axiálních fcí $f(\vec{x}) = g(r,\theta)$

Př Spočtěte Fourierovu transformaci $f(\vec{x}) = e^{-\alpha r^2}$; $\vec{x} \in \mathbb{R}^n$

$$\hat{f}(\vec{k}) = K_{n-1}\mathcal{F}\left[\int_0^\infty g(\sqrt{x^2+y^2})\, y^{n-2}\, dy\right] = K_{n-1}\mathcal{F}\left[\int_0^\infty e^{-\alpha(x^2+y^2)}\, y^{n-2}\, dy\right] =$$

$$= K_{n-1}\ \widehat{e^{-\alpha x^2}}\int_0^\infty y^{n-2} e^{-\alpha y^2}\, dy = \left| \begin{array}{l} y = \frac{1}{\sqrt{\alpha}} u^{1/2} \\ dy = \frac{1}{\sqrt{\alpha}}\frac{1}{2} u^{-1/2} \end{array} \right| =$$

$$= K_{n-1}\sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{k^2}{4\alpha}}\ \frac{1}{2}\left(\frac{1}{\sqrt{\alpha}}\right)^{n-1}\int_0^\infty u^{\frac{n}{2}-\frac{3}{2}}\, e^{-u}\, du =$$

$$= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{k^2}{4\alpha}}\ \frac{1}{2}\left(\frac{1}{\sqrt{\alpha}}\right)^{n-1}\Gamma\left(\frac{n}{2}-\frac{1}{2}\right) = \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} e^{-\frac{k^2}{4\alpha}}\ \checkmark$$

Speciálně pro $n=3$ lze použít $\hat{f}(\vec{k}) = \frac{4\pi}{k}\int_0^\infty r\, g(r)\sin(kr)\,dr$

$$\hat{f}(\vec{k}) = \frac{4\pi}{k}\int_0^\infty r\, e^{-\alpha r^2}\sin kr\,dr \overset{\text{sudé}}{=} \frac{2\pi}{k}\int_{-\infty}^{\infty} r e^{-\alpha r^2}\sin kr\,dr =$$

$$= -\frac{2\pi}{k}\,\mathrm{Im}\int_{-\infty}^{\infty} r e^{-\alpha r^2} e^{-ikr}\,dr = -\frac{2\pi}{k}\,\mathrm{Im}\,\widehat{xe^{-\alpha x^2}} =$$

$$-\frac{2\pi}{k}\,\mathrm{Im}\left(-\frac{ik}{2\alpha}\sqrt{\frac{\pi}{\alpha}}\,e^{-\frac{k^2}{4\alpha}}\right) = \frac{\pi}{\alpha}\sqrt{\frac{\pi}{\alpha}}\,e^{-\frac{k^2}{4\alpha}} = \left(\frac{\pi}{\alpha}\right)^{\frac{3}{2}} e^{-\frac{k^2}{4\alpha}} \quad \checkmark$$

Př Spočtěte FT $f(\vec{x}) = \frac{1}{\alpha^2+r^2}$; $n=3$

Pro obecné n: $\hat{f}(\vec{k}) = K_{n-1}\,\mathcal{F}\left[\int_0^\infty g(\sqrt{x^2+y^2})\,y^{n-2}\,dy\right] =$

$$= K_{n-1}\,\mathcal{F}\left[\int_0^\infty \frac{1}{\alpha^2+x^2+y^2}\,y^{n-2}\,dy\right] = K_{n-1}\int_0^\infty \frac{\pi}{\sqrt{\alpha^2+y^2}}\,e^{-k\sqrt{\alpha^2+y^2}}\,y^{n-2}\,dy$$

$$= \left|\begin{matrix} y = \alpha\sinh u \\ dy = \alpha\cosh u\,du\end{matrix}\right| = \pi K_{n-1}\,\alpha^{n-2}\int_0^\infty e^{-k\alpha\cosh u}\,\sinh^{n-2} u\,du$$

$$\Rightarrow n=3:\quad \hat{f} = \pi K_2\,\alpha\int_0^\infty e^{-k\alpha\cosh u}\sinh u\,du = \frac{\pi K_2\alpha}{k\alpha}\left[-e^{-k\alpha\cosh u}\right]_0^\infty$$

