FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

DOCTORAL THESIS

Dominik Beck

Random polytopes

Mathematical Institute of Charles University

Supervisor of the doctoral thesis: prof. RNDr. Jan Rataj, CSc.
Study programme: General Questions of Mathematics and Computer Science
Study branch: P4M8

Prague 2025

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources. It has not been used to obtain another or the same degree.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date ....

Author's signature

I would like at this point to thank all those who had confidence in me and saw the meaning in my work and my associated renunciation of everything worldly. I would like to thank especially my supervisor, Professor Jan Rataj, who supported me many times in my free research and under whose guidance I was able to give myself fully to my ideas. Without his help and direction I would not be where I am now. Many thanks also go to my family and also to my Anima, embodied in Kara Bravo, who has consistently supported me even in the most difficult times and to whom shall this work be dedicated...

> *"But if you tame me, then we shall need each other.*
> *To me, you will be unique in all the world.*
> *To you I shall be unique in all the world..."*

— Antoine de Saint-Exupéry, *The Little Prince*

Title: Random polytopes

Author: Dominik Beck

Department: Mathematical Institute of Charles University

Supervisor: prof. RNDr. Jan Rataj, CSc., Mathematical Institute of Charles University

Abstract: Our work covers the topic of moments of volumes of random simplices. We explain both combinatorial and integral-geometric treatment of the subject. The main themes throughout the work are moments of random determinants, Crofton Reduction Technique, Efon's formulae and Blaschke-Petkantschin formula.

We made a major contribution in higher dimensional generalisations of the know problems and pushed the older ideas to their limits in each of the branches mentioned. In random simplices metric moments branch, we were successful to enlarge the list of the exact volumetric moments for other three-dimensional polyhedra than to the only known three (ball, tetrahedron and cube). A new approach was developed to tackle also the volumetric moments in higher dimensions, which were inaccessible using previous methods.

A novel systematic use of the well known Crofton Reduction Technique enabled us to found other characteristics of polytopes, such as mean random distances of their interior points or the probability of a triangle formed by random interior points being obtuse.

Last but not least, in moments of random determinants branch, we found the fourth moment for a complete general case of matrix entries and the sixth moment for a special special case of centrally distributed entries. Although we solved those problems in our earlier published work, the treatment in this thesis is based solely on analytic combinatorics, making the material more broadly accessible.

Keywords: Random simplices, Mean tetrahedron volume, Crofton reduction technique, Efron formula, Random determinants, Mean distance, Polyhedra, Determinant moments, Blaschke-Petkantschin formula, Canonical section integral

Abstract (in Czech): V této práci se zabýváme momenty objemů náhodných simplexů. Vysvětlíme zde komplexní a ucelenou teorii kombinatorického a integrálně-geomtrického přístupu k tomuto problému, jehož součástí nesmí chybět momenty náhodných matic, Croftonova redukční technika, Efronovy vzorce a Blaschke-Petkantschinova formule.

Hlavní přínos této práce spočívá ve zobecnění již známých výsledků do více dimenzí, a to ve všech probíraných tématech. V kapitolách o náhodných simplexech jsme spočetli střední hodnoty objemu náhodného čtyřstěnu v tělese pro celou škálu nových mnohostěnů mimo jediných třech známých (koule, čtyřstěn, krychle). Navíc jsme odvodili analogické výsledky i ve více dimenzích.

Nový a systematický přístup ke známé Croftonově redukční metodě nám umožnil vyjádřit exaktně i další charasteriky v mnohostěnech jako například střední hodnoty vzdálenosti dvou vnitřních náhodných bodů a nebo pravděpodobnosti, že náhodný trojúhelník tvořený třemi náhodnými vnitřními body je tupoúhlý.

V neposlední řadě, co se týče momentů determinantů náhodných matic, jsme zobecnili čtvrtý moment pro obecná rozdělení prvků matice a našli i šestý moment pro rozdělení prvků s nulovou střední hodnotou. Ačkoli jsme tyto výsledky již publikovali, věnujeme se jim znovu v této práci a znovuodvozujeme je pomocí nástrojů analytické kombintoriky, čímž se téma momentů náhodných determinantů stává více přátelštější pro širší okruh čtenářstva.

Keywords (in Czech): Náhodné simplexy, Střední objem čtyřstěnu, Croftonova redukční technika, Efronova formule, Náhodné determinanty, Střední vzdálenost, Mnohostěny, Momenty determinantů, Blaschke-Petkantschinova formule, Kanonický sekční integrál

PANTA RHEI

Dominik Beck

May 30, 2025

Contents

List of symbols

$\mathcal{F}_k$	combinatorial structure of permutation tables with k rows and m_i weights 128, 130
$F_k(t)$	generating function of $f_k(n)$ 14, 121
$F_k(t,\omega)$	generating function of $f_k(n,p)$ 121, 151
G	Catalan's constant 31
$\mathcal{G}(d)$	group of all proper rigid motions in $\mathbb{R}^d$ 322
$\mathbb{G}(d,p)$	linear Grassmannian; space of p-dimensional subspaces of $\mathbb{R}^d$ passing through $\mathbf{0}$ 321
$g_k(n)$	k-moment of the determinant of B . . . .
$g_k(n,p)$	k-moment of the Gram determinant of V . . . . . .
$\mathcal{G}_k$	combinatorial structure of permutation tables with k rows and μ_i weights . . . .
$G_k(t)$	generating function of $g_k(n)$ . . . .
$G_k(t,\omega)$	generating function of $g_k(n,p)$ . . . .
$\mathbb{H}_n(K_d)$	convex hull of n points selected from K_d 11
H_k	k-th harmonic number 99, 241, 290
H'_k	k-th diharmonic number 290
$\mathrm{hull}(\cdot)$	convex hull . . . .
J	Gram matrix of U, $J = U^\top U$ 151
K°	polar body of K 322
K_d	d-dimensional convex compact body . . . .
Lang	Langford distribution 310
Li_2	dilogarithm function . . . .
$\mathcal{M}$	Mellin transform 316
$\mathbf{M}$	centrepoint (centre of mass) of K_d 234
m_r	non-central moments of X_{ij} 14, 121
N	Normal distribution 309
N_d	Standard multivariate normal distribution . 311
n_C	order of a configuration C, that is $\boldsymbol{\sigma} \cap P_d$ number of vertices 330
o_C	weight of a configuration C of P_d 329
O_d	d-dimensional orthoplex or d-cross-polytope (d-dimensional octahedron) 11
$\mathbb{P}$	probability . . . .
P_d	d-dimensional polytope . . . .
P_n	the set of all permutations on $[n]$ 123
$\mathbb{S}^{d-1}$	$(d-1)$-dimensional sphere in $\mathbb{R}^d$ with unit radius, equivalent to $\partial\mathbb{B}_d$ 321
$\mathbb{S}^{d-1}_+$	unit $(d-1)$-dimensional half-sphere in $\mathbb{R}^d$. 326
$\mathcal{SO}(d)$	special orthogonal group in $\mathbb{R}^d$ 321
T_d	(regular) d-dimensional simplex 11
T^*_d	$= \mathrm{conv}(\mathbf{e}_1, \ldots, \mathbf{e}_{d+1})$, standard d-simplex 11
$\mathbb{T}_d$	$= \mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d)$, canonical d-simplex . . 315
w_C	(symmetric) weight of a configuration C of P_d . . 330
U	(random) matrix $(X_{ij})_{n\times p}$ 15, 121, 151
Unif	Uniform distribution 309

V	(random) matrix $(Y_{ij})_{n\times p}$	
$v_n^{(k)}(K_d)$	metric moments in K_d	11
vol_d	d-dimensional volume	11
$\mathbb{x}$	collection of points $\mathbf{x}_j$	
$\mathbb{X}$	collection/sample of random points $\mathbf{X}_j$	11, 295, 301
$\mathbf{x}_j$	point (mostly in $\mathbb{R}^d$)	
$\mathbf{X}_j$	random point	
X_{ij}	(real) random variable	14, 121
Y_{ij}	central random variable $X_{ij} - m_1$	
∇	nabla vector operator, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^{\mathrm{T}}$	

Introduction

The following paragraphs summarise the core topics constituting the doctoral thesis of the author. The thesis serves as a comprehensive monograph encapsulating the findings of the author, some of which have been already published as separated papers in impacted journals, see [5, 6, 7, 8, 9, 10]. The complete list of author's publications is found in List of Publications at the end of this thesis.

Uniform random point selections and metric moments

Let K_d be a compact and convex body with $\dim K_d = d$, the so called d-body. The most trivial example is $\mathbb{B}_d$, a unit d-ball (d-dimensional ball with unit radius). Another such body is T_d, a d-simplex. Note that we can embed T_d in $\mathbb{R}^{d+1}$ such that T_d is *regular* in the following way: Let T_d^* be the convex hull of vertices which are located at the tops of the unit basis vectors $\mathbf{e}_i$, $i = 1, \ldots, d+1$ (*standard* regular d-simplex). A simple computation reveals that $\operatorname{vol}_d T_d^* = (\sqrt{d+1})/d!$ is its d-volume (d-dimensional volume). Yet another example is C_d, a regular d-cube and $O_d = \operatorname{conv}(\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_d)$, its dual, a regular d-orthoplex, which is a d-dimensional generalisation of a regular octahedron. More generally, we write P_d for a polytope of dimension d (d-polytope). Specifically, P_2 stands for a **polygon**, P_3 a **polyhedron** and P_4 a **polychoron**. Let $\mathbb{X} = (\mathbf{X}_0, \mathbf{X}_1, \ldots \mathbf{X}_n)$ be a sample of $(n+1)$ random points $\mathbf{X}_j$, $j = 0, \ldots, n$ selected uniformly and independently from the interior of K_d and let $\mathbb{H}_n(K_d) = \operatorname{conv}(\mathbb{X}) = \operatorname{conv}(\mathbf{X}_0, \ldots, \mathbf{X}_n)$ (or shortly $\mathbb{H}_n$) be the convex hull of those points. Almost surely, $\mathbb{H}_n$ is an n'-dimensional polytope, where $n' = \min\{d, n\}$. The main interest of this thesis is to study the normalised moments of random variable $\Delta_n = \operatorname{vol}_{n'} \mathbb{H}_n$. That is, we define the normalised volume $\underline{\Delta}_n = \underline{\operatorname{vol}}_{n'} \mathbb{H}_n = \operatorname{vol}_{n'} \mathbb{H}_n / (\operatorname{vol}_d K_d)^{n'/d}$ and

$$v_n^{(k)}(K_d) = \mathbb{E}\, \underline{\Delta}_n^k, \tag{1}$$

we refer to as the *metric moments* in K_d. Normalization ensures that the metric moments are scale invariant with respect to K_d. Moreover, if $n \geq d$, metric moments are in fact also affine invariant. For a d-ball, $v_n^{(k)}(\mathbb{B}_d)$ is known for any n, k and d (see Miles [48]). Obtaining $v_n^{(k)}(P_d)$ for various P_d is much more difficult. When $n = d$, our $\mathbb{H}_n$ is almost surely a d-simplex and thus $v_d^{(k)}(K_d)$ represents the volume moments of a random d-simplex (*volumetric moment*). Selected exact values when $K_d = T_d$ with $n = d$ are shown in Table 1 below. Throughout the thesis, we will see how these values can be obtained.

Probabilitity that a random triangle is obtuse

Another related problem is as follows: Let us select three points randomly uniformly from some given d-body K_d. What is the probability that the random triangle formed by those vertices is obtuse? We denote this probability as $\eta_{K_d K_d K_d}$ and call it the *obtusity probability* (in K_d) for short. Note that the obtusity probability is not a metric moment, however it can be approached by the same techniques.

$v_d^{(k)}(T_d)$	$d=1$	$d=2$	$d=3$	$d=4$	d
$k=1$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{13}{720}-\frac{\pi^2}{15015}$	$\frac{97}{27000}-\frac{2173\pi^2}{52026975}$	
$k=2$	$\frac{1}{6}$	$\frac{1}{72}$	$\frac{3}{4000}$	$\frac{1}{33750}$	$\frac{d!}{(d+1)^d(d+2)^d}$
$k=3$	$\frac{1}{10}$	$\frac{31}{9000}$	$\frac{1}{52500}\left(\frac{733}{240}+\frac{79\pi^2}{46189}\right)$	$\frac{1}{105^5}\left(\frac{5866197}{800}+\frac{63065881\pi^2}{3108248}\right)$	
$k=4$	$\frac{1}{15}$	$\frac{1063}{2469600}$	$\frac{871}{123480000}$	$\frac{2083}{96808320000}$	$*$
k	$\frac{2}{(k+2)(k+1)}$	$*$			

Table 1: Volumetric moment $v_d^{(k)}(T_d)$ of a random d-simplex in T_d

Crofton Reduction Technique

Often, we are faced with a problem in which the objective is to find the mean value of some functional which depends on random points selected from the interior of some polytope. An easier problem would be to find the mean value of the same functional, but now with some of the points being selected from the boundary of the polytope, from its edges or even being fixed at some of its vertices. We say the original problem has been *reduced*. Seemingly unrelated, those reduced problems are actually connected with the original one by a simple linear relation. Moreover, to maximalise the simplification, this procedure of reduction can be applied repeatedly. Although very powerful, the Crofton Reduction Technique (or CRT for short) still remains fairly unknown, even though the technique itself dates back more than one hundred years ago to Crofton and it is sometimes presented in textbooks on random geometry (Deltheil [23], Mathai [46]). The most influential to us was the PhD thesis of R. Sullivan [69]. Her thesis is fully devoted to CRT, which is presented there in its most general version of the so called *Crofton Differential Equation* (CDE). For even more general known version of CDE, see Ruben and Reed [61]).

In this thesis, we do not generalise further neither we use the most general version of CRT. Out of all affine transformations possible in CDE, we only consider simple scaling, which preserves uniformity of selection of random points. In Chapter 1, we introduce this (special) CRT and developed a notationally compact machinery enabling anyone to quickly determine the correct reduction equations (as demonstrated on countless examples). As a result, using our machinery, we are able not only to reproduce the famous results in just few lines, but also tackle problems yet unsolved. Those problems are the following:

$v_1^{(k)}(P_3)$	*mean distance* in polyhedra and its moments, respectively
$\eta(P_2)$	*obtusity probability* (of a random triangle selected from a polygon P_2)
$\Pi_{222}^{(k)}$	*perimeter moments* (of a random triangle selected from a disk $\mathbb{B}_2$)

Table 2: Moments studied using CRT in the thesis

Mean distance

Quantity $v_1^{(1)}(K_d)$ has the meaning of the mean distance between two random points selected uniformly and independently from K_d for any d. More generally, $v_1^{(k)}(K_d)$ are the corresponding *distance moments* in K_d.

In two dimensions, the distance moments $v_1^{(k)}(K_2)$ have been studied extensively. In fact, $v_1^{(k)}(P_2)$ is known for any polygon P_2 and any integer k (there are many partial results, see Bäsel [4] for P_2 being a regular polygon, but the same methods can be applied for any polygon as shown in our thesis).

In three dimensions, assuming K_3 is convex, Bonnet, Gusakova, Thäle and Zaporozhets [12] recently found a sharp optimal bounds on $v_1^{(1)}(K_3)$ normalised by the mean width of K_3. However, exact values for specific K_3 were scarce. The only exception was $v_1^{(1)}(\mathbb{B}^3)$ and $v_1^{(1)}(C_3)$. The value of the latter is due to Robbins and Bolis [60]. The consequence of author's investigation by applying the Crofton Reduction Technique (see Ruben and Reed [61]) is that in fact, $v_1^{(k)}(P_3)$ is always expressible in an exact form for any polyhedron P_3 and any integer k (P_3 also does not need to be convex). For example, the author showed

$$v_1^{(1)}(T_3^*) = \sqrt[3]{3}\left(\tfrac{\sqrt{2}}{7} - \tfrac{37\pi}{315} + \tfrac{4}{15}\arctan\sqrt{2} + \tfrac{113\ln 3}{210\sqrt{2}}\right) \approx 0.72946242, \tag{2}$$

$$v_1^{(1)}(O_3) = \sqrt[3]{\tfrac{3}{4}}\left(\tfrac{4}{105} + \tfrac{13\sqrt{2}}{105} - \tfrac{4\pi}{45} + \tfrac{109\ln 3}{630\sqrt{2}} + \tfrac{16\operatorname{arccot}\sqrt{2}}{315} + \tfrac{158\sqrt{2}\operatorname{argcoth}\sqrt{2}}{315}\right) \approx 0.65853. \tag{3}$$

The author also applied the Crofton Reduction Technique to obtain the values of $v_1^{(1)}(P_3)$ for all other regular polyhedra. The full investigation of $v_1^{(1)}(P_3)$ is covered in [7], which is also shown in Chapter 1 of this thesis.

Obtusity probability

In two dimensions, there are several known results. Obtusity probability was first solved in a disk by Woolhouse [77]. Later, Langford [42] found $\eta(C_2)$ and $\eta(P_2)$ for P_2 being a general rectangle. In this thesis, we generalised Langford's result to any convex polytope. For example, we found in an equilateral triangle T_2^*,

$$\eta(T_2^*) = \frac{25}{4} + \frac{\pi}{12\sqrt{3}} + \frac{393}{10}\ln\frac{\sqrt{3}}{2} \approx 0.7482. \tag{4}$$

In higher dimensions, apart from the d-ball (Buchta and Müller [17]), $\eta(K_d)$ was not known for any K_d with $d \geq 3$. Unfortunately, CRT becomes less useful in higher dimensions. Nevertheless, using CRT, we were still able to derive

$$\eta(C_3) = \frac{323338}{385875} - \frac{13G}{35} + \frac{4859\pi}{62720} - \frac{73\pi}{1680\sqrt{2}} - \frac{\pi^2}{105} + \frac{3\pi\ln 2}{224} - \frac{3\pi\ln(1+\sqrt{2})}{224}, \tag{5}$$

where G is the *Catalan's constant.*

Metric moments and random matrices

Moments of random determinants

It turns out that the even metric moments (even k) in T_d can be analyzed only by tools stemmed from the field of combinatorics. More concretely, there is a natural connection between metric moments and moments of particular random determinants. Let X_{ij} be independent and identically distributed random variables, from which we construct matrix $A = (X_{ij})_{n\times n}$. We denote moments of its entries X_{ij} as $m_r = \mathbb{E}X_{ij}^r$ and for their central moments, we write $\mu_r = \mathbb{E}\,(X_{ij} - m_1)^r$. By k-th *random determinant moment*, we mean the value $f_k(n) = \mathbb{E}\,(\det A)^k$. This value can be expressed as polynomials in m_r (or μ_r) or as expansion coefficients of the associated generating function $F_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} f_k(n)$. If X_{ij} follows the standard exponential distribution $\mathsf{Exp}(1)$ (see Table A.1 of distributions used throughout the thesis in Appendix A), that is if $m_j = j!$. Those special random determinant moments are then intimately connected with volumetric moments in T_d. For even k as shown by Reed [59]

$$v_n^{(k)}(T_n) = \left(\frac{n!}{(n+k)!}\right)^{n+1} f_k(n+1). \tag{6}$$

When $k = 4$, we get for the first ten exact determinant moments $f_4(n)$:

n	1	2	3	4	5	6	7
$f_4(n)$	24	960	51840	3511872	287953920	27988001280	3181325414400

n	8	9	10
$f_4(n)$	418846663065600	63399549828464640	10964925305310412800

Table 3: Fourth moment $f_4(n)$ of a random determinant with exponentially distributed entries

When $k = 6$, we get for the first six exact determinant moments $f_6(n)$:

n	1	2	3	4
$f_6(n)$	720	907200	1559900160	3340718899200

n	5	6
$f_6(n)$	8515130572800000	25161471058916966400

Table 4: Sixth moment of a random determinant with entries exponentially distributed

It is more convenient to first study random determinant moments without any restriction on the distribution of X_{ij}. Moments of Random determinants are discussed in Chapter 2. A natural generalisation of the problem is to consider a non-square matrix $U = (X_{ij})_{n\times p}$. Here, $f_k(n,p) = \mathbb{E}\,(\det U^\top U)^{k/2}$ are Gram determinant moments and $F_k(t,\omega) = \sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{(n-p)!}{n!p!}t^p\omega^{n-p}f_k(n,p)$ their generating functions. The exact expression for $F_2(t)$ and $F_2(t,\omega)$ can be easily derived using recurrences for any distribution of X_{ij}. By using those, Reed obtained the second metric moment $v_n^{(2)}(T_n)$ for any n in an exact form. For higher power moments, it is not that simple. In the case of fourth moment, Nyquist, Rice and Riordan [50] found the expression for $F_4(t)$ when $m_1 = 0$. The problem of finding the second and fourth moment of a random determinant was also studied by Fortet [32], Forsythe and Tukey [31] and Turan [73]. Later, Dembo [24] derived $F_4(t,\omega)$ when $m_1 = 0$. The general case for both $F_4(t)$ and $F_4(t,\omega)$ when $m_1 \neq 0$ remained unsolved. However, as it will be shown in Chapter 2 using several independent proofs (for the original one see [8]), we get

$$F_4(t) = \frac{e^{t(\mu_4-3\mu_2^2)}}{(1-\mu_2^2t)^3}\left((1+m_1\mu_3t)^4 + 6m_1^2\mu_2t\frac{(1+m_1\mu_3t)^2}{1-\mu_2^2t} + m_1^4t\frac{1+7\mu_2^2t+4\mu_2^4t^2}{(1-\mu_2^2t)^2}\right) \tag{7}$$

and

$$\begin{aligned} F_4(t,\omega) = \frac{e^{t(\mu_4-3\mu_2^2)}}{(1-\mu_2^2t)^2(1-\omega-\mu_2^2t)}\Bigg[&(1+m_1\mu_3t)^4 + \frac{6m_1^2\mu_2t(1+m_1\mu_3t)^2}{1-\mu_2^2t} + \frac{m_1^4t\left(1+7\mu_2^2t+4\mu_2^4t^2\right)}{(1-\mu_2^2t)^2} \\ &+ \frac{\omega m_1^2t}{1-\omega-\mu_2^2t}\left(\frac{2\mu_2(1+m_1\mu_3t)^2}{1-\mu_2^2t} + \frac{m_1^2\left(1+5t\mu_2^2+2t^2\mu_2^4\right)}{(1-\mu_2^2t)^2}\right) + \frac{2t^2\omega^2m_1^4\mu_2^2}{(1-\omega-\mu_2^2t)^2(1-\mu_2^2t)^2}\Bigg]. \end{aligned} \tag{8}$$

Hence, as a consequence of Reed's formula, the fourth metric moment $v_n^{(4)}(T_n)$ now also possess a closed form expression for any n. An obvious step further would be to find the sixth determinant moment ($k = 6$). However, this case is much harder to analyse. In collaboration with Aaron Potechin and Zelin Lv from Chicago University, we obtained the value of $f_6(n)$ and $F_6(t)$ when $m_1 = 0$ (see our joint work [5]). However, since the exponential distribution does not satisfy this criterion of $m_1 = 0$, one cannot apply those results on finding the values of $v_n^{(6)}(T_n)$. In order to overcome this, we developed the *marked permutation tables method*, which, coupled with the standard analytic combinatorics techniques, enabled us to express $F_6(t)$ finally also in the general case of $m_1 \neq 0$. The full investigation is however beyond the scope of this thesis. In Chapter 2, the method of marked permutation tables will be demonstrated to show yet another derivation of $F_4(t)$ and $F_4(t,\omega)$, as well as the special case of $F_6(t)$ with $\mu_3 = 0$. Note that expressing $F_6(t,\omega)$ for $m_1 \neq 0$ in general is still an open question.

Even metric moments

The knowledge of even determinant moments enables us to deduce even volumetric moments in polytopes. That is $v_d^{(k)}(K_d)$ for k even This connection is demonstrated in Chapter 3.

Odd metric moments in polytopes and integral geometry

Expressing the odd moments turns out be way harder since we can no longer rely on combinatorial techniques. In the scope of the thesis, we will further analyse the first-order metric moments $v_n^{(1)}(P_3)$ in dimension three and the volumetric moments $v_d^{(k)}(P_d)$ of any order k and in general dimensions. These quantities have the following geometrical interpretation:

$v_n^{(1)}(P_3)$	*mean convex hull volume* (including mean tetrahedron volume)
$v_d^{(k)}(P_d)$	*mean simplex d-volume* (and the corresponding moments)

Table 5: Odd metric moments considered in the thesis

First-order metric moments

In two dimensions, one of the classical problems of random geometry is to find the mean convex hull area and its moments, that is to express $v_n^{(k)}(K_2)$ for various K_2 and with $n \geq 2$. A lot of results were made in this direction. For example, Buchta and Reitzner [19] found a formula expressing $v_n^{(1)}(P_2)$ for any convex polygon P_2, a condensation of an endevour started by Buchta [13] earlier. Although the general Buchta and Reitzner's formula for $v_n^{(1)}(P_2)$ is not beyond the scope of this thesis, it is still rather technical so we omit it. A simplified, yet fully general, version of the same formula appeared in Zinani [78, p. 343].

Apart from a ball, not many exact results were known in three dimensions. Here, we are interested in expressing $v_n^{(k)}(K_3)$ with $n \geq 3$, which represents the k-th moment of a random volume of a convex hull of $(n+1)$ points. When $n = 3$, the convex hull is almost surely a tetrahedron, so $v_3^{(1)}(K_3)$ represents the *mean tetrahedron volume* and, more generally, $v_n^{(1)}(K_3)$ represents the mean volume of a convex hull of $(n+1)$ points.

By using the Euler polyhedral formula, Efron [26] showed how the *first-order volumetric moment* $v_n^{(1)}(K_3)$ with any $n \geq 3$ and K_3 being convex can be computed using an integral over cutting planes. Let a sample of random points $\mathbb{X}' = (\mathbf{X}_1', \mathbf{X}_2', \mathbf{X}_3')$ be selected uniformly and independently from the interior of K_3, then

$$v_3^{(1)}(K_3) = \frac{3}{5} - \mathbb{E}\left[\Gamma_3^+(\mathbb{X}')^2 + \Gamma_3^-(\mathbb{X}')^2\right], \tag{9}$$

where $\Gamma_3^+(\mathbb{X}') = \operatorname{vol}_3 K_3^+ / \operatorname{vol}_3 K_3$ and $\Gamma_3^-(\mathbb{X}') = \operatorname{vol}_3 K_3^- / \operatorname{vol}_3 K_3$ are the *section volume fraction* of the two parts $K_3^+ \sqcup K_3^-$ into which K_3 is divided by a *cutting plane* $\boldsymbol{\sigma}$ passing through the collection $\mathbb{X}' = (\mathbf{X}_1', \mathbf{X}_2', \mathbf{X}_3')$ of points $\mathbf{X}_j \in K_3$, $j \in \{1, 2, 3\}$. We may write $\boldsymbol{\sigma} = \mathcal{A}(\mathbb{X}')$, where $\mathcal{A}(\cdot)$ represents the affine hull. Using this formula, Buchta and Reitzner [18] calculated

$$v_3^{(1)}(T_3) = \frac{13}{720} - \frac{\pi^2}{15015} \tag{10}$$

Moreover, Buchta and Reitzner [20] derived $v_n^{(1)}(T_3)$ for any $n \geq 3$. Later, Zinanni [78] found

$$v_3^{(1)}(C_3) = \frac{3977}{216000} - \frac{\pi^2}{2160} \tag{11}$$

Until recently, the tetrahedron and the cube were the only polyhedra for which the value $v_3^{(1)}(P_3)$ was known in an exact form. Using the same Efron's formula, the author of this thesis extended the list of polyhedra for which the mean tetrahedron volume is known, first of which being the octahedron, where

$$v_3^{(1)}(O_3) = \frac{19297\pi^2}{3843840} - \frac{6619}{184320}. \tag{12}$$

The remaining polyhedra, for which the author found the exact value of $v_3^{(1)}(P_3)$, are triangular prism, square pyramid, rhombic dodecahedron, cuboctahedron, triakis tetrahedron and truncated tetrahedron. The exact derivation for all of those polyhedra is rather technical. In Chapter 5 of this thesis, we show a comprehensive derivation for only some of them.

Odd volumetric moments

The k-th volumetric moment $v_d^{(k)}(K_d)$ for odd k can no longer be solved using combinatorial techniques. Note that $v_d^{(k)}(K_d)$ represents the *mean simplex d-volume* and its moments, respectively. There is a natural overlap with the mean convex hull d-volume $v_n^{(1)}(K_d)$ when $k = 1$ and $n = d$ treated in Chapter 5. We already know that the *mean tetrahedron volume* $v_3^{(1)}(K_3)$ can be derived using Efron's formula. However, higher moments $v_3^{(k)}(K_3)$ (when $k = 3, 5, 7, \ldots$) were aparently not known prior to our work for any 3-body apart from $\mathbb{B}_3$.

In order to deduce higher volumetric moments (and volumetric moments in higher dimensions), we developed a method of Canonical section integral based on base-height splitting. The core finding is that any odd volumetric moment $v_d^{(k)}(K_d)$ can be written as some integral over even volumetric moments $v_{d-1}^{(k+1)}(\boldsymbol{\sigma} \cap K_d)$ on intersections of K_d with a hyperplane $\boldsymbol{\sigma}$. Eventually, we found $v_3^{(k)}(T_3)$, $v_3^{(k)}(C_3)$ and $v_3^{(k)}(O_3)$ up to $k = 5$.

In higher dimensions (and for higher moments of convex hulls of more than $d + 1$ points), there is no Efron's formula analog. However, the Canonical section integral can still be used to deduce various new results. For example, we found

$$v_4^{(1)}(T_4) = \frac{97}{27000} - \frac{2173\pi^2}{52026975} \approx 0.0031803708487 \tag{13}$$

and other odd volumetric moments beyond the Blaske problem. Our new method of Canonical section integral with comprehensive comments on deriving $v_d^{(k)}(P_d)$ for various polytopes P_d is discussed in Chapter 4.

Note that, apart of some figures cross-referencing, Chapter 5 on the first-order metric moments can be read independently from Chapter 4 on the odd volumetric moments. We decided to put those two chapters in this order mainly because we think our Canonical section integral approach is more elementary for readers new to the subject unaware of Efron's facet and section formulae. However, we recommend the readers who would wish to read the content in its historical order to read Chapter 5 prior to Chapter 4.

Metric moments in radial simplices

Finally, in the last Chapter 6, we study volumetric moments of simplices whose vertices are drawn from (special) radially symmetric (isotropic) distributions. Apart from re-derivation of the known Miles' results [48], we also introduce conditional radial simplices, in which one vertex is pinned. Those volumetric moments play essential role in random determinants, connecting Chapter 6 with Chapter 2.

Content overview

With all the chapters introduced, we finish the Introduction with a diagram of dependencies of the content of the chapters and appendices and how do they relate to each other.

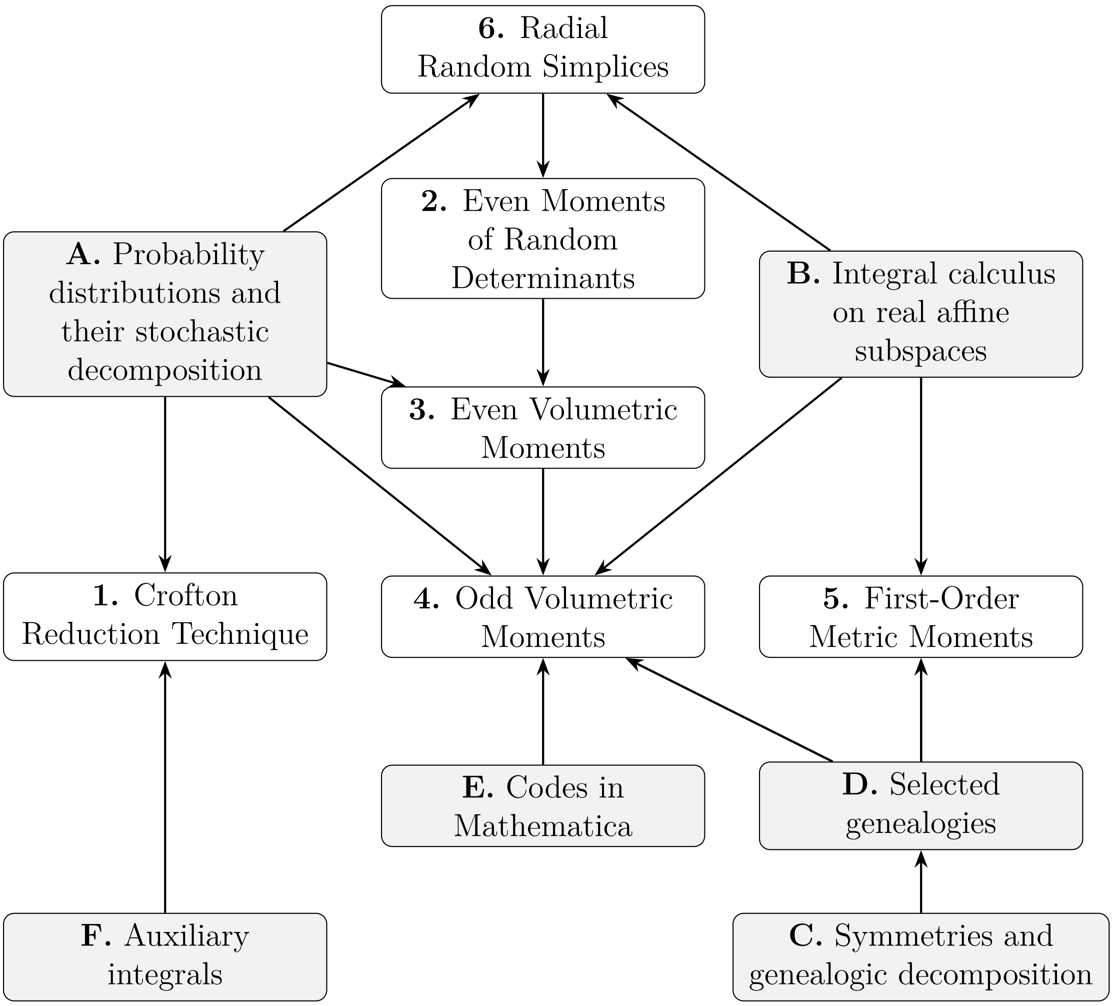


Figure 1: Logical dependencies among thesis chapters and appendices.

