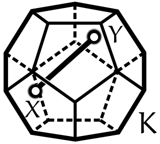
FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

Dominik Beck

Mean distance in polyhedra

Synergies between modern probability, geometric analysis and stochastic geometry, Bonn

January 2024

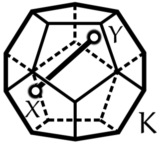


- $K \subset \mathbb{R}^d$, $\dim K = d$
- $X, Y \in K$ uniformly and independently selected

$$\Lambda(K) := \frac{\mathbb{E}\,\|X - Y\|}{\sqrt[d]{\mathrm{vol}_d\, K}}$$

Definition

Platonic solids – known and new results



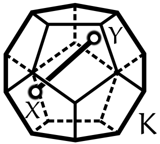
- $K \subset \mathbb{R}^d$, $\dim K = d$
- $X, Y \in K$ uniformly and independently selected

$$\Lambda(K) := \frac{\mathbb{E}\,\|X - Y\|}{\sqrt[d]{\mathrm{vol}_d\, K}}$$

K	MC	$\Lambda(K)$
ball[1]	0.6381	$\frac{18}{35}\sqrt[3]{\frac{6}{\pi}}$
icosahedron	0.6413	?
dodecahedron	0.6425	?
octahedron	0.6585	?
cube[2]	0.6617	$\frac{4}{105}+\frac{17\sqrt{2}}{105}-\frac{2\sqrt{3}}{35}-\frac{\pi}{15}+\frac{1}{5}\operatorname{arccoth}\sqrt{2}+\frac{4}{5}\operatorname{arccoth}\sqrt{3}$
tetrahedron	0.7295	?

[1] trivial

[2] David Robbins and Theodore Bolis. "Average Distance between Two Points in a Box.". In: *Amer. Math. Monthly* 85 (1978), p. 278



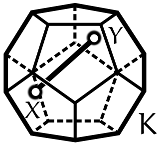
- $K \subset \mathbb{R}^d$, $\dim K = d$
- $X, Y \in K$ uniformly and independently selected

$$\Lambda(K) := \frac{\mathbb{E}\,\|X-Y\|}{\sqrt[d]{\mathrm{vol}_d\, K}}$$

K	MC	$\Lambda(K)$
ball[1]	0.6381	$\frac{18}{35}\sqrt[3]{\frac{6}{\pi}}$
icosahedron	0.6413	?
dodecahedron	0.6425	?
octahedron	0.6585	$\sqrt[3]{\frac{3}{4}}\left(\frac{4}{105}+\frac{13\sqrt{2}}{105}-\frac{4\pi}{45}+\frac{109\ln 3}{630\sqrt{2}}+\frac{16\operatorname{arccot}\sqrt{2}}{315}+\frac{158\operatorname{arccoth}\sqrt{2}}{315}\sqrt{2}\right)$
cube[2]	0.6617	$\frac{4}{105}+\frac{17\sqrt{2}}{105}-\frac{2\sqrt{3}}{35}-\frac{\pi}{15}+\frac{1}{5}\operatorname{arccoth}\sqrt{2}+\frac{4}{5}\operatorname{arccoth}\sqrt{3}$
tetrahedron	0.7295	$\sqrt[3]{3}\left(\frac{\sqrt{2}}{7}-\frac{37\pi}{315}+\frac{4}{15}\arctan\sqrt{2}+\frac{113\ln 3}{210\sqrt{2}}\right)$

[1] trivial

[2] David Robbins and Theodore Bolis. "Average Distance between Two Points in a Box.". In: *Amer. Math. Monthly* 85 (1978), p. 278



- $K \subset \mathbb{R}^d$, $\dim K = d$
- $X, Y \in K$ uniformly and independently selected

$$\Lambda(K) := \frac{\mathbb{E}\,\|X-Y\|}{\sqrt[d]{\mathrm{vol}_d\, K}}$$

K	MC	$\Lambda(K)$
ball[1]	0.6381	$\frac{18}{35}\sqrt[3]{\frac{6}{\pi}}$
icosahedron	0.6413	next slide
dodecahedron	0.6425	next slide
octahedron	0.6585	$\sqrt[3]{\frac{3}{4}}\left(\frac{4}{105}+\frac{13\sqrt{2}}{105}-\frac{4\pi}{45}+\frac{109\ln 3}{630\sqrt{2}}+\frac{16\,\mathrm{arccot}\,\sqrt{2}}{315}+\frac{158\,\mathrm{arccoth}\,\sqrt{2}}{315}\sqrt{2}\right)$
cube[2]	0.6617	$\frac{4}{105}+\frac{17\sqrt{2}}{105}-\frac{2\sqrt{3}}{35}-\frac{\pi}{15}+\frac{1}{5}\,\mathrm{arccoth}\,\sqrt{2}+\frac{4}{5}\,\mathrm{arccoth}\,\sqrt{3}$
tetrahedron	0.7295	$\sqrt[3]{3}\left(\frac{\sqrt{2}}{7}-\frac{37\pi}{315}+\frac{4}{15}\arctan\sqrt{2}+\frac{113\ln 3}{210\sqrt{2}}\right)$

[1] trivial

[2] David Robbins and Theodore Bolis. "Average Distance between Two Points in a Box.". In: *Amer. Math. Monthly* 85 (1978), p. 278

Platonic solids – new results

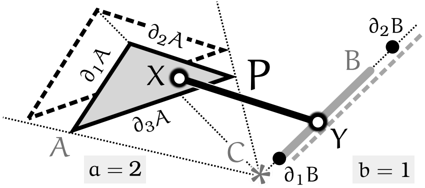
icosahedron	$\frac{1}{2}\sqrt[3]{\frac{9}{5}-\frac{3}{\sqrt{5}}}\left(\frac{197}{525}+\frac{239}{525\sqrt{5}}-\frac{44}{525}\sqrt{\frac{2(5+\sqrt{5})}{5}}-\frac{(17226+6269\sqrt{5})\pi}{157500} -\frac{(2186+1413\sqrt{5})\operatorname{arccot}\phi}{15750}+\frac{(82-75\sqrt{5})\operatorname{arccot}(\phi^2)}{5250}+\frac{4(2139+881\sqrt{5})\operatorname{arccsch}\phi}{7875} +\frac{(15969+7151\sqrt{5})\operatorname{arccoth}\phi}{12600}+\frac{(4449-1685\sqrt{5})\ln 3}{42000}-\frac{(75783+37789\sqrt{5})\ln 5}{252000}\right)$
dodecahedron	$\frac{1}{\sqrt[3]{30+14\sqrt{5}}}\left(\frac{1516}{1575}+\frac{2\sqrt{\frac{2}{5}}}{45}-\frac{124\sqrt{\frac{3}{5}}}{175}-\frac{71\sqrt{2}}{1575}-\frac{12\sqrt{3}}{35}+\frac{342}{175\sqrt{5}}+\frac{493\pi}{23625} +\frac{67\pi}{945\sqrt{5}}+\frac{(397-244\sqrt{5})\operatorname{arccot}2}{18900}+\frac{(24023+11788\sqrt{5})(\arccos\frac{2}{3}-\arccos\frac{1}{3})}{94500} -\frac{(461+212\sqrt{5})(\arccos\frac{23}{41}+\arccos\frac{39}{41})}{1000}-\frac{(1031+521\sqrt{5})\operatorname{arccosh}\frac{13}{3}}{75600}+\frac{(367+163\sqrt{5})\operatorname{arccosh}9}{16800} +\frac{(22197+8149\sqrt{5})(\operatorname{arccosh}\frac{121}{41}-\operatorname{arccosh}\frac{57}{41})}{84000}+\frac{(15763+7063\sqrt{5})(\operatorname{arccosh}\frac{7}{3}-\operatorname{arccosh}3)}{21000} +\frac{2(423+187\sqrt{5})(\operatorname{arccosh}4-\operatorname{arccosh}2)}{875}+\frac{(288889+129739\sqrt{5})\ln 3}{378000}+\frac{(109-3143\sqrt{5})\ln 5}{151200}\right)$

Special Crofton Reduction Technique

Definition. $P_{AB} = \mathbb{E}\left[P(X,Y) \,|\, X \in A, Y \in B, \text{ uniform and independent}\right]$, $L^{(p)}_{AB} = P_{AB}$ with $P = L^p = \|X - Y\|^p$, $\mathcal{A}(A)$ affine hull of A, $\mathcal{P}(\mathbb{R}^d)$ set of all polytopes in $\mathbb{R}^d$.

