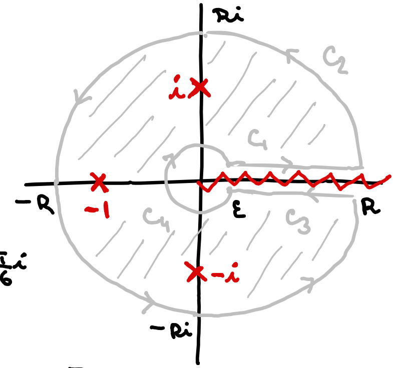
(Př) $I = \int_0^\infty \frac{\ln x}{\sqrt[3]{x}(1+x)^2(1+x^2)}\,dx \in \mathbb{R}$

$J := \oint_C \frac{\ln z}{\sqrt[3]{z}(1+z)^2(1+z^2)}\,dz$; C:



singularity: $z = \pm i$ (póly 1. řádu); $z = -1$ (pól 2. řádu)

- **REZIDUOVÁ VĚTA**

$\longrightarrow \operatorname{Res}_i f(z) = \frac{\ln z}{\sqrt[3]{z}(1+z)^2\, 2z}\Big|_i = \frac{\overbrace{\ln e^{\frac{\pi}{2}i}}^{\frac{\pi}{2}i}}{\underbrace{\sqrt[3]{e^{\frac{\pi}{2}i}}}_{e^{\frac{\pi}{6}i}}\underbrace{(1+i)^2}_{2i}\,2i} = \frac{\pi}{8i}e^{-\frac{\pi}{6}i}$

$\longrightarrow \operatorname{Res}_{-i} f(z) = \frac{\ln z}{\sqrt[3]{z}(1+z)^2\, 2z}\Big|_{-i} = \frac{\ln e^{\frac{3}{2}\pi i}}{\sqrt[3]{e^{\frac{3\pi}{2}i}}\underbrace{(1-i)^2}_{-2i}(-2i)} = \frac{3\pi}{8i}\underbrace{e^{-\frac{\pi}{2}i}}_{-i} = -\frac{3\pi}{8}$

$\longrightarrow \operatorname{Res}_{-1} f(z) = \lim_{z\to -1}\frac{1}{(2-1)!}\left[(z+1)^2 f(z)\right]^{(2-1)} = \left(\frac{\ln z}{\sqrt[3]{z}(1+z^2)}\right)'\Big|_{-1} =$

$= \frac{1}{z\sqrt[3]{z}(1+z^2)} - \frac{1}{3}\frac{\ln z}{z^{4/3}(1+z^2)} - \frac{2z\ln z}{\sqrt[3]{z}(1+z^2)^2}\Big|_{e^{\pi i}} =$

$= \frac{1}{-1\, e^{\frac{\pi i}{3}}\cdot 2} - \frac{1}{3}\frac{\ln e^{\pi i}}{e^{\frac{4\pi i}{3}}\cdot 2} - \frac{2(-1)\ln e^{\pi i}}{e^{\frac{\pi i}{3}}\cdot 2^2} = e^{-\frac{\pi}{3}i}\left(-\frac{1}{2} + \frac{\pi i}{6} + \frac{\pi i}{2}\right) = \frac{2\pi i}{3}e^{-\frac{\pi}{3}i} - \frac{1}{2}e^{-\frac{\pi}{3}i}$

$\therefore\ J = 2\pi i \sum_{\sigma\in \operatorname{Int} C}\operatorname{Res}_\sigma f(z) = 2\pi i\left(\frac{\pi}{8i}e^{-\frac{\pi}{6}i} - \frac{3\pi}{8} + \frac{2\pi i}{3}e^{-\frac{\pi}{3}i} - \frac{1}{2}e^{-\frac{\pi}{3}i}\right)$

- **PARAMETRIZACE**:

$\longrightarrow C_1: z = t + i0;\ t\in(\varepsilon,R);\ dz = dt$

$J_1 = \int_\varepsilon^R \frac{\ln(t+i0)}{\sqrt[3]{t+i0}\,(1+t)^2(1+t^2)}\,dt \xrightarrow[\varepsilon\to 0^+]{R\to\infty} \int_0^\infty \frac{\ln t}{\sqrt[3]{t}(1+t)^2(1+t^2)}\,dt = I$

$\longrightarrow C_2:\ |J_2| \le \frac{\ln R + 2\pi}{\sqrt[3]{R}(R-1)^2(R^2-1)}\,2\pi R \xrightarrow{R\to\infty} 0$

$\longrightarrow \ominus C_3: z = t - i0;\ t\in(\varepsilon,R);\ dz = dt$

$J_3 = \ominus\int_\varepsilon^R \frac{\ln(t-i0)}{\sqrt[3]{t-i0}\,(1+t)^2(1+t^2)}\,dt \xrightarrow[\varepsilon\to 0+]{R\to\infty} \ominus\int_0^\infty \frac{\ln t + 2\pi i}{\sqrt[3]{t}\,e^{\frac{\pi}{3}i}(1+t)^2(1+t^2)}\,dt =$

$= -e^{-\frac{2\pi}{3}i}\int_0^\infty \frac{\ln t}{\sqrt[3]{t}(1+t)^2(1+t^2)}\,dt - 2\pi i\, e^{-\frac{2\pi}{3}i}\underbrace{\int_0^\infty \frac{dt}{\sqrt[3]{t}(1+t)(1+t^2)}}_{I_0\in\mathbb{R}} = -e^{-\frac{2\pi}{3}i} I_0 - 2\pi i\, e^{-\frac{2\pi}{3}i} I_0$

$\longrightarrow C_4:\ |J_4| \le \frac{2\pi - \ln\varepsilon}{\sqrt[3]{\varepsilon}(1-\varepsilon)^2(1-\varepsilon^2)}\,2\pi\varepsilon \xrightarrow{\varepsilon\to 0^+} 0$

- **POROVNÁNÍ**: $\frac{\pi^2}{4}e^{-\frac{\pi}{6}i} - \frac{3\pi^2 i}{4} - \frac{4\pi^2}{3}e^{-\frac{\pi}{3}i} - \pi i\, e^{-\frac{\pi}{3}i} = I\left(1 - e^{-\frac{2\pi}{3}i}\right) - 2\pi i\, e^{-\frac{2\pi}{3}i} I_0$

$\cdot e^{\frac{2\pi}{3}i}$: $\frac{\pi^2}{4}e^{\frac{\pi}{2}i} - \frac{3\pi^2}{4}i\, e^{\frac{2\pi}{3}i} - \frac{4\pi^2}{3}e^{\frac{\pi}{3}i} - \pi i\, e^{\frac{\pi}{3}i} = I\left(e^{\frac{2\pi}{3}i} - 1\right) - 2\pi i\, I_0$

$\boxed{\text{Re}}$: $\frac{3\pi^2}{4}\underbrace{\sin\frac{2\pi}{3}}_{\frac{\sqrt{3}}{2}} - \frac{4\pi^2}{3}\underbrace{\cos\frac{\pi}{3}}_{\frac{1}{2}} + \pi\underbrace{\sin\frac{\pi}{3}}_{\frac{\sqrt{3}}{2}} = I\left(\underbrace{\cos\frac{2\pi}{3}}_{-1/2} - 1\right)$

$\boxed{\therefore\ I = \frac{4\pi^2}{9} - \frac{\pi^2\sqrt{3}}{4} - \frac{\pi\sqrt{3}}{3}}$

BONUS: $I_0 \overset{\dots}{=} \frac{4\pi}{3\sqrt{3}} - \frac{\pi}{2}$