Acknowledgments. I would like to thank Zakhar Kabluchko for a suggestion to use the base-height splitting method in order to prove Proposition 256. I also wish to thank my supervisor Jan Rataj for discussions which turned out to be essential to deduce the odd volumetric moments by using affine Grassmannians with the correct distributions on $\gamma_\perp$.

1. Crofton Reduction Technique

For then P1 and P2 will lie on two distinct sides of the polygon, and the remark we have just made shows that when on these sides P1 and P2 are to be treated as having independent uniform distributions contributing to the shape-density with weights which are readily calculated.

— David G. Kendall [39]

1.1 Preliminaries

1.1.1 Definitions

Definition 1. A polytope $A \subset \mathbb{R}^d$ of dimension $\dim A = a \in \{0, 1, 2, \ldots, d\}$ and $a-$volume $\operatorname{vol} A$ is defined as a connected and finite union of a-dimensional simplices (forming a pure simplical complex). We say a polytope is **flat** if $\dim \mathcal{A}(A) = \dim A$, where $\mathcal{A}(A)$ stands for the affine hull of A. Note that any polytope with $a = d$ is flat automatically.

Definition 2. We denote $\mathcal{P}_a(\mathbb{R}^d)$ the set of flat polytopes of dimension a in $\mathbb{R}^d$ and denote $\mathcal{P}(\mathbb{R}^d) = \bigcup_{0 \le a \le d} \mathcal{P}_a(\mathbb{R}^d)$ the set of all flat polytopes in $\mathbb{R}^d$. Finally, we denote $\mathcal{P}_+(\mathbb{R}^d) = \mathcal{P}(\mathbb{R}^d) \setminus \mathcal{P}_0(\mathbb{R}^d)$ (flat polytopes excluding points).

Definition 3. Let $A, B \in \mathcal{P}(\mathbb{R}^d)$ and $P : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, we denote $P_{AB} = \mathbb{E}\left[P(\mathbf{X}, \mathbf{Y}) \,|\, \mathbf{X} \sim \mathsf{Unif}(A), \mathbf{Y} \sim \mathsf{Unif}(B), \text{independent}\right]$. Whenever it is unambiguous, we write P_{ab} where $a = \dim A$ and $b = \dim B$ instead of P_{AB}. If there is still ambiguity, we can add additional letters after as superscripts to distinguish between various mean values P_{ab}.

Proposition 4. *For any $A \in \mathcal{P}_a(\mathbb{R}^d)$ with $a > 0$, there exist **convex** $\partial_i A \in \mathcal{P}_{a-1}(\mathbb{R})$ (**sides** of A) such that $\partial A = \bigcup_i \partial_i A$ with pairwise intersection of $\partial_i A$ having $(a-1)$-volume equal to zero.*

Remark 5. The sides of three dimensional polytopes (polyhedra) are called **faces**.

Definition 6. Let $A \in \mathcal{P}_+(\mathbb{R}^d)$. Let $\hat{\mathbf{n}}_i$ be the outer normal unit vector of $\partial_i A$ in $\mathcal{A}(A)$, then we define a **signed distance** $h_{\mathbf{C}}(\partial_i A)$ from a given point $\mathbf{C} \in \mathcal{A}(A)$ to $\partial_i A$ as the dot product $\mathbf{v}_i^\top \hat{\mathbf{n}}_i$, where $\mathbf{v}_i = \mathbf{x}_i - \mathbf{C}$ and $x_i \in \partial_i A$ arbitrary. Note that if A is convex, the signed distance coincides with the **support function** $h(A - \mathbf{C}, \hat{\mathbf{n}}_i)$ defined for any convex domain B as $h(B, \hat{\mathbf{n}}_i) = \sup_{\mathbf{b} \in B} \mathbf{b}^\top \hat{\mathbf{n}}_i$.

Remark 7. The signed distance has another geometric interpretation. Put $\mathbf{C} = \mathbf{0}$ (the origin) and $r = 1 + \varepsilon$ (with ε small). Denote $\int_{(B,A)} = \int_{B/A} - \int_{A/B}$, by linearity $\int_B = \int_A + \int_{(B,A)}$. Hence

$$\operatorname{vol} rA = \int_{rA} \mathrm{d}x = \int_A \mathrm{d}x + \int_{(rA,A)} \mathrm{d}x = \operatorname{vol} A + \varepsilon \sum_i \operatorname{vol}(\partial_i A) h_{\mathbf{0}}(\partial_i A) + O(\varepsilon^2), \tag{1.1}$$

or in other words, $\mathrm{d} \operatorname{vol} rA / \mathrm{d}r|_{r=1} = \sum_i \operatorname{vol}(\partial_i A) h_{\mathbf{0}}(\partial_i A)$ for A arbitrary (possibly non-convex).

Definition 8. Let $A \in \mathcal{P}_+(\mathbb{R}^d)$ with $a = \dim A$. Even though $\partial A \notin \mathcal{P}(\mathbb{R}^d)$ (it is not flat), we extend the definition of $P_{\partial_i A\, B}$ to $P_{\partial A\, B}$ as the weighted mean

$$P_{\partial A\, B} = \sum_i w_i P_{\partial_i A\, B} \tag{1.2}$$

with weights w_i (may be also negative) equal to

$$w_i = \frac{\operatorname{vol} \partial_i A}{a \operatorname{vol} A} h_{\mathbf{C}}(\partial_i A) \tag{1.3}$$

implicitly dependent on a point (called the scaling point) $\mathbf{C} \in \mathcal{A}(A)$.

Remark 9. Note that this definition is not dependent on the number of sides of A. That is, if we artificially splif one side $\partial_i A$ into two sides, the weighted mean stays the same. This feature enables us to extend the definition to any convex a-bodies (and their unions) as well. Let λ_{a-1} be the uniform surface measure on such body A. Then for any scaling point $\mathbf{C} \in \mathcal{A}(A)$,

$$P_{\partial AB} = \frac{1}{a \operatorname{vol} A} \int_{\partial A} P_{\mathbf{x}B} h_{\mathbf{C}}(\mathbf{x}) \lambda_{a-1}(\mathrm{d}\mathbf{x}), \tag{1.4}$$

where $h_{\mathbf{C}}(\mathbf{x})$ is the support function of A evaluated in $\mathbf{x}$ and centered at $\mathbf{y}$ and

$$P_{\mathbf{x}B} = \mathbb{E}\left[P(\mathbf{x}, \mathbf{Y}) \mid \mathbf{Y} \sim \mathsf{Unif}(B)\right] \tag{1.5}$$

by definition.

Definition 10. We say a function $P : (\mathbb{R}^d)^n \to \mathbb{R}$ is a *homogeneous functional* of order $p \in \mathbb{R}$, if there exists $\tilde{P} : (\mathbb{R}^d)^{n-1} \to \mathbb{R}$ such that $P(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n) = \tilde{P}(\mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_1, \dots, \mathbf{x}_n - \mathbf{x}_1)$ for all $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^d$ and $\tilde{P}(r\mathbf{u}_2, \dots, r\mathbf{u}_n) = r^p \tilde{P}(\mathbf{u}_2, \dots, \mathbf{u}_n)$ for all $\mathbf{u}_2, \dots, \mathbf{u}_n \in \mathbb{R}^d$ and all $r > 0$. We say P is *symmetric* if it is invariant with respect to permutations of its arguments. Finally, if P is a functional of two points, we say it is *bivariate*. If it depends of more points, we say it is *multivariate*.

Remark 11. Note that if P is symmetric, then $P_{AB} = P_{BA}$ for any domains A and B.

Example 12. If $P = L^p$, or more precisely $P(x, y) = L^p(x, y) = \|x - y\|^p$, then P is symmetric and homogeneous of $\dim P = p$ and with $\tilde{P}(x) = \|x\|^p$. Whenever $P = L^p$, we will use P_{AB} and $L^{(p)}_{AB}$ interchangeably throughout the sections on mean distances.

1.1.2 Bivariate Crofton Reduction Technique

Lemma 13 (Bivariate Crofton Reduction Technique)**.** *Let $P : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be homogeneous of order p and $A, B \in \mathcal{P}(\mathbb{R}^d)$. Provided that $\mathcal{A}(A) \cap \mathcal{A}(B)$ is non-empty, then for any $\mathbf{C} \in \mathcal{A}(A) \cap \mathcal{A}(B)$ it holds*

$$pP_{AB} = a(P_{\partial A\, B} - P_{AB}) + b(P_{A\, \partial B} - P_{AB}). \tag{1.6}$$

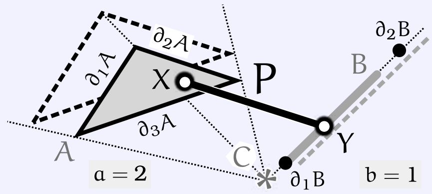


Figure 1.1: Bivariate Crofton Reduction Technique

Proof. The formula is a special case of the extension of the Crofton theorem by Ruben and Reed [61], although it is fairly simple to derive directly. Let $r = 1 + \varepsilon$ and put $\mathbf{C} = \mathbf{0}$ (the origin) without loss of generality. The key is to express $P_{rA,rB}$ in two different ways:

- By definition,

$$\begin{aligned} P_{rA,rB} &= \mathbb{E}\left[P(\mathbf{X}, \mathbf{Y}) \mid \mathbf{X} \sim \mathsf{Unif}(rA), \mathbf{Y} \sim \mathsf{Unif}(rB)\right] \\ &= \mathbb{E}\left[P(r\mathbf{X}', r\mathbf{Y}') \mid \mathbf{X}' \in A, \mathbf{Y}' \sim \mathsf{Unif}(B)\right] \\ &= r^p\, \mathbb{E}\left[P(\mathbf{X}', \mathbf{Y}') \mid \mathbf{X}' \sim \mathsf{Unif}(A), \mathbf{Y}' \sim \mathsf{Unif}(B)\right] \\ &= r^p P_{AB} = P_{AB} + \varepsilon p P_{AB} + O(\varepsilon^2). \end{aligned}$$

- On the other hand,

$$\begin{aligned} &\operatorname{vol} rA \operatorname{vol} rB\, P_{rA,rB} = \operatorname{vol} rA \operatorname{vol} rB\, \mathbb{E}\left[P(\mathbf{X},\mathbf{Y}) \mid \mathbf{X} \sim \mathsf{Unif}(rA), \mathbf{Y} \sim \mathsf{Unif}(rB)\right] \\ &= \int_{rA}\int_{rB} P(x,y)\, \mathrm{d}x\mathrm{d}y = \int_{A}\int_{B} P(x,y)\, \mathrm{d}x\mathrm{d}y + \int_{(rA,A)}\int_{B} P(x,y)\, \mathrm{d}x\mathrm{d}y \\ &+ \int_{A}\int_{(rB,B)} P(x,y)\, \mathrm{d}x\mathrm{d}y + \int_{(rA,A)}\int_{(rB,B)} P(x,y)\, \mathrm{d}x\mathrm{d}y \\ &= \operatorname{vol} A \operatorname{vol} B\, P_{AB} + \varepsilon \operatorname{vol} B \sum_i \operatorname{vol}(\partial_i A) h_O(\partial_i A) P_{\partial_i A\, B} \\ &+ \varepsilon \operatorname{vol} A \sum_j \operatorname{vol}(\partial_j B) h_O(\partial_j B) P_{A\, \partial_j B} + O(\varepsilon^2). \end{aligned}$$

Comparing the ε terms of both expressions and using Remark 7, we get the statement of the lemma. If either of $\dim A$ or $\dim B$ is zero, the lemma holds too. ■

To find the expectation of P, in the first step, we choose $A = K$ and $B = K$, where K is a given d-polytope. Since the affine hulls of both A and B fill the whole space $\mathbb{R}^d$, any point in $\mathbb{R}^d$ can be selected for $\mathbf{C}$. We then employ the reduction technique to express P_{AB} in $P_{A'B'}$ where A' and B' have smaller dimensions then A and B. The pairs of various A' and B' we encounter we call **configurations**. The process is repeated until the affine hull intersection of A' and B' is empty. In that case, we have reached an **irreducible** configuration.

1.1.3 Multivariate Crofton Reduction Technique

Let us instead consider multivariate functionals P (dependent on more that only two points). One example is area, volume or obtusity. CRT naturally generalises.

Definition 14. Let $P = P(x_1, x_2, \dots, x_n)$ be a homogenous function of n points. We define $P_{A_1 A_2 \dots A_n} = \mathbb{E}\left[P(\mathbf{X}_1, \dots, \mathbf{X}_n) \,|\, \mathbf{X}_1 \sim \mathsf{Unif}(A_1), \dots, \mathbf{X}_n \sim \mathsf{Unif}(A_n)\right]$, where $A_j, j = 1, \dots, n$ are flat domains from which the points $\mathbf{X}_j$ are selected randomly uniformly (according to distribution $\mathsf{Unif}(A_j)$).

Lemma 15 (Multivariate Crofton Reduction Technique)**.** *Let $P : (\mathbb{R}^d)^n \to \mathbb{R}$ be homogeneous of order p and $A_1, \dots, A_n \in \mathcal{P}(\mathbb{R}^d)$, $a_i = \dim A_i$, then for any $\mathbf{C} \in \bigcap_{1 \le i \le n} \mathcal{A}(A_i)$ (scaling point) it holds*

$$
\begin{aligned}
pP_{A_1 A_2 \dots A_n} = a_1(P_{\partial A_1\, A_2 \dots A_n} - P_{A_1 \dots A_n}) + a_2(P_{A_1 \partial A_2 \dots A_n} - P_{A_1 \dots A_n}) \\
+ \dots + a_n(P_{A_1 A_2 \dots \partial A_n} - P_{A_1 \dots A_n}).
\end{aligned}
\tag{1.7}
$$

Remark 16. Symmetry of P in points $\mathbf{X}_1, \dots, \mathbf{X}_n$ is not required for CRT to hold. However, we often assume so. As a result, $P_{A_1, \dots, A_n}$ will be invariant with respect to permutations of $A_1, \dots, A_n$.

1.1.4 Functional Crofton Reduction Technique

The most general version of Crofton Reduction Technique is availible for functions of homogeneous functionals.

Definition 17. Let P be a multivariate functional of points $\mathbf{X}_i, i = 1, \dots, n$ selected uniformly from domains A_i. Then for any function $\psi : \mathbb{R} \to \mathbb{R}$, we define

$$
\psi_{A_1 A_2 \dots A_n} = \mathbb{E}\left[\psi(P)\right] = \mathbb{E}\left[\psi(P(\mathbf{X}_1, \dots, \mathbf{X}_n)) \mid \mathbf{X}_i \sim \mathsf{Unif}(A_i)\right]. \tag{1.8}
$$

If ψ is moreover differentiable, we denote

$$
\psi^*_{A_1 A_2 \dots A_n} = \mathbb{E}\left[P\psi'(P)\right] = \mathbb{E}\left[P\psi'(P(\mathbf{X}_1, \dots, \mathbf{X}_n)) \mid \mathbf{X}_i \sim \mathsf{Unif}(A_i)\right]. \tag{1.9}
$$

Lemma 18 (Functional Crofton Reduction Technique)**.** *Let $P : (\mathbb{R}^d)^n \to \mathbb{R}$ be homogeneous of order p and $A_1, \dots, A_n \in \mathcal{P}(\mathbb{R}^d)$, $a_i = \dim A_i$ and there exists $\mathbf{C} \in \bigcap_{1 \le i \le n} \mathcal{A}(A_i)$. Then for any differentiable function $\psi : \mathbb{R} \to \mathbb{R}$, we have*

$$
\begin{aligned}
p\,\psi^*_{A_1 A_2 \dots A_n} = a_1(\psi_{\partial A_1\, A_2 \dots A_n} - \psi_{A_1 \dots A_n}) + a_2(\psi_{A_1 \partial A_2 \dots A_n} - \psi_{A_1 \dots A_n}) \\
+ \dots + a_n(\psi_{A_1 A_2 \dots \partial A_n} - \psi_{A_1 \dots A_n}).
\end{aligned}
\tag{1.10}
$$

Proof. We show how we can derive the the lemma for analytic functions. Let $Q = P^k$, then Q is homogeneous of order kp, so Equation (1.7) turns into

$$
kp\,Q_{A_1, \dots, A_n} = a_1(Q_{\partial A_1, A_2, \dots, A_n}) + \dots + a_n(Q_{A_1, A_2, \dots, \partial A_n}). \tag{1.11}
$$

Note that the left hand side may be written as $\mathbb{E}\left[pP\frac{\partial Q}{\partial P} \mid \mathbf{X}_i \in \mathsf{Unif}(A_i)\right]$. Any analytic function of P can be written in the form $\psi(P) = \sum_{k=0}^{\infty} \alpha_k P^k$ for some constants α_k. Multiplying Equation (1.11) by α_k and summing up over all k and by linearity, we get Equation (1.10), which finishes the proof. The lemma however extends beyond analytic functions. See Ruben and Reed [61] for more general treatment. ■

1.1.5 CRT for distributions, Dirac kernel method

A direct consequence of the functional Crofton Reduction Technique is the ability of relating distributions between each other via simple differential equations. We have the following result:

Definition 19. Let P be a multivariate functional of points $\mathbf{X}_i, i = 1, \dots, n$ selected uniformly from domains A_i. Viewed as a random variable, we write for the Cumulative Density Function (CDF) of P,

$$F_{A_1A_2\dots A_n}(\lambda) = \mathbb{P}\left[P \le \lambda\right] = \mathbb{P}\left[P(\mathbf{X}_1, \dots, \mathbf{X}_n) \le \lambda \mid \mathbf{X}_i \sim \mathsf{Unif}(A_i)\right] \tag{1.12}$$

and for its Probability Density Function (PDF), $f_{A_1A_2\dots A_n}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} F_{A_1A_2\dots A_n}(\lambda)$.

In what follows, we assume that the PDF always almost surely exists and it is (piecewise) continuous.

Lemma 20 (Distributional Crofton Reduction Technique)**.** *Let* $P : (\mathbb{R}^d)^n \to \mathbb{R}$ *be homogeneous of order* p *and* $A_1, \dots, A_n \in \mathcal{P}(\mathbb{R}^d)$, $a_i = \dim A_i$ *and there exist* $\mathbf{C} \in \bigcap_{1 \le i \le n} \mathcal{A}(A_i)$. *Then we have for the CDF of the random variable* P,

$$\begin{aligned} -p\,\lambda F'_{A_1A_2\dots A_n}(\lambda) &= a_1(F_{\partial A_1\, A_2\dots A_n}(\lambda) - F_{A_1\dots A_n}(\lambda)) \\ &\quad + a_2(F_{A_1 \partial A_2\dots A_n}(\lambda) - F_{A_1\dots A_n}(\lambda)) + \cdots \\ &\quad + a_n(F_{A_1A_2\dots \partial A_n}(\lambda) - F_{A_1\dots A_n}(\lambda)), \end{aligned} \tag{1.13}$$

or equivalently, by differentiation, we get for its PDF,

$$\begin{aligned} -p\,(\lambda f_{A_1A_2\dots A_n}(\lambda))' &= a_1(f_{\partial A_1\, A_2\dots A_n}(\lambda) - f_{A_1\dots A_n}(\lambda)) \\ &\quad + a_2(f_{A_1 \partial A_2\dots A_n}(\lambda) - f_{A_1\dots A_n}(\lambda)) + \cdots \\ &\quad + a_n(f_{A_1A_2\dots \partial A_n}(\lambda) - f_{A_1\dots A_n}(\lambda)). \end{aligned} \tag{1.14}$$

Proof. By definition, $F_{A_1,\dots,A_n}(\lambda) = \mathbb{P}\left[P(\mathbf{X}_1, \dots, \mathbf{X}_n) \le \lambda\right]$. Scaling the domains A_i by some positive $r = 1 + \varepsilon$ and by homogeneity of P,

$$\begin{aligned} F_{rA_1,\dots,rA_n}(\lambda) &= \mathbb{P}\left[P(r\mathbf{X}_1, \dots, r\mathbf{X}_n) \le \lambda\right] = \mathbb{P}\left[P(\mathbf{X}_1, \dots, \mathbf{X}_n) \le r^{-p}\lambda\right] \\ &= F_{A_1,\dots,A_n}(r^{-p}\lambda) = F_{A_1,\dots,A_n}(\lambda) - p\lambda\varepsilon F'_{A_1,\dots,A_n}(\lambda) + o(\varepsilon), \end{aligned} \tag{1.15}$$

which gives the left-hand side of Equation (1.13) (the coefficient of ε). The right side is obtained by expanding $\mathbb{P}\left[P(r\mathbf{X}_1, \dots, r\mathbf{X}_n) \le \lambda\right]$ in ε as a sum over ∂A_i boundaries, which is an argument equivalent to the one shown in the proof of Lemma 13. ■

Alternatively (and perhaps less rigorously), Lemma 20 is a direct consequence of the Functional Crofton Reduction Technique. We just select $\psi(P) = \mathbb{1}_{P\leq\lambda}$, then

$$\mathbb{E}\left[\psi(P)\right] = \mathbb{E}\left[\mathbb{1}_{P\leq\lambda}\right] = \mathbb{P}\left[P \leq \lambda\right] = F(\lambda). \tag{1.16}$$

The right hand side of Equation (1.13) is obvious since we now have $\psi_{A_1,\ldots,A_n} = F_{A_1,\ldots,A_n}(\lambda)$. To show the left hand side, we have, formally

$$\psi^*(P) = P\psi'(P) = -P\delta(P-\lambda) = -\lambda\delta(P-\lambda), \tag{1.17}$$

where δ is the *Dirac delta function*. Note that, formally, the probability density function (PDF) $f(\lambda)$ of random variable P can be written as

$$f(\lambda) = \mathbb{E}\left[\delta(P-\lambda)\right] \tag{1.18}$$

from which $\mathbb{E}\left[\psi^*(P)\right] = -\lambda f(\lambda)$, so $\psi^*_{A_1A_2\ldots A_n} = -\lambda f_{A_1A_2\ldots A_n}(\lambda)$.

1.1.6 CRT for joint densities of more functionals

Definition 21. Let $P = P(\mathbf{X}_1,\ldots,\mathbf{X}_n)$ and $P' = P'(\mathbf{X}_1,\ldots,\mathbf{X}_n)$ be multivariate functionals of points $\mathbf{X}_i, i = 1,\ldots,n$ selected uniformly from domains A_i. Viewed as random variables, we write $f_{A_1A_2\ldots A_n}(\lambda,\lambda')$ for their Joint Probability Density Function (JPDF). That is, for any measurable $M \subset \mathbb{R}^2$, we have

$$\mathbb{P}\left[(P,P')^\top \in M\right] = \int_M f(\lambda,\lambda')\mathrm{d}\lambda\mathrm{d}\lambda'. \tag{1.19}$$

Similarly, for their Joint Cumulative Density Function (JCDF),

$$F_{A_1A_2\ldots A_n}(\lambda,\lambda') = \mathbb{P}\left[P \leq \lambda, P' \leq \lambda' \mid \mathbf{X}_i \sim \mathsf{Unif}(A_i)\right]. \tag{1.20}$$

Note that $F_{A_1,\ldots,A_n}(\lambda,\lambda') = \int_{-\infty}^{\lambda'}\int_{-\infty}^{\lambda} f_{A_1,\ldots,A_n}(t,t')\,\mathrm{d}t\mathrm{d}t'$.

Similarly as in the case of the ordinary one-variable Distributional Crofton Reduction Technique, we can relate the joint CDF with the CDFs containing the boundaries. We state the following lemma (without proof)

Lemma 22 (Joint Distributional Crofton Reduction Technique)**.** *Let $P, P' : (\mathbb{R}^d)^n \to \mathbb{R}$ be homogeneous functionals of order p and p', respectively, and $A_1, \ldots, A_n \in \mathcal{P}(\mathbb{R}^d)$, $a_i = \dim A_i$ and there exists $\mathbf{C} \in \bigcap_{1\le i\le n} \mathcal{A}(A_i)$. Then we have for the JCDF of the random variables P, P',*

$$\begin{aligned} &- p\,\lambda \frac{F_{A_1A_2\ldots A_n}(\lambda,\lambda')}{\partial\lambda} - p'\,\lambda' \frac{F_{A_1A_2\ldots A_n}(\lambda,\lambda')}{\partial\lambda'} \\ &= a_1(F_{\partial A_1\,A_2\ldots A_n}(\lambda,\lambda') - F_{A_1\ldots A_n}(\lambda,\lambda')) \\ &+ a_2(F_{A_1\partial A_2\ldots A_n}(\lambda,\lambda') - F_{A_1\ldots A_n}(\lambda,\lambda')) + \cdots \\ &+ a_n(F_{A_1A_2\ldots\partial A_n}(\lambda,\lambda') - F_{A_1\ldots A_n}(\lambda,\lambda')), \end{aligned} \tag{1.21}$$

or equivalently, by differentiation, we get for its JPDF,

$$\begin{aligned} &- p\,\frac{\partial(\lambda f_{A_1A_2\ldots A_n}(\lambda,\lambda'))}{\partial\lambda} - p'\,\frac{\partial(\lambda' f_{A_1A_2\ldots A_n}(\lambda,\lambda'))}{\partial\lambda'} \\ &= a_1(f_{\partial A_1\,A_2\ldots A_n}(\lambda,\lambda') - f_{A_1\ldots A_n}(\lambda,\lambda')) \\ &+ a_2(f_{A_1\partial A_2\ldots A_n}(\lambda,\lambda') - f_{A_1\ldots A_n}(\lambda,\lambda')) + \cdots \\ &+ a_n(f_{A_1A_2\ldots\partial A_n}(\lambda,\lambda') - f_{A_1\ldots A_n}(\lambda,\lambda')). \end{aligned} \tag{1.22}$$

1.2 Overview of functionals

What follows is an overview of the functionals treated in this thesis. Detailed derivations are found in subsequent sections dedicated to each functional. We also discuss which results were known and which are novel.

1.2.1 Distance

Denoted as $L = L(\mathbf{X},\mathbf{Y}) = \|\mathbf{X}-\mathbf{Y}\|$, the distance between two random points $\mathbf{X},\mathbf{Y} \sim \mathsf{Unif}(K_d)$, where $K_d \subset \mathbb{R}^d$, is among the most natural bivariate homogeneous symmetric functionals we might think of. Note that the order of L is exactly one. In order to get higher order functionals, we may put $P = L^p$, which has order p. In the following sections, we often just assume, if not stated differently, that $P = L^p$. The expected value of P is then the p-th moment of random distance of two points $\mathbf{X},\mathbf{Y}$. In our notation, we write for the moments

$$L_{dd}^{(p)} = L_{K_dK_d}^{(p)} = \mathbb{E}\left[L(\mathbf{X},\mathbf{Y})^p \middle| \mathbf{X},\mathbf{Y} \sim \mathsf{Unif}(K_d)\right]. \tag{1.23}$$

Those moments are related with the metric moments defined in the Introduction via the following normalisation

$$v_1^{(p)}(K_d) = \frac{L_{dd}^{(p)}}{(\mathrm{vol}_d\, K_d)^{p/d}}. \tag{1.24}$$

Two dimensions

The functional of distance in two dimensions is fairly understood and has been extensively studied (see Bäsel [4] and references therein). Recently, Uve Bäsel [4] expressed $L_{22}^{(p)}$ in P_2 being a regular polygon in $\mathbb{R}^2$ (that is, $d = 2$). We will discuss how we can re-derive those results using Crofton Reduction Technique (CRT). First-order metric moments of distance in selected K_2 are shown in Table 1.1 below.

K_2		numerical value	$v_1^{(1)}(K_2)$
$\mathbb{B}_2$	disk	0.510826	$\frac{128}{45\pi^{3/2}}$
C_2	square	0.521405	$\frac{2}{15} + \frac{\sqrt{2}}{15} + \frac{1}{3}\operatorname{argsinh}(1)$
T_2^*	equilateral triangle	0.554364	$\frac{4+3\ln 3}{10\sqrt[4]{3}}$

Table 1.1: Mean distance in various 2-bodies with unit area

Three dimensions

Let K_3 be a polyhedron. Even-power moments $L_{33}^{(p)}$ are trivial to compute. When p is odd, the value $L_{33}^{(p)}$ has been known in the exact form only for K_3 being a ball (trivial) or (for $p = 1$) a unit cube [60], known as the so called Robbins constant

$$\mathbb{E}\,[L] = \frac{4}{105} + \frac{17\sqrt{2}}{105} - \frac{2\sqrt{3}}{35} - \frac{\pi}{15} + \frac{1}{5}\operatorname{argcoth}\sqrt{2} + \frac{4}{5}\operatorname{argcoth}\sqrt{3} \approx 0.66170718. \tag{1.25}$$

Recently, Bonnet, Gusakova, Thäle and Zaporozhets [12] found a sharp optimal bound on the *normalised mean distance* $\Gamma_{33} = L_{33}/V_1(K_3)$ in convex and compact K_3, where $V_1(K_3) = 2\int_{\mathbb{S}^2} \|\operatorname{proj}_{\hat{\mathbf{n}}} K_3\|\,\mathrm{d}\hat{\mathbf{n}}$ is the first intrinsic volume of K_3. A special case of their result in three dimensions gives $\frac{5}{28} < \Gamma_{33} < \frac{1}{3}$.

As stated in [12], although the first intrinsic volume is easy to express in any polyhedron, number of examples for which an exact formula for L_{33} is available is rather limited. We will show that this might not be the case and indeed one can find L_{33} (and all natural moments $L_{33}^{(p)}$) in an exact form easily for any K_3 being a polyhedron. The main result of our own investigation is thus the following theorem:

Theorem 23. *For any given polyhedron, the mean distance between two of its inner points selected at random can always be expressed in terms of elementary functions of the location of its vertices and sides. The same holds for all other natural moments.*

Remark 24. By elementary functions, we mean a closed field of functions containing radicals, exponential, trigonometric, and hyperbolic functions and their inverses.

The theorem is solely based on the Crofton Reduction Technique, see [25, 61], which under certain conditions enables us to express $L_{33}^{(p)}$ as some linear combination of $L_{AB}^{(p)} = \mathbb{E}\,[L^p|\,\mathbf{X} \in A, \mathbf{Y} \in B]$ over domains A and B with smaller dimension than that of K_3. The theorem then follows from the observation that we are able to decompose all the corresponding terms into computable double integrals, as we will see in Section 1.4.2. In fact, very recently, using different methods, Ciccariello [21] showed that the so-called chord-length distribution, which is related to the distribution of L, can also be expressed in terms of elementary functions in any polyhedron K_3.

Exact mean distances in regular polyhedra

The table below summarises all new results of exact mean distance in various polyhedra. For completeness, the previously known cases of a ball and a cube have been added as well. Each solid K_3 has $\operatorname{vol} K_3 = 1$. This normalisation ensures the right column displays exactly the first distance metric moment $v_1^{(1)}(K_3) = L_{33}/\sqrt[3]{\operatorname{vol}_3 K_3}$. As usual, $\phi = (1+\sqrt{5})/2$ is the Golden ratio.