Lemma (SCRT)[3]**.** Let $P : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be homogeneous of order p (that is, $P(rX, rY) = |r|^p P(X,Y)$) and $A, B \in \mathcal{P}(\mathbb{R}^d)$, $a = \dim A$, $b = \dim B$, then for any $C \in \mathcal{A}(A) \cap \mathcal{A}(B)$ (common scaling point)

$$pP_{AB} = a(P_{\partial A\, B} - P_{AB}) + b(P_{A\, \partial B} - P_{AB}).$$



where we define $P_{\partial A\, B} = \sum_i w_i P_{\partial_i A\, B}$

with weights $w_i = \dfrac{\operatorname{vol} \partial_i A}{a \operatorname{vol} A} h_{A-C}(\hat{n}_i)$

with $\hat{n}_i$ outer normal to $\partial_i A$ lying in $\mathcal{A}(A)$

[3] H Ruben and WJ Reed. "A more general form of a theorem of Crofton". In: *Journal of Applied Probability* (1973), pp. 479–482.

General Crofton Reduction Technique

Definition. $P_{A_1 A_2 \ldots A_n} = \mathbb{E}\left[P(X_1, \ldots, X_n) \,|\, X_1 \in A_1, \ldots, X_n \in A_n, \text{ uniform and independent}\right]$.

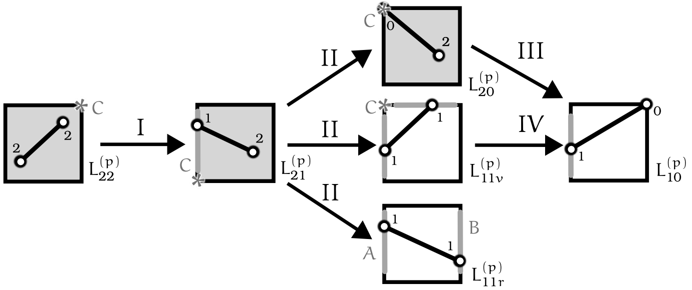
Lemma (GCRT)[4]**.** Let $P : (\mathbb{R}^d)^n \to \mathbb{R}$ be homogeneous of order p and $A_1, \ldots, A_n \in \mathcal{P}(\mathbb{R}^d)$, $a_i = \dim A_i$, then for any $C \in \bigcap_{1 \leqslant i \leqslant n} \mathcal{A}(A_i)$ (scaling point)

$$pP_{A_1 A_2 \ldots A_n} = a_1 (P_{\partial A_1 A_2 \ldots A_n} - P_{A_1 \ldots A_n}) + a_2 (P_{A_1 \partial A_2 \ldots A_n} - P_{A_1 \ldots A_n}) + \cdots + a_n (P_{A_1 A_2 \ldots \partial A_n} - P_{A_1 \ldots A_n}).$$

[4] H Ruben and WJ Reed. "A more general form of a theorem of Crofton". In: *Journal of Applied Probability* (1973), pp. 479–482.

Square line picking

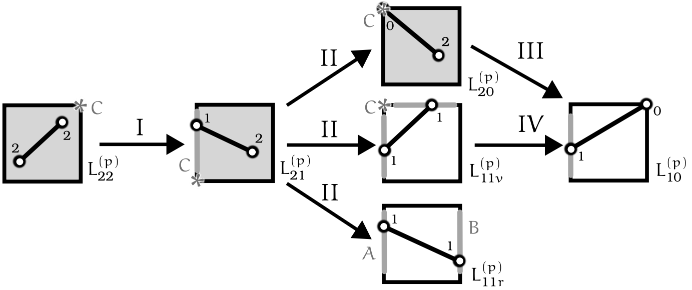
$P_{ab} = L_{ab}^{(p)}$



Square line picking

$P_{ab} = L_{ab}^{(p)}$

$$\text{I:}\quad pP_{22} = 2\cdot 2(P_{21} - P_{22})$$
$$\text{II:}\quad pP_{21} = 2(P_{11}-P_{21})+1(P_{20}-P_{21}),$$
$$P_{11} = \tfrac{1}{2}P_{11v} + \tfrac{1}{2}P_{11r}$$
$$\text{III:}\quad pP_{20} = 2(P_{10} - P_{20})$$
$$\text{IV:}\quad pP_{11v} = 2(P_{10} - P_{11v})$$



Square line picking

$P_{ab} = L_{ab}^{(p)}$

$$\text{I}: \; pP_{22} = 2 \cdot 2(P_{21} - P_{22})$$

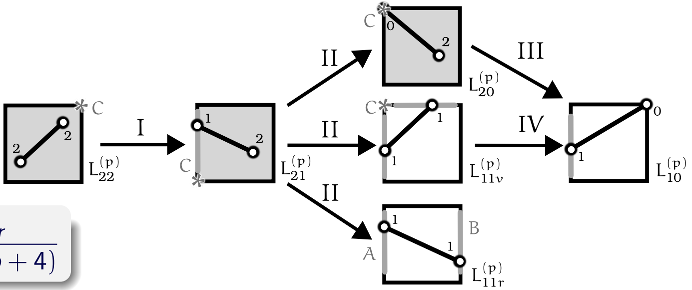
$$\text{II}: \; pP_{21} = 2(P_{11} - P_{21}) + 1(P_{20} - P_{21}),$$

$$P_{11} = \tfrac{1}{2} P_{11v} + \tfrac{1}{2} P_{11r}$$

$$\text{III}: \; pP_{20} = 2(P_{10} - P_{20})$$

$$\text{IV}: \; pP_{11v} = 2(P_{10} - P_{11v})$$

$$P_{22} = \frac{16 P_{10}}{(p+2)(p+3)(p+4)} + \frac{4 P_{11r}}{(p+3)(p+4)}$$



Square line picking $P_{ab} = L_{ab}^{(p)}$

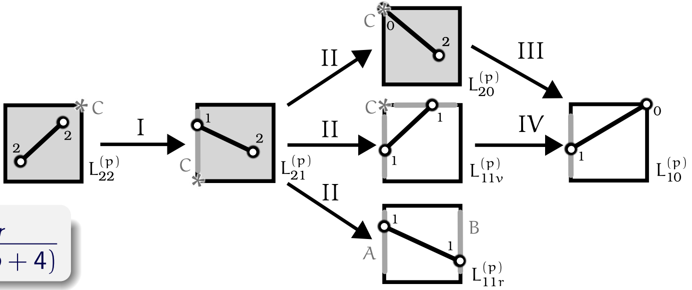
$$\text{I:}\quad pP_{22} = 2\cdot 2(P_{21} - P_{22})$$

$$\text{II:}\quad pP_{21} = 2(P_{11} - P_{21}) + 1(P_{20} - P_{21}),$$

$$P_{11} = \tfrac{1}{2}P_{11v} + \tfrac{1}{2}P_{11r}$$

$$\text{III:}\quad pP_{20} = 2(P_{10} - P_{20})$$

$$\text{IV:}\quad pP_{11v} = 2(P_{10} - P_{11v})$$

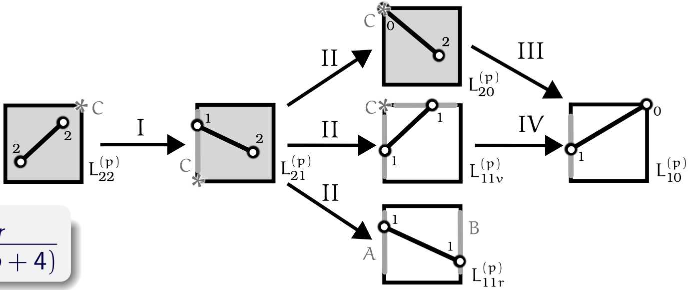


$$P_{22} = \frac{16P_{10}}{(p+2)(p+3)(p+4)} + \frac{4P_{11r}}{(p+3)(p+4)}$$

$$\left.\begin{aligned} P_{10} &= \int_0^1 (1+x^2)^{p/2}\mathrm{d}x = {}_2F_1\left(\frac{1}{2}, -\frac{p}{2}; \frac{3}{2}; -1\right) \\ P_{11r} &= 2\int_0^1 (1-u)(1+u^2)^{p/2}\,\mathrm{d}u \end{aligned}\right\}$$

Square line picking $\quad P_{ab} = L_{ab}^{(p)}$

$$\text{I:}\quad pP_{22} = 2\cdot 2(P_{21}-P_{22})$$
$$\text{II:}\quad pP_{21} = 2(P_{11}-P_{21})+1(P_{20}-P_{21}),$$
$$P_{11} = \tfrac{1}{2}P_{11v}+\tfrac{1}{2}P_{11r}$$
$$\text{III:}\quad pP_{20} = 2(P_{10}-P_{20})$$
$$\text{IV:}\quad pP_{11v} = 2(P_{10}-P_{11v})$$



$$P_{22} = \frac{16P_{10}}{(p+2)(p+3)(p+4)} + \frac{4P_{11r}}{(p+3)(p+4)}$$

$$P_{10} = \int_0^1 (1+x^2)^{p/2}\mathrm{d}x = {}_2F_1\left(\frac{1}{2},-\frac{p}{2};\frac{3}{2};-1\right)$$
$$P_{11r} = 2\int_0^1 (1-u)(1+u^2)^{p/2}\,\mathrm{d}u$$

$$\therefore P_{22} = 8\frac{1-2^{\frac{p}{2}+1}+(p+4)\,{}_2F_1\left(\frac{1}{2},-\frac{p}{2};\frac{3}{2};-1\right)}{(p+2)(p+3)(p+4)}$$