$$= \frac{\pi K_2}{k}\,e^{-k\alpha} = \frac{2\pi^2}{k}\,e^{-k\alpha}$$

Jiné řešení přes $n=3$ vzorec : $\hat{f}(\vec{k}) = \frac{4\pi}{k}\int_0^\infty r\,g(r)\sin(kr)\,dr$

$$\widehat{\frac{1}{r^2+\alpha^2}} \overset{3D}{=} \frac{4\pi}{k}\int_0^\infty r\,\frac{1}{r^2+\alpha^2}\sin(kr)\,dr \overset{\text{sym.}}{=} \frac{2\pi}{k}\int_{-\infty}^{\infty}\frac{r\sin(kr)}{r^2+\alpha^2}\,dr =$$

$$= -\frac{2\pi}{k}\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{r}{r^2+\alpha^2}\,e^{-ikr}\,dr = -\frac{2\pi}{k}\,\mathrm{Im}\,\widehat{\frac{x}{x^2+\alpha^2}} =$$

$$= -\frac{2\pi}{k}\,\mathrm{Im}\left(-i\pi e^{-\alpha|k|}\,\mathrm{sgn}\,k\right) = \frac{2\pi^2}{k}\,e^{-\alpha|k|}\,\mathrm{sgn}|k| \overset{k=\|\vec{k}\|}{=} \frac{2\pi^2}{k}\,e^{-\alpha k}$$

DŮSLEDEK : $\widecheck{\frac{1}{k^2+\alpha^2}} \underset{\vec{k}\to-\vec{k}}{\overset{3D}{=}} \frac{1}{(2\pi)^3}\,\widehat{\frac{1}{k^2+\alpha^2}} = \frac{1}{(2\pi)^3}\left(\frac{2\pi^2}{r}\,e^{-\alpha r}\right) = \frac{1}{4\pi r}\,e^{-\alpha r}$

BONUS: $n=2$: $\widehat{\frac{1}{\alpha^2+r^2}} = \pi K_1\int_0^\infty e^{-k\alpha\cosh u}\,du = 2\pi K_0(k\alpha)$

Př: **RIESZŮV POTENCIÁL** Najděte $\widehat{r^\lambda}$ ... $\lambda \in (-n, \frac{1-n}{2})$

Sol $$\widehat{r^\lambda} = \frac{(2\pi)^{\frac{n}{2}}}{k^{\frac{n}{2}-1}} \int_0^\infty r^{\frac{n}{2}+\lambda} J_{\frac{n}{2}-1}(kr)\,dr \overset{\text{subst.}}{=} \left| \begin{aligned} r &= \frac{t}{k} \\ dr &= \frac{dt}{k} \end{aligned} \right| =$$

$$= \frac{(2\pi)^{\frac{n}{2}}}{k^{n+\lambda}} \int_0^\infty t^{\frac{n}{2}+\lambda} J_{\frac{n}{2}-1}(t)\,dt = c(\lambda, n)\, \frac{1}{k^{n+\lambda}}$$

Trik : Konstantu $c(\lambda,n)$ můžeme najít jiným způsobem:

$$\langle r^\lambda, \hat{\varphi} \rangle = \langle \widehat{r^\lambda}, \varphi \rangle = c(\lambda,n) \langle k^{-n-\lambda}, \varphi \rangle \quad ; \forall \varphi \in \mathcal{S}(\mathbb{R})$$

Připomenutí : $\varphi(\vec{x}) = e^{-r^2}$ je $\hat{\varphi}(\vec{k}) = \pi^{\frac{n}{2}} e^{-\frac{k^2}{4}}$; číli

- $$\langle r^\lambda, \hat{\varphi} \rangle = \pi^{\frac{n}{2}} \int_{\mathbb{R}^n} r^\lambda e^{-\frac{r^2}{4}} d\vec{x} = \pi^{\frac{n}{2}} K_n \int_0^\infty r^{\lambda+n-1} e^{-\frac{r^2}{4}} dr =$$

$$= \left| \begin{aligned} r &= 2\sqrt{u} \\ dr &= \frac{1}{\sqrt{u}} du \end{aligned} \right| = K_n \pi^{\frac{n}{2}} 2^{\lambda+n-1} \underbrace{\int_0^\infty u^{\frac{\lambda}{2}+\frac{n}{2}-1} e^{-u} du}_{\Gamma(\frac{\lambda}{2}+\frac{n}{2})}$$