K_3	$v_1^{(1)}(K_3)$
ball 0.63807479	$\frac{18}{35}\sqrt[3]{\frac{6}{\pi}}$
icosahedron 0.64131249	$\frac{1}{2}\sqrt[3]{\frac{9}{5}-\frac{3}{\sqrt{5}}}\left(\frac{197}{525}+\frac{239}{525\sqrt{5}}-\frac{44}{525}\sqrt{2+\frac{2}{\sqrt{5}}}-\frac{\left(17226+6269\sqrt{5}\right)\pi}{157500} - \frac{\left(2186+1413\sqrt{5}\right)\operatorname{arccot}\phi}{15750}+\frac{\left(82-75\sqrt{5}\right)\operatorname{arccot}\left(\phi^2\right)}{5250}+\frac{4\left(2139+881\sqrt{5}\right)\operatorname{argcsch}\phi}{7875} + \frac{\left(15969+7151\sqrt{5}\right)\operatorname{argcoth}\phi}{12600}+\frac{\left(4449-1685\sqrt{5}\right)\ln 3}{42000}-\frac{\left(75783+37789\sqrt{5}\right)\ln 5}{252000}\right)$
dodecahedron 0.64252068	$\frac{1}{\sqrt[3]{30+14\sqrt{5}}}\left(\frac{1516}{1575}+\frac{2\sqrt{\frac{2}{5}}}{45}-\frac{124\sqrt{\frac{3}{5}}}{175}-\frac{71\sqrt{2}}{1575}-\frac{12\sqrt{3}}{35}+\frac{342}{175\sqrt{5}}+\frac{493\pi}{23625} + \frac{67\pi}{945\sqrt{5}}+\frac{\left(397-244\sqrt{5}\right)\operatorname{arccot}2}{18900}+\frac{\left(24023+11788\sqrt{5}\right)\left(\arccos\frac{2}{3}-\arccos\frac{1}{3}\right)}{94500} - \frac{\left(461+212\sqrt{5}\right)\left(\arccos\frac{23}{41}+\arccos\frac{39}{41}\right)}{1000}-\frac{\left(1031+521\sqrt{5}\right)\operatorname{argcosh}\frac{13}{3}}{75600} + \frac{\left(367+163\sqrt{5}\right)\operatorname{argcosh}9}{16800}+\frac{\left(22197+8149\sqrt{5}\right)\left(\operatorname{argcosh}\frac{121}{41}-\operatorname{argcosh}\frac{57}{41}\right)}{84000} + \frac{\left(15763+7063\sqrt{5}\right)\left(\operatorname{argcosh}\frac{7}{3}-\operatorname{argcosh}3\right)}{21000}+\frac{\left(288889+129739\sqrt{5}\right)\ln 3}{378000} + \frac{2\left(423+187\sqrt{5}\right)(\operatorname{argcosh}4-\operatorname{argcosh}2)}{875}+\frac{\left(109-3143\sqrt{5}\right)\ln 5}{151200}\right)$
octahedron 0.65853073	$\sqrt[3]{\frac{3}{4}}\left(\frac{4}{105}+\frac{13\sqrt{2}}{105}-\frac{4\pi}{45}+\frac{109\ln 3}{630\sqrt{2}}+\frac{16\operatorname{arccot}\sqrt{2}}{315}+\frac{158\operatorname{argcoth}\sqrt{2}}{315}\sqrt{2}\right)$
cube, [60] 0.66170718	$\frac{4}{105}+\frac{17\sqrt{2}}{105}-\frac{2\sqrt{3}}{35}-\frac{\pi}{15}+\frac{1}{5}\operatorname{argcoth}\sqrt{2}+\frac{4}{5}\operatorname{argcoth}\sqrt{3}$
tetrahedron 0.72946242	$\sqrt[3]{3}\left(\frac{\sqrt{2}}{7}-\frac{37\pi}{315}+\frac{4}{15}\arctan\sqrt{2}+\frac{113\ln 3}{210\sqrt{2}}\right)$

Table 1.2: Mean distance in various solids of unit volume, $\phi = (1+\sqrt{5})/2$ is the Golden ratio.

Normalised mean distance

We could select normalisation in which $V_1(K_3) = 1$ rather than $\operatorname{vol} K_3 = 1$. In order to express the normalised mean distance Γ_{33}, we just rescale our values in Table 1.2 by $\sqrt[3]{\operatorname{vol} K_3}/V_1(K_3)$. Both $\operatorname{vol} K_3$ and $V_1(K_3)$ can be expressed easily. The following Table 1.3 shows the volume of the regular polyhedra with edge length equal to l. To express $V_1(K_3)$, we use the formula $V_1(K_3) = \frac{1}{2\pi}\sum_i l_i(\pi-\delta_i)$, where the sum is carried over all edges E_i of K_3 having length l_i and dihedral angle δ_i. The following table shows the value of $V_1(K_3)$ for the five regular polyhedra (Platonic solids) with common edge length $l_i = l$ for all i.

When K_3 is a ball, $\sqrt[3]{\operatorname{vol} K_3}/V_1(K_3) = \frac{1}{6}\sqrt[3]{\frac{\pi}{6}}$ trivially. Finally, performing the scaling, in Table 1.5 we show numerical values of Γ_{33} for the same solids K_3 as in Table 1.2. The lower and the upper bound of Γ_{33} for K_3 convex compact (based

K_3	tetrahedron	cube	octahedron	dodecahedron	icosahedron
$\frac{\operatorname{vol} K_3}{l^3}$	$\frac{\sqrt{2}}{12}$	1	$\frac{\sqrt{2}}{3}$	$\frac{15+7\sqrt{5}}{4}$	$\frac{5\left(3+\sqrt{5}\right)}{12}$

Table 1.3: First intrinsic volume of Platonic solids with unit edge length

K_3	tetrahedron	cube	octahedron	dodecahedron	icosahedron
$\frac{V_1(K_3)}{l}$	$\frac{3\arccos\left(-\frac{1}{3}\right)}{\pi}$	6	$\frac{6\arccos\frac{1}{3}}{\pi}$	$\frac{15\arctan 2}{\pi}$	$\frac{15\arcsin\frac{2}{3}}{\pi}$

Table 1.4: First intrinsic volume of Platonic solids with unit edge length

on [12]) are set to 5/28 and 1/3, respectively.

K_3	lower bound	tetrahedron	octahedron	cube
Γ_{33}	0.17857143	0.19601928	0.21800285	0.22056906
K_3	icosahedron	dodecahedron	ball	upper bound
Γ_{33}	0.23872552	0.23963024	0.25714286	0.33333333

Table 1.5: Normalised mean distance in Platonic solids with unit first intrinsic volume

1.2.2 Triangle area

Let $\mathbf{X},\mathbf{Y},\mathbf{Z} \sim \mathsf{Unif}(K_d), K_d \subset \mathbb{R}^d$. We denote $S = S(\mathbf{X},\mathbf{Y},\mathbf{Z})$ as the area of a triangle whose vertices are points $\mathbf{X},\mathbf{Y},\mathbf{Z}$. The area functional S is trivariate, symmetric and homogeneous of order two. For its moments, we write

$$S_{ddd}^{(k)} = \mathbb{E}\left[S(\mathbf{X},\mathbf{Y},\mathbf{Z})^k \mid \mathbf{X},\mathbf{Y},\mathbf{Z} \sim \mathsf{Unif}(K_d)\right]. \tag{1.26}$$

The question of obtaining S_{ddd} makes sense only when $d \geq 2$. Further more, we can normalise it such that the solid from which the points are picked is of unit d-volume. As a result, we get the metric moment of area

$$v_2^{(p)}(K_d) = \frac{S_{ddd}^{(p)}}{(\operatorname{vol}_d K_d)^{2p/d}}. \tag{1.27}$$

First-order metric moments of area for selected K_d are shown in Table 1.6 below. Apart from the d-ball (Miles [48]), $v_2^{(1)}(K_d)$ is not known for any K_d with $d \geq 3$.

1.2.3 Obtusity indicator

We can use CRT to deduce the probability $\eta(K_d)$ that a random triangle whose vertices $\mathbf{X},\mathbf{Y},\mathbf{Z}$ are independently selected from $\mathsf{Unif}(K_d), K_d \subset \mathbb{R}^d$ is obtuse. In order to do that, the only thing we need is to consider a trivariate functional

K_d		numerical value	$v_2^{(1)}(K_d)$
$\mathbb{B}_2$	disk	0.0739	$\frac{35}{48\pi^2}$
$\mathbb{B}_3$	ball	0.1413	$\frac{9}{154}\sqrt[3]{\frac{9\pi}{2}}$
T_2	triangle	0.0833	$\frac{1}{12}$

Table 1.6: Mean triangle area in various bodies with unit d-volume

$\eta = \eta(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ being equal to one when the random triangle is obtuse and zero otherwise. We shall call this functional the *obtusity indicator*. It is symmetric, trivarite and homogeneous of order zero. In our convention,

$$\eta(K_d) = \eta_{K_d K_d K_d} = \eta_{ddd} = \mathbb{E}\left[\eta(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \mid \mathbf{X}, \mathbf{Y}, \mathbf{Z} \sim \mathsf{Unif}(K_d)\right]. \tag{1.28}$$

Note that a triangle is obtuse when exactly one internal angle is obtuse. Hence, we can decompose the obtusity indicator almost surely as follows

$$\eta(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \eta^*(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) + \eta^*(\mathbf{Y}, \mathbf{Z}, \mathbf{X}) + \eta^*(\mathbf{Z}, \mathbf{X}, \mathbf{Y}), \tag{1.29}$$

where we denoted $\eta^*(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ as the obtusity indicator that are equal to one when the obtuse angle is located at the first vertex $\mathbf{X}$. Furthermore, we can write out this indicator in terms of a dot product as

$$\eta^*(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbb{1}_{(\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X})<0} \tag{1.30}$$

since $(\mathbf{Y} - \mathbf{X})^\top(\mathbf{Z} - \mathbf{X}) = \|\mathbf{Y} - \mathbf{X}\|\|\mathbf{Z} - \mathbf{X}\| \cos\alpha$, where α is the angle at vertex $\mathbf{X}$ of the triangle $\mathbf{XYZ}$. Therefore,

$$\eta(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbb{1}_{(\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X})<0} + \mathbb{1}_{(\mathbf{Z}-\mathbf{Y})^\top(\mathbf{X}-\mathbf{Y})<0} + \mathbb{1}_{(\mathbf{X}-\mathbf{Z})^\top(\mathbf{Y}-\mathbf{Z})<0}. \tag{1.31}$$

Taking expectation and by symmetry, we get for the obtusity probability

$$\eta(K_d) = 3\,\mathbb{P}\left[(\mathbf{Y} - \mathbf{X})^\top(\mathbf{Z} - \mathbf{X}) < 0 \mid \mathbf{X}, \mathbf{Y}, \mathbf{Z} \sim \mathsf{Unif}(K_d)\right]. \tag{1.32}$$

In a given configuration $\mathbf{X} \sim \mathsf{Unif}(A), \mathbf{Y} \sim \mathsf{Unif}(B), \mathbf{Z} \sim \mathsf{Unif}(C)$, we write

$$\eta_{ABC} = \mathbb{E}\left[\eta(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \mid \mathbf{X} \sim \mathsf{Unif}(A), \mathbf{Y} \sim \mathsf{Unif}(B), \mathbf{Z} \sim \mathsf{Unif}(C)\right]. \tag{1.33}$$

Additionally, we indicate by * the position of the obtuse vertex, so

$$\begin{aligned} \eta_{A^*BC} &= \mathbb{E}\left[\eta^*(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \mid \mathbf{X} \sim \mathsf{Unif}(A), \mathbf{Y} \sim \mathsf{Unif}(B), \mathbf{Z} \sim \mathsf{Unif}(C)\right] \\ &= \mathbb{P}\left[(\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) < 0 \mid \mathbf{X} \sim \mathsf{Unif}(A), \mathbf{Y} \sim \mathsf{Unif}(B), \mathbf{Z} \sim \mathsf{Unif}(C)\right], \end{aligned} \tag{1.34}$$

similarly for η_{AB^*C} and η_{ABC^*}. Hence, we may write the expected value of the obtusity indicator in any configuration as

$$\eta_{ABC} = \eta_{A^*BC} + \eta_{AB^*C} + \eta_{ABC^*}. \tag{1.35}$$

Random triangle obtusity probability $\eta_{ddd} = \eta(K_d)$ in selected K_d's is shown in Table 1.7 below. In there, G is the *Catalan's constant*

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.91596559417721901505460351493238 41\ldots \tag{1.36}$$

K_d		numerical value	$\eta(K_d)$
$\mathbb{B}_2$	disk, [77]	0.7197	$\frac{9}{8} - \frac{4}{\pi^2}$
$\mathbb{B}_3$	ball, [34, 17]	0.5286	$\frac{37}{70}$
C_2	square, [42]	0.7252	$\frac{97}{150} + \frac{\pi}{40}$
T_2^*	equilateral triangle	0.7482	$\frac{25}{4} + \frac{\pi}{12\sqrt{3}} + \frac{393}{10} \ln \frac{\sqrt{3}}{2}$
C_3	cube	0.5427	$\frac{323338}{385875} - \frac{13G}{35} + \frac{4859\pi}{62720} - \frac{73\pi}{1680\sqrt{2}} - \frac{\pi^2}{105} + \frac{3\pi \ln 2}{224} - \frac{3\pi \ln(1+\sqrt{2})}{224}$

Table 1.7: Probability that a random triangle in K_d is obtuse

In two dimensions, there are several known results. Obtusity probability was first solved in a disk by Woolhouse [77] as a corollary to the Silvester problem. Later, Langford [42] found $\eta(K_2)$ for K_2 being a general rectangle. Our table only shows the exact result for the special case $K_2 = C_2$. Without stating a complete proof, we believe it is easy to generalise Langford's result to any convex polygon. This is demonstrated in Section 1.6.1 on $\eta(T_2^*)$, where T_2^* is an equilateral triangle.

In higher dimensions, apart from the d-ball (Hall [34] and Buchta and Müller [17]), $\eta(K_d)$ was not known for any K_d with $d \geq 3$. In Section 1.7.2, we newly found the obtusity probability in the unit cube C_3 (also included in Table 1.7).

1.2.4 Perimeter and related functionals of a triangle

Let us (independently) select vertices $\mathbf{X} \sim \mathsf{Unif}(A), \mathbf{Y} \sim \mathsf{Unif}(B), \mathbf{Z} \sim \mathsf{Unif}(C)$ of a triangle $\mathbf{XYZ}$ from regions A, B, C with dimensions a, b, c as usual. We denote $L = |\mathbf{XY}|$, $L' = |\mathbf{XZ}|$ and $L'' = |\mathbf{YZ}|$ its (random) side-lengths and $\Theta = |\angle\mathbf{XZY}|$, $\Theta' = |\angle\mathbf{XYZ}|$ and $\Theta'' = |\angle\mathbf{YXZ}|$ the corresponding (random) sizes of its internal angles. Then, we denote its perimeter as

$$\Pi = \Pi(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = |\mathbf{XY}| + |\mathbf{XZ}| + |\mathbf{YZ}| = L + L' + L''. \tag{1.37}$$

The perimeter is a trivariate symmetric homogeneous functional of order one. Correspondingly, using our notation, we write for its moments

$$\Pi_{ABC}^{(k)} = \mathbb{E}\left[\Pi(\mathbf{X},\mathbf{Y},\mathbf{Z})^k \mid \mathbf{X}\sim \mathsf{Unif}(A), \mathbf{Y}\sim \mathsf{Unif}(B), \mathbf{Z}\sim \mathsf{Unif}(C)\right] \tag{1.38}$$

The question to determine the second perimeter moment $\Pi_{222}^{(2)}$ in the unit disk was first proposed by Finch [28] who obtained its numerical estimate. Although the problem of finding exact perimeter moments may seem natural and elementary, there were essentially no results known (even in the case of the disk). However, by CRT, we are able to obtain its exact value and also the higher moments (Equation (1.420)). The exact values of perimeter moments we found in the unit disk are shown in Table 1.8 below. In there, $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ is the *Apéry's constant* . See Section 1.6.3 for detailed calculation.

$\mathbb{B}_2$	numerical value	$\Pi_{222}^{(k)}$
$k=-1$	0.416744	$\frac{64}{15\pi} - \frac{64\ln 2}{15\pi}$
$k=1$	2.7162	$\frac{128}{15\pi}$
$k=2$	8.0271	$3 + \frac{3383}{72\pi^2} + \frac{35\zeta(3)}{16\pi^2}$
$k=3$	25.2395	$\frac{93584}{1225\pi} + \frac{1024\ln 2}{245\pi}$
$k=4$	83.2737	$\frac{49}{2} + \frac{1029\zeta(3)}{32\pi^2} + \frac{9745549}{18000\pi^2}$
$k=5$	285.644	$\frac{62912704}{72765\pi} + \frac{32768\ln 2}{693\pi}$

Table 1.8: Random triangle perimeter moments $\Pi_{222}^{(k)}$ in the unit disk $\mathbb{B}_2$

First moment

By symmetry, we immediately know that

$$\mathbb{E}[\Pi] = \mathbb{E}[L + L' + L''] = 3\mathbb{E}[L] \tag{1.39}$$

in any K_d. Therefore, the first perimeter moment is trivially deduced from the first moment of distance.

Second moment

The second perimeter moment turns out to be non-trivial. Taking expectation of

$$\Pi^2 = (L + L' + L'')^2 = L^2 + L'^2 + L''^2 + 2LL' + 2LL'' + 2L'L'', \tag{1.40}$$

we get, by symmetry

$$\mathbb{E}[\Pi^2] = 3\mathbb{E}[L^2] + 6\mathbb{E}[LL']\,. \tag{1.41}$$

The functional $LL' = |\mathbf{XY}||\mathbf{XZ}|$ is not symmetric with respect to every permutation of points $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$. However, we may define another functional, namely the symmetric polynomial

$$T = T(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = LL' + LL'' + L'L'' \tag{1.42}$$

such that $\mathbb{E}[T] = 3\mathbb{E}[LL']$. The functional T is symmetric and homogeneous of order two.

Third moment

Let us consider the following symmetric and homogeneous polynomials of order three

$$J = L^2L' + LL'^2 + L^2L'' + LL''^2 + L'^2L'' + L'L''^2, \qquad K = LL'L''. \tag{1.43}$$

By symmetry, we have $\mathbb{E}[J] = 6\mathbb{E}[LL'^2]$. Using those polynomials, we can write for the third power of perimeter,

$$\Pi^3 = (L + L' + L'')^3 = L^3 + L'^3 + L''^3 + 3J + 6K, \tag{1.44}$$

from which, taking the expectation,

$$\mathbb{E}[\Pi^3] = 3\mathbb{E}[L^3] + 3\mathbb{E}[J] + 6\mathbb{E}[K] = 3\mathbb{E}[L^3] + 18\mathbb{E}[L^2L'] + 6\mathbb{E}[LL'L'']. \tag{1.45}$$

Both $\mathbb{E}[J]$ and $\mathbb{E}[K]$ are non-trivial to obtain.

Fourth moment

Let us consider the following symmetric and homogeneous polynomials of order four

$$\begin{aligned} U &= L^3L' + LL'^3 + L^3L'' + LL''^3 + L'^3L'' + L'L''^3, \\ V &= L^2L'L'' + LL'^2L'' + LL'L''^2 \\ W &= L^2L'^2 + L^2L''^2 + L'^2L''^2 \end{aligned} \tag{1.46}$$

By symmetry, we have $\mathbb{E}[U] = 6\mathbb{E}[L^3L']$, $\mathbb{E}[V] = 3\mathbb{E}[L^2L'L'']$ and $\mathbb{E}[W] = 3\mathbb{E}[L^2L'^2]$. The fourth power of perimeter is then

$$\Pi^4 = L^4 + L'^4 + L''^4 + 4U + 12V + 6W. \tag{1.47}$$

Thus, the fourth moment of perimeter is then expressible as

$$\begin{aligned} \mathbb{E}[\Pi^4] &= 3\mathbb{E}[L^4] + 4\mathbb{E}[U] + 12\mathbb{E}[V] + 6\mathbb{E}[W] \\ &= 3\mathbb{E}[L^4] + 24\mathbb{E}[L^3L'] + 36\mathbb{E}[L^2L'L''] + 18\mathbb{E}[L^2L'^2]. \end{aligned} \tag{1.48}$$

Although $\mathbb{E}[U]$ and $\mathbb{E}[V]$ are non-trivial to obtain, interestingly, the first moment of W is trivial and it can be actually obtained from the fourth moment of distance and the second moment of area. Let $R = \Pi/2$. By Heron's formula, we have for the area S of the random triangle $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ with side-lengths L, L', L'',

$$S = \sqrt{R(R-L)(R-L')(R-L'')}. \tag{1.49}$$

Squaring this identity and by rearranging, we get

$$2W = 16S^2 + L^4 + L'^4 + L''^4 \tag{1.50}$$

so, taking expectation, we get

$$\mathbb{E}[W] = 8\mathbb{E}[S^2] + \tfrac{3}{2}\mathbb{E}[L^4] \qquad \text{or} \qquad \mathbb{E}[L^2L'^2] = \tfrac{8}{3}\mathbb{E}[S^2] + \tfrac{1}{2}\mathbb{E}[L^4]. \tag{1.51}$$

1.3 Bivariate functionals in two dimensions

1.3.1 Equilateral triangle

Let us have a bivariate symmetric homogeneous functional P of order p dependent on two random points picked from and equilateral triangle. We put $P = L^p$ (assumed implicitly throughout this section). For our triangle, we can take

$$T_2^* = \operatorname{conv}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \subset \mathbb{R}^3 \tag{1.52}$$

with area $\operatorname{vol}_2 T_2^* = \sqrt{3}/2$ and side-length $l = \sqrt{2}$. Additional, for a given $i = 1, 2, 3$, we denote E_i as an edge of T_2^* opposite to vertex $\mathbf{e}_i$. For the definition of various mean values $P_{ab} = L_{ab}^{(p)}$, see Figure 1.2. We also included the position of the scaling point $\mathbf{C}$ in cases reduction is possible. The arrows indicate which configurations reduce to which. Each arrow is labeled by a roman numeral corresponding to a given reduction equation in the system of reduction equations.

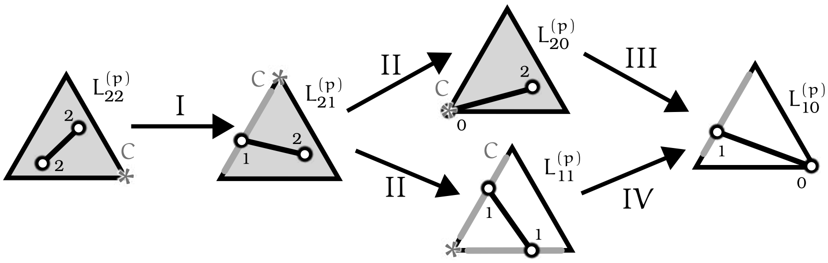


Figure 1.2: All different $L_{ab}^{(p)}$ configurations in an equilateral triangle

Reduction system

The full system obtained by CRT is

$$\begin{aligned}
\mathbf{I}:\ pP_{22} &= 2 \cdot 2(P_{21} - P_{22}) \\
\mathbf{II}:\ pP_{21} &= 2(P_{11} - P_{21}) + 1(P_{20} - P_{21}), \\
\mathbf{III}:\ pP_{20} &= 2(P_{10} - P_{20}) \\
\mathbf{IV}:\ pP_{11} &= 2(P_{10} - P_{11}).
\end{aligned}$$

By the use of symmetry, the terms can be given as follows: $P_{22} = P_{T_2^* T_2^*}$, $P_{21} = P_{T_2^* E_1}$, $P_{20} = P_{T_2^* \mathbf{e}_1}$, $P_{11} = P_{\mathbf{e}_1 \mathbf{e}_2}$, $P_{10} = P_{E_1 \mathbf{e}_1}$. Our linear system is solved by

$$P_{22} = \frac{24 P_{10}}{(4+p)(3+p)(2+p)}. \tag{1.53}$$

The remaining configuration (10) is irreducible (no scaling point available).

$\mathbf{P_{10}}$

In configuration (10), one point is drawn uniformly from an edge of T_2^* while the other is fixed at one of the opposite vertices. We can parametrise the points as

$$\mathbf{X} = \mathbf{e}_1, \mathbf{Y} = \mathbf{e}_2 + t(\mathbf{e}_3 - \mathbf{e}_2), \qquad t \in (0,1), \tag{1.54}$$

from which $L = \|\mathbf{X} - \mathbf{Y}\| = \sqrt{2 - 2t + 2t^2}$ and thus for $P = L^p$,

$$L_{10}^{(p)} = \mathbb{E}\left[\|\mathbf{X} - \mathbf{Y}\|^p\right] = \int_0^1 (2 - 2t + 2t^2)^{p/2}\ \mathrm{d}t. \tag{1.55}$$

This integral is straightforward. For example, when $p = 1$, we get

$$L_{10} = \frac{4 + 3\ln 3}{4\sqrt{2}}. \tag{1.56}$$

$\mathbf{P_{22}}$

Substituting P_{10} into Equation (1.53) with $P = L^p$, we get for general $p > -2$ (not necessarily an integer),

$$L_{22}^{(p)} = \frac{24\int_0^1 (2 - 2t + 2t^2)^{p/2}\ \mathrm{d}t}{(4+p)(3+p)(2+p)}. \tag{1.57}$$

Normalising the result, we get in an equilateral triangle with unit area,

$$v_1^{(p)}(T_2^*) = \frac{L_{22}^{(p)}}{(\mathrm{vol}_2\, T_2^*)^{p/2}} = \frac{24\int_0^1 \left(\frac{4}{\sqrt{3}}(1 - t + t^2)\right)^{p/2}\ \mathrm{d}t}{(4+p)(3+p)(2+p)}. \tag{1.58}$$

For example, plugging $p = 1$, we obtain for the mean distance between two random points in the unit equilateral triangle,

$$v_1^{(1)}(T_2^*) = \frac{4 + 3\ln 3}{10\sqrt[4]{3}} \approx 0.554364. \tag{1.59}$$

Note that, in the equilateral triangle with unit side-length, we have

$$L_{22}|_{l=1} = \frac{L_{22}|_{l=\sqrt{2}}}{\sqrt{2}} = \frac{4 + 3\ln 3}{20}. \tag{1.60}$$

Distance density

The density $f_{22}(\lambda)$ of the random distance L between two interior points in T_2^* can be recovered from moments using inverse Mellin transform (see appendix A.5). It is convenient to first rescale our triangle so its side-length is one (and hence $\lambda \in (0,1)$). Rescaled Equation (1.57) yields

$$\mathcal{M}[f_{22}] = \frac{L_{22}^{(p-1)}}{\sqrt{2}^{p-1}} = \frac{24\int_0^1 (1 - t + t^2)^{p/2}\ \mathrm{d}t}{(4+p)(3+p)(2+p)}. \tag{1.61}$$

Taking the inverse Mellin transform, we get, formally,

$$f_{22}(\lambda) = 24\,\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3\biggl[\int_0^1 \delta(\lambda - \sqrt{1-t+t^2})\,\mathrm{d}t\biggr] \tag{1.62}$$

From Table A.5 (see Appendix A),

$$\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3\delta(\lambda-\alpha) = \frac{\lambda(\alpha-\lambda)^2}{2\alpha^4}\mathbb{1}_{\lambda<\alpha}, \tag{1.63}$$

via which we can deduce

$$f_{22}(\lambda) = 12\lambda\int_0^1 \frac{(\sqrt{1-t+t^2}-\lambda)^2}{(1-t+t^2)^2}\mathbb{1}_{\lambda<\sqrt{1-t+t^2}}\,\mathrm{d}t. \tag{1.64}$$

Note that since $t\in(0,1)$, we have $\sqrt{1-t+t^2}\in(\sqrt{3}/2,1)$ and thus we can write $\mathbb{1}_{\lambda<\sqrt{1-t+t^2}} = 1-\mathbb{1}_{\lambda\geq\sqrt{1-t+t^2}} = 1-\mathbb{1}_{\lambda\geq\sqrt{1-t+t^2}}\mathbb{1}_{\lambda\geq\frac{\sqrt{3}}{2}}$. Hence,

$$\begin{aligned} f_{22}(\lambda) = {}& 12\lambda\int_0^1 \frac{(\sqrt{1-t+t^2}-\lambda)^2}{(1-t+t^2)^2}\,\mathrm{d}t \\ & - 12\lambda\left[\int_0^1 \frac{(\sqrt{1-t+t^2}-\lambda)^2}{(1-t+t^2)^2}\mathbb{1}_{\lambda<\sqrt{1-t+t^2}}\,\mathrm{d}t\right]\mathbb{1}_{\lambda\geq\frac{\sqrt{3}}{2}}. \end{aligned} \tag{1.65}$$

To calculate the integrals, we substitute $t=\frac{1}{2}+\frac{\sqrt{3}}{2}\tan\theta$, by symmetry,

$$f_{22}(\lambda) = 16\lambda\sqrt{3}\int_0^{\frac{\pi}{6}}\Bigl(1-\tfrac{2\lambda\cos\theta}{\sqrt{3}}\Bigr)^2\mathrm{d}t - 16\lambda\sqrt{3}\left[\int_0^{\arccos\frac{\sqrt{3}}{2\lambda}}\Bigl(1-\tfrac{2\lambda\cos\theta}{\sqrt{3}}\Bigr)^2\mathrm{d}t\right]\mathbb{1}_{\lambda\geq\frac{\sqrt{3}}{2}}. \tag{1.66}$$

and hence, immediately, we finally get for the density on $\lambda\in(0,1)$,

$$\begin{aligned} f_{22}(\lambda) = {}& 8\lambda\left[\tfrac{\pi}{\sqrt{3}}\left(1+\tfrac{2\lambda^2}{3}\right)-4\lambda\left(1-\tfrac{\lambda}{4}\right)\right] \\ & + 8\lambda\sqrt{3}\left[3\sqrt{\tfrac{4\lambda^2}{3}-1}-2\left(1+\tfrac{2\lambda^2}{3}\right)\arccos\tfrac{\sqrt{3}}{2\lambda}\right]\mathbb{1}_{\lambda\geq\frac{\sqrt{3}}{2}}. \end{aligned} \tag{1.67}$$

1.3.2 Square

Let us have a bivariate symmetric homogeneous functional P of order p dependent on two random points picked from K being a square. That is, $K=C_2$ with vertices $V_1[0,0]$, $V_2[1,0]$, $V_3[1,1]$, $V_4[0,1]$ and edges connecting them E_{12}, E_{23}, E_{34}, E_{41} ($E_{ij}=\overline{V_iV_j}$). Note that the edge length is $a=1$ and the area $\operatorname{vol}K=1$ so the mean of P is already normalised. We put $P=L^p$. For the definition of various mean values $P_{ab}=L_{ab}^{(p)}$, see Figure 1.3. We also included the position of the scaling point $\mathbf{C}$ in cases reduction is possible. The arrows indicate which configurations reduce to which. Each arrow is labeled by a roman numeral corresponding to a given reduction equation in the system of reduction equations.

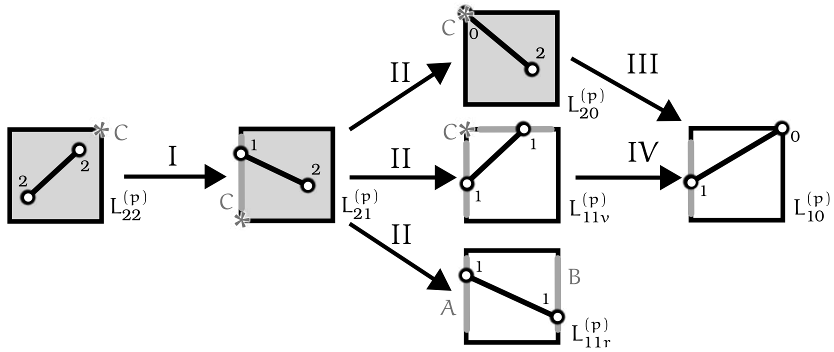


Figure 1.3: All different $L_{ab}^{(p)}$ configurations encountered for K being a square

Reduction system

The full system obtained by CRT is

$$\begin{aligned}
\textbf{I}: pP_{22} &= 2\cdot 2(P_{21}-P_{22})\\
\textbf{II}: pP_{21} &= 2(P_{11}-P_{21})+1(P_{20}-P_{21}),\\
\textbf{III}: pP_{20} &= 2(P_{10}-P_{20})\\
\textbf{IV}: pP_{11v} &= 2(P_{10}-P_{11v})
\end{aligned}$$

with

$$P_{11}=\tfrac{1}{2}P_{11v}+\tfrac{1}{2}P_{11r}.$$

By the use of symmetry, the terms can be given as follows: $P_{22}=P_{KK}$, $P_{21}=P_{KE_{41}}, P_{20}=P_{KV_4}, P_{11v}=P_{E_{34}E_{41}}, P_{11r}=P_{E_{23}E_{41}}, P_{10}=P_{V_3E_{41}}$. Our linear system is solved by

$$P_{22}=\frac{16P_{10}}{(4+p)(3+p)(2+p)}+\frac{4P_{11r}}{(4+p)(3+p)}. \tag{1.68}$$

The remaining configurations (10) and (11r) are irreducible (no scaling point available).