Square line picking

$P_{ab} = L_{ab}^{(p)}$

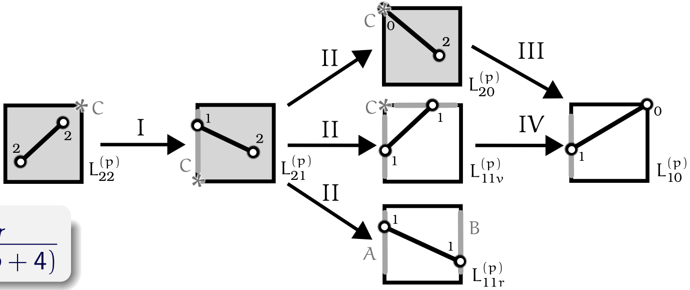
$$\text{I:}\quad pP_{22} = 2\cdot 2(P_{21} - P_{22})$$

$$\text{II:}\quad pP_{21} = 2(P_{11} - P_{21}) + 1(P_{20} - P_{21}),$$

$$P_{11} = \tfrac{1}{2}P_{11v} + \tfrac{1}{2}P_{11r}$$

$$\text{III:}\quad pP_{20} = 2(P_{10} - P_{20})$$

$$\text{IV:}\quad pP_{11v} = 2(P_{10} - P_{11v})$$



$$P_{22} = \frac{16P_{10}}{(p+2)(p+3)(p+4)} + \frac{4P_{11r}}{(p+3)(p+4)}$$

$$P_{10} = \int_0^1 (1+x^2)^{p/2}\mathrm{d}x = {}_2F_1\left(\frac{1}{2}, -\frac{p}{2}; \frac{3}{2}; -1\right)$$

$$P_{11r} = 2\int_0^1 (1-u)(1+u^2)^{p/2}\,\mathrm{d}u$$

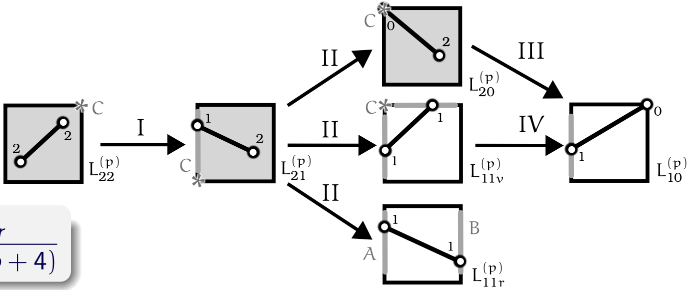
$$\therefore P_{22} = 8\frac{1 - 2^{\frac{p}{2}+1} + (p+4)\,{}_2F_1\left(\frac{1}{2}, -\frac{p}{2}; \frac{3}{2}; -1\right)}{(p+2)(p+3)(p+4)}$$

$$\text{IMT: } f(\lambda) = \begin{cases} 2\lambda(\pi - 4\lambda + \lambda^2) & , 0 \leqslant \lambda < 1 \\ 2\lambda\left(\pi - 2 - \lambda^2 + 4\sqrt{\lambda^2 - 1} - 4\arccos\left(\frac{1}{\lambda}\right)\right) & , 1 \leqslant \lambda \leqslant \sqrt{2} \end{cases}$$

Square line picking

$P_{ab} = L_{ab}^{(p)}$

$$\text{I: } pP_{22} = 2\cdot 2(P_{21} - P_{22})$$
$$\text{II: } pP_{21} = 2(P_{11} - P_{21}) + 1(P_{20} - P_{21}),$$
$$P_{11} = \tfrac{1}{2}P_{11v} + \tfrac{1}{2}P_{11r}$$
$$\text{III: } pP_{20} = 2(P_{10} - P_{20})$$
$$\text{IV: } pP_{11v} = 2(P_{10} - P_{11v})$$



$$P_{22} = \frac{16P_{10}}{(p+2)(p+3)(p+4)} + \frac{4P_{11r}}{(p+3)(p+4)}$$

$$P_{10} = \int_0^1 (1+x^2)^{p/2}\,\mathrm{d}x = {}_2F_1\left(\frac{1}{2}, -\frac{p}{2}; \frac{3}{2}; -1\right)$$
$$P_{11r} = 2\int_0^1 (1-u)(1+u^2)^{p/2}\,\mathrm{d}u$$
$$\therefore P_{22} = 8\,\frac{1 - 2^{\frac{p}{2}+1} + (p+4)\,{}_2F_1\left(\frac{1}{2}, -\frac{p}{2}; \frac{3}{2}; -1\right)}{(p+2)(p+3)(p+4)}$$

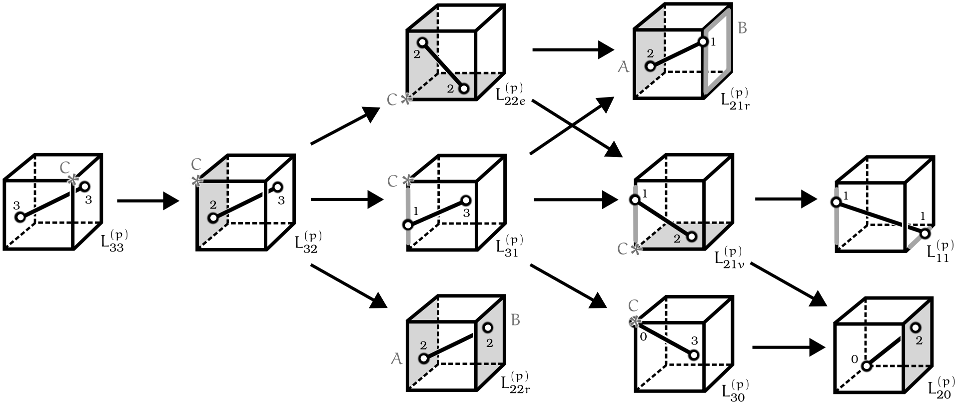
$$\text{IMT: } f(\lambda) = \begin{cases} 2\lambda(\pi - 4\lambda + \lambda^2) & , 0 \leqslant \lambda < 1 \\ 2\lambda\left(\pi - 2 - \lambda^2 + 4\sqrt{\lambda^2 - 1} - 4\arccos\left(\frac{1}{\lambda}\right)\right) & , 1 \leqslant \lambda \leqslant \sqrt{2} \end{cases}$$

- Exact P_{22} in regular polygons are known[5]

[5] Uwe Bäsel. "The moments of the distance between two random points in a regular polygon". In: *arXiv preprint arXiv:2101.03815* (2021)

Cube line picking

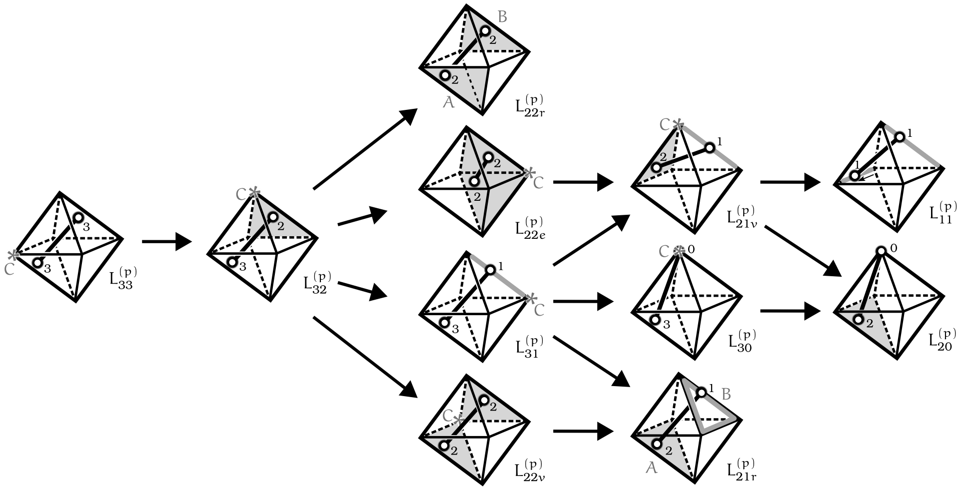
$P_{ab} = L_{ab}^{(p)}$ configurations



$$L_{33} = \frac{3L_{11}}{35} + \frac{3L_{20}}{35} + \frac{8L_{21r}}{35} + \frac{L_{22r}}{7}.$$