- $$\langle k^{-n-\lambda}, \varphi \rangle = \int_{\mathbb{R}^n} k^{-n-\lambda} e^{-k^2} d\vec{k} = K_n \int_0^\infty k^{-\lambda-1} e^{-k^2} dk =$$

$$= \left| \begin{aligned} k &= \sqrt{u} \\ dk &= \tfrac{1}{2} u^{-\frac{1}{2}} \end{aligned} \right| = \frac{1}{2} K_n \underbrace{\int_0^\infty u^{-\frac{\lambda}{2}-1} e^{-u} du}_{\Gamma(-\frac{\lambda}{2})}$$

$$\therefore\ c(\lambda,n) = \frac{K_n \pi^{\frac{n}{2}} 2^{\lambda+n-1} \Gamma(\frac{\lambda+n}{2})}{K_n \frac{1}{2} \Gamma(-\frac{\lambda}{2})}$$

$$\boxed{\therefore\ \widehat{r^\lambda} = \frac{\pi^{\frac{n}{2}} 2^{\lambda+n} \Gamma(\frac{\lambda+n}{2})}{k^{\lambda+n} \Gamma(-\frac{\lambda}{2})}}$$

27.4.23

Př: Řešte Helmholtzovu PDR $\Delta u - \alpha^2 u = f$ ve 3D

Pišme $u(\vec{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{u}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}\, d\vec{k}$; obdobně $\hat{f}$

$$\Rightarrow \quad \Delta u - \alpha^2 u - f = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \left(-k^2\hat{u} - \alpha^2\hat{u} - \hat{f}\right) e^{i\vec{k}\cdot\vec{x}}\, d\vec{k} = 0$$

$$\therefore \quad \hat{u} = -\frac{\hat{f}}{k^2+\alpha^2}$$ ⇒ to je součin obrazů

$$\Rightarrow \quad \hat{u} = -\hat{f}\, \widehat{\frac{e^{-\alpha r}}{4\pi r}} \quad \Rightarrow \quad u = -f * \frac{e^{-\alpha r}}{4\pi r}$$

čili $u(\vec{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} f(\vec{y})\, \frac{e^{-\alpha\|\vec{x}-\vec{y}\|}}{\|\vec{x}-\vec{y}\|}\, d\vec{y}$

BONUS: Poissonova PDR $\Delta u = f$: $\alpha \to 0^+$

$$u(\vec{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(\vec{y})}{\|\vec{x}-\vec{y}\|}\, d\vec{y}$$

Př: Spočtěte Fourierovu transformaci $f(\vec{x}) = \chi_{[0,R]}(r)$ ve 3D [DÚ]

Př: Spočtěte FT $f(\vec{x}) = e^{-\alpha r}$; $n=3$ dle vzorce pro $n=3$. [T12]

Př* Spočtěte FT $f(\vec{x}) = \frac{1}{\sqrt{\alpha^2+r^2}}$; $n=3$

Řešení dle $n=3$ vzorce : $\hat{f}(\vec{k}) = \frac{4\pi}{k}\int_0^\infty r\, g(r) \sin(kr)\, dr$

$$\hat{f}(\vec{k}) = \frac{4\pi}{k}\int_0^\infty \frac{r\sin(kr)}{\sqrt{\alpha^2+r^2}}\, dr = -\frac{4\pi}{k}\frac{\partial}{\partial k}\int_0^\infty \frac{\cos(kr)}{\sqrt{\alpha^2+r^2}}\, dr \overset{r=\alpha t}{=} -\frac{4\pi}{k}\frac{\partial}{\partial k}\int_0^\infty \frac{\cos(\alpha k t)}{\sqrt{1+t^2}}\, dt$$

$$= -\frac{4\pi}{k}\frac{\partial}{\partial k} K_0(\alpha k) = -\frac{4\pi}{k}\frac{\partial}{\partial k}\int_0^\infty e^{-\alpha k\cosh u}\, du = \frac{4\pi\alpha}{k}\int_0^\infty e^{-\alpha k\cosh u}\cosh u\, du = \frac{4\pi\alpha K_1(\alpha k)}{k}$$