$\mathbf{P_{10}}$

In configuration (10), one point is drawn uniformly from an edge of C_2 while the other is fixed at one of the opposite vertices. We can parametrise the points as

$$\mathbf{X}=[0,1-t],\mathbf{Y}=[1,1],\qquad t\in(0,1) \tag{1.69}$$

and thus for $P=L^p$,

$$L_{10}^{(p)}=\mathbb{E}\left[\|\mathbf{X}-\mathbf{Y}\|^p\right]=\int_0^1(1+t^2)^{p/2}\,\mathrm{d}t. \tag{1.70}$$

For example, when $p=1$, we get

$$L_{10}=\tfrac{1}{\sqrt{2}}+\tfrac{1}{2}\operatorname{argsinh}(1). \tag{1.71}$$

$\mathbf{P_{11r}}$

In configuration $(11r)$, one point is drawn uniformly from an edge of C_2 while the other is drawn from the opposite edge (Those edges are denoted as A and B in Figure 1.3). We can parametrise the points as

$$\mathbf{X} = [0, x], \mathbf{Y} = [1, y], \qquad x \in (0,1), y \in (0,1) \tag{1.72}$$

and thus for $P = L^p$, using this parametrization,

$$L_{11r}^{(p)} = \mathbb{E}\left[\|\mathbf{X} - \mathbf{Y}\|^p\right] = \int_0^1 \int_0^1 (1 + (x-y)^2)^{p/2} \, \mathrm{d}x\mathrm{d}y. \tag{1.73}$$

Via the change of variables $u = x - y, v = y$ and integrating out v,we get

$$L_{11r}^{(p)} = 2\int_0^1 (1-u)(1+u^2)^{p/2}\mathrm{d}u = 2L_{10}^{(p)} - \frac{2^{\frac{p}{2}+2} - 2}{p+2}. \tag{1.74}$$

For example, when $p = 1$, we get

$$L_{11r} = \frac{2}{3} - \frac{\sqrt{2}}{3} + \operatorname{argsinh}(1). \tag{1.75}$$

$\mathbf{P_{22}}$

Substituting P_{10} and P_{11r} into Equation (1.68) with $P = L^p$, we get for general $p > -2$ (not necessarily an integer),

$$v_1^{(p)}(C_2) = L_{22}^{(p)} = \frac{8(1 - 2^{\frac{p}{2}+1})}{(4+p)(3+p)(2+p)} + \frac{8\int_0^1 (1+t^2)^{p/2}\mathrm{d}t}{(3+p)(2+p)}. \tag{1.76}$$

Plugging $p = 1$, we obtain for the mean distance between two random points in the unit square,

$$v_1^{(1)}(C_2) = L_{22} = \frac{2}{15} + \frac{\sqrt{2}}{15} + \frac{1}{3}\operatorname{argsinh}(1) \approx 0.5214054331647207. \tag{1.77}$$

Distance density

The density $f_{22}(\lambda)$ of the random distance L between two interior points in C_2 can be recovered from moments using inverse Mellin transform (see appendix A.5). By Equation (1.76), we have

$$\mathcal{M}[f_{22}] = L_{22}^{(p-1)} = \frac{8(1 - 2^{\frac{p+1}{2}})}{(3+p)(2+p)(1+p)} + \frac{8\int_0^1 (1+t^2)^{\frac{p-1}{2}}\mathrm{d}t}{(2+p)(1+p)}. \tag{1.78}$$

Taking the inverse Mellin transform, we get, formally,

$$f_{22}(\lambda) = 8\,\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3\left[\delta(\lambda - 1) - 2\delta(\lambda - \sqrt{\tfrac{1}{2}})\right] + 8\mathcal{I}_1\mathcal{I}_2\left[\int_0^1 \delta(\lambda - \sqrt{1+t^2})\mathrm{d}t\right] \tag{1.79}$$

From Table A.5 (see Appendix A),

$$\mathcal{I}_1\mathcal{I}_2\delta(\lambda-\alpha) = \lambda\alpha^{-3}(\alpha-\lambda)\mathbb{1}_{\lambda<\alpha}, \qquad \mathcal{I}_1\mathcal{I}_2\mathcal{I}_3\delta(\lambda-\alpha) = \frac{\lambda(\alpha-\lambda)^2}{2\alpha^4}\mathbb{1}_{\lambda<\alpha}, \tag{1.80}$$

via which we can deduce

$$f_{22}(\lambda) = 4\lambda(1-\lambda)^2\mathbb{1}_{\lambda<1} - 2\lambda(\sqrt{2}-\lambda)^2 + 8\lambda\int_0^1 \frac{\sqrt{1+t^2}-\lambda}{(1+t^2)^{3/2}}\mathbb{1}_{\lambda<\sqrt{1+t^2}}\mathrm{d}t. \tag{1.81}$$

Hence, $f_{22}(\lambda)$ is nonzero only when $\lambda \in (0,\sqrt{2})$. Note that since $t \in (0,1)$, we can write for $\lambda \in (0,\sqrt{2})$ that $\mathbb{1}_{\lambda<\sqrt{1+t^2}} = 1-\mathbb{1}_{\lambda\geq\sqrt{1+t^2}} = 1-\mathbb{1}_{\lambda\geq\sqrt{1+t^2}}\mathbb{1}_{\lambda\geq 1}$. Similarly, we write $\mathbb{1}_{\lambda<1} = 1-\mathbb{1}_{\lambda\geq 1}$ and thus

$$\begin{aligned} f_{22}(\lambda) &= 4\lambda(1-\lambda)^2 - 2\lambda(\sqrt{2}-\lambda)^2 + 8\lambda\int_0^1 \frac{\sqrt{1+t^2}-\lambda}{(1+t^2)^{3/2}}\mathrm{d}t \\ &\quad - \left[4\lambda(1-\lambda)^2 + 8\lambda\int_0^1 \frac{\sqrt{1+t^2}-\lambda}{(1+t^2)^{3/2}}\mathbb{1}_{\lambda\geq\sqrt{1+t^2}}\mathrm{d}t\right]\mathbb{1}_{\lambda\geq 1}. \end{aligned} \tag{1.82}$$

To calculate the integrals, we substitute $t = \tan\theta$,

$$\begin{aligned} f_{22}(\lambda) &= 4\lambda(1-\lambda)^2 - 2\lambda(\sqrt{2}-\lambda)^2 + 8\lambda\int_0^{\pi/4}(1-\lambda\cos\theta)\mathrm{d}\theta \\ &\quad - \left[4\lambda(1-\lambda)^2 + 8\lambda\int_0^{\arccos(1/\lambda)}(1-\lambda\cos\theta)\mathrm{d}\theta\right]\mathbb{1}_{\lambda\geq 1} \end{aligned} \tag{1.83}$$

and hence, immediately, we finally get for the density on $\lambda \in (0,\sqrt{2})$,

$$f_{22}(\lambda) = 2\pi\lambda - 2(4-\lambda)\lambda^2 - 4\lambda\left[(\lambda-1)^2 - 2\sqrt{\lambda^2-1} + 2\arccos\left(\frac{1}{\lambda}\right)\right]\mathbb{1}_{\lambda\geq 1}. \tag{1.84}$$

1.3.3 Disk

Consider a bivariate symmetric homogeneous functional P of order p dependent on two random points picked uniformly from the unit disk $\mathbb{B}_2 = \{\mathbf{x}\in\mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1\}$ with area $\operatorname{vol}_2\mathbb{B}_2 = \pi$. Additionally, we require P to be *rotationally* symmetric with respect to the origin. That is, for any $\mathbf{x},\mathbf{y}\in\mathbb{B}_2$ and any orthogonal matrix R we have $P(R\mathbf{x},R\mathbf{y}) = P(\mathbf{x},\mathbf{y})$. This assumption is satisfied by the choice $P = L^p$ (which is implicitly assumed in this section). Table 1.9 below shows various explicit $L_{22}^{(p)}$ distance moments for selected p's (from Equation (1.93)).

$L_{22}^{(-1)}$	$L_{22}^{(0)}$	$L_{22}^{(1)}$	$L_{22}^{(2)}$	$L_{22}^{(3)}$	$L_{22}^{(4)}$	$L_{22}^{(5)}$	$L_{22}^{(6)}$	$L_{22}^{(7)}$	$L_{22}^{(8)}$	$L_{22}^{(9)}$	$L_{22}^{(10)}$
$\frac{16}{3\pi}$	1	$\frac{128}{45\pi}$	1	$\frac{2048}{525\pi}$	$\frac{5}{3}$	$\frac{16384}{2205\pi}$	$\frac{7}{2}$	$\frac{524288}{31185\pi}$	$\frac{42}{5}$	$\frac{4194304}{99099\pi}$	22

Table 1.9: Mean distance moments $L_{22}^{(p)}$ between two random points in $\mathbb{B}_2$

Reduction system

According to our convention, we write

$$P_{ab} = \mathbb{E}\left[P(\mathbf{X},\mathbf{Y}) \mid \mathbf{X} \sim \mathsf{Unif}(A), \mathbf{Y} \sim \mathsf{Unif}(B)\right], \tag{1.85}$$

where $a = \dim A$, $b = \dim B$ and the concrete selection of A, B is deduced from the reduction diagram in Figure 1.4 below. In this diagram, we also included the position of the scaling point $\mathbf{C}$ in cases reduction is possible. The arrows indicate which configurations reduce to which. Each arrow is labeled by a roman numeral corresponding to a given reduction equation in the system of reduction equations.

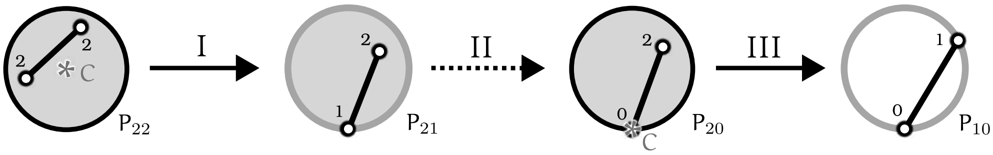


Figure 1.4: All different P_{ab} sub-configurations in $\mathbb{B}_2$

Reduction system

The full system obtained by CRT is

$$\begin{aligned} \mathbf{I}:\ pP_{22} &= 2\cdot 2(P_{21} - P_{22}) \\ \mathbf{II}:\ P_{21} &= P_{20}. \\ \mathbf{III}:\ pP_{20} &= 2(P_{10} - P_{20}), \end{aligned}$$

where the equation **II** follows from the rotational symmetry of P. The solution of our system is

$$P_{22} = \frac{8P_{10}}{(4+p)(2+p)}. \tag{1.86}$$

$\mathbf{P_{10}}$

In configuration (10), one point $\mathbf{X}$ is drawn uniformly from the boundary $\partial\mathbb{B}_2$ while the other $\mathbf{Y}$ is fixed at the boundary. Keep in mind that P_{10} is defined via generalization of Remark 9 as a mean weighted by the support function

$$P_{10} = \frac{1}{2\,\mathrm{vol}_2\,\mathbb{B}_2}\int_{\partial\mathbb{B}_2} P(\mathbf{x},\mathbf{y})h_{\mathbf{y}}(\mathbf{x})\lambda_1(\mathrm{d}\mathbf{x}), \tag{1.87}$$

where the support function $h_{\mathbf{y}}(\mathbf{x})$ of $\mathbb{B}_2$ evaluated in $\mathbf{x}$ and centered at $\mathbf{y} \in \partial\mathbb{B}_2$ (arbitrary fixed point) is given explicitly [69, p. 58] as

$$h_{\mathbf{y}}(\mathbf{x}) = \tfrac{1}{2}\|\mathbf{x}-\mathbf{y}\|^2. \tag{1.88}$$

Parametrising the integral using polar coordinates with their center located at $\mathbf{y}$,

$$\mathbf{x} = [2\sin\varphi\cos\varphi, 2\sin^2\varphi + 1], \mathbf{y} = [0,-1], \qquad \varphi \in [0,\pi). \tag{1.89}$$

We have $\mathrm{d}\mathbf{x} = 2(\cos(2\varphi), \sin(2\varphi))\mathrm{d}\varphi$ and hence, for the uniform measure on $\partial\mathbb{B}_2$,

$$\lambda_1(\mathrm{d}\mathbf{x}) = \|\mathrm{d}\mathbf{x}\| = 2\mathrm{d}\varphi. \tag{1.90}$$

Next, note that $\|\mathbf{x} - \mathbf{y}\| = 2\sin\varphi$ and thus $h_{\mathbf{y}}(\mathbf{x}) = 2\sin^2\varphi$. Furthermore, since $P = L^p$, we get

$$P(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|^p = (2\sin\varphi)^p. \tag{1.91}$$

Overall, putting everything together and by using symmetry in φ, we get

$$L_{10}^{(p)} = \mathbb{E}\left[\|\mathbf{X} - \mathbf{Y}\|^p\right] = \frac{1}{\pi}\int_0^{\pi/2} (2\sin\varphi)^{2+p}\ \mathrm{d}\varphi. \tag{1.92}$$

$\mathbf{P_{22}}$

Substituting P_{10} into Equation (1.86) with $P = L^p$, we get for general $p > -2$ (not necessarily an integer),

$$L_{22}^{(p)} = \frac{8\int_0^{\pi/2}(2\sin\varphi)^{2+p}\ \mathrm{d}\varphi}{\pi(4+p)(2+p)} = \frac{(4+p)(1+p)!}{\Gamma(\frac{p}{2}+3)^2} = \frac{2^{2p+6}\Gamma(\frac{p+3}{2})^2}{\pi(2+p)(4+p)(2+p)!}. \tag{1.93}$$

Plugging $p = 1$, the mean distance between two random points in the unit disk in various configurations is shown in Table 1.10.

L_{22}	L_{21}	L_{20}	L_{10}
$\frac{128}{45\pi}$	$\frac{32}{9\pi}$	$\frac{32}{9\pi}$	$\frac{16}{3\pi}$

Table 1.10: Mean distance in $\mathbb{B}_2$ in various configurations

Note that L_{22} can be normalised to the first metric moment as

$$v_1^{(1)}(\mathbb{B}_2) = \frac{L_{22}}{\sqrt{\pi}} = \frac{128}{45\pi^{3/2}} \approx 0.510826. \tag{1.94}$$

Distance density

The density $f_{22}(\lambda)$ of the random distance L between two interior points in $\mathbb{B}_2$ can be recovered from moments using inverse Mellin transform (see appendix A.5). By Equation (1.93), we have

$$\mathcal{M}[f_{22}] = L_{22}^{(p-1)} = \frac{8\int_0^{\pi/2}(2\sin\varphi)^{1+p}\ \mathrm{d}\varphi}{\pi(3+p)(1+p)}. \tag{1.95}$$

Taking the inverse Mellin transform, we get, formally,

$$f_{22}(\lambda) = \frac{8}{\pi}\mathcal{I}_1\mathcal{I}_3\left[\int_0^{\pi/2}(2\sin\varphi)^2\,\delta(\lambda - 2\sin\varphi)\ \mathrm{d}\varphi\right]. \tag{1.96}$$

From Table A.5 (see Appendix A),

$$\mathcal{I}_1\mathcal{I}_3\delta(\lambda - \alpha) = \frac{\lambda}{2\alpha^4}(\alpha^2 - \lambda^2)\mathbb{1}_{\lambda<\alpha}, \tag{1.97}$$

via which we can deduce

$$f_{22}(\lambda) = \frac{4}{\pi}\int_0^{\pi/2}\left(1 - \frac{\lambda^2}{4\sin^2\varphi}\right)\mathbb{1}_{\lambda<2\sin\varphi}\,\mathrm{d}\varphi. \tag{1.98}$$

This integral is trivial, we obtain that $\lambda \in (0,2)$ and we have there

$$f_{22}(\lambda) = \tfrac{4\lambda}{\pi}\left(\arccos\left(\tfrac{\lambda}{2}\right) - \tfrac{\lambda}{2}\sqrt{1-\tfrac{\lambda^2}{4}}\right). \tag{1.99}$$

1.3.4 General regular polygons

Let K be a regular n-sided polygon ($n \geq 3$) with vertices $V_i = [\cos\frac{2\pi i}{n}, \sin\frac{2\pi k}{n}]$, where $i = 0,1,2,\dots,n-1$, so the polygon is circumscribed by a circle with radius one. For edges, we write $E_i = \overline{V_iV_{i+1}}$ with convention $V_n = V_0$. Note that for the area, we have $\operatorname{vol} K = \frac{n}{2}\sin\frac{2\pi}{n}$ and for edge length $l = \operatorname{vol} E_i = 2\sin\frac{\pi}{n}$. Let $P = L^p$, then P is symmetric and homogenous of order p. In order to express P_{22}, we again use the Crofton Reduction technique. First, we select $\mathbf{C} = [0,0]$ as our first scaling point. That way, $pP_{22} = 4(P_{21} - P_{22})$, where P_{21} is a configuration with $A = K$ and B is an (arbitrary) edge of K, we thus choose $B = E_0$. Our next scaling point is $\mathbf{C} = V_0 = [1,0]$, we have $pP_{21} = 2(P_{11} - P_{21}) + (P_{20} - P_{11})$. So far,

$$P_{22} = \frac{4(2P_{11} + P_{20})}{(4+p)(3+p)}, \tag{1.100}$$

where $P_{20} = P_{KV_0}$ and

$$P_{11} = \frac{2}{n}\sec\left(\frac{\pi}{n}\right)\sum_{i=1}^{n-2}\sin\left(\frac{\pi i}{n}\right)\sin\left(\frac{\pi(i+1)}{n}\right)P_{E_iE_0}. \tag{1.101}$$

The last relation follows from Crofton Reduction Technique and Definition 8 with $h_{\mathbf{C}}(\partial_i K) = 2\sin\frac{\pi i}{n}\sin\frac{\pi(i+1)}{n}$, where $\partial_i K = E_i$ with outer normal orientation. To reduce P_{20}, we choose $\mathbf{C} = V_0$ and get $pP_{20} = 2(P_{10} - P_{20})$, where (the weights are the same)

$$P_{10} = \frac{2}{n}\sec\left(\frac{\pi}{n}\right)\sum_{i=1}^{n-2}\sin\left(\frac{\pi i}{n}\right)\sin\left(\frac{\pi(i+1)}{n}\right)P_{E_iV_0}. \tag{1.102}$$

Finally, we can also reduce $P_{E_iE_0}$ into linear combination of $P_{E_iE_0}$. However, the reduction is dependent on whether n is even or odd. Formally, let us write $pP_{E_iE_0} = 2(P_{E_i\partial E_0} - P_{E_iE_0})$ regardless of parity of n, so overall

$$P_{22} = \frac{16\sec(\pi/n)}{n(4+p)(3+p)(2+p)}\sum_{i=1}^{n-2}\sin\left(\frac{\pi i}{n}\right)\sin\left(\frac{\pi(i+1)}{n}\right)(2P_{E_i\partial E_0} + P_{E_iV_0}). \tag{1.103}$$

Irreducible terms

First, we shall compute the irreducible terms $P_{E_iV_0}$. These terms can be written as an integral over all points X selected uniformly from E_i. A natural parametrisation is of course $X = V_i + s(V_{i+1} - V_i)$, $s \in (0, l)$. Thus

$$P_{E_iV_0} = \frac{1}{l} \int_0^l \|V_i + s(V_{i+1} - V_i) - V_0\|^p \, \mathrm{d}s. \tag{1.104}$$

However, we may parametrise X in polar coordinates centered in V_0 instead. Let us write $r(\varphi) = \|X - V_0\|$ for $X \in E_i$, given that $X - V_0$ points in the direction of the polar angle φ. Note that the polar angle of vertex V_i is $\varphi_i = \frac{\pi}{2} + \frac{\pi i}{n}$. Straightforward calculation reveals that

$$r(\varphi) = 2\sin\left(\frac{\pi i}{n}\right)\sin\left(\frac{\pi(i+1)}{n}\right)\sec\left(\frac{2\pi i - n\varphi + \pi}{n}\right) \tag{1.105}$$

and

$$\mathrm{d}s = \sqrt{r(\varphi)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2}\, \mathrm{d}\varphi = 2\sin\left(\frac{\pi i}{n}\right)\sin\left(\frac{\pi(i+1)}{n}\right)\sec^2\left(\frac{2\pi i - n\varphi + \pi}{n}\right)\, \mathrm{d}\varphi. \tag{1.106}$$

Therefore,

$$P_{E_iV_0} = 2^p \sin^{1+p}\left(\frac{\pi i}{n}\right)\sin^{1+p}\left(\frac{\pi(i+1)}{n}\right)\csc\left(\frac{\pi}{n}\right)\int_{\varphi_i}^{\varphi_{i+1}} \sec^{2+p}\left(\frac{2\pi i - n\varphi + \pi}{n}\right)\, \mathrm{d}\varphi. \tag{1.107}$$

Using reparametrisation $(2\pi i - n\varphi + \pi)/n = \zeta$, we get

$$P_{E_iV_0} = 2^p \sin^{1+p}\left(\frac{\pi i}{n}\right)\sin^{1+p}\left(\frac{\pi(i+1)}{n}\right)\csc\left(\frac{\pi}{n}\right)\int_{\pi i/n}^{\pi(i+1)/n} \csc^{2+p}\zeta\, \mathrm{d}\zeta. \tag{1.108}$$

By definition and contrary to its usual meaning, we put $P_{E_0V_0} = P_{E_nV_0} = 0$.

Odd number of sides

If n is odd, we select for all i the following scaling point

$$C_i = \mathcal{A}(E_i) \cap \mathcal{A}(E_0) = \cos\left(\frac{\pi}{n}\right)\sec\left(\frac{\pi i}{n}\right)\left[\cos\left(\frac{\pi(i+1)}{n}\right), \sin\left(\frac{\pi(i+1)}{n}\right)\right] \tag{1.109}$$

from which

$$P_{E_i\partial E_0} = \frac{1}{2}\csc\left(\frac{\pi}{n}\right)\sec\left(\frac{\pi i}{n}\right)\left(\sin\left(\frac{\pi(i+1)}{n}\right)P_{E_iV_0} - \sin\left(\frac{\pi(i-1)}{n}\right)P_{E_iV_1}\right). \tag{1.110}$$

We can simplify Equation (1.103) in the following way: Note that, rotating K by $2\pi/n$, we see that $P_{E_iV_1} = P_{E_{i-1}V_0}$, using which we can deduce, after splitting and then by shifting the summation from i to $i+1$,

$$P_{22} = \frac{-16\sec(\pi/n)}{n(4+p)(3+p)(2+p)} \sum_{i=1}^{n-2} \frac{\sin^2(\pi i/n)\sin^2(\pi(i+1)/n)}{\cos(\pi i/n)\cos(\pi(i+1)/n)} P_{E_iV_0}, \tag{1.111}$$

from which immediately in total, for odd n and $p > -2$ arbitrary,

$$P_{22} = \frac{-2^{4+p}\sec(\frac{\pi}{n})\csc(\frac{\pi}{n})}{n(4+p)(3+p)(2+p)} \sum_{i=1}^{n-2} \frac{\sin^{3+p}(\frac{\pi i}{n})\sin^{3+p}(\frac{\pi(i+1)}{n})}{\cos(\frac{\pi i}{n})\cos(\frac{\pi(i+1)}{n})} \int_{\frac{\pi i}{n}}^{\frac{\pi(i+1)}{n}} \csc^{2+p}\zeta\, \mathrm{d}\zeta. \tag{1.112}$$

Even number of sides

If n is even, then $P_{E_{n/2}E_0}$ is irreducible since E_0 and $E_{n/2}$ are parallel and we need two parameters to describe the position of points which are drawn from those sides. Nevertheless, we can always integrate out one of the parameters (or use the overlap formula, and adaption of Proposition 27 from the next section in two dimensions) to deduce that

$$P_{E_{n/2}E_0} = \frac{1}{l^2}\int_{-l}^{l}(h^2+x^2)^{p/2}|l-x|\ \mathrm{d}x, \tag{1.113}$$

where $h = 2\cos(\pi/n)$ is the separation between $E_{n/2}$ and E_0. Identifying $\frac{1}{l}\int_0^l(h^2+x^2)^{p/2}\ \mathrm{d}x$ as $P_{E_{n/2}V_0}$ and solving the remaining integral, we get

$$P_{E_{n/2}E_0} = 2P_{E_{n/2}V_0} - \frac{2^{p+1}\csc^2\left(\frac{\pi}{n}\right)}{2+p}\left(1-\cos^{2+p}\left(\frac{\pi}{n}\right)\right), \tag{1.114}$$

The Equation (1.103) is still valid provided we treat $P_{E_{n/2}\partial E_0}$ only formally as the solution of the equation $pP_{E_{n/2}E_0} = 2(P_{E_{n/2}\partial E_0} - P_{E_{n/2}E_0})$. That is,

$$P_{E_{n/2}\partial E_0} = \tfrac{2+p}{2}P_{E_{n/2}E_0} = (2+p)L_{E_{n/2}V_0} - 2^p\csc^2\left(\tfrac{\pi}{n}\right)\left(1-\cos^{2+p}\left(\tfrac{\pi}{n}\right)\right). \tag{1.115}$$

Exploiting symmetries $P_{E_iV_0} = P_{E_{n-i-1}V_0}$ and $P_{E_i\partial E_0} = P_{E_{n-i}\partial E_0}$ and and shifting $i \to i+1$, we get from Equation (1.103) for any even n and $p > -2$,

$$\begin{aligned} P_{22} = {} & \frac{32\sec\left(\frac{\pi}{n}\right)}{n(4+p)(3+p)(2+p)}\Bigg(\cos\left(\frac{\pi}{n}\right)\left(3+p+\cot^2\left(\frac{\pi}{n}\right)\right)P_{E_{n/2}V_0} \\ & - 2^p\frac{\cos(\frac{\pi}{n})}{\sin^2(\frac{\pi}{n})}\left(1-\cos^{2+p}\left(\frac{\pi}{n}\right)\right) - \sum_{i=1}^{\frac{n}{2}-2}\frac{\sin^2(\pi i/n)\sin^2(\pi(i+1)/n)}{\cos(\pi i/n)\cos(\pi(i+1)/n)}P_{E_iV_0}\Bigg). \end{aligned} \tag{1.116}$$

Arbitrary number of sides

Alternatively, if we redefine $P_{E_i\partial E_0}$ to be equal to $\lim_{x\to i}P_{E_x\partial E_0}$, where

$$P_{E_x\partial E_0} = \tfrac{1}{2}\csc\left(\tfrac{\pi}{n}\right)\sec\left(\tfrac{\pi x}{n}\right)\left(\sin\left(\tfrac{\pi(x+1)}{n}\right)P_{E_xV_0} - \sin\left(\tfrac{\pi(x-1)}{n}\right)P_{E_xV_1}\right). \tag{1.117}$$

and

$$P_{E_xV_0} = 2^p\sin^{1+p}\left(\frac{\pi x}{n}\right)\sin^{1+p}\left(\frac{\pi(x+1)}{n}\right)\csc\left(\frac{\pi}{n}\right)\int_{\pi x/n}^{\pi(x+1)/n}\csc^{2+p}\zeta\ \mathrm{d}\zeta \tag{1.118}$$

for $x \in \mathbb{R}$. One can show, by taking the limit, that we get the same expression for $P_{E_{n/2}V_0}$ when n is even. Thus, using this redefinition of $P_{E_i\partial E_0}$, Equation (1.103) is valid for $n \geq 3$ regardless of n being even or odd.

Even moments

Let us briefly discuss the case of even moments. Uwe Bäsel [4] found the values of $L_{22}^{(p)}$ for $p=2$ and $p=4$, namely

$$L_{22}^{(2)} = \frac{1}{3}\left(2+\cos\frac{2\pi}{n}\right), \qquad L_{22}^{(4)} = \frac{1}{90}\left(77+64\cos\frac{2\pi}{n}+9\cos\frac{4\pi}{n}\right). \tag{1.119}$$

Using Equation (1.103) with redefinition $P_{E_i\partial E_0} = \lim_{x\to i} P_{E_x\partial E_0}$, we can rederive those formulae easily. For example, when $p=2$, we get from Equation (1.118),

$$L_{E_xV_0}^{(2)} = \frac{5}{3}+\frac{1}{3}\cos\frac{2\pi}{n}-\cos\frac{2\pi x}{n}-\cos\frac{2\pi(x+1)}{n} \tag{1.120}$$

and from Equation (1.117), after simplifications,

$$L_{E_i\partial E_0}^{(2)} = \frac{8}{3}+\frac{4}{3}\cos\frac{2\pi}{n}-\cos\frac{2\pi(i-1)}{n}-2\cos\frac{2\pi i}{n}-\cos\frac{2\pi(i+1)}{n}. \tag{1.121}$$

Therefore, plugging those into Equation (1.103) with $p=2$,

$$\begin{aligned} L_{22}^{(2)} = \frac{2\sec(\pi/n)}{15n}\sum_{i=1}^{n-2}\sin\left(\frac{\pi i}{n}\right)\sin\left(\frac{\pi(i+1)}{n}\right)\Bigg(7+3\cos\frac{2\pi}{n} \\ -2\cos\frac{2\pi(i-1)}{n}-5\cos\frac{2\pi i}{n}-3\cos\frac{2\pi(i+1)}{n}\Bigg). \end{aligned} \tag{1.122}$$

Finally, we can sum this series using CAS software (*Mathematica* of *Maple*). As a result, we are able to deduce the following simple formulae for higher even moments (the formulae are valid for $n \geq 3$ regardless of n being odd or even),

$$L_{22}^{(6)} = \tfrac{1}{420}\left(628+661\cos\tfrac{2\pi}{n}+164\cos\tfrac{4\pi}{n}+17\cos\tfrac{6\pi}{n}\right)-\tfrac{\delta_{n3}}{50} \tag{1.123}$$

$$L_{22}^{(8)} = \tfrac{1}{1575}\left(4921+5936\cos\tfrac{2\pi}{n}+1974\cos\tfrac{4\pi}{n}+368\cos\tfrac{6\pi}{n}+31\cos\tfrac{8\pi}{n}\right)-\tfrac{16\delta_{3n}}{175}+\tfrac{2\delta_{4n}}{225} \tag{1.124}$$

$$\begin{aligned} L_{22}^{(10)} = \tfrac{1}{4158}\Big(30476+40162\cos\tfrac{2\pi}{n}+16072\cos\tfrac{4\pi}{n}+4093\cos\tfrac{6\pi}{n} \\ +628\cos\tfrac{8\pi}{n}+45\cos\tfrac{10\pi}{n}\Big)-\tfrac{15\delta_{3n}}{49}+\tfrac{4\delta_{4n}}{63}-\tfrac{2\delta_{5n}}{441} \end{aligned} \tag{1.125}$$

$$\begin{aligned} L_{22}^{(12)} = \tfrac{1}{840840}\Big(15673314+21975552\cos\tfrac{2\pi}{n}+10006023\cos\tfrac{4\pi}{n}+3122432\cos\tfrac{6\pi}{n} \\ +661402\cos\tfrac{8\pi}{n}+87296\cos\tfrac{10\pi}{n}+5461\cos\tfrac{12\pi}{n}\Big)-\tfrac{2847\delta_{3n}}{3080}+\tfrac{668\delta_{4n}}{2205}-\tfrac{17+\sqrt{5}}{441}\delta_{5n}+\tfrac{\delta_{6n}}{392}, \end{aligned} \tag{1.126}$$

$$\begin{aligned} L_{22}^{(14)} = \tfrac{1}{308880}\Big(15540360+22811745\cos\tfrac{2\pi}{n}+11429660\cos\tfrac{4\pi}{n}+4126221\cos\tfrac{6\pi}{n} \\ +1081192\cos\tfrac{8\pi}{n}+198713\cos\tfrac{10\pi}{n}+23124\cos\tfrac{12\pi}{n}+1285\cos\tfrac{14\pi}{n}\Big) \\ -\tfrac{384\delta_{3n}}{143}+\tfrac{1816\delta_{4n}}{1485}-\tfrac{3057+331\sqrt{5}}{14256}\delta_{5n}+\tfrac{11\delta_{6n}}{360}-\tfrac{\delta_{7n}}{648}, \end{aligned} \tag{1.127}$$

where δ_{jk} is the *Kronecker delta*. Based on the obtained results, we state the following conjecture:

Conjecture 25. *Let K be a regular n-sided polygon ($n \geq 3$) circumscribed by a circle with unit radius. Then for p being a positive even integer,*

$$L_{22}^{(p)} = \sum_{j=0}^{p/2} \left(a_{jp} \cos\left(\frac{2\pi j}{n}\right) + b_{jp}\delta_{jn} \right) \tag{1.128}$$

for some numbers a_{jp} and b_{jp}.

> *Remark* 26. Note that, since $n \geq 3$, the values of b_{0p}, b_{1p} and b_{2p} are not given uniquely as we can subtract them from a_{jp}'s.

Using Equation (1.103), we also found the following limit

$$\lim_{p\to -2^+} (2+p) L_{22}^{(p)} = \frac{4\pi}{n \sin \frac{2\pi}{n}}, \tag{1.129}$$

which is in agreement with general statement valid for any compact convex K, that

$$\lim_{p\to -2^+} (2+p) L_{22}^{(p)} = \frac{2\pi}{\operatorname{vol} K}, \tag{1.130}$$

which is a special case of Corollary 299.1 with $d = 2$.