Octahedron line picking

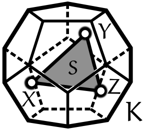
$P_{ab} = L_{ab}^{(p)}$ configurations



[5] D. B. "Mean distance in polyhedra". In: *arXiv preprint arXiv:2309* (2023)

Mean triangle area

p=2

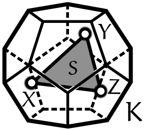


- $K \subset \mathbb{R}^d$, $\dim K = d$
- $X, Y, Z \in K$ uniformly and independently selected
- $S = \|\Delta_{XYZ}\|$ triangle surface area

$$\Sigma(K) := \frac{\mathbb{E}\,\|\Delta_{XYZ}\|}{(\mathrm{vol}_d\, K)^{2/d}}$$

Mean triangle area

p=2



- $K \subset \mathbb{R}^d$, $\dim K = d$
- $X, Y, Z \in K$ uniformly and independently selected
- $S = \|\Delta_{XYZ}\|$ triangle surface area

$$\Sigma(K) := \frac{\mathbb{E}\,\|\Delta_{XYZ}\|}{(\mathrm{vol}_d\, K)^{2/d}}$$

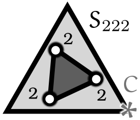
K	MC	$\Sigma(K)$
disk	0.0739	$\frac{35}{48\pi^2}$
ball	0.1413	$\frac{9}{154}\sqrt[3]{\frac{9\pi}{2}}$
triangle	0.0833	$\frac{1}{12}$

- A general formula for $\Sigma(P)$ is known for $P \subset \mathbb{R}^2$ being any convex polygon[6].
- Apart from the ball, $\Sigma(K)$ is not known for any higher dimensional K.

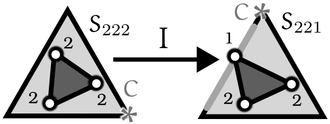
[6] Christian Buchta and Matthias Reitzner. "Equiaffine inner parallel curves of a plane convex body and the convex hulls of randomly chosen points". In: *Probability Theory and Related Fields* 108.3 (1997), pp. 385–415

Mean triangle area

p=2

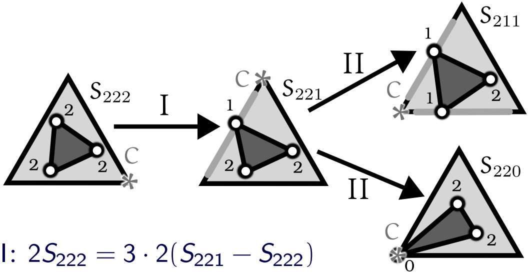


Mean triangle area p=2



I: $2S_{222} = 3 \cdot 2(S_{221} - S_{222})$

Mean triangle area p=2

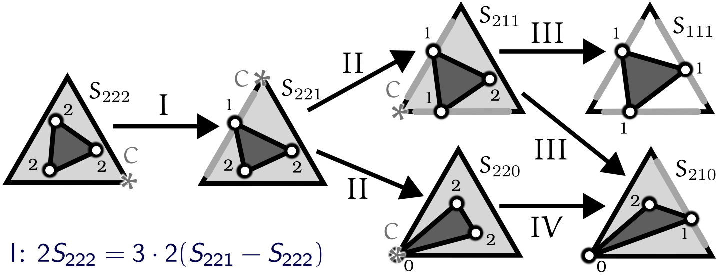


$$\text{I:}\ 2S_{222} = 3 \cdot 2(S_{221} - S_{222})$$

$$\text{II:}\ 2S_{221} = 2 \cdot 2(S_{211} - S_{221}) + 1(S_{220} - S_{221}),$$

Mean triangle area

p=2



$$\text{I}: \; 2S_{222} = 3 \cdot 2(S_{221} - S_{222})$$

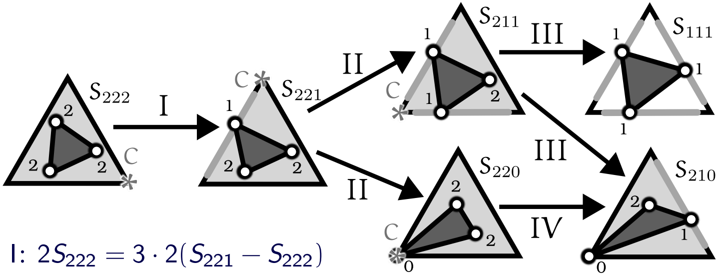
$$\text{II}: \; 2S_{221} = 2 \cdot 2(S_{211} - S_{221}) + 1(S_{220} - S_{221}),$$

$$\text{III}: \; 2S_{211} = 2(S_{111} - S_{211}) + 2 \cdot 1(S_{210} - S_{211}),$$

$$\text{IV}: \; 2S_{220} = 2 \cdot 2(S_{210} - S_{220})$$

Mean triangle area

p=2



I: $2S_{222} = 3 \cdot 2(S_{221} - S_{222})$

II: $2S_{221} = 2 \cdot 2(S_{211} - S_{221}) + 1(S_{220} - S_{221})$,

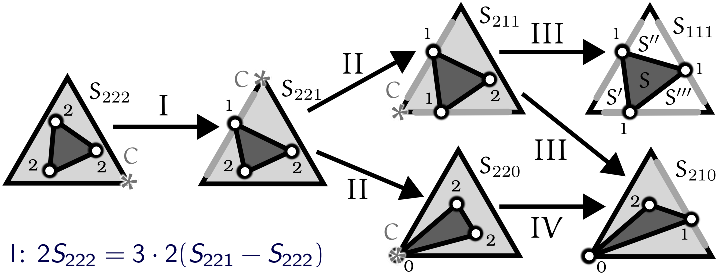
III: $2S_{211} = 2(S_{111} - S_{211}) + 2 \cdot 1(S_{210} - S_{211})$,

IV: $2S_{220} = 2 \cdot 2(S_{210} - S_{220})$

$$\therefore S_{222} = \frac{1}{7} S_{111} + \frac{3}{14} S_{210}$$

Mean triangle area

p=2



I: $2S_{222} = 3 \cdot 2(S_{221} - S_{222})$

II: $2S_{221} = 2 \cdot 2(S_{211} - S_{221}) + 1(S_{220} - S_{221})$,

III: $2S_{211} = 2(S_{111} - S_{211}) + 2 \cdot 1(S_{210} - S_{211})$,

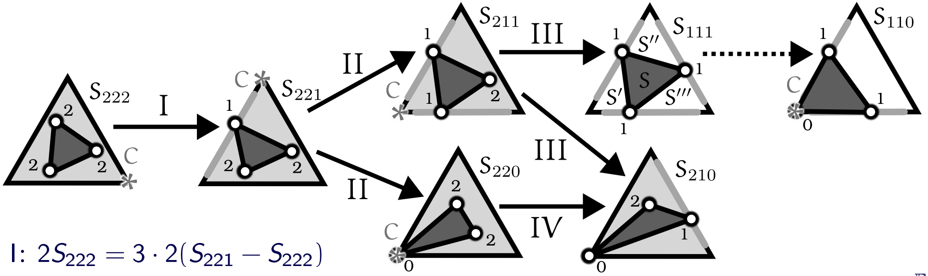
IV: $2S_{220} = 2 \cdot 2(S_{210} - S_{220})$

$S_{111} : 1 = S + S' + S'' + S'''$

$$\therefore S_{222} = \frac{1}{7} S_{111} + \frac{3}{14} S_{210}$$

Mean triangle area

p=2



I: $2S_{222} = 3 \cdot 2(S_{221} - S_{222})$

II: $2S_{221} = 2 \cdot 2(S_{211} - S_{221}) + 1(S_{220} - S_{221})$,

III: $2S_{211} = 2(S_{111} - S_{211}) + 2 \cdot 1(S_{210} - S_{211})$,

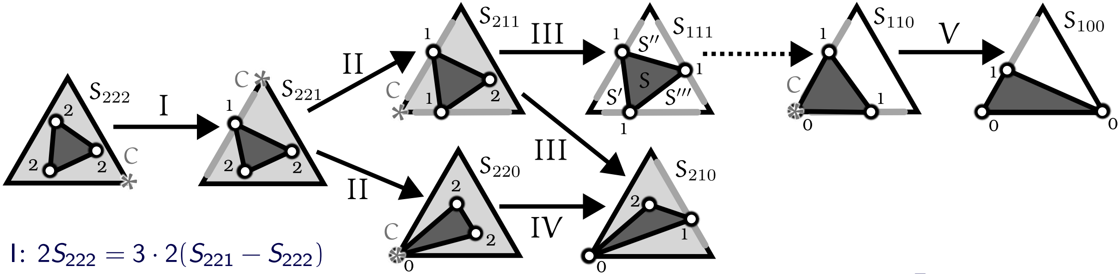
IV: $2S_{220} = 2 \cdot 2(S_{210} - S_{220})$