(Př)* Spočtěte FT $f(\vec{x}) = e^{-\alpha r}$; $\vec{x}\in\mathbb{R}^n$; $\alpha>0$ $\downarrow$ SLOŽITÁ METODA

Pomocí vzorce z mezivýpočtu: $\hat{f}(\vec{k}) = K_{n-1}\int_0^\infty\int_0^\pi g(r)\, e^{-ikr\cos\theta}\, r^{n-1}\sin^{n-2}\theta\, d\theta\, dr$

$$\widehat{e^{-\alpha r}} = K_{n-1}\int_0^\infty\int_0^\pi e^{-\alpha r} e^{-ikr\cos\theta}\, r^{n-1}\sin^{n-2}\theta\, d\theta\, dr$$

Trik: $r^{n-2}\sin^{n-2}\theta = \left(-\frac{\partial}{\partial\beta}\right)^{n-2} e^{-\beta r\sin\theta}\Big|_{\beta=0}$; $r e^{-\alpha r} = \left(-\frac{\partial}{\partial\alpha}\right) e^{-\alpha r}$

$$\therefore\ \hat{f}(\vec{x}) = (-1)^{n-1}\partial_\alpha\partial_\beta^{n-2}\int_0^\pi\int_0^\infty e^{-\alpha r} e^{-ikr\cos\theta} e^{-\beta r\sin\theta}\, dr\, d\theta\Big|_{\beta=0} =$$

$$= (-1)^{n-1}\partial_\alpha\partial_\beta^{n-2}\int_0^\pi \frac{d\theta}{\alpha+ik\cos\theta+\beta\sin\theta}\Big|_{\beta=0} = \left|\begin{array}{c}\text{weierstrass}\\ t=\operatorname{tg}\frac{\theta}{2}\end{array}\right|$$

$$= (-1)^{n-1}\partial_\alpha\partial_\beta^{n-2}\int_0^\infty \frac{\frac{2}{1+t^2}\,dt}{\alpha+ik\frac{1-t^2}{1+t^2}+\beta\frac{2t}{1+t^2}}\Big|_{\beta=0} = (-1)^{n-1}\partial_\alpha\partial_\beta^{n-2}\int_0^\infty\frac{2\,dt}{\alpha(1+t^2)+ik(1-t^2)+2\beta t}\Big|_{\beta=0}$$

$$\overset{ⓒ}{=} (-1)^{n-1}\partial_\alpha\partial_\beta^{n-2}(-1)\sum \operatorname{Res}\frac{2\ln z}{\alpha(1+z^2)+ik(1-z^2)+2\beta z}\Big|_{\beta=0} =$$

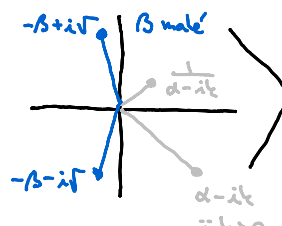
$$= (-1)^n\partial_\alpha\partial_\beta^{n-2}\left(\frac{\ln\sigma_+}{\sigma_+(\alpha-ik)+\beta}+\frac{\ln\sigma_-}{\sigma_-(\alpha-ik)+\beta}\right)\Big|_{\beta=0} =$$

$$\left\langle D = b^2-4ac = (2\beta)^2-4(\alpha-ik)(\alpha+ik) = -4(\alpha^2+k^2-\beta^2) < 0 \Rightarrow \sigma_\pm = \frac{-\beta\pm i\sqrt{\alpha^2+k^2-\beta^2}}{\alpha-ik}\right\rangle$$

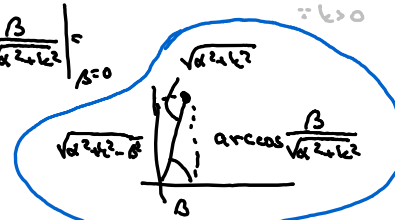
$$= (-1)^n\partial_\alpha\partial_\beta^{n-2}\frac{1}{i\sqrt{\alpha^2+k^2-\beta^2}}\underbrace{(\ln\sigma_+-\ln\sigma_-)}_{i(\arg\sigma_+-\arg\sigma_-)} = 2(-1)^n\partial_\alpha\partial_\beta^{n-2}\frac{\pi-\arg(-\beta+i\sqrt{\alpha^2+k^2-\beta^2})}{\sqrt{\alpha^2+k^2-\beta^2}}\Big|_{\beta=0}$$