Odd moments

When $p = -1$, we got

$$L_{E_i V_0}^{(-1)} = \frac{\csc(\pi/n)}{2} \ln \frac{\tan(\pi(i+1)/n)}{\tan(\pi i/n)}, \tag{1.131}$$

from which, immediately when n is odd,

$$L_{22}^{(-1)} = \frac{-4\sec(\pi/n)\csc(\pi/n)}{3n} \sum_{i=1}^{n-2} \frac{\sin^2(\pi i/n)\sin^2(\pi(i+1)/n)}{\cos(\pi i/n)\cos(\pi(i+1)/n)} \ln \frac{\tan(\pi(i+1)/n)}{\tan(\pi i/n)}. \tag{1.132}$$

Limit behaviour for large number of sides

In order to extract the limiting properties of P_{22}, we let n go to infinity but at the same time hold $\sigma_i = i/n$ as fixed. We denote $\varepsilon = 1/n$. By Taylor Expansion of Equation (1.108),

$$P_{E_i V_0} = 2^p \Big(1 + \tfrac{p\pi\varepsilon \cot(\pi\sigma_i)}{2} - \tfrac{p\pi^2\varepsilon^2(3-p-(1+p)\cos(2\pi\sigma_i))}{12} \csc^2(\pi\sigma_i)\Big) \sin^p(\pi\sigma_i) + O(\varepsilon^3), \tag{1.133}$$

from which

$$P_{E_i \partial E_0} = 2^{p-1}(2+p)\Big(1 - \tfrac{p\pi^2\varepsilon^2(6-p-(4+p)\cos(2\pi\sigma_i))}{24} \csc^2(\pi\sigma_i)\Big) \sin^p(\pi\sigma_i) + O(\varepsilon^3). \tag{1.134}$$

All together, by Equation (1.103),

$$\begin{aligned} P_{22} = {} & \tfrac{2^{4+p}\varepsilon}{(4+p)(3+p)(2+p)} \textstyle\sum_{i=1}^{n-2} \Big((3+p)\sin^{2+p}(\pi\sigma_i) + \tfrac{3\pi\varepsilon}{4}(2+p)\sin^p(\pi\sigma_i)\sin(2\pi\sigma_i) \\ & + \tfrac{p\pi^2\varepsilon^2}{24}\left(-12 - 2p + p^2 + (4+p)^2\cos(2\pi\sigma_i)\right)\sin^p(\pi\sigma_i) + O(\varepsilon^3)\Big). \end{aligned} \tag{1.135}$$

Let $f(x)$ be any sufficiently smooth function. Note that for any n, again denoting $\varepsilon = 1/n$ and $\sigma_i = i/n$,

$$\int_0^1 f(x)\ \mathrm{d}x = \sum_{i=0}^{n-1} \int_{\sigma_i}^{\sigma_i+\varepsilon} f(x)\ \mathrm{d}x, \tag{1.136}$$

so by expanding $f(x)$ into Taylor series around $x = \sigma_i$ and integrating, we get

$$\int_0^1 f(x)\mathrm{d}x = \varepsilon \sum_{i=0}^{n-1} \left(f(\sigma_i) + \frac{\varepsilon}{2} f'(\sigma_i) + \frac{\varepsilon^2}{6} f''(\sigma_i) + \frac{\varepsilon^3}{24} f'''(\sigma_i) + \frac{\varepsilon^4}{120} f^{(4)}(\sigma_i) + O(\varepsilon^5) \right). \tag{1.137}$$

Replacing f with f', f'' and so on, we can, by linear combination, invert this relation to

$$\varepsilon \sum_{i=0}^{n-1} f(\sigma_i) = \int_0^1 f(x) - \frac{\varepsilon}{2} f'(x) + \frac{\varepsilon^2}{12} f''(x) - \frac{\varepsilon^4}{720} f^{(4)}(x) + O(\varepsilon^5)\ \mathrm{d}x, \tag{1.138}$$

which is not surprising since it is essentially the Euler-Maclaurin formula. Since our sums run from 1 to $n-2$, so by subtracting terms with $i = 0$ and $i = n-1$ to the right hand side and performing necessary Taylor expansions, we get

$$\begin{aligned} \varepsilon \sum_{i=1}^{n-2} f(\sigma_i) = \int_0^1 f(x)\ \mathrm{d}x - \frac{\varepsilon}{2}(f(0) + 3f(1)) - \frac{\varepsilon^2}{12}\left(f'(0) - 13f'(1)\right) \\ - \frac{\varepsilon^3}{2} f''(1) + \frac{\varepsilon^4}{720}\left(f^{(3)}(0) + 119 f^{(3)}(1)\right) + O(\varepsilon^5). \end{aligned} \tag{1.139}$$

Therefore, summing Equation (1.135) using this relation, we get the following formula

$$P_{22} = \left(1 - \frac{p\pi^2}{3n^2} + O\left(\frac{1}{n^4}\right)\right) P_{22d}, \tag{1.140}$$

where

$$P_{22d} = L_{22d}^{(p)} = \frac{2^{4+p}\Gamma\left(\frac{3+p}{2}\right)}{(2+p)(4+p)\sqrt{\pi}\,\Gamma\left(2+\frac{p}{2}\right)} \tag{1.141}$$

is the mean distance p-th moment in a unit disk (Equation (1.93)). This approximation is valid for all $p > -2$. Using the same technique, we are able to further improve the estimate to

$$P_{22} = \left(1 - \frac{p\pi^2}{3n^2} + \frac{p^2(8+11p)\pi^4}{180(1+p)n^4} + \frac{p\left(16 + p\left(8 + p - 15p^2\right)\right)\pi^6}{1890(1+p)n^6} + O\left(\frac{1}{n^8}\right)\right) P_{22d} \tag{1.142}$$

with the property that the expansion is valid for $p > k - 5$ if it is truncated at $1/n^k$ term. The reason why the approximation is not correct there is because f is not sufficiently smooth at the endpoints for low p. However, we can treat those cases separately. Most notably, when $p = 1$, we got

$$L_{22}^{(1)} = \left(1 - \frac{\pi^2}{3n^2} + \frac{19\pi^4}{360n^4} + O\left(\frac{1}{n^6}\right)\right) P_{22d}. \tag{1.143}$$

1.4 Bivariate functionals in three dimensions

1.4.1 Ball

Consider a bivariate symmetric homogeneous functional P of order p dependent on two random points picked uniformly from the unit ball $\mathbb{B}_3 = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| \leq 1\}$ with volume $\mathrm{vol}_3 \mathbb{B}_3 = 4\pi/3$. Additionally, we require P to be *rotationally* symmetric with respect to the origin. That is, for any $\mathbf{x}, \mathbf{y} \in \mathbb{B}_3$ and any orthogonal matrix R we have $P(R\mathbf{x}, R\mathbf{y}) = P(\mathbf{x}, \mathbf{y})$. This assumption is satisfied by the choice $P = L^p$ (which is implicitly assumed in this section). Table 1.11 below shows various explicit $L_{33}^{(p)}$ distance moments for selected p's (from Equation (1.148)).

$L_{33}^{(-2)}$	$L_{33}^{(-1)}$	$L_{33}^{(0)}$	$L_{33}^{(1)}$	$L_{33}^{(2)}$	$L_{33}^{(3)}$	$L_{33}^{(4)}$	$L_{33}^{(5)}$	$L_{33}^{(6)}$	$L_{33}^{(7)}$	$L_{33}^{(8)}$	$L_{33}^{(9)}$
$\frac{9}{4}$	$\frac{6}{5}$	1	$\frac{36}{35}$	$\frac{6}{5}$	$\frac{32}{21}$	$\frac{72}{35}$	$\frac{32}{11}$	$\frac{64}{15}$	$\frac{4608}{715}$	$\frac{768}{77}$	$\frac{1024}{65}$

Table 1.11: Mean distance moments $L_{33}^{(p)}$ between two random points in $\mathbb{B}_3$

Reduction system

According to our convention, we write

$$P_{ab} = \mathbb{E}\left[P(\mathbf{X}, \mathbf{Y}) \mid \mathbf{X} \sim \mathsf{Unif}(A), \mathbf{Y} \sim \mathsf{Unif}(B)\right], \tag{1.144}$$

where $a = \dim A$, $b = \dim B$ and the concrete selection of A, B is deduced from the reduction diagram in Figure 1.5 below. In this diagram, we also included the position of the scaling point $\mathbf{C}$ in cases reduction is possible. The arrows indicate which configurations reduce to which. Each arrow is labeled by a roman numeral corresponding to a given reduction equation in the system of reduction equations.

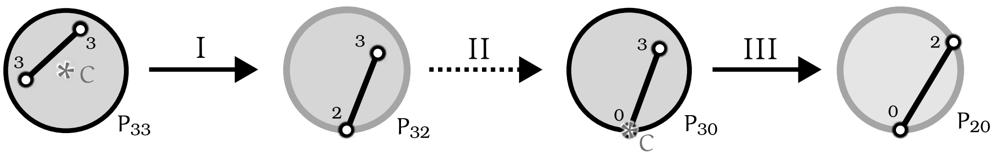


Figure 1.5: All different P_{ab} sub-configurations in $\mathbb{B}_3$

The full system obtained by CRT is

$$\begin{aligned} \mathbf{I}: \ pP_{33} &= 2 \cdot 3(P_{32} - P_{33}) \\ \mathbf{II}: \ P_{32} &= P_{30}. \\ \mathbf{III}: \ pP_{30} &= 3(P_{20} - P_{30}), \end{aligned}$$

where the equation **II** follows from the rotational symmetry of P. The solution of our system is

$$P_{33} = \frac{18P_{20}}{(6+p)(3+p)}. \tag{1.145}$$

$\mathbf{P_{20}}$

In configuration (20), one point $\mathbf{X}$ is drawn uniformly from the boundary $\partial\mathbb{B}_3$ while the other $\mathbf{Y}$ is fixed at the boundary. Keep in mind that P_{20} is defined via generalization of Remark 9 as a mean weighted by the support function

$$P_{20} = \frac{1}{3\,\mathrm{vol}_3\,\mathbb{B}_3} \int_{\partial\mathbb{B}_3} P(\mathbf{x},\mathbf{y}) h_{\mathbf{y}}(\mathbf{x}) \lambda_2(\mathrm{d}\mathbf{x}), \tag{1.146}$$

where the support function $h_{\mathbf{y}}(\mathbf{x})$ of $\mathbb{B}_3$ evaluated in $\mathbf{x}$ and centered at $\mathbf{y} \in \partial\mathbb{B}_3$ (arbitrary fixed point) is given explicitly as $h_{\mathbf{y}}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}-\mathbf{y}\|^2$. Parametrising the integral using spherical coordinates (θ, φ) with their center located at $\mathbf{y}$,

$$\begin{aligned} \mathbf{x} &= [2\sin\theta\cos\theta\cos\varphi, 2\sin\theta\cos\theta\sin\varphi, 2\cos^2\theta+1], && \theta \in [0, \pi/2), \\ \mathbf{y} &= [0, -1], && \varphi \in [0, 2\pi). \end{aligned}$$

Note that $\|\mathbf{x}-\mathbf{y}\| = 2\cos\theta$ and thus $h_{\mathbf{y}}(\mathbf{x}) = 2\cos^2\theta$. Furthermore, since $P = L^p$, we get $P(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|^p = (2\cos\theta)^p$. The uniform measure on $\partial\mathbb{B}_3$ is given by $\lambda_2(\mathrm{d}\mathbf{x}) = 4\sin\theta\cos\theta\mathrm{d}\theta\mathrm{d}\varphi$. Alternatively, integrating out the axial angle φ (which P does not depend on), we get $\lambda_2(\mathrm{d}\mathbf{x}) = 8\pi\sin\theta\cos\theta\mathrm{d}\theta$. Overall,

$$L_{20}^{(p)} = \mathbb{E}\left[\|\mathbf{X}-\mathbf{Y}\|^p\right] = \frac{1}{2}\int_0^{\pi/2} (2\cos\theta)^{3+p}\sin\theta\ \mathrm{d}\theta = \frac{2^{p+2}}{p+4}. \tag{1.147}$$

$\mathbf{P_{33}}$

Substituting P_{20} into Equation (1.145) with $P = L^p$, we get for general $p > -3$ (not necessarily an integer),

$$L_{33}^{(p)} = \frac{72 \cdot 2^p}{(6+p)(4+p)(3+p)}. \tag{1.148}$$

Plugging $p = 1$, the mean distance between two random points in the unit disk in various configurations is shown in Table 1.12.

L_{33}	L_{32}	L_{30}	L_{20}
$\frac{36}{35}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{8}{5}$

Table 1.12: Mean distance in $\mathbb{B}_3$ in various configurations

Note that L_{33} can be normalised to the first metric moment as

$$v_1^{(1)}(\mathbb{B}_3) = \frac{L_{33}}{\sqrt[3]{4\pi/3}} = \frac{18}{35}\sqrt[3]{\frac{6}{\pi}} \approx 0.63807479. \tag{1.149}$$

Distance density

The density $f_{33}(\lambda)$ of the random distance L between two interior points in $\mathbb{B}_3$ can be recovered from moments using inverse Mellin transform (see appendix A.5). By Equation (1.148), we have

$$\mathcal{M}[f_{33}] = L_{33}^{(p-1)} = \frac{72 \cdot 2^{p-1}}{(5+p)(3+p)(2+p)}. \tag{1.150}$$

Taking the inverse Mellin transform, we get, formally,

$$f_{33}(\lambda) = 72\,\mathcal{I}_2\mathcal{I}_3\mathcal{I}_5\delta(\lambda - 2). \tag{1.151}$$

From Table A.5 (see Appendix A), we immediately obtain

$$f_{33}(\lambda) = \tfrac{3}{16}\lambda^2(2-\lambda)^2(4+\lambda)\mathbb{1}_{\lambda<2}. \tag{1.152}$$

Interestingly, this density function is much simpler than $f_{22}(\lambda)$ in the unit disk.

1.4.2 General and special polyhedra

If $K = P_3$ is a polyhedron, the following configurations are irreducible in $\mathbb{R}^3$:

- A is a polygon and B is a point
- A and B are two skew line segments
- A and B are two parallel polygons or one polygon and one line segment parallel to it

Polygon and a point

In the first case, A is a polygon and B a point. Denote $\operatorname{proj}_A(\cdot)$ a perpendicular projection onto $\mathcal{A}(A)$. Next, denote h the distance between B and $\mathcal{A}(A)$. With $k = x - \operatorname{proj}_A B$ where $x \in A$, we have that

$$L_{AB}^{(p)} = \frac{1}{\operatorname{vol} A}\int_A (h^2+k^2)^{p/2}\,\mathrm{d}k \tag{1.153}$$

is expressible in terms of elementary functions. To see this, write and $\partial_i A, i = 1,\dots,s$ for the sides of the polygon A, oriented such that the path through the vertices of A is counterclockwise. Then, by inclusion/exclusion, and switching to polar coordinates

$$L_{AB}^{(p)} = \frac{1}{\operatorname{vol} A}\sum_{i=1}^{s}\int_{T_i}(h^2+k^2)^{p/2}\,\mathrm{d}k = \frac{1}{\operatorname{vol} A}\sum_{i=1}^{s}\int_{\alpha_i}^{\beta_i}\int_0^{h_i/\cos\varphi}(h^2+r^2)^{p/2}r\,\mathrm{d}r\mathrm{d}\varphi \tag{1.154}$$

where T_i is a signed triangle whose one vertex is the point $\operatorname{proj} B$ and the other two vertices are the consecutive endpoints of $\partial_i A$. Rescaling the vector k by h, we can rewrite each integral in the sum in a standard way

$$\int_{T_i}(h^2+k^2)^{p/2}\,\mathrm{d}k = h^{2+p}\left(I_{00}^{(p)}\left(h_i/h,\beta_i\right) - I_{00}^{(p)}\left(h_i/h,\alpha_i\right)\right) \tag{1.155}$$

where α_i and β_i are their respective polar angles (in counterclockwise order) and h_i is the perpendicular distance from $\operatorname{proj} B$ to $\partial_i A$. The polar angles are defined

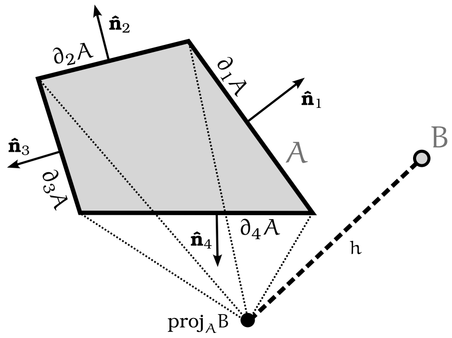


Figure 1.6: Point-polygon triangle decomposition

such that the closest point on the line $\mathcal{A}(\partial_i A)$ from $\operatorname{proj} B$ has its value equal to zero, increasing in the clockwise direction (see Figure 1.6). The integral is positive if the angle of consecutive vertices of the polygon increased and negative if it decreased.
Summing all contributions, we finally get our **point-polygon formula**

$$L_{AB}^{(p)} = \frac{h^{2+p}}{\operatorname{vol} A} \sum_{i=1}^{s} \left(I_{00}^{(p)}(h_i/h, \beta_i) - I_{00}^{(p)}(h_i/h, \alpha_i) \right). \tag{1.156}$$

Two skewed line segments

The second case is in fact equivalent with the first. If A and B are two skew line segments, write $A - B = \{u - v \mid u \in A, v \in B\}$ (which is a parallelogram). Then, by shifting, we get for any homogeneous P, denoting O as the origin

$$P_{AB} = P_{O,A-B}. \tag{1.157}$$

So we can always reduce this problem to the polygon and a point problem treated before.

Overlap formula

From now on, in case of no ambiguity, we often write simply proj instead of proj_A for the perpendicular projection operator onto $\mathcal{A}(A)$.

Proposition 27. *Let $A, B \in \mathcal{P}_+(\mathbb{R}^3)$, $a = 2$, $b \in \{1, 2\}$, such that $\mathcal{A}(A)$ and $\mathcal{A}(B)$ are parallel with perpendicular separation vector s having length $h = \|s\|$. Let $P(x, y)$ be homogeneous and let k be a vector lying in the projection plane $\mathcal{A}(A)$, then*

$$\begin{aligned} P_{AB} &= \frac{1}{\operatorname{vol} A \operatorname{vol} B} \int_A \int_B \tilde{P}(x - y) \, \mathrm{d}x \mathrm{d}y \\ &= \frac{1}{\operatorname{vol} A \operatorname{vol} B} \int_{\mathbb{R}^2} \tilde{P}(s + k) \operatorname{vol} A \cap (\operatorname{proj} B + k) \, \mathrm{d}k. \end{aligned} \tag{1.158}$$

Especially, for $P = L^p$, we get $L_{AB}^{(p)} = \frac{1}{\operatorname{vol} A \operatorname{vol} B} \int_{\mathbb{R}^2} (h^2 + k^2)^{p/2} \operatorname{vol} A \cap (\operatorname{proj} B + k) \, \mathrm{d}k$.

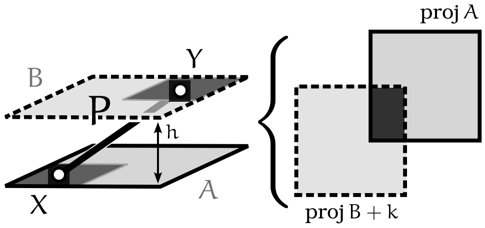


Figure 1.7: Overlap formula

Remark 28. Since $\mathrm{vol}\, A \cap (\mathrm{proj}\, B + k)$ is a piece-wise polynomial function of degree at most two on polygonal domains, the double integral is expressible in terms of elementary functions for any integer $p > -3$.

Proof. Let $A, B \subset \mathbb{R}^d$ be compact domains with dimensions a and b, respectively, and P be symmetric homogeneous functional $\mathbb{R}^d \to \mathbb{R}$ of order $p > -3$. Let $A_B(z) = A \cap (B + z)$, $c = \max_{z \in \mathbb{R}^d} \dim A_B(z)$ and $\mathbf{C} = \{z \in \mathbb{R}^d \mid \dim A_B(z) = c\}$. Then, by substitution $y = x + z$ and by Fubini's theorem,

$$P_{AB} = \frac{1}{\mathrm{vol}\, A\, \mathrm{vol}\, B} \int_A \int_B \tilde{P}(x - y)\ \mathrm{d}x\mathrm{d}y = \frac{1}{\mathrm{vol}\, A\, \mathrm{vol}\, B} \int_C \tilde{P}(z)\, \mathrm{vol}\, A_B(z)\ \mathrm{d}z. \tag{1.159}$$

When A, B are parallel in $d = 3$, the proposition follows. ■

Definition 29. An *overlap diagram* of A (face) and B (parallel face or edge) consists of partitions of $\mathbb{R}^2$ into open subdomains D_i where $\mathrm{vol}\, A \cap (\mathrm{proj}\, B + \mathbf{k})$ can be expressed as a single polynomial function in $\mathbf{k} \in \mathbb{R}^2$ of degree at most two. Since A and B are polygons or a polygon and a *polyline* (a piece-wise straight curve), respectively. These subdomains D_i are also polygonal (polylinial, respectively). When there is no ambiguity, we denote those subdomains D_i by numbers corresponding to the number of sides of the polygon $A \cap (\mathrm{proj}\, B + \mathbf{k})$ of intersection in case B is a face, or the number of line segments of the polyline $A \cap (\mathrm{proj}\, B + \mathbf{k})$ of intersection when B is an edge, respectively

Remark 30. For brevity, we often write $\mathrm{vol}\, A \cap \mathrm{proj}\, B + \mathbf{k}$ as a shorthand for $\mathrm{vol}\, A \cap (\mathrm{proj}\, B + \mathbf{k})$.

Auxiliary integrals

Apart from rotations and reflections, integrals encountered in this section have the following form ($h > 0$)

$$I_f^{(p)}(h, \zeta, \gamma) = \int_{D(\zeta,\gamma)} f(x, y) \left(h^2 + x^2 + y^2\right)^{p/2}\ \mathrm{d}x\mathrm{d}y, \tag{1.160}$$

where $D(\zeta, \gamma)$ is the *fundamental triangle domain* with vertices $[0, 0]$, $[\zeta, 0]$, $[\zeta, \zeta \tan \gamma]$ ($\zeta > 0$, $0 < \gamma < \pi/2$) and $f(x, y)$ is a polynomial in x and y of degree at most two (quadratic in x and y). We can write $f(x, y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2$. Based on x and y terms, we have the following

$$\begin{aligned} I_f^{(p)}(h, \zeta, \gamma) = {} & a_{00} I_{00}^{(p)}(h, \zeta, \gamma) + a_{10} I_{10}^{(p)}(h, \zeta, \gamma) + a_{01} I_{01}^{(p)}(h, \zeta, \gamma) \\ & + a_{20} I_{20}^{(p)}(h, \zeta, \gamma) + a_{11} I_{11}^{(p)}(h, \zeta, \gamma) + a_{02} I_{02}^{(p)}(h, \zeta, \gamma), \end{aligned} \tag{1.161}$$

where

$$I_{ij}^{(p)}(h,\zeta,\gamma) = \int_{D(\zeta,\gamma)} x^i y^j \left(h^2 + x^2 + y^2\right)^{p/2} \, \mathrm{d}x\mathrm{d}y. \tag{1.162}$$

The parameters of those integrals are not optimal. We only need to consider the case $h = 1$. To see this, denote

$$I_{ij}^{(p)}(q,\gamma) = I_{ij}^{(p)}(1,q,\gamma) = \int_{D(q,\gamma)} x^i y^j \left(1 + x^2 + y^2\right)^{p/2} \, \mathrm{d}x\mathrm{d}y. \tag{1.163}$$

By scaling $x \to hx$, $y \to hy$, we can write

$$I_{ij}^{(p)}(h,\zeta,\gamma) = h^{2+p+i+j} I_{ij}^{(p)}(\zeta/h,\gamma). \tag{1.164}$$

Thus, with $q = \zeta/h$,

$$\begin{aligned} I_f^{(p)}(h,\zeta,\gamma) = h^{2+p}\Big[& a_{00} I_{00}^{(p)}(q,\gamma) + a_{10} h I_{10}^{(p)}(q,\gamma) + a_{01} h I_{01}^{(p)}(q,\gamma) \\ & + a_{20} h^2 I_{20}^{(p)}(q,\gamma) + a_{11} h^2 I_{11}^{(p)}(q,\gamma) + a_{02} h^2 I_{02}^{(p)}(q,\gamma)\Big]. \end{aligned} \tag{1.165}$$

Selected values of the auxiliary integrals $I_{ij}^{(p)}(q,\gamma)$ and the methods how we can derive them are found in Appendix F.

General polyhedra

Theorem 31. *Let $K \in \mathcal{P}(\mathbb{R}^3)$, $E_j, F_k, j \in \{1,\ldots,e\}, k \in \{1,\ldots,f\}$, denote the edges and faces of K, respectively, and let $P : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be symmetric and homogeneous of order $p > -3$. Then*

$$P_{KK} = \frac{2}{(6+p)(5+p)} \left(\sum_{k<k'} P_{F_k F_{k'}} w_{F_k F_{k'}} + \sum_j P_{K E_j} w_{K E_j} \right), \tag{1.166}$$

with weights w_{AB} (independent on P and p) given as follows: We fix $\mathbf{C}$ any point in $\mathbb{R}^3$, $\mathbf{C}_k$ any point on $\mathcal{A}(F_k)$ and D_j any point on $\mathcal{A}(E_j)$. Denote $F_{k(j)}, F_{k'(j)}$ the two faces on which lies the edge E_j, then

$$w_{F_k F_{k'}} = \frac{\operatorname{vol} F_k \operatorname{vol} F_{k'}}{\operatorname{vol}^2 K} \left(h_{\mathbf{C}}(F_k) h_{\mathbf{C}_k}(F_{k'}) + h_{\mathbf{C}}(F_{k'}) h_{\mathbf{C}_{k'}}(F_k)\right), \tag{1.167}$$

$$w_{K E_j} = \frac{\operatorname{vol} K \operatorname{vol} E_j}{\operatorname{vol}^2 K} \left(h_{\mathbf{C}}(F_{k(j)}) h_{\mathbf{C}_{k(j)}}(D_j) + h_{\mathbf{C}}(F_{k'(j)}) h_{\mathbf{C}_{k'(j)}}(D_j)\right). \tag{1.168}$$

Proof. Use the Crofton Reduction Technique twice. ■

Remark 32. Note that the weights are not unique as they depend on the position of scaling points.

Remark 33. Note that if $P = L^p$ and for any polyhedron K, all terms P_{AB} in Equation (1.166) are either further reducible or A and B are parallel. In both cases, we can express $L_{AB}^{(p)}$ in terms of auxiliary integrals. Theorem 23 follows.

Nonparallel polyhedra

For polyhedra which have some special properties, we are able to further reduce Theorem 31 above.

Definition 34. Let $\mathcal{P}^*(\mathbb{R}^3)$ denote the set of all polyhedra having the property that affine hulls of any of its three faces of meet at a single point. We call them **nonparallel polyhedra**. Also, we denote $\mathcal{P}^*_{\text{convex}}(\mathbb{R}^3)$ a subset of those which are convex.

Theorem 35. *Let $K \in \mathcal{P}^*(\mathbb{R}^3)$ and $V_i, E_j, F_k, i \in \{1,\dots,v\}, j \in \{1,\dots,e\}, k \in \{1,\dots,f\}$, denote the vertices, edges and faces of K, respectively, and P be symmetric and homogeneous of order $p > -3$. Then*

$$P_{KK} = \frac{12}{(6+p)(5+p)(4+p)(3+p)} \left(\sum_{\substack{ik \\ V_i \notin \mathcal{A}(F_k)}} P_{V_i F_k} w_{V_i F_k} + \sum_{\substack{j<j' \\ \mathcal{A}(E_j) \cap \mathcal{A}(E_{j'}) = \emptyset}} P_{E_j E_{j'}} w_{E_j E_{j'}} \right) \tag{1.169}$$

for some weights w_{AB} which are independent on P and p.

Proof. Since no pair of faces nor edges are parallel, we can further reduce $P_{F_k F_{k'}}$ and P_{KE_j} from Theorem 31 twice. The weights are easily computable by choosing appropriate scaling points. Note that again the weights are not unique and depend on the selection of those scaling points. For example, let $\mathbf{C} \in \mathcal{A}(F_k) \cap \mathcal{A}(F_{k'})$, $k < k'$. Then by CRT, we get

$$P_{F_k F_{k'}} = \frac{2}{4+p} \left(P_{\partial F_k F_{k'}} + P_{F_k \partial F_{k'}} \right). \tag{1.170}$$

Note that both $P_{\partial F_k F_{k'}}$ and $P_{F_k \partial F_{k'}}$ are expressible as some linear combination of $P_{E_i F_k}$ with $\mathcal{A}(E_i) \cap \mathcal{A}(F_k)$. Finally, we can reduce even this term. Let $\mathbf{C}' \in \mathcal{A}(E_i) \cap \mathcal{A}(F_k)$, then

$$P_{E_i F_k} = \frac{1}{2+p} \left(P_{\partial E_i F_k} + 2 P_{E_i \partial F_k} \right), \tag{1.171}$$

which in turn is expressible as a linear combination of $P_{V_i F_k}$ and $P_{E_i E_{i'}}$ with $V_i \notin \mathcal{A}(F_k)$ and $\mathcal{A}(E_i) \cap \mathcal{A}(E_{i'}) = \emptyset$. The reduction of terms P_{KE_j} is similar. ■

Nonparallel convex polyhedra

In the case of convex nonparallel polyhedra, we can find very simple relations for weights w_{AB}. First, we start with a known formula (a special case of Proposition 298 with $d = 3$ and the factor of 2 absorbed into integration over the whole sphere $\mathbb{S}^2$ rather than the half-sphere $\mathbb{S}^2_+$)

Lemma 36. *Let K be a convex and compact set in $\mathbb{R}^3$ and P symmetric homogeneous of order $p > -3$, then*

$$P_{KK} = \frac{1/\operatorname{vol}^2 K}{(4+p)(3+p)} \int_{\mathbb{S}^2} \int_{\hat{\mathbf{n}}_\perp} \tilde{P}(\hat{\mathbf{n}}) \operatorname{vol}_1(\boldsymbol{\sigma} \cap K)^{4+p} \,\mathrm{d}\mathbf{y}\mathrm{d}\hat{\mathbf{n}}, \tag{1.172}$$

where the integration in carried over all directions $\hat{\mathbf{n}}$ on the unit sphere $\mathbb{S}^2$ with surface measure $\mathrm{d}\hat{\mathbf{n}}$ having $\int_{\mathbb{S}^2} \mathrm{d}\hat{\mathbf{n}} = 4\pi$ and over all points $\mathbf{y}$ on plane $\hat{\mathbf{n}}_\perp$ passing through the origin and being perpendicular to $\hat{\mathbf{n}}$. Finally, $\operatorname{vol}_1(\boldsymbol{\sigma} \cap K)$ denotes the length of the intersection of K and the line $\boldsymbol{\sigma}$ passing through point $\mathbf{y}$ in the direction of unit vector $\hat{\mathbf{n}}$.

Corollary 36.1. *By Fubini's theorem,*

$$\lim_{p\to -3^+}(3+p)P_{KK} = \frac{1}{\operatorname{vol} K}\int_{\mathbb{S}^2} \tilde{P}(\hat{\mathbf{n}})\,\mathrm{d}\hat{\mathbf{n}} \tag{1.173}$$

> *Remark* 37. Similar formulae as the Lemma above are available in higher dimensions as well and can be deduced from Blaschke-Petkantschin formula (see Appendix B).

Theorem 38. *Let $K \in \mathcal{P}^*_{\text{convex}}(\mathbb{R}^3)$ and V_i, E_j, F_k, P, w_{AB} be defined exactly as in Theorem 35. Denote h_{ik} the distance between V_i and $\mathcal{A}(F_k)$, similarly denote $h_{jj'}$ the distance between O and $\mathcal{A}(E_j - E_{j'})$ and $\theta_{jj'}$ the angle between E_j and $E_{j'}$ (on the same plane under perpendicular projection). Then*

$$\begin{aligned} P_{KK} = \frac{12/\operatorname{vol} K}{(6+p)(5+p)(4+p)(3+p)}\Bigg(&\sum_{\substack{ik\\ V_i\notin\mathcal{A}(F_k)}} P_{V_iF_k} n_{V_iF_k} \operatorname{vol} F_k h_{ik} \\ &+ \sum_{\substack{j<j'\\ \mathcal{A}(E_j)\cap\mathcal{A}(E_{j'})=\emptyset}} P_{E_jE_{j'}} n_{E_jE_{j'}} \operatorname{vol} E_j \operatorname{vol} E_{j'} h_{jj'} \sin\theta_{jj'}\Bigg), \end{aligned} \tag{1.174}$$

with weights n_{AB} satisfying the following projection relation: Choose a direction $\hat{\mathbf{n}}$ and project K onto a plane perpendicular to it. Then the weights corresponding to vertex-face pairs which overlap and to pairs of edges which cross add up to one. Symbolically,

$$1 = \sum_{\substack{ik\\ V_i\notin\mathcal{A}(F_k)}} n_{V_iF_k}\mathbb{1}_{\hat{\mathbf{n}}\in V_iF_k} + \sum_{\substack{j<j'\\ \mathcal{A}(E_j)\cap\mathcal{A}(E_{j'})=\emptyset}} n_{E_jE_{j'}}\mathbb{1}_{\hat{\mathbf{n}}\in E_jE_{j'}}, \tag{1.175}$$

where $\mathbb{1}_{\hat{\mathbf{n}}\in AB} = 1$ if there are points $x \in A, y \in B$ such that $x - y$ is parallel with $\hat{\mathbf{n}}$, otherwise $\mathbb{1}_{\hat{\mathbf{n}}\in AB} = 0$. On top of that, the extreme case where one of the points x, y lies on the boundary of A or B leaves the value $\mathbb{1}_{\hat{\mathbf{n}}\in AB}$ undefined.