$S_{111} : 1 = S + S' + S'' + S''' \xrightarrow{\mathbb{E}} S_{111} = 1 - 3S_{110}$

$$\therefore S_{222} = \tfrac{1}{7} S_{111} + \tfrac{3}{14} S_{210}$$

Mean triangle area

p=2



$$\text{I: } 2S_{222} = 3 \cdot 2(S_{221} - S_{222})$$

$$\text{II: } 2S_{221} = 2 \cdot 2(S_{211} - S_{221}) + 1(S_{220} - S_{221}),$$

$$\text{III: } 2S_{211} = 2(S_{111} - S_{211}) + 2 \cdot 1(S_{210} - S_{211}),$$

$$\text{IV: } 2S_{220} = 2 \cdot 2(S_{210} - S_{220})$$

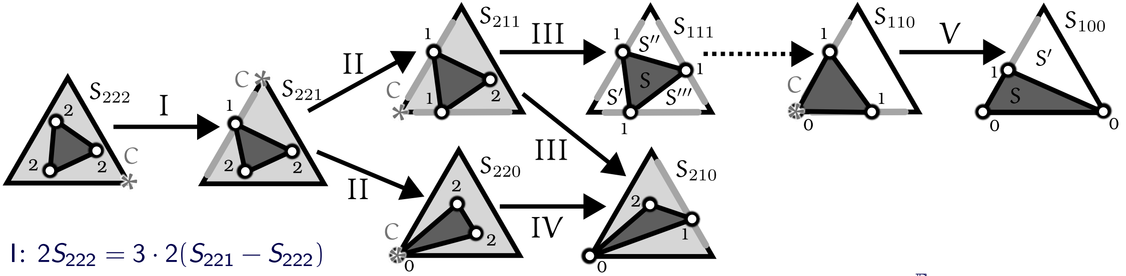
$$\text{V: } 2S_{110} = 2 \cdot 1(S_{100} - S_{110})$$

$$S_{111} : 1 = S + S' + S'' + S''' \xrightarrow{\mathbb{E}} S_{111} = 1 - 3S_{110}$$

$$\therefore S_{222} = \frac{1}{7} S_{111} + \frac{3}{14} S_{210}$$

Mean triangle area

p=2



I: $2S_{222} = 3 \cdot 2(S_{221} - S_{222})$

II: $2S_{221} = 2 \cdot 2(S_{211} - S_{221}) + 1(S_{220} - S_{221}),$

III: $2S_{211} = 2(S_{111} - S_{211}) + 2 \cdot 1(S_{210} - S_{211}),$

IV: $2S_{220} = 2 \cdot 2(S_{210} - S_{220})$

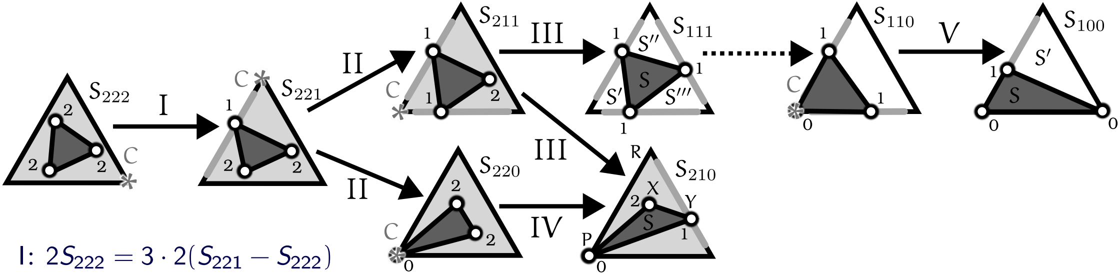
V: $2S_{110} = 2 \cdot 1(S_{100} - S_{110})$

$$\therefore S_{222} = \frac{1}{7} S_{111} + \frac{3}{14} S_{210}, \quad S_{110} = \frac{2}{3} S_{100}$$

$S_{111} : 1 = S + S' + S'' + S''' \xrightarrow{\mathbb{E}} S_{111} = 1 - 3S_{110}$

$S_{100} : 1 = S' + S''$

Mean triangle area p=2



I: $2S_{222} = 3 \cdot 2(S_{221} - S_{222})$

II: $2S_{221} = 2 \cdot 2(S_{211} - S_{221}) + 1(S_{220} - S_{221}),$

III: $2S_{211} = 2(S_{111} - S_{211}) + 2 \cdot 1(S_{210} - S_{211}),$

IV: $2S_{220} = 2 \cdot 2(S_{210} - S_{220})$

V: $2S_{110} = 2 \cdot 1(S_{100} - S_{110})$

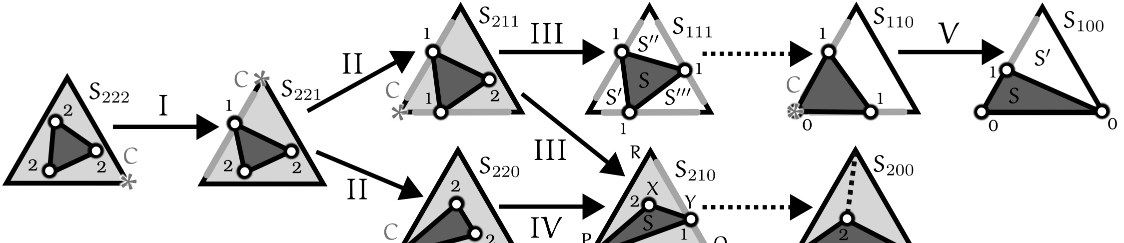
$S_{111} : 1 = S + S' + S'' + S''' \xrightarrow{\mathbb{E}} S_{111} = 1 - 3S_{110}$

$S_{100} : 1 = S' + S'' \xrightarrow{\mathbb{E}} S_{100} = \frac{1}{2}$

$\therefore S_{222} = \frac{1}{7} S_{111} + \frac{3}{14} S_{210}, \quad S_{110} = \frac{2}{3} S_{100}$

Mean triangle area

p=2



I: $2S_{222} = 3 \cdot 2(S_{221} - S_{222})$

II: $2S_{221} = 2 \cdot 2(S_{211} - S_{221}) + 1(S_{220} - S_{221})$,

III: $2S_{211} = 2(S_{111} - S_{211}) + 2 \cdot 1(S_{210} - S_{211})$,

IV: $2S_{220} = 2 \cdot 2(S_{210} - S_{220})$

V: $2S_{110} = 2 \cdot 1(S_{100} - S_{110})$

$S_{111} : 1 = S + S' + S'' + S''' \xrightarrow{\mathbb{E}} S_{111} = 1 - 3S_{110}$

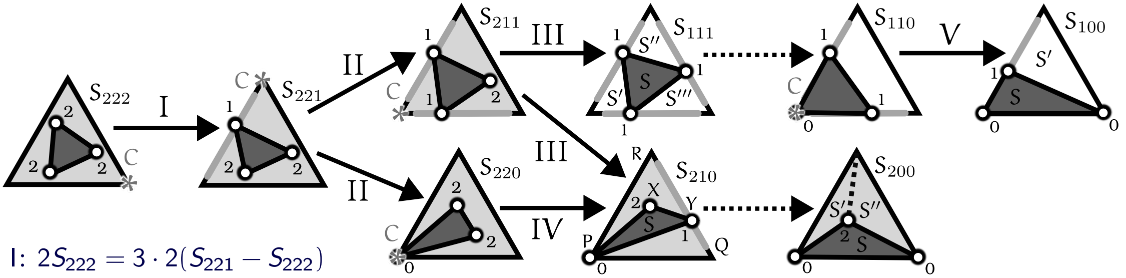
$S_{100} : 1 = S' + S'' \xrightarrow{\mathbb{E}} S_{100} = \frac{1}{2}$

$$S' = \|\Delta_{PYR}\| \to S_{210} = \mathbb{E}\left[\mathbb{E}\left[S|S'\right]\right] = \mathbb{E}\left[S_{200}S'^2 + S_{200}(1-S')^2\right] = \frac{2}{3}S_{200}$$

$$\therefore\ S_{222} = \tfrac{1}{7}S_{111} + \tfrac{3}{14}S_{210}, \qquad S_{110} = \tfrac{2}{3}S_{100}$$