$\because |\sigma_+| = |\sigma_-|$

$$\left\langle\begin{array}{l}\arg\left(\frac{-\beta+i\sqrt{\alpha^2+k^2-\beta^2}}{\alpha-ik}\right) = \arg(-\beta+i\sqrt{\alpha^2+k^2-\beta^2})+\arg\frac{1}{\alpha-ik}\\ \arg\left(\frac{-\beta-i\sqrt{\alpha^2+k^2-\beta^2}}{\alpha-ik}\right) = \underbrace{\arg(-\beta-i\sqrt{\alpha^2+k^2-\beta^2})}_{2\pi-\arg(-\beta+i\sqrt{\alpha^2+k^2-\beta^2})}+\arg\frac{1}{\alpha-ik}\end{array}\right\rangle$$



$$= 2(-1)^n\partial_\alpha\partial_\beta^{n-2}\frac{\arccos\frac{\beta}{\sqrt{\alpha^2+k^2}}}{\sqrt{\alpha^2+k^2-\beta^2}}\Big|_{\beta=0} = (-1)^{n+1}\partial_\alpha\partial_\beta^{n-1}\arccos^2\frac{\beta}{\sqrt{\alpha^2+k^2}}\Big|_{\beta=0}$$



[najdu Taylorovu řadu a mám výsledek
$\longrightarrow$ tato metoda ale velmi složitá]

(Př)* Spočtěte FT $f(\vec{x}) = e^{-\alpha r}$; $\vec{x} \in \mathbb{R}^n$; $\alpha > 0$ (Znovu!)

Ansatz: DLE PŘEDCHOZÍHO POSTUPU $\widehat{e^{-\alpha r}} = \frac{c(\alpha, n)}{(\alpha^2 + k^2)^{\frac{n+1}{2}}}$

ULTIMÁTNÍ TRIK: $f, \varphi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle$

$\longrightarrow$ Volím $\varphi(\vec{x}) = r^\lambda \Rightarrow \widehat{r^\lambda} = \frac{\pi^{\frac{n}{2}} 2^{\lambda+n} \Gamma\left(\frac{\lambda+n}{2}\right)}{k^{\lambda+n} \Gamma\left(-\frac{\lambda}{2}\right)}$;

- $\langle \widehat{e^{-\alpha r}}, r^\lambda \rangle \overset{\text{Ansatz}}{=} c(\alpha, n) \int_{\mathbb{R}^n} \frac{r^\lambda}{(\alpha^2 + r^2)^{\frac{n+1}{2}}} \, d\vec{x} = \mathbin{//} d\vec{x} = K_n r^{n-1} dr$

$$= c(\alpha, n) K_n \int_0^\infty \frac{r^{\lambda+n-1}}{(\alpha^2 + r^2)^{\frac{n+1}{2}}} \, dr = \left| \begin{array}{c} \text{substituce} \\ r = \alpha \sqrt{\frac{t}{1-t}} \end{array} \right| =$$

$$= c(\alpha, n) K_n \tfrac{1}{2} \alpha^{\lambda-1} \underbrace{\int_0^1 t^{\frac{\lambda+n}{2}-1} (1-t)^{-\frac{\lambda}{2}-\frac{1}{2}} \, dt}_{B\left(\frac{\lambda+n}{2}, \frac{1-\lambda}{2}\right)} = \frac{K_n c(\alpha, n) \alpha^{\lambda-1} \Gamma\left(\frac{\lambda+n}{2}\right) \Gamma\left(\frac{1-\lambda}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$$