Proof. The key observation is that the weights are independent of the choice of the function P as long it is symmetric and homogeneous. Let $\varepsilon > 0$ be small and $\hat{\mathbf{n}}$ be a fixed unit vector, $\Omega_\varepsilon = \pi\varepsilon^2 + O(\varepsilon^4)$ then denotes a solid angle with apex half angle equal to ε. We define $R^{(p)}(\varepsilon, \hat{\mathbf{n}}, x, y) = \|x - y\|^p$ if the angle between $\hat{\mathbf{n}}$ and $x - y$ is smaller than ε and zero otherwise. Alternatively, denote $\mathbf{C}(\varepsilon, V, \hat{\mathbf{n}})$ a double-cone region whose vertex is V, apex angle 2ε and the axis has direction $\hat{\mathbf{n}}$. Then for any domains A and B,

$$R^{(p)}_{AB}(\varepsilon, \hat{\mathbf{n}}) = \int_A\int_B R^{(p)}(\varepsilon, \hat{\mathbf{n}}, x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_A\int_{B\cap C(\varepsilon,x,\hat{\mathbf{n}})} \|x-y\|^p\,\mathrm{d}y\,\mathrm{d}x. \tag{1.176}$$

Note that R is symmetric and homogeneous in x, y of order p. Hence, by Lemma 36,

$$\lim_{p\to -3^+}(3+p)R^{(p)}_{KK}(\varepsilon, \hat{\mathbf{n}}) = \frac{1}{\operatorname{vol} K}\int_{\mathbb{S}^2}\tilde{R}^{(-3)}(\varepsilon, \hat{\mathbf{n}}, \hat{\mathbf{n}})\,\mathrm{d}\hat{\mathbf{n}} = \frac{2\Omega_\varepsilon}{\operatorname{vol} K} + O(\varepsilon^4). \tag{1.177}$$

On the other hand, via Theorem 35,

$$\lim_{p\to-3^+}(3+p)R^{(p)}_{KK}(\varepsilon,\hat{\mathbf{n}}) = 2\left(\sum_{\substack{ik\\ V_i\notin\mathcal{A}(F_k)}} R^{(-3)}_{V_iF_k}(\varepsilon,\hat{\mathbf{n}})w_{V_iF_k} + \sum_{\substack{j<j'\\ \mathcal{A}(E_j)\cap\mathcal{A}(E_{j'})=\emptyset}} R^{(-3)}_{E_jE_{j'}}(\varepsilon,\hat{\mathbf{n}})w_{E_jE_{j'}}\right). \tag{1.178}$$

We are able to express $R^{(-3)}_{V_iF_k}(\varepsilon,\hat{\mathbf{n}})$ and $R^{(-3)}_{E_jE_{j'}}(\varepsilon,\hat{\mathbf{n}})$ in the following way:

$$R^{(-3)}_{V_iF_k}(\varepsilon,\hat{\mathbf{n}}) = \frac{\Omega_\varepsilon \mathbb{1}_{\hat{\mathbf{n}}\in V_iF_k}}{\operatorname{vol} F_k h_{ik}} + O(\varepsilon^4), \quad R^{(-3)}_{E_jE_{j'}}(\varepsilon,\hat{\mathbf{n}}) = \frac{\Omega_\varepsilon \mathbb{1}_{\hat{\mathbf{n}}\in E_jE_{j'}}}{\operatorname{vol} E_j \operatorname{vol} E_{j'} h_{jj'}\sin\theta_{jj'}} + O(\varepsilon^4). \tag{1.179}$$

We will prove only the first equality as the other one is get simply by shifting (edge-edge configuration is equivalent to vertex-face configuration by means of Equation (1.157)). Let $V_i\notin\mathcal{A}(F_k)$ for some (polygonal) face F_k and vertex V_i. We denote by r the distance between V_i and the point of intersection of $\mathcal{A}(F_k)$ and the line passing through the vertex V_i in the direction of $\hat{\mathbf{n}}$. Note that the perpendicular distance h_{ik} between V_i and $\mathcal{A}(F_k)$ is independent on the direction of $\hat{\mathbf{n}}$. Since ε is small, we can write

$$R^{(p)}_{V_iF_k}(\varepsilon,\hat{\mathbf{n}}) = \frac{1}{\operatorname{vol} F_k}\int_{F_k} R^{(p)}(\varepsilon,\hat{\mathbf{n}},x,V_i)\,\mathrm{d}x = \frac{r^p \operatorname{vol} F_k\cap C(\varepsilon,V_i,\hat{\mathbf{n}})}{\operatorname{vol} F_k} + O(\varepsilon^4) \tag{1.180}$$

Assuming $\hat{\mathbf{n}}\in V_iF_k$, the point of intersection lies in the interior of F_k. Hence, for sufficiently small ε, we get that $V_i\cap C(\varepsilon,V_i,\hat{\mathbf{n}})$ is an ellipse with area

$$\operatorname{vol} V_i\cap C(\varepsilon,V_i,\hat{\mathbf{n}}) = \mathbb{1}_{\hat{\mathbf{n}}\in V_iF_k}\frac{\Omega_\varepsilon r^3}{h_{ik}} + O(\varepsilon^4) \tag{1.181}$$

Hence

$$R^{(p)}_{V_iF_k}(\varepsilon,\hat{\mathbf{n}}) = \frac{1}{\operatorname{vol} F_k}\int_{F_k} R^{(p)}(\varepsilon,\hat{\mathbf{n}},x,V_i)\,\mathrm{d}x = \frac{r^{3+p}\Omega_\varepsilon \mathbb{1}_{\hat{\mathbf{n}}\in V_iF_k}}{\operatorname{vol} F_k h_{ik}} + O(\varepsilon^4) \tag{1.182}$$

when $p=-3$, the dependency on r vanishes. Finally, comparing this relation with Equation (1.177), we get the equation for weights

$$\frac{1}{\operatorname{vol} K} = \sum_{\substack{ik\\ V_i\notin\mathcal{A}(F_k)}} \frac{w_{V_iF_k}\mathbb{1}_{\hat{\mathbf{n}}\in V_iF_k}}{\operatorname{vol} F_k h_{ik}} + \sum_{\substack{j<j'\\ \mathcal{A}(E_j)\cap\mathcal{A}(E_{j'})=\emptyset}} \frac{w_{E_jE_{j'}}\mathbb{1}_{\hat{\mathbf{n}}\in E_jE_{j'}}}{\operatorname{vol} E_j \operatorname{vol} E_{j'} h_{jj'}\sin\theta_{jj'}} \tag{1.183}$$

valid for any $\hat{\mathbf{n}}$ for which all the values $\mathbb{1}_{\hat{\mathbf{n}}\in AB}$ are well defined. Lastly, defining auxiliary weight n_{AB} via

$$w_{V_iF_k} = \frac{\operatorname{vol} F_k h_{ik} n_{V_iF_k}}{\operatorname{vol} K}, \qquad w_{E_jE_{j'}} = \frac{\operatorname{vol} E_j \operatorname{vol} E_{j'} h_{jj'} n_{E_jE_{j'}} \sin\theta_{jj'}}{\operatorname{vol} K}, \tag{1.184}$$

we get

$$1 = \sum_{\substack{ik\\ V_i\notin\mathcal{A}(F_k)}} n_{V_iF_k}\mathbb{1}_{\hat{\mathbf{n}}\in V_iF_k} + \sum_{\substack{j<j'\\ \mathcal{A}(E_j)\cap\mathcal{A}(E_{j'})=\emptyset}} n_{E_jE_{j'}}\mathbb{1}_{\hat{\mathbf{n}}\in E_jE_{j'}}. \tag{1.185}$$

This constrain alone enables us to determine admissible weights for any convex nonparallel polyhedron via set of linear equations got by varying the direction of $\hat{\mathbf{n}}$. ∎

Tetrahedron

As an example, we express the random distance moments in the case of a tetrahedron. There are two possible ways how a planar projection of a tetrahedron could look like (almost surely) with respect to the number of intersecting pairs of edges and vertices/faces in the projection (see Figure 1.8).

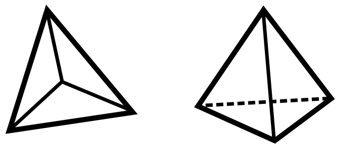


Figure 1.8: Tetrahedron projection orientations

In the first case, one vertex covers one face. There are no other vertex/face nor edge/edge coverings. Similarly, in the second case, one edge is covered by another edge. There are again no other coverings. Thus, in order to satisfy Equation (1.175), we can simply choose $n_{V_iF_k} = n_{E_jE_{j'}} = 1$ for each vertex V_i, face F_k and edges $E_j, E_{j'}$. Hence, by Theorem 38,

$$\begin{aligned} P_{KK} = \frac{12/\operatorname{vol} K}{(6+p)(5+p)(4+p)(3+p)} \Bigg(& \sum_{\substack{ik \\ V_i \notin \mathcal{A}(F_k)}} P_{V_iF_k} \operatorname{vol} F_k h_{ik} \\ + & \sum_{\substack{j<j' \\ \mathcal{A}(E_j)\cap\mathcal{A}(E_{j'})=\emptyset}} P_{E_jE_{j'}} \operatorname{vol} E_j \operatorname{vol} E_{j'} h_{jj'} \sin\theta_{jj'} \Bigg). \end{aligned} \tag{1.186}$$

Regular polyhedra

To apply our general method, we shall derive the mean distance in all five regular polyhedra (also known as Platonic solids). Among those solids, only the tetrahedron is nonparallel convex, so Theorem 38 applies here. Hence, we used this theorem to find the mean distance in a general (possibly irregular) tetrahedron. In the following sections, we calculate the mean distance in all other Platonic solids (including the regular tetrahedron again). Since they are an example of parallel polyhedra, we cannot use Theorem 38 due to presence of irreducible configurations of type face-face and edge-face. However, we can still calculate the mean distance. The calculation relies the Overlap formula as well as the symmetries of those regular polyhedra which drastically reduce the number of configurations needed to be considered. Throughout this section, we denote ν the area of (any) face of K and l the length (any) of its edge. These values makes sense because K is a regular polyhedron. Furthermore, $\phi = \frac{1+\sqrt{5}}{2}$ is the Golden ratio.

1.4.3 Regular tetrahedron

Let us have P bivariate symmetric homogeneous of order p dependent on two random points picked from K a regular tetrahedron given by vertices $V_1[1,0,0]$, $V_2[0,1,0]$, $V_3[0,0,1]$, $V_4[1,1,1]$, edges connecting them E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{34} ($E_{ij} = \overline{V_iV_j}$, where $i \neq j$) and with opposite faces F_1, F_2, F_3, F_4. Note that the edge length is $a = \sqrt{2}$ and the volume $\operatorname{vol} K = 1/3$, so if we want to express the mean of P in a tetrahedron of unit volume, we must multiply all our results by $3^{p/3}$. We put $P = L^p$. For the definition of various mean values $P_{ab} = L_{ab}^{(p)}$, see Figure 1.9. We also included the position of the scaling point $\mathbf{C}$ in cases reduction is possible. The arrows indicate which configurations reduce to which.

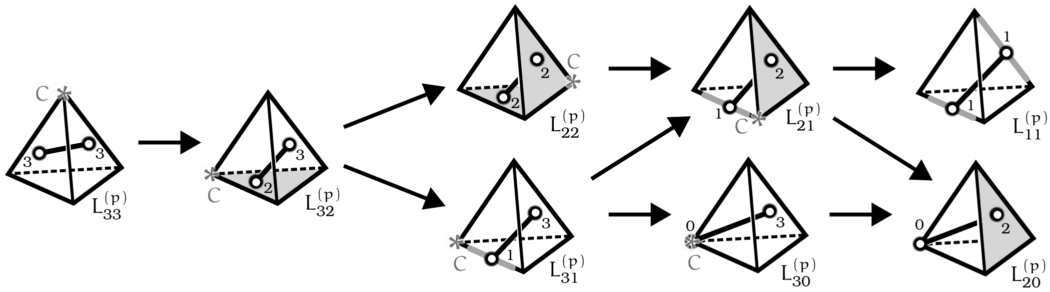


Figure 1.9: All different $L_{ab}^{(p)}$ configurations encountered for K being a regular tetrahedron

Based on CRT, let us write our reduction system of equations:

$$\begin{aligned}
pP_{33} &= 6(P_{32} - P_{33}) \\
pP_{32} &= 3(P_{22} - P_{32}) + 2(P_{31} - P_{32}) \\
pP_{22} &= 4(P_{21} - P_{22}) \\
pP_{31} &= 3(P_{21} - P_{31}) + 1(P_{30} - P_{31}) \\
pP_{21} &= 2(P_{11} - P_{21}) + 1(P_{20} - P_{21}) \\
pP_{30} &= 3(P_{20} - P_{30}),
\end{aligned}$$

where $P_{33} = P_{KK}$ and by symmetry, we can put $P_{32} = P_{KF_1}, P_{31} = P_{KE_{12}}, P_{30} = P_{KV_1}, P_{22} = P_{F_1F_2}, P_{21} = P_{F_4V_{14}}, P_{20} = P_{F_4V_4}, P_{11} = P_{E_{12}E_{34}}$. This linear system has a solution

$$P_{33} = \frac{72(3P_{11} + 2P_{20})}{(6+p)(5+p)(4+p)(3+p)}. \tag{1.187}$$

To demonstrate our technique for irreducible configurations, we derive the value of L_{33}. That means, we choose $P = L^p$ with $p = 1$.

$\mathbf{L_{20}}$

By (1.156), by symmetry and using $\operatorname{vol} F_4 = \sqrt{3}/2, h_1 = 1/\sqrt{6}, h = 2/\sqrt{3}$,

$$P_{20} = L_{F_4V_4}^{(p)} = \frac{6h^{2+p}}{\operatorname{vol} F_4} I_{00}^{(p)}\left(\frac{\sqrt{2}}{4}, \frac{\pi}{3}\right). \tag{1.188}$$

Using the recursion relations,

$$I_{00}^{(1)}\left(\frac{\sqrt{2}}{4}, \frac{\pi}{3}\right) = \frac{1}{16\sqrt{2}} - \frac{\pi}{9} + \frac{1}{3}\arcsin\sqrt{\frac{2}{3}} + \frac{25}{96\sqrt{2}} \operatorname{argsinh}\frac{1}{\sqrt{3}}, \tag{1.189}$$

so, further using $\arcsin\sqrt{2/3} = \arctan\sqrt{2}$ and $\operatorname{argsinh}(1/\sqrt{3}) = \frac{1}{2}\ln 3$,

$$L_{20} = \frac{\sqrt{2}}{3} - \frac{32\pi}{27} + \frac{32}{9}\arctan\sqrt{2} + \frac{25\ln 3}{18\sqrt{2}}. \tag{1.190}$$

L_{11}

By shifting (1.157), we get $L_{11} = L_{AB}$, where B is the origin and A is a parallelogram with vertices $[1,0,-1],[0,1,-1]$, $[-1,0,-1]$, $[0,-1,-1]$. Therefore, by the point-polygon formula (1.156) with $h=1$ and $\operatorname{vol} A = 2$,

$$L_{11} = \frac{8h^3}{\operatorname{vol} A} I_{00}^{(1)}\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right), \tag{1.191}$$

where by recurrences,

$$I_{00}^{(1)}\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right) = \frac{1}{6\sqrt{2}} - \frac{\pi}{12} + \frac{1}{3}\arcsin\frac{1}{\sqrt{3}} + \frac{7}{12\sqrt{2}}\operatorname{argsinh}\frac{1}{\sqrt{3}}. \tag{1.192}$$

Hence, writing $\arcsin\left(1/\sqrt{3}\right) = \frac{\pi}{2} - \arctan\sqrt{2}$ and $\operatorname{argsinh}(1/\sqrt{3}) = \frac{1}{2}\ln 3$,

$$L_{11} = \frac{\sqrt{2}}{3} + \frac{\pi}{3} - \frac{4}{3}\arctan\sqrt{2} + \frac{7\ln 3}{6\sqrt{2}}. \tag{1.193}$$

L_{33}

Substituting L_{20} and L_{11} into Equation (1.187) with $P = L^p$ and $p=1$, we get, finally

$$L_{33} = \frac{3}{35}(3L_{11} + 2L_{20}) = \frac{\sqrt{2}}{7} - \frac{37\pi}{315} + \frac{4}{15}\arctan\sqrt{2} + \frac{113\ln 3}{210\sqrt{2}}.$$

Or, re-scaling to the unit volume tetrahedron,

$$v_1^{(1)}(T_3) = \sqrt[3]{3}\left(\frac{\sqrt{2}}{7} - \frac{37\pi}{315} + \frac{4}{15}\arctan\sqrt{2} + \frac{113\ln 3}{210\sqrt{2}}\right) \approx 0.72946242, \tag{1.194}$$

which is an *exact* expression of an approximation given by Weisstein [75]. Similarly, we would proceed in the case of the second moment:

$$v_1^{(2)}(T_3) = \frac{9}{10\sqrt[3]{3}}. \tag{1.195}$$

Alternatively, we can express the result as the normalised mean distance Γ_{KK}. Since $V_1(K) = 3\sqrt{2}\arccos\left(-\frac{1}{3}\right)/\pi$ (see Table 1.4 with $a = \sqrt{2}$), we have

$$\Gamma_{KK} = \frac{L_{33}}{V_1(K)} = \frac{\pi\left(\frac{\sqrt{2}}{7} - \frac{37\pi}{315} + \frac{4}{15}\arctan\sqrt{2} + \frac{113\ln 3}{210\sqrt{2}}\right)}{3\sqrt{2}\arccos\left(-\frac{1}{3}\right)} \approx 0.19601928. \tag{1.196}$$

Of course, using the reduction technique, we could get other moments (replacing $I_{ij}^{(1)}$ by $I_{ij}^{(p)}$ integrals), and even for a general edge-length tetrahedron.

General moments

It is convenient to express our distance moments $L_{33}^{(k)}$ in regular tetrahedron with edge-length $a = 1$. These moments are deduced from our previous formulae via rescaling by $2^{-p/2}$. By Equation (1.187) and by rescaling,

$$L_{33}^{(p)}\Big|_{a=1} = \frac{\left(\frac{1}{2}\right)^{\frac{p}{2}} 72(3P_{11}+2P_{20})}{(6+p)(5+p)(4+p)(3+p)} = \frac{72\left[12\left(\frac{1}{2}\right)^{\frac{p}{2}} I_{00}^{(p)}\left(\frac{\sqrt{2}}{2},\frac{\pi}{4}\right) + \frac{32}{\sqrt{3}}\left(\frac{2}{3}\right)^{\frac{p}{2}} I_{00}^{(p)}\left(\frac{\sqrt{2}}{4},\frac{\pi}{3}\right)\right]}{(6+p)(5+p)(4+p)(3+p)}. \tag{1.197}$$

Writing out the auxiliary integrals $I_{00}^{(p)}$ explicitly, we get

$$\begin{aligned} L_{33}^{(p)}\Big|_{a=1} &= \frac{24}{(2+p)(3+p)(4+p)(5+p)(6+p)}\left[-9\pi\left(\frac{1}{2}\right)^{\frac{p}{2}} - \frac{32\pi}{\sqrt{3}}\left(\frac{2}{3}\right)^{\frac{p}{2}}\right. \\ &\left. + 36\left(\frac{1}{2}\right)^{\frac{p}{2}}\int_0^{\frac{\pi}{4}}\left(1+\tfrac{1}{2}\sec^2\varphi\right)^{\frac{p}{2}+1}\mathrm{d}\varphi + 32\sqrt{3}\left(\frac{2}{3}\right)^{\frac{p}{2}}\int_0^{\frac{\pi}{3}}\left(1+\tfrac{1}{8}\sec^2\varphi\right)^{\frac{p}{2}+1}\mathrm{d}\varphi\right]. \end{aligned} \tag{1.198}$$

Density

The density can be recovered from moments using inverse Mellin transform (see appendix A.5). For the density $f_{33}(\lambda)$ of the random distance between two points in a tetrahedron with unit edge-length $a = 1$, we have by Equation (1.198)

$$\begin{aligned} \mathcal{M}[f] = L_{33}^{(p-1)}\Big|_{a=1} &= \frac{24}{(1+p)(2+p)(3+p)(4+p)(5+p)}\left[-9\pi\left(\frac{1}{2}\right)^{\frac{p-1}{2}} - \frac{32\pi}{\sqrt{3}}\left(\frac{2}{3}\right)^{\frac{p-1}{2}}\right. \\ &\left. + 36\left(\frac{1}{2}\right)^{\frac{p-1}{2}}\int_0^{\frac{\pi}{4}}\left(1+\tfrac{1}{2}\sec^2\varphi\right)^{\frac{p+1}{2}}\mathrm{d}\varphi + 32\sqrt{3}\left(\frac{2}{3}\right)^{\frac{p-1}{2}}\int_0^{\frac{\pi}{3}}\left(1+\tfrac{1}{8}\sec^2\varphi\right)^{\frac{p+1}{2}}\mathrm{d}\varphi\right]. \end{aligned} \tag{1.199}$$

Taking the inverse Mellin transform, we get, formally,

$$\begin{aligned} f_{33}(\lambda) = 24\,\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3\mathcal{I}_4\mathcal{I}_5 &\left[-9\pi\delta(\lambda-\sqrt{\tfrac{1}{2}}) - \frac{32\pi}{\sqrt{3}}\delta(\lambda-\sqrt{\tfrac{2}{3}})\right. \\ &+ 36\int_0^{\frac{\pi}{4}}\left(1+\tfrac{1}{2}\sec^2\varphi\right)\delta\left(\lambda-\sqrt{\tfrac{1}{2}}\sqrt{1+\tfrac{1}{2}\sec^2\varphi}\right)\mathrm{d}\varphi \\ &\left.+ 32\sqrt{3}\int_0^{\frac{\pi}{3}}\left(1+\tfrac{1}{8}\sec^2\varphi\right)\delta\left(\lambda-\sqrt{\tfrac{2}{3}}\sqrt{1+\tfrac{1}{8}\sec^2\varphi}\right)\mathrm{d}\varphi\right] \end{aligned} \tag{1.200}$$

From Table A.5 (see Appendix A),

$$\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3\mathcal{I}_4\mathcal{I}_5\delta(\lambda-\alpha) = \frac{\lambda(\alpha-\lambda)^4}{24\alpha^6}\mathbb{1}_{\lambda<\alpha}. \tag{1.201}$$

via which we can deduce for $\lambda \in (0,1)$ that

$$\begin{aligned} f_{33}(\lambda) &= -72\pi\lambda\left(\sqrt{\tfrac{1}{2}}-\lambda\right)^4\mathbb{1}_{\lambda<\sqrt{\frac{1}{2}}} - 36\pi\lambda\sqrt{3}\left(\sqrt{\tfrac{2}{3}}-\lambda\right)^4\mathbb{1}_{\lambda<\sqrt{\frac{2}{3}}} \\ &+ 72\lambda\int_0^{\frac{\pi}{4}}\left(1-\frac{\sqrt{2}\lambda}{\sqrt{1+\frac{1}{2}\sec^2\varphi}}\right)^4\mathbb{1}_{\lambda<\sqrt{\frac{1}{2}}\sqrt{1+\frac{\sec^2(\varphi)}{2}}}\,d\varphi \\ &+ 48\sqrt{3}\lambda\int_0^{\frac{\pi}{3}}\left(1-\frac{\sqrt{\frac{3}{2}}\lambda}{\sqrt{1+\frac{1}{8}\sec^2\varphi}}\right)^4\mathbb{1}_{\lambda<\sqrt{\frac{2}{3}}\sqrt{1+\frac{1}{8}\sec^2\varphi}}\,d\varphi. \end{aligned} \tag{1.202}$$

Substituting $l = \sqrt{\frac{1}{2}}\sqrt{1+\frac{1}{2}\sec^2\varphi}$ and $l = \sqrt{\frac{2}{3}}\sqrt{1+\frac{1}{8}\sec^2\varphi}$ respectively, by inclusion/exclusion and after some simplifications, we get

$$\begin{aligned} f_{33}(\lambda) &= 24\sqrt{2}\pi\lambda^2 - 72\sqrt{3}\pi\lambda^3 + 144\left(2+\sqrt{2}\arctan\sqrt{2}\right)\lambda^4 - 6\left(6+5\sqrt{3}\pi\right)\lambda^5 \\ &+ 72\pi\lambda\left(\sqrt{\tfrac{1}{2}}-\lambda\right)^4 \mathbb{1}_{\lambda>\sqrt{\frac{1}{2}}} + 36\pi\lambda\sqrt{3}\left(\sqrt{\tfrac{2}{3}}-\lambda\right)^4 \mathbb{1}_{\lambda>\sqrt{\frac{2}{3}}} \\ &- \mathbb{1}_{\lambda>\frac{\sqrt{3}}{2}} \int_{\sqrt{3}/2}^{\lambda} \frac{144\lambda\left(5l^2-3\right)(l-\lambda)^4}{l^3\left(2l^2-1\right)\left(3l^2-2\right)\sqrt{4l^2-3}}\,\mathrm{d}l. \end{aligned} \tag{1.203}$$

Calculating the remaining integral is not hard. We got for all $\lambda \in (0,1)$,

$$\begin{aligned} f_{33}(\lambda) &= 24\sqrt{2}\pi\lambda^2 - 72\sqrt{3}\pi\lambda^3 + 144\left(2+\sqrt{2}\arctan\sqrt{2}\right)\lambda^4 - 6\left(6+5\sqrt{3}\pi\right)\lambda^5 \\ &+ 72\pi\lambda\left(\sqrt{\tfrac{1}{2}}-\lambda\right)^4 \mathbb{1}_{\lambda>\sqrt{\frac{1}{2}}} + 36\pi\lambda\sqrt{3}\left(\sqrt{\tfrac{2}{3}}-\lambda\right)^4 \mathbb{1}_{\lambda>\sqrt{\frac{2}{3}}} \\ &- 12\lambda\mathbb{1}_{\lambda>\frac{\sqrt{3}}{2}}\Bigg[21\lambda^2\sqrt{4\lambda^2-3} + \sqrt{3}\left(4+36\lambda^2+9\lambda^4\right)\arctan\left(\sqrt{12\lambda^2-9}\right) \\ &- 12\sqrt{2}\lambda\left(2+3\lambda^2\right)\arctan\left(\sqrt{8-\tfrac{6}{\lambda^2}}\right) - 24\sqrt{2}\lambda\left(1+2\lambda^2\right)\arctan\left(\sqrt{2-\tfrac{3}{2\lambda^2}}\right) \\ &+ 6\left(1+12\lambda^2+4\lambda^4\right)\arctan\left(\sqrt{4\lambda^2-3}\right) - 3\sqrt{3}\lambda^2\left(12+5\lambda^2\right)\arccos\left(\tfrac{\sqrt{3}}{2\lambda}\right)\Bigg]. \end{aligned} \tag{1.204}$$

1.4.4 Cube

We present a re-derivation of the Robbins constant for K being a cube via our method. Here, we demonstrate the Crofton Reduction Technique including the overlap formula. A standard way how to choose its vertices is $[0,0,0]$, $[1,0,0]$, $[0,1,0]$, $[0,0,1]$, $[0,1,1]$, $[1,0,1]$, $[1,1,0]$, $[1,1,1]$. Under this choice, the edge length $l=1$, face area $\nu=1$ and the volume $\operatorname{vol} K = 1$. We put $P = L^p$. For the definition of various mean values $P_{ab} = L_{ab}^{(p)}$, see Figure 1.10. Note that in L_{21r} configuration, we let B to be four edges (boundary of an opposite face) rather than just one edge.
Performing the reduction, we get the set of equations, where

$$\begin{aligned} pP_{33} &= 6(P_{32}-P_{33}), \\ pP_{32} &= 3(P_{22}-P_{32}) + 2(P_{31}-P_{32}), \\ pP_{22e} &= 4(P'_{21}-P_{22e}), \\ pP_{31} &= 3(P_{21}-P_{31}) + 1(P_{30}-P_{31}), \\ pP_{21v} &= 2(P_{11}-P_{21v}) + 1(P_{20}-P_{21v}), \\ pP_{30} &= 3(P_{20}-P_{30}) \end{aligned}$$

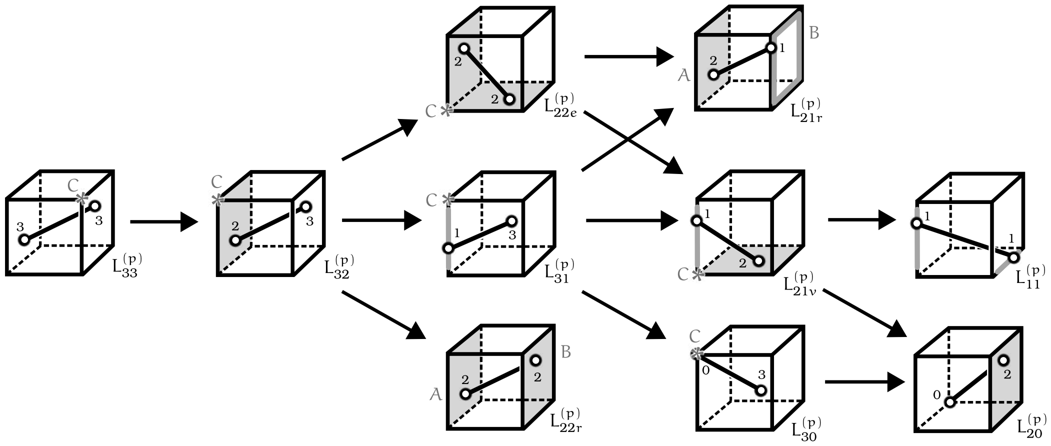


Figure 1.10: All different $L_{ab}^{(p)}$ configurations encountered for K being a cube

with

$$
\begin{aligned}
P_{22} &= \tfrac{2}{3}P_{22e} + \tfrac{1}{3}P_{22r}, \\
P_{21} &= \tfrac{1}{3}P_{21v} + \tfrac{2}{3}P_{21r}, \\
P'_{21} &= \tfrac{1}{2}P_{21v} + \tfrac{1}{2}P_{21r}.
\end{aligned}
$$

Solving the system, we get

$$P_{33} = \frac{72(P_{11}+P_{20})}{(3+p)(4+p)(5+p)(6+p)} + \frac{48P_{21r}}{(4+p)(5+p)(6+p)} + \frac{6P_{22r}}{(5+p)(6+p)}. \tag{1.205}$$

When $p = 1$, we get for the mean distance

$$L_{33} = \frac{1}{35}(3L_{11} + 3L_{20} + 8L_{21r} + 5L_{22r}). \tag{1.206}$$

$\mathbf{L_{20}}$

Without loss of generality, we can write $L_{20} = L_{AB}$, where A is the cube's upper face defined as a square with vertices $[0,0,1]$, $[1,0,1]$, $[1,1,1]$, $[0,1,1]$ and B is the origin $[0,0,0]$. Domains A and B are separated by distance $h = 1$. The face A is having area $\operatorname{vol} A = 1$. By (1.156) and by symmetry,

$$L_{20} = \frac{2}{\operatorname{vol} A} I_{00}^{(1)}\left(1, \frac{\pi}{4}\right). \tag{1.207}$$

Using recurrence relations (see Table F.1 in Appendix),

$$L_{20} = \frac{1}{\sqrt{3}} - \frac{\pi}{18} + \frac{4}{3}\operatorname{argcoth}\sqrt{3}. \tag{1.208}$$

$\mathbf{L_{11}}$

The value L_{11} can be defined as a mean distance between egde $E_1 = \overline{[0,0,0][0,1,0]}$ and edge $E_2 = \overline{[0,1,1][1,1,1]}$. Shifting E_1 by vector $-E_2$ (See shifting relation (1.157)), we can rewrite this as $L_{11} = L_{AB}$, where again A is the upper face of the cube and B is the origin. Hence $L_{11} = L_{20}$.

$\mathbf{L_{21r}}$

Since the reduction technique cannot be applied on AB being parallel, we use the overlap formula with A being one face of the cube. In case of L_{21r}, the other domain B is an opposite edge. By symmetry, we can add to this edge also three other edges opposite to A (see Figure 1.10). Hence, B is a boundary of the face opposite to A with length $\operatorname{vol} B = 4$. Let $k = (x, y)$ then $\operatorname{proj} A = \operatorname{proj} B$ is a square with vertices $[\frac{1}{2}, \frac{1}{2}]$, $[-\frac{1}{2}, \frac{1}{2}]$, $[-\frac{1}{2}, -\frac{1}{2}]$, $[\frac{1}{2}, -\frac{1}{2}]$. In order to $\operatorname{proj} A$ and $\operatorname{proj} B + k$ have nonzero intersection, k must be confined in the region $-1 < x < 1 \wedge -1 < y < 1$. By symmetry, we can chose k to lie in the fundamental triangle domain $D(1, \pi/4)$ (we then multiply the values by 8).