Mean triangle area

p=2



I: $2S_{222} = 3 \cdot 2(S_{221} - S_{222})$

II: $2S_{221} = 2 \cdot 2(S_{211} - S_{221}) + 1(S_{220} - S_{221})$,

III: $2S_{211} = 2(S_{111} - S_{211}) + 2 \cdot 1(S_{210} - S_{211})$,

IV: $2S_{220} = 2 \cdot 2(S_{210} - S_{220})$

V: $2S_{110} = 2 \cdot 1(S_{100} - S_{110})$

$S_{111} : 1 = S + S' + S'' + S''' \xrightarrow{\mathbb{E}} S_{111} = 1 - 3S_{110}$

$S_{100} : 1 = S' + S'' \xrightarrow{\mathbb{E}} S_{100} = \frac{1}{2}$

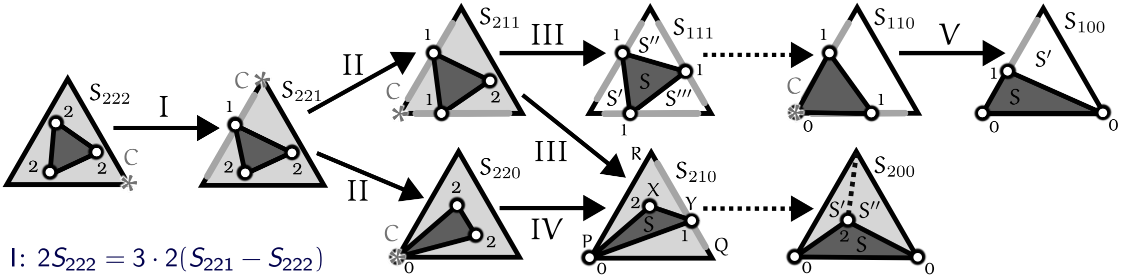
$S' = \|\Delta_{PYR}\| \to S_{210} = \mathbb{E}\,[\mathbb{E}\,[S|S']] = \mathbb{E}\,\left[S_{200} S'^2 + S_{200}(1-S')^2\right] = \frac{2}{3} S_{200}$

$\therefore S_{222} = \frac{1}{7} S_{111} + \frac{3}{14} S_{210}, \quad S_{110} = \frac{2}{3} S_{100}$

$S_{200} : 1 = S' + S'' + S'''$

Mean triangle area

p=2



I: $2S_{222} = 3\cdot 2(S_{221} - S_{222})$

II: $2S_{221} = 2\cdot 2(S_{211} - S_{221}) + 1(S_{220} - S_{221})$,

III: $2S_{211} = 2(S_{111} - S_{211}) + 2\cdot 1(S_{210} - S_{211})$,

IV: $2S_{220} = 2\cdot 2(S_{210} - S_{220})$

V: $2S_{110} = 2\cdot 1(S_{100} - S_{110})$

$$\therefore S_{222} = \tfrac{1}{7}S_{111} + \tfrac{3}{14}S_{210}, \quad S_{110} = \tfrac{2}{3}S_{100}$$

$S_{111}: 1 = S + S' + S'' + S''' \xrightarrow{\mathbb{E}} S_{111} = 1 - 3S_{110}$

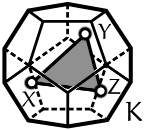
$S_{100}: 1 = S' + S'' \xrightarrow{\mathbb{E}} S_{100} = \frac{1}{2}$

$$S' = \|\Delta_{PYR}\| \to S_{210} = \mathbb{E}\left[\mathbb{E}\left[S|S'\right]\right] = \mathbb{E}\left[S_{200}S'^2 + S_{200}(1-S')^2\right] = \frac{2}{3}S_{200}$$

$S_{200}: 1 = S' + S'' + S''' \xrightarrow{\mathbb{E}} S_{200} = \frac{1}{3}$

$S_{222} = \frac{1}{12}$ | $S_{221} = \frac{1}{9}$ | $S_{211} = \frac{17}{108}$ | $S_{220} = \frac{4}{27}$ | $S_{111} = \frac{1}{4}$ | $S_{210} = \frac{2}{9}$ | $S_{110} = \frac{1}{4}$ | $S_{200} = \frac{1}{3}$ | $S_{100} = \frac{1}{2}$

Triangle obtusity probability

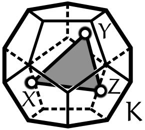


- $K \subset \mathbb{R}^d$, $\dim K = d$
- $X, Y, Z \in K$ uniformly and independently selected
- $\eta(X, Y, Z) = 1_{\Delta_{XYZ} \text{ is obtuse}}$

$$H(K) = \mathbb{E}\,\eta(X, Y, Z)$$

Triangle obtusity probability

p=0



- $K \subset \mathbb{R}^d$, $\dim K = d$
- $X, Y, Z \in K$ uniformly and independently selected
- $\eta(X, Y, Z) = 1_{\Delta_{XYZ} \text{ is obtuse}}$

$$H(K) = \mathbb{E}\eta(X, Y, Z)$$

K	MC	$H(K)$
disk[7]	0.7197	$\frac{9}{8} - \frac{4}{\pi^2}$
ball[8]	0.5286	$\frac{37}{70}$
square[9]	0.7252	$\frac{97}{150} + \frac{\pi}{40}$
triangle	0.7482	?

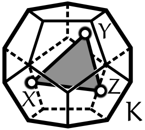
[7] WSB Woolhouse. "Some additional observations on the four-point problem". In: *Mathematical Questions and their Solutions from the Educational Times* 7 (1867), p. 81

[8] Christian Buchta and Josef Müller. "Random polytopes in a ball". In: *Journal of applied probability* 21.4 (1984), pp. 753–762, SOLVED IN ANY DIMENSIONS

[9] Eric Langford. "The probability that a random triangle is obtuse". In: *Biometrika* 56.3 (1969), pp. 689–690, SOLVED FOR RECTANGLE IN GENERAL

Triangle obtusity probability

p=0



- $K \subset \mathbb{R}^d$, $\dim K = d$
- $X, Y, Z \in K$ uniformly and independently selected
- $\eta(X, Y, Z) = 1_{\Delta_{XYZ} \text{ is obtuse}}$

$$H(K) = \mathbb{E}\eta(X, Y, Z)$$

K	MC	$H(K)$
disk[7]	0.7197	$\frac{9}{8} - \frac{4}{\pi^2}$
ball[8]	0.5286	$\frac{37}{70}$
square[9]	0.7252	$\frac{97}{150} + \frac{\pi}{40}$
triangle	0.7482	$\frac{25}{4} + \frac{\pi}{12\sqrt{3}} + \frac{393}{10} \ln \frac{\sqrt{3}}{2}$

[7] WSB Woolhouse. "Some additional observations on the four-point problem". In: *Mathematical Questions and their Solutions from the Educational Times* 7 (1867), p. 81

[8] Christian Buchta and Josef Müller. "Random polytopes in a ball". In: *Journal of applied probability* 21.4 (1984), pp. 753–762, SOLVED IN ANY DIMENSIONS

[9] Eric Langford. "The probability that a random triangle is obtuse". In: *Biometrika* 56.3 (1969), pp. 689–690, SOLVED FOR RECTANGLE IN GENERAL

- $H(P)$ expressible for any polygon $P \subset \mathbb{R}^2$ (consequence of CRT, ongoing work).
- **Conjecture (B.D., 2024):** $H([0,1]^3) = p + \pi^2 q + Cr$ for some rationals p, q, r and the Catalan constant C.

Rectangle

Theorem. (Langford[10]) Probability of a triangle obtusity in a rectangle

Let K be a rectangle with aspect ratio $a \geqslant 1$. The probability $P(a)$ that a random triangle picked from K is obtuse is given by

$$P(a) = \begin{cases} \frac{1}{3} + \frac{47}{300}\left(a^2 + \frac{1}{a^2}\right) + \frac{1}{80}\pi\left(a^3 + \frac{1}{a^3}\right) - \frac{1}{5}\ln a\left(a^2 - \frac{1}{a^2}\right), & 1 \leqslant a \leqslant 2 \\ \frac{1}{3} + \frac{1}{a^2}\left(\frac{\pi}{80a} + \frac{47}{300} + \frac{\ln a}{5}\right) + \frac{47a^2}{300} - \frac{1}{5}a^2\ln a + \frac{a^3}{40}\arcsin\frac{2}{a} \\ \quad + \left(\frac{a^2}{10} - \frac{3}{5a^2}\right)\operatorname{arccosh}\frac{a^2-2}{2} + \frac{a\sqrt{a^2-4}}{150}\left(-31 + \frac{63}{a^2} + \frac{64}{a^4}\right), & a > 2 \end{cases}$$

[10] Eric Langford. "The probability that a random triangle is obtuse". In: *Biometrika* 56.3 (1969), pp. 689–690