- $\langle e^{-\alpha r}, \widehat{r^\lambda} \rangle = \frac{\pi^{\frac{n}{2}} 2^{\lambda+n} \Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)} \int_{\mathbb{R}^n} e^{-\alpha r} r^{-\lambda-n} \, d\vec{x} = \mathbin{//} d\vec{x} = K_n r^{n-1} dr$

$$= \frac{\pi^{\frac{n}{2}} 2^{\lambda+n} \Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)} K_n \int_0^\infty e^{-\alpha r} r^{-\lambda-1} \, dr \overset{\text{sub.}}{=} \frac{\pi^{\frac{n}{2}} 2^{\lambda+n} \Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)} K_n \alpha^\lambda \underbrace{\int_0^\infty e^{-t} t^{-\lambda-1} \, dt}_{\Gamma(-\lambda)}$$

$\longrightarrow$ Čili $c(\alpha, n) = \frac{\pi^{\frac{n}{2}} 2^{\lambda+n+1} \alpha \Gamma\left(\frac{n+1}{2}\right) \Gamma(-\lambda)}{\Gamma\left(\frac{1-\lambda}{2}\right) \Gamma\left(-\frac{\lambda}{2}\right)} \Rightarrow$ nesmí záviset na λ!

$\longrightarrow$ Máme k dispozici duplikační formuli:

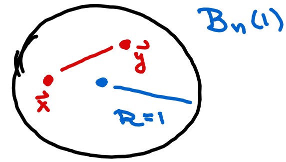
$$\Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-1} \Gamma(z) \Gamma\left(z + \tfrac{1}{2}\right) \mathbin{//} z \to -\tfrac{\lambda}{2}$$

$$\therefore\ c(\alpha, n) = \frac{\pi^{\frac{n}{2}} 2^{\lambda+n+1} \alpha \Gamma\left(\frac{n+1}{2}\right) \frac{1}{\sqrt{\pi}} 2^{-\lambda-1} \Gamma\left(-\frac{\lambda}{2}\right) \Gamma\left(\frac{1-\lambda}{2}\right)}{\Gamma\left(\frac{1-\lambda}{2}\right) \Gamma\left(-\frac{\lambda}{2}\right)} = \pi^{\frac{n-1}{2}} 2^n \alpha \Gamma\left(\frac{n+1}{2}\right)$$

$$\therefore\ \boxed{\widehat{e^{-\alpha r}} = \frac{2^n \pi^{\frac{n-1}{2}} \alpha \Gamma\left(\frac{n+1}{2}\right)}{(\alpha^2 + k^2)^{\frac{n+1}{2}}}} \xrightarrow{\text{důsledek}} \therefore\ \widehat{\frac{\alpha}{(\alpha^2 + r^2)^{\frac{n+1}{2}}}} = \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} e^{-\alpha k}$$

2.5.23

Př. Energie jednotkové koule



Model: $E_n(\vec x,\vec y) = \dfrac{e^{-\alpha\|\vec x-\vec y\|}}{\|\vec x-\vec y\|}$

Celková energie: $e_n(\alpha) = \displaystyle\int_{B_n(1)}\int_{B_n(1)} \frac{e^{-\alpha\|\vec x-\vec y\|}}{\|\vec x-\vec y\|}\, d\vec y\, d\vec x$??

Trik: $\displaystyle\int_{B_n(1)} \frac{e^{-\alpha\|\vec x-\vec y\|}}{\|\vec x-\vec y\|}\, d\vec y = \int_{\mathbb{R}^n} \underbrace{\frac{e^{-\alpha\|\vec x-\vec y\|}}{\|\vec x-\vec y\|}}_{f(\vec x-\vec y)} \underbrace{\Theta(1-\|\vec y\|)}_{g(\vec y)}\, d\vec y = (f*g)(\vec x)$

$$\therefore\ e_n(\alpha) = \int_{\mathbb{R}^n} (f*g)(\vec x)\, g(\vec x)\, d\vec x = \int_{\mathbb{R}^n} \widehat{f*g}\ \check{g}\, d\vec x = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{f*g}\ \hat g\, d\vec k$$

$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat f\, \hat g^2\, d\vec k \overset{\text{radiální funkce}}{=} \frac{K_n}{(2\pi)^n}\int_0^\infty k^{n-1}\hat f(k)\,\hat g^2(k)\, dk$$

Pro případ $n=3$:

$f(\vec x) = \dfrac{e^{-\alpha r}}{r} \Rightarrow \hat f(\vec k) = \dfrac{4\pi}{\alpha^2+k^2}$; $g(\vec x) = \Theta(1-r)$, čili

$$\hat g(k) = \frac{4\pi}{k}\int_0^1 r\sin(kr)\,dr = -\frac{4\pi}{k}\partial_k\int_0^1 \cos(kr)\,dr = -\frac{4\pi}{k}\partial_k \frac{\sin k}{k} = 4\pi\frac{\sin k - k\cos k}{k^3}$$

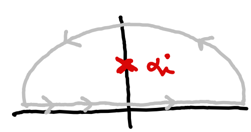
$$\therefore\ e_3(\alpha) = 32\pi\int_0^\infty \frac{(\sin k - k\cos k)^2}{k^4(\alpha^2+k^2)}\,dk \overset{\text{sym.}}{=} 16\pi\int_{-\infty}^\infty \frac{(\sin k - k\cos k)^2}{k^4(\alpha^2+k^2)}\,dk$$

$$= 16\pi\int_{-\infty}^\infty \frac{\sin^2 k - 2k\sin k\cos k + k^2\cos^2 k}{k^4(\alpha^2+k^2)}\,dk = 8\pi\int_{-\infty}^\infty \frac{1+k^2+(k^2-1)\cos 2k - 2k\sin 2k}{k^4(\alpha^2+k^2)}\,dk$$

$$= 8\pi\,\mathrm{Re}\int_{-\infty}^\infty \frac{1+k^2+(k^2-1)e^{2ik}+2kie^{2ik}+\frac{2}{3}ik^3}{k^4(\alpha^2+k^2)}\,dk$$

tento člen kompenzuje rozvoj v nule aby byl $O(k^4)$

$$= 8\pi\,\mathrm{Re}\,2\pi i\,\frac{1+k^2+(k^2-1)e^{2ik}+2kie^{2ik}+\frac{2}{3}ik^3}{k^4\cdot 2k}\Bigg|_{\alpha i}$$



$$= 8\pi\,\mathrm{Re}\,2\pi i\,\frac{1-\alpha^2-(1+\alpha^2)e^{-2\alpha} - 2\alpha e^{-2\alpha}+\frac{2}{3}\alpha^3}{2i\alpha^5}$$

$$= 8\pi^2\,\frac{1-\alpha^2+\frac{2}{3}\alpha^3-(1+\alpha)^2e^{-2\alpha}}{\alpha^5}$$

$$\boxed{e_3(\alpha) = \int_{B_3(1)}\int_{B_3(1)} \frac{e^{-\alpha\|\vec x-\vec y\|}}{\|\vec x-\vec y\|}\,d\vec x\,d\vec y = \frac{8\pi^2}{\alpha^5}\left(1-\alpha^2+\frac{2}{3}\alpha^3-(1+\alpha)^2e^{-2\alpha}\right)}$$

Bonus 2: RHS pro malé α: $8\pi^2\left(\frac{4}{15}-\frac{2\alpha}{9}+\frac{4\alpha^2}{35}-\frac{2\alpha^3}{45}\right)+O(\alpha^4)$

LHS pro malé α: $\displaystyle\int_{B_3(1)}\int_{B_3(1)}\frac{1}{\|\vec x-\vec y\|}-\alpha+\frac{\alpha^2}{2}\|\vec x-\vec y\|-\frac{\alpha^3}{6}\|\vec x-\vec y\|^2\,d\vec x\,d\vec y+O(\alpha^4)$

Porovnání: $\displaystyle\int_{B_3(1)}\int_{B_3(1)}\|\vec x-\vec y\|\,d\vec x\,d\vec y=\frac{64\pi^2}{35}=\left(\frac{4\pi}{3}\right)^2\frac{36}{35}$

Bonus 3: Díky analytickému prodloužení $\alpha\to-i\alpha$:

$$\int_{B(1)}\int_{B_3(1)}\frac{e^{i\alpha\|\vec x-\vec y\|}}{\|\vec x-\vec y\|}\,d\vec x\,d\vec y=\frac{8\pi^2 i}{\alpha^5}\left(1+\alpha^2+\frac{2i}{3}\alpha^3-(1-\alpha i)^2e^{2\alpha i}\right)$$