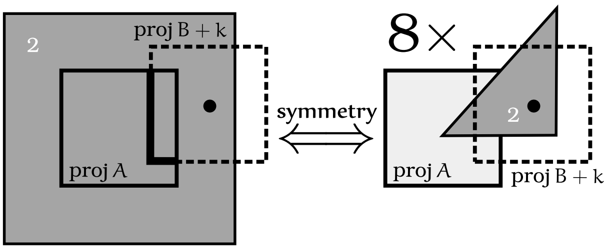


Figure 1.11: Overlap of the opposite face and edges of a cube

Setting up the integral,

$$L_{21r} = \frac{8}{\operatorname{vol} A \operatorname{vol} B} \int_D \sqrt{h^2 + k^2} \operatorname{vol} A \cap \operatorname{proj} B + k \, \mathrm{d}k, \tag{1.209}$$

where $h = 1, \operatorname{vol} A = 1, \operatorname{vol} B = 4$ and $D = D(1, \pi/4)$ is a domain in Figure 1.11 on the right (labeled with the number 2). In this domain, we can write for the length of the polyline of intersection

$$\operatorname{vol} A \cap \operatorname{proj} B + k = 2 - x - y, \tag{1.210}$$

which gives us in terms of our auxiliary integrals

$$L_{21r} = \frac{8}{\operatorname{vol} A \operatorname{vol} B} \left[2I_{00}^{(1)}\left(1, \frac{\pi}{3}\right) - I_{10}^{(1)}\left(1, \frac{\pi}{4}\right) - I_{01}^{(1)}\left(1, \frac{\pi}{4}\right) \right] \tag{1.211}$$

Via recursions (see Table F.1 in Appendix), we get

$$L_{21r} = \frac{7}{6\sqrt{2}} - \frac{1}{\sqrt{3}} - \frac{\pi}{9} + \frac{1}{4} \operatorname{argcoth} \sqrt{2} + \frac{5}{3} \operatorname{argcoth} \sqrt{3}. \tag{1.212}$$

$\mathbf{L_{22r}}$

Again, we use the overlap formula for AB being opposite faces. By symmetry, we again integrate $\operatorname{vol} A \cap \operatorname{proj} B + k$ over one eighth of all positions of k (see Figure 1.12).
Setting up the integral,

$$L_{22r} = \frac{8}{\operatorname{vol} A \operatorname{vol} B} \int_D \sqrt{h^2 + k^2} \operatorname{vol} A \cap \operatorname{proj} B + k \, \mathrm{d}k, \tag{1.213}$$

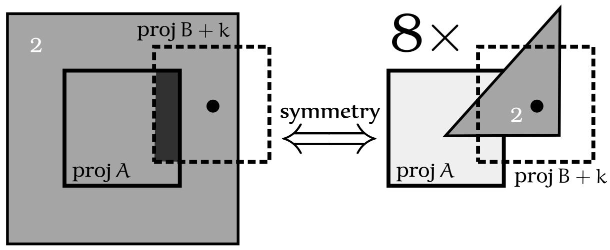


Figure 1.12: Overlap of the opposite faces of a cube

where $h = 1, \operatorname{vol} A = \operatorname{vol} B = 1$ and $D = D(1, \pi/4)$ is a fundamental triangle domain (labeled 2 in Figure 1.15 on the right). In this domain, we have for the polygon of intersection

$$\operatorname{vol} A \cap \operatorname{proj} B + k = (1-x)(1-y), \tag{1.214}$$

and therefore

$$L_{22r} = \frac{8}{\operatorname{vol} A \operatorname{vol} B}\left[I_{00}^{(1)}\left(1, \frac{\pi}{4}\right) - I_{10}^{(1)}\left(1, \frac{\pi}{4}\right) - I_{01}^{(1)}\left(1, \frac{\pi}{4}\right) + I_{11}^{(1)}\left(1, \frac{\pi}{4}\right)\right]. \tag{1.215}$$

Going through all recursions, we get, after simplifications

$$L_{22r} = \frac{4}{15} + \frac{\sqrt{2}}{5} - \frac{4}{5\sqrt{3}} - \frac{2\pi}{9} + \operatorname{argcoth}\left(\sqrt{2}\right) + \frac{4}{3}\operatorname{argcoth}\sqrt{3}. \tag{1.216}$$

L_{33}

Putting everything together by using (1.219), we finally arrive at Robin's constant

$$v_1^{(1)}(C_3) = L_{33} = \frac{4}{105} + \frac{17\sqrt{2}}{105} - \frac{2\sqrt{3}}{35} - \frac{\pi}{15} + \frac{1}{5}\operatorname{argcoth}\sqrt{2} + \frac{4}{5}\operatorname{argcoth}\sqrt{3} \approx 0.66170718. \tag{1.217}$$

1.4.5 Regular octahedron

A standard way how to select vertices of an regular octahedron the vertices is $[\pm1, 0, 0]$, $[\pm1, 0, 0]$, $[\pm1, 0, 0]$. Under this choice, the edge length is $l = \sqrt{2}$, the area of each face is $\nu = \sqrt{3}/2$ and the volume of K is $\operatorname{vol} K = 4/3$. Again, we put $P = L^p$. For the definition of various mean values $P_{ab} = L_{ab}^{(p)}$, see Figure 1.13. We also included the position of the scaling point $\mathbf{C}$ in cases when the reduction is possible.

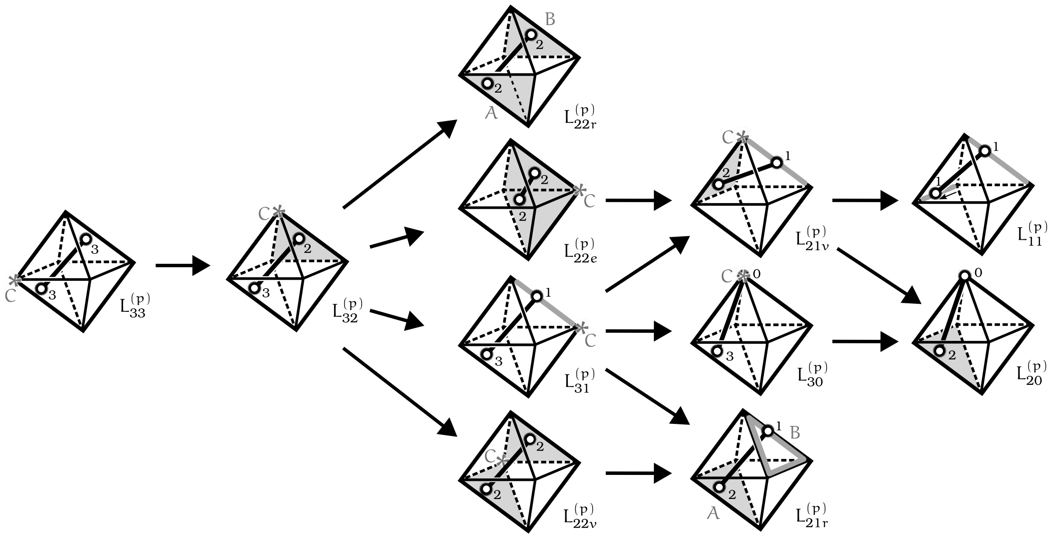


Figure 1.13: All different $L_{ab}^{(p)}$ configurations encountered for K being a regular octahedron

Performing the reduction, we get the set of equations, where

$$\begin{aligned}
pP_{33} &= 6(P_{32} - P_{33}),\\
pP_{32} &= 3(P_{22} - P_{32}) + 2(P_{31} - P_{32}),\\
pP_{22e} &= 4(P_{21v} - P_{22e}),\\
pP_{22v} &= 4(P_{21r} - P_{22v}),\\
pP_{31} &= 3(P_{21} - P_{31}) + 1(P_{30} - P_{31}),\\
pP_{21v} &= 2(P_{11} - P_{21v}) + 1(P_{20} - P_{21v}),\\
pP_{30} &= 3(P_{20} - P_{30})
\end{aligned}$$

with

$$\begin{aligned}
P_{22} &= \tfrac{1}{4}P_{22e} + \tfrac{1}{4}P_{22r} + \tfrac{1}{2}P_{22v},\\
P_{21} &= \tfrac{1}{2}P_{21v} + \tfrac{1}{2}P_{21r}.
\end{aligned}$$

Solving the system, we get for any bivariate functional P,

$$P_{33} = \frac{72(P_{20} + P_{11})}{(3+p)(4+p)(5+p)(6+p)} + \frac{54P_{21r}}{(4+p)(5+p)(6+p)} + \frac{9P_{22r}}{2(5+p)(6+p)}. \tag{1.218}$$

When $p = 1$, we get for the mean length

$$L_{33} = \frac{3}{140}(4L_{20} + 4L_{11} + 12L_{21r} + 5L_{22r}). \tag{1.219}$$

$\mathbf{L_{20}}$

More precisely, we can write $L_{20} = L_{AB}$, where face A has vertices $[-1, 0, 0]$, $[0, -1, 0]$, $[0, 0, -1]$ and $B = [0, 0, 1]$ (see Figure 1.13). Vertex B is separated from $\mathcal{A}(A)$ by distance $h = 2/\sqrt{3}$. By (1.156) and by symmetry,

$$L_{20} = \frac{2h^3}{\operatorname{vol} A}\left(I_{00}^{(1)}\left(\frac{\sqrt{2}}{2}, \frac{\pi}{3}\right) - I_{00}^{(1)}\left(\frac{\sqrt{2}}{4}, \frac{\pi}{3}\right)\right), \tag{1.220}$$

where $\operatorname{vol} A = \nu = \sqrt{3}/2$ is the area of A. Using recurrence relations on $I_{00}^{(1)}(\cdot,\cdot)$,

$$I_{00}^{(1)}\left(\frac{\sqrt{2}}{2}, \frac{\pi}{3}\right) = \frac{1}{4} - \frac{\pi}{36} + \frac{7 \operatorname{argsinh} 1}{12\sqrt{2}} = \frac{1}{4} - \frac{\pi}{36} + \frac{7 \operatorname{argcoth} \sqrt{2}}{12\sqrt{2}}. \tag{1.221}$$

The other $I_{00}^{(1)}$ integral is already given by (1.189) (we just write $\arcsin\sqrt{2/3} = \pi/2 - \operatorname{arccot}\sqrt{2}$), hence

$$L_{20} = \tfrac{8}{9} - \tfrac{\sqrt{2}}{9} - \tfrac{8\pi}{27} - \tfrac{25 \ln 3}{54\sqrt{2}} + \tfrac{32}{27} \operatorname{arccot}\sqrt{2} + \tfrac{28}{27}\sqrt{2} \operatorname{argcoth}\sqrt{2} \tag{1.222}$$

$\mathbf{L_{11}}$

By shifting (1.157), we get $L_{11} = L_{AB}$, where B is the origin and A is a parallelogram with vertices $[1,0,1]$, $[0,1,1]$, $[0,2,0]$, $[1,1,0]$. Therefore, by the point-polygon formula (1.156) with $h = 2/\sqrt{3}$ and $\operatorname{vol} A = \sqrt{3}$,

$$L_{11} = \frac{h^3}{\operatorname{vol} A}\left(4 I_{00}^{(1)}\left(\frac{\sqrt{2}}{4}, \frac{\pi}{3}\right) + 2 I_{00}^{(1)}\left(\frac{\sqrt{2}}{2}, \frac{\pi}{3}\right)\right), \tag{1.223}$$

Hence, since the I's are already given by (1.189) and (1.221), we get, simplifying,

$$L_{11} = \frac{4}{9} + \frac{\sqrt{2}}{9} + \frac{4\pi}{27} + \frac{25 \ln 3}{54\sqrt{2}} - \frac{32}{27} \operatorname{arccot}\sqrt{2} + \frac{14}{27}\sqrt{2} \operatorname{argcoth}\sqrt{2} \tag{1.224}$$

$\mathbf{L_{21r}}$

Since the reduction technique cannot be applied on AB being parallel, we use the overlap formula with A being one face of the octahedron. By symmetry, we can choose B as all three opposite edges to A instead of just one, the mean value stays the same (see Figure 1.13). This choice makes the overlap formula simpler. To compute $\operatorname{vol} A \cap \operatorname{proj} B + k$, we slide the projection B across A. To get L_{21r}, we then integrate over the length of their intersection with respect to all vectors k. By symmetry, we can integrate over just one sixth of all sliding domains (see Figure 1.14 for our overlap diagram, in which white numbers represent the number of line segments in the AB projection intersection with respect to position of the shift vector k – black dot).

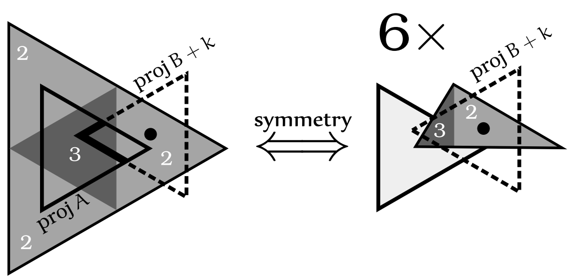


Figure 1.14: Overlap of the opposite face and edges of an octahedron

Hence, setting up the integral,

$$L_{21r} = \frac{6}{\operatorname{vol} A \operatorname{vol} B} \int_D \sqrt{h^2 + k^2} \operatorname{vol} A \cap \operatorname{proj} B + k \; \mathrm{d}k, \tag{1.225}$$

where $h = 2/\sqrt{3}$, $\operatorname{vol} A = \sqrt{3}/2$, $\operatorname{vol} B = 3\sqrt{2}$ and D is a domain in Figure 1.14 on the right consisted of two subdomains D_j where $j \in \{2,3\}$ denotes the number of line segments of the intersection $A \cap (\operatorname{proj} B + k)$, which is a polyline. We have $D = D_2 \sqcup D_3$. Let $k = (x, y)$ with the origin coinciding with the centroid of $\operatorname{proj} A$ triangle with vertices $[0, \frac{\sqrt{6}}{3}]$, $[-\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}]$, $[-\frac{\sqrt{6}}{2}, -\frac{\sqrt{2}}{2}]$ (see Figure 1.14). Let us denote $v_j = \operatorname{vol} A \cap \operatorname{proj} B + k$ for all $k \in D_j$, then we have for the subdomains:

- D_3 is a triangle with vertices $[0, 0]$, $[\frac{\sqrt{6}}{6}, 0]$, $[\frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}]$ in which $v_3 = \sqrt{2}$
- D_2 is a triangle with vertices $[\frac{\sqrt{6}}{6}, 0]$, $[\frac{2\sqrt{6}}{3}, 0]$, $[\frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}]$ in which $v_2 = \frac{4\sqrt{2}}{3} - \frac{2x}{\sqrt{3}}$

Note that in general, $\operatorname{vol} A \cap \operatorname{proj} B + k$ is **linear** in (x, y) in the subdomains. By inclusion/exclusion, we can write our integral as

$$\begin{aligned} L_{21r} &= \tfrac{6}{\operatorname{vol} A \operatorname{vol} B} \Big[\textstyle\iint_{D_3 \cup D_2} v_2 \sqrt{h^2+x^2+y^2}\, \mathrm{d}x\mathrm{d}y + \iint_{D_3} (v_3 - v_2) \sqrt{h^2+x^2+y^2}\, \mathrm{d}x\mathrm{d}y \Big] \\ &= \tfrac{6}{\operatorname{vol} A \operatorname{vol} B} \Big[\textstyle\iint_{D_3 \cup D_2} \Big(\frac{4\sqrt{2}}{3} - \frac{2x}{\sqrt{3}}\Big) \sqrt{h^2+x^2+y^2}\mathrm{d}x\mathrm{d}y + \iint_{D_3} \Big(\frac{2x}{\sqrt{3}} - \frac{\sqrt{2}}{3}\Big) \sqrt{h^2+x^2+y^2}\mathrm{d}x\mathrm{d}y \Big]. \end{aligned} \tag{1.226}$$

Note that the second integral over domain 3 is already in the form of an integral over standard fundamental triangle domain since $D_3 = D(\frac{\sqrt{6}}{6}, \frac{\pi}{3})$. The first integral over domain 3∪2 can be written in such manner after rotation and reflection. To obtain the correct transformation, we let φ' to start (be zero) for the half-line connecting the origin with point $[\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{2}]$, increasing in the clockwise direction. That is $\varphi = \pi/3 - \varphi'$ and thus $x = r\cos(\frac{\pi}{3} - \varphi')$ and $y = r\sin(\frac{\pi}{3} - \varphi')$. Expanding out the trigonometric functions and writing $x' = r\cos\varphi'$ and $y' = r\cos\varphi'$, we get

$$\begin{aligned} x &= r\cos\frac{\pi}{3}\cos\varphi' + r\sin\frac{\pi}{3}\sin\varphi' = x'\cos\frac{\pi}{3} + y'\sin\frac{\pi}{3} = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y', \\ y &= r\sin\frac{\pi}{3}\cos\varphi' - r\cos\frac{\pi}{3}\sin\varphi' = x'\sin\frac{\pi}{3} - y'\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y' \end{aligned} \tag{1.227}$$

and so

$$v_2 = \frac{4\sqrt{2}}{3} - \frac{2x}{\sqrt{3}} = \frac{4\sqrt{2}}{3} - \frac{x'}{\sqrt{3}} - y'. \tag{1.228}$$

Our integration domain $D_3 \sqcup D_2$ in (x', y') is simply $D(\frac{\sqrt{6}}{3}, \frac{\pi}{3})$. Note that $x^2 + y^2$ is invariant with respect to this transformation so $x^2 + y^2 = x'^2 + y'^2$. By scaling with h, we can write L_{21r} in terms of the auxiliary integrals as

$$\begin{aligned} L_{21r} = \tfrac{6h^3}{\operatorname{vol} A \operatorname{vol} B} \Bigg[& \tfrac{4\sqrt{2}}{3} I_{00}^{(1)}\left(\tfrac{\sqrt{6}}{3h}, \tfrac{\pi}{3}\right) - \tfrac{h}{\sqrt{3}} I_{10}^{(1)}\left(\tfrac{\sqrt{6}}{3h}, \tfrac{\pi}{3}\right) - h I_{01}^{(1)}\left(\tfrac{\sqrt{6}}{3h}, \tfrac{\pi}{3}\right) \\ & + \tfrac{2h}{\sqrt{3}} I_{10}^{(1)}\left(\tfrac{\sqrt{6}}{6h}, \tfrac{\pi}{3}\right) - \tfrac{\sqrt{2}}{3} I_{00}^{(1)}\left(\tfrac{\sqrt{6}}{6h}, \tfrac{\pi}{3}\right) \Bigg] \end{aligned} \tag{1.229}$$

with $\frac{\sqrt{6}}{3h} = \frac{\sqrt{2}}{2}$ and $\frac{\sqrt{6}}{6h} = \frac{\sqrt{2}}{4}$. Via recursions (see Table F.1 in Appendix), we get

$$L_{21r} = -\tfrac{10}{27} + \tfrac{47}{54\sqrt{2}} - \tfrac{16\pi}{81} + \tfrac{143\ln 3}{648\sqrt{2}} + \tfrac{32}{81}\operatorname{arccot}\sqrt{2} + \tfrac{85}{81}\sqrt{2}\operatorname{argcoth}\sqrt{2} \tag{1.230}$$

$\mathbf{L_{22r}}$

Again, we use the overlap formula for A and B being opposite faces of K. By symmetry, we again integrate $\operatorname{vol} A \cap \operatorname{proj} B + k$ over one sixth of all positions of

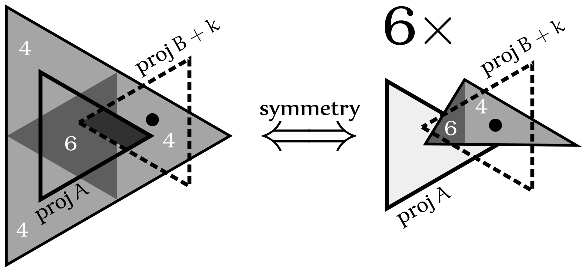


Figure 1.15: Overlap of the opposite faces of an octahedron

vector k (see Figure 1.15, in which white numbers represent the number of sides of a polygon of intersection of AB projections with respect to position of the shift vector k – black dot).

Setting up the integral,

$$L_{22r} = \frac{6}{\operatorname{vol} A \operatorname{vol} B} \int_D \sqrt{h^2 + k^2} \operatorname{vol} A \cap \operatorname{proj} B + k \, \mathrm{d}k, \tag{1.231}$$

where $h = 2/\sqrt{3}, \operatorname{vol} A = \operatorname{vol} B = \nu = \sqrt{3}/2$ and D is a domain in Figure 1.15 on the right consisted of two subdomains labeled 6 and 4 according to the number of sides of the intersection (which is a polygon). That is, $D = D_6 \sqcup D_4$. Let $k = (x, y)$ and denote $v_j = \operatorname{vol} A \cap \operatorname{proj} B + k$ for those k which lie in $\in D_j$, then the subdomain

- D_6 is again a triangle with vertices $[0, 0]$, $[\frac{\sqrt{6}}{6}, 0]$, $[\frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}]$ in which $v_6 = \frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{2}x^2 - \frac{\sqrt{3}}{2}y^2$
- D_4 is a triangle with vertices $[\frac{\sqrt{6}}{6}, 0]$, $[\frac{2\sqrt{6}}{3}, 0]$, $[\frac{\sqrt{6}}{6}, \frac{\sqrt{2}}{2}]$ in which $v_4 = \frac{4}{3\sqrt{3}} - \frac{2\sqrt{2}x}{3} + \frac{x^2}{2\sqrt{3}} - \frac{\sqrt{3}y^2}{2}$

Domains $6, 4$ coincide with $3, 2$ in L_{21r} case, that is $D_6 = D_3$ and $D_4 = D_2$. Note that in general, $\operatorname{vol} A \cap \operatorname{proj} B + k$ is **quadratic** in (x, y) in the subdomains. By inclusion/exclusion, we can write the integral as

$$\begin{aligned} L_{22r} &= \tfrac{6}{\operatorname{vol} A \operatorname{vol} B}\Big[\textstyle\iint_{D_6 \cup D_4} v_4 \sqrt{h^2 + x^2 + y^2} \, \mathrm{d}x\mathrm{d}y + \iint_{D_6} (v_6 - v_4) \sqrt{h^2 + x^2 + y^2} \, \mathrm{d}x\mathrm{d}y \Big] \\ &= \tfrac{6}{\operatorname{vol} A \operatorname{vol} B}\Big[\textstyle\iint_{D_6 \cup D_4} \left(\frac{4}{3\sqrt{3}} - \frac{2\sqrt{2}x}{3} + \frac{x^2}{2\sqrt{3}} - \frac{\sqrt{3}y^2}{2} \right) \sqrt{h^2 + x^2 + y^2} \, \mathrm{d}x\mathrm{d}y \\ &\qquad + \textstyle\iint_{D_6} \left(-\frac{1}{3\sqrt{3}} + \frac{2\sqrt{2}x}{3} - \frac{2x^2}{\sqrt{3}} \right) \sqrt{h^2 + x^2 + y^2} \, \mathrm{d}x\mathrm{d}y \Big]. \end{aligned} \tag{1.232}$$

Again, the integral over domain 6 is in a standard form. The other integral must be first transformed using $x = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$, $y = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y'$, which gives

$$v_4 = \frac{4}{3\sqrt{3}} - \frac{2\sqrt{2}x}{3} + \frac{x^2}{2\sqrt{3}} - \frac{\sqrt{3}y^2}{2} = \frac{4}{3\sqrt{3}} - \frac{\sqrt{2}x'}{3} - \sqrt{\frac{2}{3}}y' - \frac{x'^2}{\sqrt{3}} + x'y' \tag{1.233}$$

and therefore

$$L_{22r} = \frac{6h^3}{\operatorname{vol} A \operatorname{vol} B}\Bigg[\tfrac{4}{3\sqrt{3}} I_{00}^{(1)}\left(\tfrac{\sqrt{2}}{2}, \tfrac{\pi}{3}\right) - \tfrac{\sqrt{2}h}{3} I_{10}^{(1)}\left(\tfrac{\sqrt{2}}{2}, \tfrac{\pi}{3}\right) - \sqrt{\tfrac{2}{3}} h I_{01}^{(1)}\left(\tfrac{\sqrt{2}}{2}, \tfrac{\pi}{3}\right)$$
$$- \tfrac{h^2}{\sqrt{3}} I_{20}^{(1)}\left(\tfrac{\sqrt{2}}{2}, \tfrac{\pi}{3}\right) + h^2 I_{11}^{(1)}\left(\tfrac{\sqrt{2}}{2}, \tfrac{\pi}{3}\right)$$
$$- \tfrac{1}{3\sqrt{3}} I_{00}^{(1)}\left(\tfrac{\sqrt{2}}{4}, \tfrac{\pi}{3}\right) + \tfrac{2\sqrt{2}h}{3} I_{10}^{(1)}\left(\tfrac{\sqrt{2}}{4}, \tfrac{\pi}{3}\right) - \tfrac{2h^2}{\sqrt{3}} I_{20}^{(1)}\left(\tfrac{\sqrt{2}}{4}, \tfrac{\pi}{3}\right)\Bigg]. \tag{1.234}$$

Going through all recursions, we get, after simplifications

$$L_{22r} = \frac{8}{45} + \frac{\sqrt{2}}{9} - \frac{32\pi}{135} + \frac{293\ln 3}{270\sqrt{2}} - \frac{64}{135}\operatorname{arccot}\sqrt{2} + \frac{124}{135}\sqrt{2}\operatorname{argcoth}\sqrt{2} \tag{1.235}$$

L_{33}

Putting everything together by using (1.219), we finally arrive at

$$L_{33} = \frac{4}{105} + \frac{13\sqrt{2}}{105} - \frac{4\pi}{45} + \frac{109\ln 3}{630\sqrt{2}} + \frac{16\operatorname{arccot}\sqrt{2}}{315} + \frac{158\operatorname{argcoth}\sqrt{2}}{315}\sqrt{2}. \tag{1.236}$$

Rescaling, we get our mean distance in a regular octahedron having unit volume

$$v_1^{(1)}(O_3) = \sqrt[3]{\tfrac{3}{4}}\left(\tfrac{4}{105} + \tfrac{13\sqrt{2}}{105} - \tfrac{4\pi}{45} + \tfrac{109\ln 3}{630\sqrt{2}} + \tfrac{16\operatorname{arccot}\sqrt{2}}{315} + \tfrac{158\operatorname{argcoth}\sqrt{2}}{315}\sqrt{2}\right) \approx 0.65853073. \tag{1.237}$$

1.4.6 Regular icosahedron

Regular icosahedron shares many features with regular octahedron. We have already seen that the Crofton Reduction Technique itself is very powerful to reduce the the mean distance until two domains from which we select two points have empty affine hull. As a consequence, the only remaining terms in the icosahedron expansion are the parallel edge-face and parallel face-face configurations. Note that these two parallel configurations have the same overlap diagram as the octahedron has.

Let $\phi = (1+\sqrt{5})/2$ be the Golden ratio. A standard selection of vertices is $[\pm\phi, \pm 1, 0]$ and all of their cyclic permutations. That way, our edges have length $l = 2$. The volume is equal to $\operatorname{vol} K = 10(3+\sqrt{5})/3$ and the face area $\nu = \sqrt{3}$. Again, we put $P = L^p$. For the definition of various mean values $P_{ab} = L_{ab}^{(p)}$, see Figure 1.16.

Performing the reduction, we get the set of equations:

$$\begin{aligned}
pP_{33} &= 6(P_{32} - P_{33}),\\
pP_{32} &= 3(P_{22} - P_{32}) + 2(P_{31} - P_{32}),\\
pP_{31} &= 3(P_{21} - P_{31}) + 1(P_{30} - P_{31}),\\
pP_{22e} &= 4(P_{21v} - P_{22e}),\\
pP_{22v} &= 4(P_{21d} - P_{22v}),
\end{aligned}$$

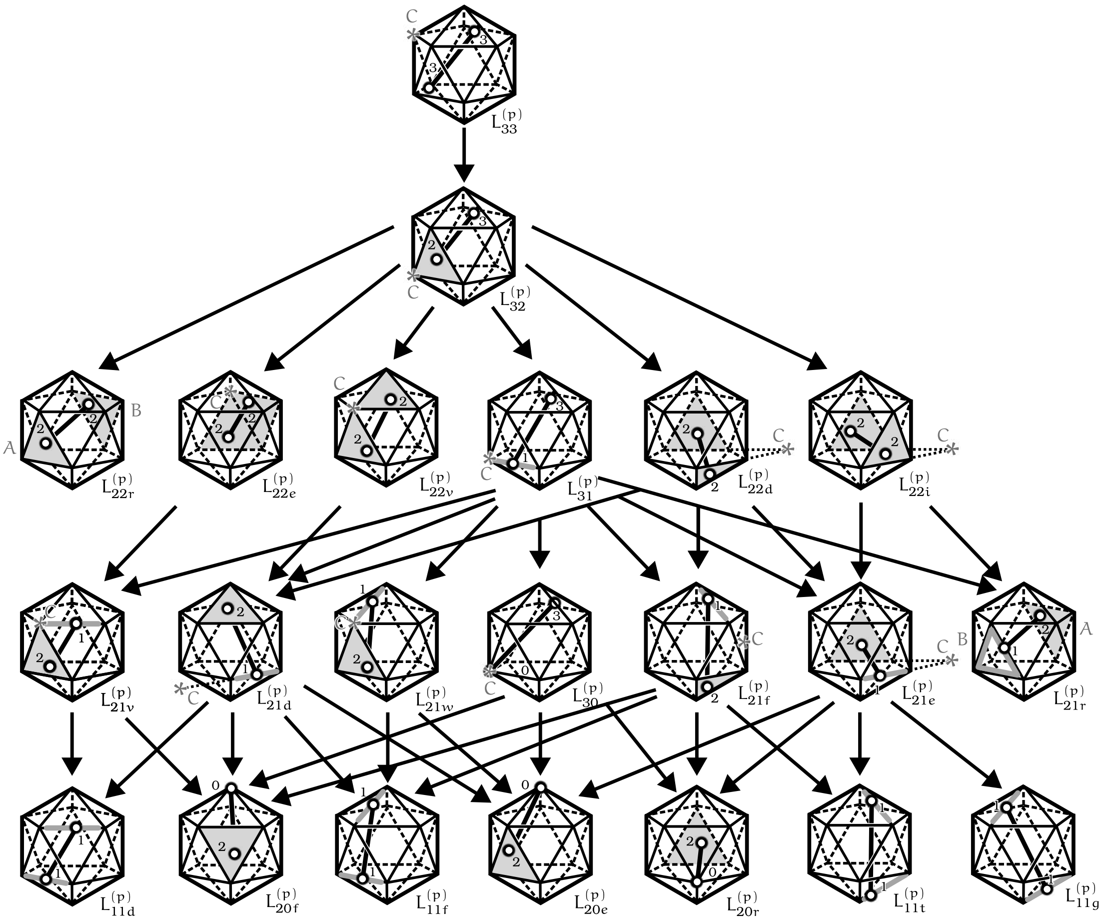


Figure 1.16: All different $L^{(p)}_{ab}$ configurations encountered for K being a regular icosahedron

$$
\begin{aligned}
pP_{22d} &= 4(P'_{21} - P_{22d}),\\
pP_{22i} &= 4(P''_{21} - P_{22i}),\\
pP_{30} &= 3(P_{20} - P_{30}),\\
pP_{21v} &= 2(P_{11d} - P_{21v}) + 1(P_{20u} - P_{21v}),\\
pP_{21w} &= 2(P_{11f} - P_{21w}) + 1(P_{20l} - P_{21w}),\\
pP_{21f} &= 2(P_{11} - P_{21f}) + 1(P'_{20} - P_{21f}),\\
pP_{21d} &= 2(P'_{11} - P_{21d}) + 1(P''_{20} - P_{21d}),\\
pP_{21e} &= 2(P''_{11} - P_{21e}) + 1(P'''_{20} - P_{21e}),
\end{aligned}
$$

with

$$
\begin{aligned}
P_{22} &= \frac{2P_{22d}}{5} + \frac{P_{22r}}{10} + \frac{P_{22v}}{5} + \frac{P_{22e}}{10\phi^2} + \frac{\phi^2 P_{22i}}{10},\\
P_{21} &= \frac{\phi P_{21e}}{5} + \frac{P_{21f}}{2\phi\sqrt{5}} + \frac{P_{21r}}{5} + \frac{P_{21d}}{5} + \frac{P_{21v}}{5\phi^2} + \frac{P_{21w}}{10\phi},\\
P'_{21} &= \frac{P_{21f}}{2} - \frac{\phi P_{21d}}{2} + \frac{\phi^2 P_{21e}}{2},\\
P''_{21} &= \phi^2 P_{21r} - \phi P_{21e},\\
P_{20} &= \frac{P_{20f}}{2\phi^2} + \frac{P_{20e}}{2\phi} + \frac{P_{20r}}{2},
\end{aligned}
$$

$$
\begin{aligned}
P'_{20} &= \phi P_{20r} - \frac{P_{20f}}{\phi},\\
P''_{20} &= \phi^2 P_{20e} - \phi P_{20f},\\
P'''_{20} &= \phi^2 P_{20r} - \phi P_{20e},\\
P_{11} &= 2P_{11t} - P_{11f},\\
P'_{11} &= \phi^2 P_{11f} - \phi P_{11d},\\
P''_{11} &= \phi^2 P_{11g} - \phi P_{11t}.
\end{aligned}
$$