Rectangle

Theorem. (Langford[10]) Probability of a triangle obtusity in a rectangle

Let K be a rectangle with aspect ratio $a \geqslant 1$. The probability $P(a)$ that a random triangle picked from K is obtuse is given by

$$P(a) = \begin{cases} \frac{1}{3} + \frac{47}{300}\left(a^2 + \frac{1}{a^2}\right) + \frac{1}{80}\pi\left(a^3 + \frac{1}{a^3}\right) - \frac{1}{5}\ln a\left(a^2 - \frac{1}{a^2}\right), & 1 \leqslant a \leqslant 2 \\ \frac{1}{3} + \frac{1}{a^2}\left(\frac{\pi}{80a} + \frac{47}{300} + \frac{\ln a}{5}\right) + \frac{47a^2}{300} - \frac{1}{5}a^2\ln a + \frac{a^3}{40}\arcsin\frac{2}{a} \\ \quad + \left(\frac{a^2}{10} - \frac{3}{5a^2}\right)\operatorname{arccosh}\frac{a^2-2}{2} + \frac{a\sqrt{a^2-4}}{150}\left(-31 + \frac{63}{a^2} + \frac{64}{a^4}\right), & a > 2 \end{cases}$$

- **Corollary.** Probability a triangle is obtuse in a square is $P(1) = \frac{97}{150} + \frac{\pi}{40}$

[10] Eric Langford. "The probability that a random triangle is obtuse". In: *Biometrika* 56.3 (1969), pp. 689–690

Triangle

- We have found the following new result for picking an obtuse triangle in a general triangle

Triangle

- We have found the following new result for picking an obtuse triangle in a general triangle

Theorem. Probability of a triangle being obtuse in a triangle

Let K be a triangle ABC with inner angles α, β, γ, respectively. Denote $P(\alpha, \beta)$ the probability that a random triangle picked from K is obtuse. Then $P(\alpha, \beta)$ is expressible in terms of logarithms and trigonometric functions of α, β, γ as

$$P(\alpha, \beta) = \begin{cases} P_{ac}(\alpha, \beta), & ABC \text{ acute} \\ P_{ob}(\alpha, \beta), & ABC \text{ obtuse at } C \end{cases}$$

(The functions P_{ac} and P_{ob} are on the next slide)

Triangle

- We have found the following new result for picking an obtuse triangle in a general triangle

Theorem. Probability of a triangle being obtuse in a triangle

Let K be a triangle ABC with inner angles α, β, γ, respectively. Denote $P(\alpha, \beta)$ the probability that a random triangle picked from K is obtuse. Then $P(\alpha, \beta)$ is expressible in terms of logarithms and trigonometric functions of α, β, γ as

$$P(\alpha, \beta) = \begin{cases} P_{ac}(\alpha, \beta), & ABC \text{ acute} \\ P_{ob}(\alpha, \beta), & ABC \text{ obtuse at } C \end{cases}$$

(The functions P_{ac} and P_{ob} are on the next slide)

- **Corollary.** Probability a triangle is obtuse in an equilateral triagle is $P(\pi/3, \pi/3) = \frac{25}{4} + \frac{\pi}{12\sqrt{3}} + \frac{393}{10} \ln \frac{\sqrt{3}}{2}$.

Triangle

- We have found the following new result for picking an obtuse triangle in a general triangle

Theorem. Probability of a triangle being obtuse in a triangle

Let K be a triangle ABC with inner angles α, β, γ, respectively. Denote $P(\alpha, \beta)$ the probability that a random triangle picked from K is obtuse. Then $P(\alpha, \beta)$ is expressible in terms of logarithms and trigonometric functions of α, β, γ as

$$P(\alpha, \beta) = \begin{cases} P_{ac}(\alpha, \beta), & ABC \text{ acute} \\ P_{ob}(\alpha, \beta), & ABC \text{ obtuse at } C \end{cases}$$

(The functions P_{ac} and P_{ob} are on the next slide)

- **Corollary.** Probability a triangle is obtuse in an equilateral triagle is $P(\pi/3, \pi/3) = \frac{25}{4} + \frac{\pi}{12\sqrt{3}} + \frac{393}{10} \ln \frac{\sqrt{3}}{2}$. (a **minimum value** among all triangles)

Triangle in a triangle

$$
\begin{aligned}
&\frac{1}{1920}\\
&(-128\,(Csc[\gamma]^4\,Log[Sin[\alpha]\,Sin[\beta]]+Csc[\beta]^4\,Log[Sin[\alpha]\,Sin[\gamma]]+Csc[\alpha]^4\,Log[Sin[\beta]\,Sin[\gamma]])\,Sin[\alpha]^2\\
&\quad Sin[\beta]^2\,Sin[\gamma]^2+Csc[\alpha]^2\,Csc[\beta]^2\,Csc[\gamma]^2\\
&\quad(-2\,(-36+20\,Cos[2\alpha]+11\,Cos[4\alpha]-2\,Cos[2\,(\alpha-2\beta)]-2\,Cos[4\alpha-2\beta]-\\
&\qquad 16\,Cos[2\,(\alpha-\beta)]+Cos[4\,(\alpha-\beta)]+20\,Cos[2\beta]+11\,Cos[4\beta]-16\,Cos[2\,(2\alpha+\beta)]+\\
&\qquad Cos[4\,(2\alpha+\beta)]-2\,Cos[2\,(3\alpha+\beta)]-16\,Cos[2\,(\alpha+2\beta)]+Cos[4\,(\alpha+2\beta)]-\\
&\qquad 2\,Cos[2\,(\alpha+3\beta)]-2\,Cos[6\alpha+4\beta]-2\,Cos[4\alpha+6\beta]+20\,Cos[2\gamma]+11\,Cos[4\gamma])+\\
&\quad \alpha\,(40\,Sin[2\alpha]-5\,Sin[4\alpha]+4\,Sin[2\,(\alpha-2\beta)]+4\,Sin[4\alpha-2\beta]-20\,Sin[2\,(\alpha-\beta)]-\\
&\qquad Sin[4\,(\alpha-\beta)]-20\,Sin[2\,(2\alpha+\beta)]+8\,Cos[\beta]\,Sin[3\,(2\alpha+\beta)]-Sin[4\,(2\alpha+\beta)])-\\
&\quad \beta\,(4\,Sin[2\,(\alpha-2\beta)]+4\,Sin[4\alpha-2\beta]-20\,Sin[2\,(\alpha-\beta)]-Sin[4\,(\alpha-\beta)]-40\,Sin[2\beta]+\\
&\qquad 5\,Sin[4\beta]+20\,Sin[2\,(\alpha+2\beta)]-8\,Cos[\alpha]\,Sin[3\,(\alpha+2\beta)]+Sin[4\,(\alpha+2\beta)])+\\
&\quad (\pi-\alpha-\beta)\,(20\,Sin[2\,(2\alpha+\beta)]+Sin[4\,(2\alpha+\beta)]-4\,Sin[2\,(3\alpha+\beta)]+20\,Sin[2\,(\alpha+2\beta)]+Sin[\\
&\qquad 4\,(\alpha+2\beta)]-4\,Sin[2\,(\alpha+3\beta)]-4\,Sin[6\alpha+4\beta]-4\,Sin[4\alpha+6\beta]+40\,Sin[2\gamma]-5\,Sin[4\gamma]))+\\
&2\,Csc[\gamma]^2\,(64\,(6\,Log[Sin[\alpha]]\,Sec[\alpha]^2\,Sin[\beta]^2\,Tan[\alpha]^2+Sin[\alpha]^2\,(-1+Tan[\beta]^2))+\\
&\quad Sec[\beta]^2\,((33+Cos[4\alpha]\,(-17+Cos[4\beta])+(-17+8\,Cos[2\alpha])\,Cos[4\beta])\,Sec[\alpha]^2+\\
&\qquad 8\,(-1+Tan[\alpha]^2+48\,Log[Sin[\beta]]\,Sin[\alpha]^2\,Tan[\beta]^2)))+\\
&2\,Csc[\beta]^2\,(Sec[\alpha]^2\,((33-8\,Cos[2\gamma]-17\,Cos[4\gamma]+Cos[4\alpha]\,(-17+8\,Cos[2\gamma]+Cos[4\gamma]))\,Sec[\gamma]^2-\\
&\qquad 64\,Sin[\gamma]^2\,(Cos[2\alpha]-6\,Log[Sin[\alpha]]\,Tan[\alpha]^2))+384\,Log[Sin[\gamma]]\,Sec[\gamma]^2\,Sin[\alpha]^2\,Tan[\gamma]^2)+\\
&2\,Csc[\alpha]^2\,(Sec[\beta]^2\,((33+Cos[4\beta]\,(-17+Cos[4\gamma])-17\,Cos[4\gamma])\,Sec[\gamma]^2-64\,Sin[\gamma]^2\\
&\qquad (Cos[2\beta]-6\,Log[Sin[\beta]]\,Tan[\beta]^2))-64\,Sec[\gamma]^2\,Sin[\beta]^2\,(Cos[2\gamma]-6\,Log[Sin[\gamma]]\,Tan[\gamma]^2))+\\
&64\,(18+4\,Sec[\beta]\,Sec[\gamma]\,Sin[\alpha]\,Tan[\alpha]+(9-5\,Cos[2\alpha]+Cos[2\beta]-5\,Cos[2\gamma])\\
&\quad Csc[\alpha]\,Csc[\gamma]\,Log[Sin[\beta]]\,Sec[\beta]\,Tan[\beta]^2+4\,Tan[\beta]\,Tan[\gamma]+\\
&\quad Csc[\beta]\,((9+Cos[2\alpha]-5\,Cos[2\beta]-5\,Cos[2\gamma])\,Csc[\gamma]\,Log[Sin[\alpha]]\,Sec[\alpha]\,Tan[\alpha]^2+\\
&\qquad (9-5\,Cos[2\alpha]-5\,Cos[2\beta]+Cos[2\gamma])\,Csc[\alpha]\,Log[Sin[\gamma]]\,Sec[\gamma]\,Tan[\gamma]^2)))
\end{aligned}
$$