Fourierova trans.: $\alpha\to\beta$ čili $\displaystyle\int_{-\infty}^{\infty}@\,e^{-i\alpha\beta}\,d\alpha$:

$$2\pi\int_{B(1)}\int_{B_3(1)}\underbrace{\frac{\delta(\beta-\|\vec x-\vec y\|)}{\|\vec x-\vec y\|}}_{\delta(\beta-\|\vec x-\vec y\|)/\beta}\,d\vec x\,d\vec y=8\pi^2 i\int_{-\infty}^{\infty}\frac{1+\alpha^2+\frac{2i}{3}\alpha^3-(1-\alpha i)^2e^{2\alpha i}}{\alpha^5}e^{-i\alpha\beta}\,d\alpha$$

$$=8\pi i\lim_{\varepsilon\to0^+}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon}\frac{1+\alpha^2+\frac{2i}{3}\alpha^3}{\alpha^5}e^{-i\alpha\beta}\,d\alpha-\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon}\frac{(1-\alpha i)^2}{\alpha^5}e^{-i\alpha(\beta-2)}\,d\alpha$$

$$=8\pi^2 i\left(2\pi i\chi_{(-\infty,0)}(\beta)\,\mathrm{Res}_0\frac{\left(1+\alpha^2+\frac{2i}{3}\alpha^3\right)}{\alpha^5}e^{-i\alpha\beta}-2\pi i\chi_{(-\infty,2)}(\beta)\,\mathrm{Res}_0\frac{(1-\alpha i)^2}{\alpha^5}e^{-i\alpha(\beta-2)}\right)$$

$$=-16\pi^3\left(\chi_{(-\infty,0)}(\beta)\frac{1}{4!}\left(\left(1+\alpha^2+\frac{2i}{3}\alpha^3\right)e^{-i\alpha\beta}\right)^{(4)}_{\alpha}-\chi_{(-\infty,2)}(\beta)\left((1-\alpha i)^2e^{-i\alpha(\beta-2)}\right)^{(4)}_{\alpha}\right)\Bigg|_{\alpha=0}=$$

$$=-\frac{16\pi^3}{4!}\Big[\chi_{(-\infty,0)}(\beta)\left((-i\beta)^4+\binom{4}{2}(-i\beta)^2 2+\binom{4}{3}(-i\beta)4i\right)$$
$$-\chi_{(-\infty,2)}(\beta)\left((-i(\beta-2))^4+\binom{4}{1}(-i(\beta-2))^3(-2i)+\binom{4}{2}(-i(\beta-2))^2(-2)\right)\Big]=$$

$$=-\frac{2\pi^3}{3}\left[\chi_{(-\infty,0)}(\beta)(\beta^4-12\beta^2+16\beta)-\chi_{(-\infty,2)}(\beta)\left(\underbrace{(\beta-2)^4+8(\beta-2)^3+12(\beta-2)^2}_{\beta^4-12\beta^2+16\beta}\right)\right]$$

$$=\frac{2\pi^3}{3}(\beta^4-12\beta^2+16\beta)\chi_{(0,2)}(\beta)$$

Pravděpodobnostní význam:

$$F(\beta)=P(\|\vec X-\vec Y\|\le\beta)=\left(\frac{3}{4\pi}\right)^2\int_{B(1)}\int_{B_3(1)}\theta(\beta-\|\vec x-\vec y\|)\,d\vec x\,d\vec y$$

$$\hookrightarrow f(\beta)=\frac{dF(\beta)}{d\beta}=\frac{9}{16\pi^2}\int_{B_3(1)}\int_{B_3(1)}\delta(\beta-\|\vec x-\vec y\|)\,d\vec x\,d\vec y=\frac{9\beta}{16\pi^2}\frac{\pi^3}{3}(\beta^4-12\beta^2+16\beta)\chi_{(0,2)}(\beta)$$

$=\frac{3}{16}(\beta^5-12\beta^3+16\beta^2)\chi_{(0,2)}(\beta)=\frac{3\beta^2}{16}(2-\beta)^2(\beta+4)\chi_{(0,2)}(\beta)$ rozdělení vzdálenosti dvou bodů v jednotkové kouli