Figure 1: The function P_{ac}

$$
\begin{aligned}
&\frac{1}{480}\,(8\,Cos[2\alpha]+8\,(19+7\,Cos[2\beta])\,Sec[\alpha]^2\,Sec[\beta]^2-8\,(-44+Cos[2\beta]+10\,Sec[\beta]^2)+\\
&2\,(52-57\,Cos[2\beta]-12\,Cos[4\beta]+Cos[6\beta])\,Csc[\gamma]\,Sec[\beta]^3\,Sin[\alpha]+\\
&16\,\beta\,(-2\,Cos[\alpha+2\beta]+Cos[\alpha]\,Cos[2\alpha+4\beta])\,Csc[\beta]^2\,Csc[\gamma]^2\,Sin[\alpha]^3+\\
&8\,Cot[\beta]\,Sin[2\alpha]+16\,\alpha\,(-2\,Cos[2\alpha+\beta]+Cos[\beta]\,Cos[4\alpha+2\beta])\,Csc[\alpha]^2\,Csc[\gamma]^2\,Sin[\beta]^3+\\
&Cot[\alpha]\,Sec[\beta]^3\,(74\,Sin[\beta]-21\,Sin[3\beta]+Sin[5\beta])+96\,Cot[\beta]\,Sec[\alpha]^2\,Tan[\alpha]-\\
&4\,(23-12\,Cos[2\beta]+13\,Cos[4\beta])\,Csc[\beta]\,Sec[\beta]^3\,Tan[\alpha]+4\,Log[Sin[\alpha]]\,Tan[\alpha]^2\\
&\quad (4\,(7+Cos[2\alpha]-5\,Cos[2\beta]-3\,Cos[2\gamma])\,Csc[\beta]\,Csc[\gamma]\,Sec[\alpha]+4\,(15+4\,Cos[2\alpha]+Cos[4\alpha])\\
&\qquad Csc[\gamma]^2\,Sec[\alpha]^2\,Sin[\beta]^2-(-21+4\,Cos[2\alpha]+Cos[4\alpha])\,(1+Cot[\beta]\,Tan[\alpha])^2)+\\
&32\,Log[Cos[\beta]]\,(-5\,Csc[\gamma]^2\,Sin[\alpha]^2\,Sin[\beta]^2+Tan[\alpha]^2\,(1+Cot[\beta]\,Tan[\alpha])\,(5+3\,Cot[\beta]\,Tan[\alpha]))+\\
&4\,(19-28\,Cos[2\beta]+Cos[4\beta])\,Csc[\gamma]^2\,Tan[\beta]^2+4\,Log[Sin[\beta]]\,Tan[\beta]^2\\
&\quad (-4\,(-7+5\,Cos[2\alpha]-Cos[2\beta]+3\,Cos[2\gamma])\,Csc[\alpha]\,Csc[\gamma]\,Sec[\beta]+4\,(15+4\,Cos[2\beta]+Cos[4\beta])\\
&\qquad Csc[\gamma]^2\,Sec[\beta]^2\,Sin[\alpha]^2-(-21+4\,Cos[2\beta]+Cos[4\beta])\,(1+Cot[\alpha]\,Tan[\beta])^2)+\\
&32\,Log[Cos[\alpha]]\,(-5\,Csc[\gamma]^2\,Sin[\alpha]^2\,Sin[\beta]^2+Tan[\beta]^2\,(1+Cot[\alpha]\,Tan[\beta])\,(5+3\,Cot[\alpha]\,Tan[\beta]))+\\
&32\,Log[Sin[\gamma]]\,((Csc[\alpha]^4+Csc[\beta]^4)\,Sin[\alpha]^2\,Sin[\beta]^2\,Sin[\gamma]^2-\\
&\quad (3\,Cot[\alpha]^2-4\,Cot[\alpha]\,Cot[\beta]+3\,Cot[\beta]^2)\,(Tan[\alpha]+Tan[\beta])^4-2\,Tan[\gamma]^2))
\end{aligned}
$$

Figure 2: The function P_{ob}

Triangle in a disk

Method: CRT

- Consider a disk D of unit radius in which we pick three points randomly uniformly forming a triangle with side lengths a, b, c and perimeter $P = a + b + c$. We solved $\mathbb{E}\left[P^2\right]$ proposed by Finch[11] and many other mean values:

[11] Steven Finch. "Perimeter variance of uniform random triangles". In: *arXiv preprint arXiv:1007.0261* (2010).

- Consider a disk D of unit radius in which we pick three points randomly uniformly forming a triangle with side lengths a, b, c and perimeter $P = a + b + c$. We solved $\mathbb{E}\left[P^2\right]$ proposed by Finch[11] and many other mean values:

Function	Monte Carlo	Exact mean value
ab	0.8378	$\frac{3383}{432\pi^2} + \frac{35\zeta(3)}{96\pi^2}$
P^2	8.027	$3 + \frac{3383}{72\pi^2} + \frac{35\zeta(3)}{16\pi^2}$
abc	0.7531	$\frac{8(173+64\ln 2)}{735\pi}$
ab^2	0.9442	$\frac{4672}{1575\pi}$
P^3	23.351	$\frac{776848}{11025\pi} + \frac{1024\ln 2}{245\pi}$
a/b	1.5108	$\frac{289}{27\pi^2} + \frac{7\zeta(3)}{2\pi^2}$

[11] Steven Finch. "Perimeter variance of uniform random triangles". In: *arXiv preprint arXiv:1007.0261* (2010).

List of new results of mean solid/characteristic picking

- Mean distance in polyhedra
  - tetrahedron
  - octahedron
  - icosahedron
  - dodecahedron
- Mean distance in polygons
- Probability of a triangle being obtuse in a given triangle

- Means of triangle characteristics in a unit disk
  - perimeter squared
  - perimeter cubed
  - product of two sides
  - product of three sides
  - product of a side and a square of another
  - quotient of two sides

Thank you for your attention!

[Woo67] WSB Woolhouse. "Some additional observations on the four-point problem". In: *Mathematical Questions and their Solutions from the Educational Times* 7 (1867), p. 81.

[Lan69] Eric Langford. "The probability that a random triangle is obtuse". In: *Biometrika* 56.3 (1969), pp. 689–690.

[RR73] H Ruben and WJ Reed. "A more general form of a theorem of Crofton". In: *Journal of Applied Probability* (1973), pp. 479–482.

[RB78] David Robbins and Theodore Bolis. "Average Distance between Two Points in a Box.". In: *Amer. Math. Monthly* 85 (1978), p. 278.

[BM84] Christian Buchta and Josef Müller. "Random polytopes in a ball". In: *Journal of applied probability* 21.4 (1984), pp. 753–762.

[BR97] Christian Buchta and Matthias Reitzner. "Equiaffine inner parallel curves of a plane convex body and the convex hulls of randomly chosen points". In: *Probability Theory and Related Fields* 108.3 (1997), pp. 385–415.

[Fin10] Steven Finch. "Perimeter variance of uniform random triangles". In: *arXiv preprint arXiv:1007.0261* (2010).

[Bäs21] Uwe Bäsel. "The moments of the distance between two random points in a regular polygon". In: *arXiv preprint arXiv:2101.03815* (2021).

[B23] D. B. "Mean distance in polyhedra". In: *arXiv preprint arXiv:2309* (2023).