Solving the system, we get, after simplifications,

$$
\begin{aligned}
P_{33} = &\frac{18(12\phi^2 P_{11d} - 12\phi^4 P_{11f} + 4\phi^8 P_{11g} + 12\phi^2 P_{11t} - 6\phi^4 P_{20e} + 4\phi^8 P_{20r} + 6P_{20f})}{5\phi^4(3+p)(4+p)(5+p)(6+p)}\\
&+ \frac{108\phi^2 P_{21r}}{5(4+p)(5+p)(6+p)} + \frac{9P_{22r}}{5(5+p)(6+p)}.
\end{aligned} \tag{1.238}
$$

When $p = 1$, we get for the mean distance

$$
\begin{aligned}
L_{33} = &\frac{3L_{22r}}{70} - \frac{9L_{11f}}{175} - \frac{9L_{20e}}{350} + \frac{9L_{20f}}{350\phi^4} + \frac{9L_{11d}}{175\phi^2}\\
&+ \frac{9L_{11t}}{175\phi^2} + \frac{18\phi^2 L_{21r}}{175} + \frac{3\phi^4 L_{11g}}{175} + \frac{3\phi^4 L_{20r}}{175}.
\end{aligned} \tag{1.239}
$$

L_{11d}

Let $A' = \overline{[1,0,\phi][-1,0,\phi]}$ and $B' = \overline{[0,\phi,1][\phi,1,0]}$ be edges of K, then $L_{11d} = L_{A'B'}$. By shifting, $L_{A'B'} = L_{OA}$, where $O = [0,0,0]$ is the origin and $A = A' - B'$ is a polygon with vertices $[1,-\phi,\frac{1}{\phi}], [-\frac{1}{\phi},-1,\phi], [-\phi^2,-1,\phi], [-1,-\phi,\frac{1}{\phi}]$ (a parallelogram) having area $\operatorname{vol} A = \sqrt{10-2\sqrt{5}}$. Projecting O onto $\mathcal{A}(A)$, we obtain $\operatorname{proj}_A O = [0,-1-\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}]$ and separation $h = \sqrt{2+\frac{2}{\sqrt{5}}}$. Point-Polygon formula yields

$$
L_{11d} = \tfrac{2h^3}{\operatorname{vol} A}\left(2I_{00}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{2\pi}{5}\right) - I_{00}^{(1)}\left(\tfrac{1}{2},\tfrac{\pi}{5}\right) + I_{00}^{(1)}\left(\tfrac{1}{2},\tfrac{2\pi}{5}\right)\right) \approx 2.0431430525135. \tag{1.240}
$$

Explicitly, after series of simplifications on $I_{00}^{(1)}(\cdot,\cdot)$ by recursion formulae, we obtain

$$
\begin{aligned}
L_{11d} = &\tfrac{5}{6} + \tfrac{1}{2\sqrt{5}} + \tfrac{1}{15}(2\pi)\left(3+\sqrt{5}\right) - \tfrac{8}{15}\left(3+\sqrt{5}\right)\operatorname{arccot}\phi\\
&- \tfrac{8}{15}\left(3+\sqrt{5}\right)\operatorname{arccot}\left(\phi^2\right) + \tfrac{1}{60}\left(31-3\sqrt{5}\right)\ln 3 + \tfrac{13}{120}\left(3+\sqrt{5}\right)\ln 5.
\end{aligned} \tag{1.241}
$$

L_{11g}

Let A' be the same edge as in L_{11d} and $B' = \overline{[1,0,\phi][-1,0,\phi]}$, then $L_{11g} = L_{A'B'}$. By shifting, $L_{A'B'} = L_{OA}$, where $O = [0,0,0]$ and $A = A' - B'$ is a polygon with vertices $[0,0,2\phi],[1,-\phi,\phi^2],[-1,-\phi,\phi^2],[-2,0,2\phi]$ having area $\operatorname{vol} A = 2\sqrt{3}$. Projecting O onto $\mathcal{A}(A)$, we obtain $\operatorname{proj}_A O = [0,-\frac{2\phi}{3},\frac{2\phi^3}{3}]$ and separation $h = \sqrt{\frac{14}{3}+2\sqrt{5}}$. Point-Polygon formula yields

$$L_{11g} = \frac{2h^3}{\operatorname{vol} A}\left(2I_{00}^{(1)}\left(\frac{1}{2\phi^2},\frac{\pi}{3}\right)+I_{00}^{(1)}\left(\frac{1}{\phi^2},\frac{\pi}{3}\right)\right) \approx 3.1806727116118. \tag{1.242}$$

Explicitly, after series of simplifications,

$$\begin{aligned} L_{11g} &= \tfrac{1}{9}+\tfrac{\sqrt{5}}{9}+\tfrac{2}{9}\sqrt{2\left(5+\sqrt{5}\right)}+\tfrac{4}{45}\left(9+4\sqrt{5}\right)\pi-\tfrac{16}{27}\left(9+4\sqrt{5}\right)\operatorname{arccot}\phi \\ &\quad+\left(\tfrac{43}{27}+\tfrac{2\sqrt{5}}{3}\right)\operatorname{argcoth}\phi+\tfrac{2}{27}\left(23+9\sqrt{5}\right)\operatorname{argcsch}\phi-\left(\tfrac{43}{108}+\tfrac{\sqrt{5}}{6}\right)\ln 5. \end{aligned} \tag{1.243}$$

$\mathbf{L_{11f}}$

Let A' be the same edge as in L_{11d} and $B' = \overline{[\phi,1,0][\phi,-1,0]}$, then $L_{11f} = L_{A'B'}$. By shifting, $L_{A'B'} = L_{OA}$, where $O = [0,0,0]$ and $A = A' - B'$ is a polygon with vertices $[-\frac{1}{\phi},-1,\phi],[-\frac{1}{\phi},1,\phi],[-\phi^2,1,\phi],[-\phi^2,-1,\phi]$ having area $\operatorname{vol} A = 4$. Projecting O onto $\mathcal{A}(A)$, we obtain $\operatorname{proj}_A O = [0,0,\phi]$ and separation $h = \phi$. Point-Polygon formula yields

$$\begin{aligned} L_{11f} = \tfrac{2h^3}{\operatorname{vol} A}\Bigg(& I_{00}^{(1)}\left(\tfrac{1}{\phi},\arctan\left(\phi^2\right)\right)+I_{00}^{(1)}\left(\phi,\arctan\left(\tfrac{1}{\phi^2}\right)\right)-I_{00}^{(1)}\left(\tfrac{1}{\phi^2},\arctan\phi\right) \\ & -I_{00}^{(1)}\left(\tfrac{1}{\phi},\arctan\tfrac{1}{\phi}\right)\Bigg) \approx 2.3977565034445. \end{aligned} \tag{1.244}$$

Explicitly, after series of simplifications,

$$\begin{aligned} L_{11f} &= \tfrac{5}{6}+\tfrac{\sqrt{5}}{6}-\tfrac{\pi}{96}\left(1+\sqrt{5}\right)^3+\tfrac{1}{2}\left(2+\sqrt{5}\right)\operatorname{arccot}\phi-\tfrac{1}{6}\left(2+\sqrt{5}\right)\operatorname{arccot}\left(\phi^2\right) \\ &\quad+\tfrac{1}{24}\left(39+17\sqrt{5}\right)\operatorname{argcoth}\phi+\tfrac{1}{48}\left(1-5\sqrt{5}\right)\ln 3-\tfrac{1}{96}\left(17+11\sqrt{5}\right)\ln 5. \end{aligned} \tag{1.245}$$

$\mathbf{L_{11t}}$

Again, let A' be the same edge as in L_{11d} and $B' = \overline{[1,0,-\phi][\phi,1,0]}$, then $L_{11t} = L_{A'B'}$. By shifting, $L_{A'B'} = L_{OA}$, where $O = [0,0,0]$ and $A = A' - B'$ is a polygon with vertices $[0,0,2\phi],[-\frac{1}{\phi},-1,\phi],[-\phi^2,-1,\phi],[-2,0,2\phi]$ having area $\operatorname{vol} A = \sqrt{2\left(5+\sqrt{5}\right)}$. Projecting O onto $\mathcal{A}(A)$, we obtain $\operatorname{proj}_A O = [0,-1-\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}]$ and separation $h = \sqrt{2+\frac{2}{\sqrt{5}}}$. Point-Polygon formula yields

$$L_{11t} = \frac{2h^3}{\operatorname{vol} A}\left(I_{00}^{(1)}\left(\frac{1}{2},\frac{\pi}{5}\right)-I_{00}^{(1)}\left(\frac{1}{2},\frac{2\pi}{5}\right)+I_{00}^{(1)}\left(\phi,\frac{\pi}{5}\right)\right) \approx 2.8940519649490. \tag{1.246}$$

Explicitly, after series of simplifications,

$$\begin{aligned} L_{11t} &= \tfrac{4}{3}\sqrt{1+\tfrac{2}{\sqrt{5}}}-\tfrac{1}{6}-\tfrac{\sqrt{5}}{6}-\tfrac{8\pi}{75}\left(1+\sqrt{5}\right)+\tfrac{8}{15}\left(1+\sqrt{5}\right)\operatorname{arccot}\phi \\ &\quad+\tfrac{4}{15}\left(8+3\sqrt{5}\right)\operatorname{argcsch}\phi-\tfrac{13}{120}\left(1+\sqrt{5}\right)\ln 5. \end{aligned} \tag{1.247}$$

L$_{\mathbf{20e}}$

Let A be the face of K with vertices $[1, 0, \phi]$, $[-1, 0, \phi]$, $[0, \phi, 1]$ (an equilateral triangle) and let B be vertex $[\phi, -1, 0]$, then $L_{20e} = L_{AB}$. Projecting B onto $\mathcal{A}(A)$, we obtain $\operatorname{proj}_A B = [\phi, -\frac{1}{3}, \frac{2\phi^2}{3}]$ and separation $h = 2\phi/\sqrt{3}$. By Point-Polygon formula,

$$L_{20e} = \tfrac{2h^3}{\nu}\Bigg(I_{00}^{(1)}\left(\frac{1}{2\phi^2}, \frac{\pi}{3}\right) - I_{00}^{(1)}\left(\frac{1}{2\phi^2}, \arctan\left(\sqrt{15} + 2\sqrt{3}\right)\right) + I_{00}^{(1)}\left(\tfrac{\phi^2}{2}, \arctan\left(\sqrt{15} - 2\sqrt{3}\right)\right)\Bigg) \approx 2.688729552544. \tag{1.248}$$

Explicitly, after series of simplifications,

$$L_{20e} = \tfrac{7}{9} + \tfrac{\sqrt{5}}{9} + \tfrac{8\pi}{27}\left(2 + \sqrt{5}\right) - \tfrac{16}{9}\left(2 + \sqrt{5}\right)\operatorname{arccot}\phi + \tfrac{1}{27}\left(104 + 47\sqrt{5}\right)\operatorname{argcoth}\phi - \tfrac{1}{108}\left(112 + 61\sqrt{5}\right)\ln 5. \tag{1.249}$$

L$_{\mathbf{20r}}$

Let A be the same face of K as in the section on L_{20e} and let B be vertex $[1, 0, -\phi]$, then $L_{20r} = L_{AB}$. Projecting B onto $\mathcal{A}(A)$, we obtain $\operatorname{proj}_A B = [1, \frac{2\phi}{3}, \frac{\sqrt{5}\phi}{3}]$ and separation $h = 2\phi^2/\sqrt{3}$. By Point-Polygon formula,

$$L_{20r} = \tfrac{2h^3}{\nu}\left(I_{00}^{(1)}\left(\tfrac{1}{\phi^2}, \tfrac{\pi}{3}\right) - I_{00}^{(1)}\left(\tfrac{1}{2\phi^2}, \tfrac{\pi}{3}\right)\right) \approx 3.28394367574. \tag{1.250}$$

Explicitly, after series of simplifications,

$$L_{20r} = \tfrac{4}{9}\sqrt{2\left(5 + \sqrt{5}\right)} - \tfrac{1}{9} - \tfrac{\sqrt{5}}{9} - \tfrac{16\pi}{135}\left(9 + 4\sqrt{5}\right) + \tfrac{16}{27}\left(9 + 4\sqrt{5}\right)\operatorname{arccot}\phi - \left(\tfrac{43}{27} + \tfrac{2\sqrt{5}}{3}\right)\operatorname{argcoth}\phi + \tfrac{4}{27}\left(23 + 9\sqrt{5}\right)\operatorname{argcsch}\phi + \left(\tfrac{43}{108} + \tfrac{\sqrt{5}}{6}\right)\ln 5. \tag{1.251}$$

L$_{\mathbf{20f}}$

Let A be the same face of K as in the section on L_{20e} and let B be vertex $[\phi, 1, 0]$, then $L_{20f} = L_{AB}$. Projecting B onto $\mathcal{A}(A)$, we obtain $\operatorname{proj}_A B = [\phi, \frac{\phi^3}{3}, \frac{2\phi}{3}]$ and separation $h = 2/\sqrt{3}$. By Point-Polygon formula,

$$L_{20f} = \tfrac{2h^3}{\nu}\Bigg(I_{00}^{(1)}\left(\tfrac{\phi^2}{2}, \tfrac{\pi}{3}\right) - I_{00}^{(1)}\left(\tfrac{\sqrt{5}}{2}, \arctan\sqrt{\tfrac{3}{5}}\right) - I_{00}^{(1)}\left(\tfrac{\phi^2}{2}, \arctan\left(\sqrt{15} - 2\sqrt{3}\right)\right)\Bigg) \approx 2.2472771159735. \tag{1.252}$$

Explicitly, after series of simplifications,

$$L_{20f} = \tfrac{10}{9} + \tfrac{2\sqrt{5}}{9} - \tfrac{8\pi}{27} + \tfrac{32}{27}\operatorname{arccot}\phi + \tfrac{16}{27}\operatorname{arccot}(\phi^2) - \tfrac{17}{54}\sqrt{5}\ln 3 + \left(\tfrac{1}{2} + \tfrac{5\sqrt{5}}{27}\right)\ln 5. \tag{1.253}$$

$\mathbf{L_{21r}}$

Let A be a face of K and B be a boundary of the opposite face. In icosahedron K, two faces are separated by the distance $2\phi^2/\sqrt{3}$. Since the overlap diagram of these faces is the same as the one associated to two opposite faces of an octahedron (see Figure 1.14), the coefficients of the expansion of irreducible L_{21r} term into auxiliary integrals $I_{ij}^{(1)}$ *match.* However, this is only valid provided the edge length is $\sqrt{2}$. Since our icosahedron K has $l = 2$, we first rescale our icosahedron by $1/\sqrt{2}$. In the final step, since the mean distance scales linearly, we have just rescale L_{ab} back by multiplying it by $\sqrt{2}$. Hence, by using Equation (1.229),

$$L_{21r} = L_{21r}\Big|_{l=2} = \sqrt{2}L_{21r}\Big|_{l=\sqrt{2}} = \sqrt{2}\frac{6h^3}{\operatorname{vol} A \operatorname{vol} B}\Bigg[\tfrac{4\sqrt{2}}{3}I_{00}^{(1)}\left(\tfrac{\sqrt{6}}{3h},\tfrac{\pi}{3}\right) - \tfrac{h}{\sqrt{3}}I_{10}^{(1)}\left(\tfrac{\sqrt{6}}{3h},\tfrac{\pi}{3}\right) - hI_{01}^{(1)}\left(\tfrac{\sqrt{6}}{3h},\tfrac{\pi}{3}\right) + \tfrac{2h}{\sqrt{3}}I_{10}^{(1)}\left(\tfrac{\sqrt{6}}{6h},\tfrac{\pi}{3}\right) - \tfrac{\sqrt{2}}{3}I_{00}^{(1)}\left(\tfrac{\sqrt{6}}{6h},\tfrac{\pi}{3}\right)\Bigg] \approx 3.1819213671057, \tag{1.254}$$

where $h = \sqrt{2}\phi^2/\sqrt{3}$, $\operatorname{vol} A = \sqrt{3}/2$ and $\operatorname{vol} B = 3\sqrt{2}$ are the rescaled icosahedron opposite faces separation, rescaled face area and face perimeter, respectively. Contrary to the octahedron case, we now have $\frac{\sqrt{6}}{3h} = 1/\phi^2$ and $\frac{\sqrt{6}}{6h} = 1/(2\phi^2)$. Via recursions, we get after some simplifications,

$$L_{21r} = \tfrac{227}{108} + \tfrac{107\sqrt{5}}{108} - \tfrac{25}{27}\sqrt{10 + \tfrac{22}{\sqrt{5}}} - \tfrac{8\pi}{135}\left(9 + 4\sqrt{5}\right) + \tfrac{16}{81}\left(9 + 4\sqrt{5}\right)\operatorname{arccot}\phi + \left(\tfrac{1043}{324} + \tfrac{13\sqrt{5}}{9}\right)\operatorname{argcoth}\phi + \left(\tfrac{179}{81} + \tfrac{7\sqrt{5}}{9}\right)\operatorname{argcsch}\phi - \tfrac{1043+468\sqrt{5}}{1296}\ln 5. \tag{1.255}$$

$\mathbf{L_{22r}}$

Again, Overlap diagram of L_{21r} configuration matches that of an octahedron. Immediately from Equation (1.234), by rescaling and replacing $\sqrt{2}/2$ by $1/\phi^2$ and $\sqrt{2}/4$ by $1/(2\phi^2)$ in the first argument of $I_{ij}^{(p)}$ integrals, we get

$$L_{22r} = \sqrt{2}\frac{6h^3}{\operatorname{vol} A \operatorname{vol} B}\Bigg[\tfrac{4}{3\sqrt{3}}I_{00}^{(1)}\left(\tfrac{1}{\phi^2},\tfrac{\pi}{3}\right) - \tfrac{\sqrt{2}h}{3}I_{10}^{(1)}\left(\tfrac{1}{\phi^2},\tfrac{\pi}{3}\right) - \sqrt{\tfrac{2}{3}}hI_{01}^{(1)}\left(\tfrac{1}{\phi^2},\tfrac{\pi}{3}\right) - \tfrac{h^2}{\sqrt{3}}I_{20}^{(1)}\left(\tfrac{1}{\phi^2},\tfrac{\pi}{3}\right) + h^2I_{11}^{(1)}\left(\tfrac{1}{\phi^2},\tfrac{\pi}{3}\right) - \tfrac{1}{3\sqrt{3}}I_{00}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{3}\right) + \tfrac{2\sqrt{2}h}{3}I_{10}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{3}\right) - \tfrac{2h^2}{\sqrt{3}}I_{20}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{3}\right)\Bigg] \approx 3.12998447304770, \tag{1.256}$$

where $h = \sqrt{2}\phi^2/\sqrt{3}$ and $\operatorname{vol} A = \operatorname{vol} B = \sqrt{3}/2$. Explicitly, after some simplifications,

$$L_{22r} = \tfrac{4}{9}\left(3 + 2\sqrt{5}\right)\sqrt{10 + \tfrac{22}{\sqrt{5}}} - \tfrac{271}{45} - \tfrac{119}{9\sqrt{5}} + \tfrac{16\pi}{675}\left(78 + 35\sqrt{5}\right) - \tfrac{32}{135}\left(67 + 30\sqrt{5}\right)\operatorname{arccot}\phi + \left(\tfrac{611}{45} + \tfrac{164\sqrt{5}}{27}\right)\operatorname{argcoth}\phi - \tfrac{28}{135}\left(9 + 5\sqrt{5}\right)\operatorname{argcsch}\phi - \left(\tfrac{611}{180} + \tfrac{41\sqrt{5}}{27}\right)\ln 5. \tag{1.257}$$

$\mathbf{L_{33}}$

Putting everything together by using (1.239), we finally arrive at

$$L_{33} = \frac{197}{525} + \frac{239}{525\sqrt{5}} - \frac{44}{525}\sqrt{2+\frac{2}{\sqrt{5}}} - \frac{\left(17226+6269\sqrt{5}\right)\pi}{157500} - \frac{\left(2186+1413\sqrt{5}\right)\operatorname{arccot}\phi}{15750}$$
$$+ \frac{\left(82-75\sqrt{5}\right)\operatorname{arccot}\left(\phi^2\right)}{5250} + \frac{\left(15969+7151\sqrt{5}\right)\operatorname{argcoth}\phi}{12600} + \frac{4\left(2139+881\sqrt{5}\right)\operatorname{argcsch}\phi}{7875}$$
$$+ \frac{\left(4449-1685\sqrt{5}\right)\ln 3}{42000} - \frac{\left(75783+37789\sqrt{5}\right)\ln 5}{252000} \approx 1.66353152568500. \tag{1.258}$$

Rescaling, we get our mean distance in a regular icosahedron having unit volume

$$v_1^{(1)}(\text{icosahedron}) = \frac{L_{33}}{\sqrt[3]{\frac{10}{3}\left(3+\sqrt{5}\right)}} \approx 0.64131248551. \tag{1.259}$$

1.4.7 Regular dodecahedron

Finaly, we will calculte the mean distance in the regular dodecahedron. Let us choose the vertices as $[\pm\phi, \pm\phi, \pm\phi]$, $[0, \pm 1, \pm\phi^2]$ and all their cyclic permutations ($\phi = (1+\sqrt{5})/2$ as usual). Under this choice, each edge has length $l = 2$ and each face has area $\nu = \sqrt{25+10\sqrt{5}}$.
Performing CRT, we get the configurations shown in Figure 1.17. Even though there are less configurations than for the icosahedron, the dodecahedron has more complicated overlap diagram (see Figure 1.18, there is ten-fold symmetry with respect to rotation and reflection). Distance moments are again connected through CRT via the following set of reduction equations

$$\begin{aligned}
pP_{33} &= 6(P_{32} - P_{33}),\\
pP_{32} &= 3(P_{22} - P_{32}) + 2(P_{31} - P_{32}),\\
pP_{31} &= 3(P_{21} - P_{31}) + 1(P_{30} - P_{31}),\\
pP_{22e} &= 4(P'_{21} - P_{22e}),\\
pP_{22i} &= 4(P''_{21} - P_{22i}),\\
pP_{30} &= 3(P_{20} - P_{30}),\\
pP_{21v} &= 1(P_{20e} - P_{21v}) + 2(P_{11} - P_{21v}),\\
pP_{21f} &= 2(P'_{11} - P_{21f}) + 1(P'_{20} - P_{21f}),\\
pP_{21d} &= 2(P''_{11} - P_{21d}) + 1(P''_{20} - P_{21d})
\end{aligned}$$

with

$$\begin{aligned}
P_{22} &= \frac{1}{6\phi}\left(\sqrt{5}P_{22e} + \phi P_{22r} + \phi^2\sqrt{5}P_{22i}\right),\\
P_{21} &= \frac{P_{21r}}{3} + \frac{P_{21d}}{3} + \frac{\phi P_{21f}}{6} + \frac{P_{21v}}{6\phi^2},\\
P'_{21} &= \frac{1}{\phi\sqrt{5}}\left(P_{21v} + \phi^2 P_{21d}\right),
\end{aligned}$$

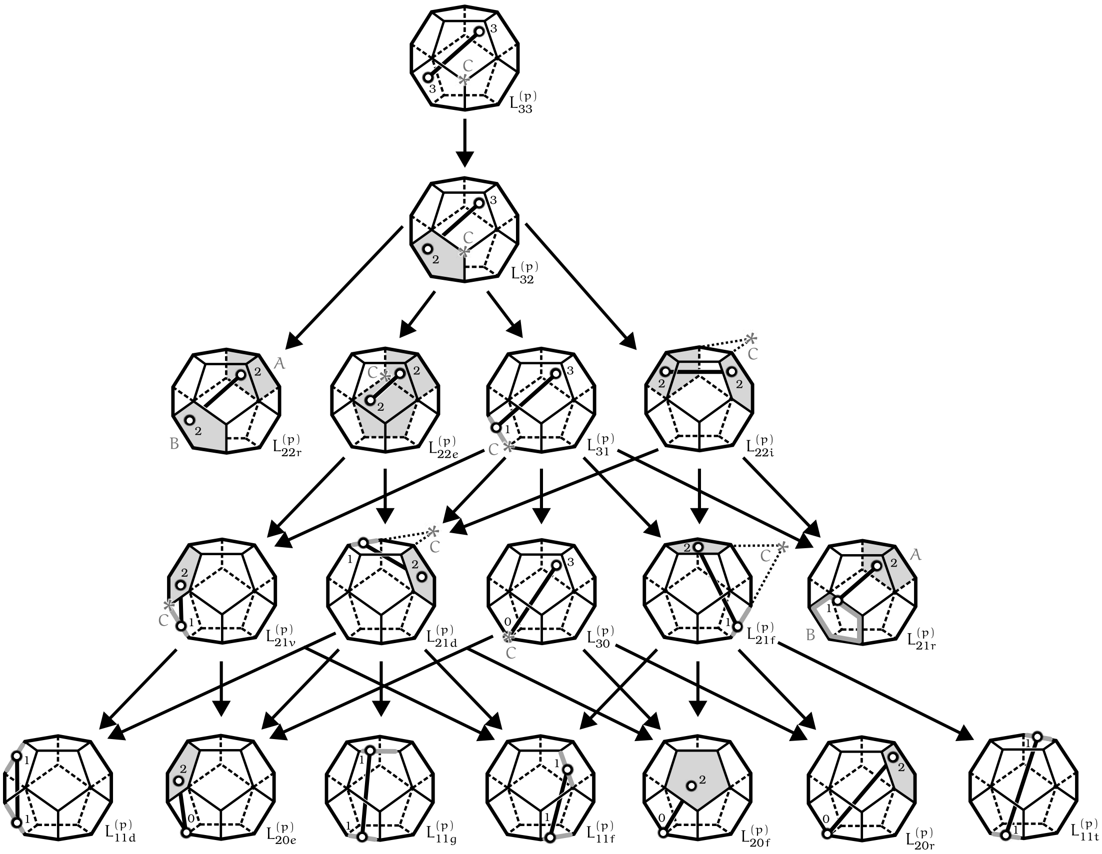


Figure 1.17: All different $L_{ab}^{(p)}$ configurations encountered for K being a regular icosahedron

$$
\begin{aligned}
P''_{21} &= \frac{1}{\sqrt{5}}\left(\phi P_{21f} + \phi P_{21r} - P_{21d}\right),\\
P_{20} &= \frac{1}{2\phi^2}\Big(P_{20e} + \phi P_{20f} + \phi^2 P_{20r}\Big),\\
P'_{20} &= \phi^2 P_{20r} - \phi P_{20f},\\
P''_{20} &= \phi^2 P_{20f} - \phi P_{20e},\\
P_{11} &= \frac{1}{\phi\sqrt{5}}\left(2P_{11d} + \phi P_{11f}\right),\\
P'_{11} &= \frac{1}{\sqrt{5}}\left(2\phi P_{11t} - P_{11f}\right),\\
P''_{11} &= \frac{1}{\sqrt{5}}\left(\phi P_{11g} + \phi P_{11f} - P_{11d}\right).
\end{aligned}
$$

Solving the system, we get, after simplifications,

$$
\begin{aligned}
P_{33} = &\frac{12\Big(2\sqrt{5}P_{11d} + 5\phi P_{20e} + 2\phi^3 P_{11g} - 2\phi^4\sqrt{5}P_{11f} - 5\phi^5 P_{20f} + 4\sqrt{5}\phi^6 P_{20r} + 2\phi^9 P_{11t}\Big)}{\phi^4\sqrt{5}\,(3+p)(4+p)(5+p)(6+p)}\\
&+ \frac{60\phi P_{21r}}{\sqrt{5}(4+p)(5+p)(6+p)} + \frac{3P_{22r}}{(5+p)(6+p)}.
\end{aligned}
\tag{1.260}
$$

When $p = 1$, we get for the mean distance

$$L_{33} = \frac{L_{11d}}{35\phi^4} + \frac{L_{20e}}{14\sqrt{5}\phi^3} + \frac{L_{11g}}{35\sqrt{5}\phi} - \frac{L_{11f}}{35} + \frac{L_{22r}}{14} - \frac{\phi L_{20f}}{14\sqrt{5}} + \frac{2\phi L_{21r}}{7\sqrt{5}} + \frac{2\phi^2 L_{20r}}{35} + \frac{\phi^5 L_{11t}}{35\sqrt{5}}. \tag{1.261}$$

$\mathbf{L_{11d}}$

Let $A' = \overline{[0,\phi^2,1][0,\phi^2,-1]}$ and $B' = \overline{[\phi,\phi,-\phi][1,0,-\phi^2]}$ be edges of K, then $L_{11d} = L_{A'B'}$. By shifting, $L_{A'B'} = L_{OA}$, where $O = [0,0,0]$ is the origin and $A = A' - B'$ is a polygon with vertices $[-\phi,1,\phi^2], [-1,\phi^2,\sqrt{5}\phi], [-1,\phi^2,\phi], [-\phi,1,1/\phi]$ having area $\operatorname{vol} A = 2\sqrt{3}$. Projecting O onto $\mathcal{A}(A)$, we obtain $\operatorname{proj}_A O = [-1-\frac{\sqrt{5}}{3}, \frac{2}{3}, 0]$ and separation $h = (1+\sqrt{5})/\sqrt{3}$. Point-Polygon formula yields

$$L_{11d} = \frac{2h^3}{\operatorname{vol} A}\left(I_{00}^{(1)}\left(\tfrac{1}{2\phi^2}, \tfrac{\pi}{3}\right) + I_{00}^{(1)}\left(\tfrac{\sqrt{5}}{2}, \tfrac{\pi}{3}\right) - I_{00}^{(1)}\left(\tfrac{1}{2\phi^2}, \arctan\left(\sqrt{3}\left(2+\sqrt{5}\right)\right)\right) - I_{00}^{(1)}\left(\tfrac{\sqrt{5}}{2}, \arctan\sqrt{\tfrac{3}{5}}\right)\right) \approx 3.1367199950978. \tag{1.262}$$

Explicitly, after series of simplifications,

$$L_{11d} = \tfrac{10\sqrt{2}}{9} - \tfrac{\sqrt{5}}{3} + \tfrac{5\sqrt{10}}{9} - \tfrac{5}{9} - \tfrac{2\pi}{27}\left(2+\sqrt{5}\right) - \tfrac{17}{108}\left(5+2\sqrt{5}\right)\ln 3 - \tfrac{(4+7\sqrt{5})\ln 5}{108} + \tfrac{4}{27}\left(2+\sqrt{5}\right)\left(2\operatorname{arccot} 2 + 2\operatorname{arccot}\sqrt{2} - \arccos\tfrac{2}{3}\right) + \tfrac{17}{108}\left(5+2\sqrt{5}\right)\operatorname{argcosh}\tfrac{13}{3}. \tag{1.263}$$

$\mathbf{L_{11g}}$

Let A' be the same edge as in L_{11d} and $B' = \overline{[\phi,-\phi,\phi][\phi^2,-1,0]}$, then $L_{11g} = L_{A'B'}$. By shifting, $L_{A'B'} = L_{OA}$, where $O = [0,0,0]$ and $A = A' - B'$ is a polygon with vertices $[-\phi,\phi^3,-1/\phi], [-\phi^2,\sqrt{5}\phi,1], [-\phi^2,\sqrt{5}\phi,-1], [-\phi,\phi^3,-\phi^2]$ having area $\operatorname{vol} A = \sqrt{10-2\sqrt{5}}$. Projecting O onto $\mathcal{A}(A)$, we obtain $\operatorname{proj}_A O = [-1-\frac{3}{\sqrt{5}}, 2+\frac{4}{\sqrt{5}}, 0]$ and separation $h = \sqrt{10+\frac{22}{\sqrt{5}}}$. Point-Polygon formula yields

$$L_{11g} = \frac{2h^3}{\operatorname{vol} A}\left(2I_{00}^{(1)}\left(\tfrac{1}{2\phi^4}, \tfrac{2\pi}{5}\right) - I_{00}^{(1)}\left(\tfrac{1}{2\phi^2}, \tfrac{\pi}{5}\right) + I_{00}^{(1)}\left(\tfrac{1}{2\phi^2}, \tfrac{2\pi}{5}\right)\right) \approx 4.60478605392525. \tag{1.264}$$

Explicitly, after series of simplifications,

$$L_{11g} = \tfrac{5}{6} - \tfrac{1}{3\sqrt{2}} + \tfrac{11}{6\sqrt{5}} + \tfrac{1}{\sqrt{10}} - \left(\tfrac{47}{5} + \tfrac{21}{\sqrt{5}}\right)\pi + \tfrac{1}{120}\left(219+97\sqrt{5}\right)\ln 3 + \tfrac{2}{15}\left(47+21\sqrt{5}\right)\left(\arccos\tfrac{2}{3} + 2\arccos\tfrac{1}{\sqrt{41}} + 2\arccos\tfrac{3}{\sqrt{41}} - 2\operatorname{arccot}\sqrt{2}\right) + \frac{1}{120}\left(219+97\sqrt{5}\right)\left(\operatorname{argcosh}\frac{7}{3} - \operatorname{argcosh} 3\right) + \frac{1}{60}\left(91+33\sqrt{5}\right)\left(\operatorname{argcosh}\frac{9}{\sqrt{41}} - \operatorname{argcosh}\frac{7}{\sqrt{41}}\right). \tag{1.265}$$

$\mathbf{L_{11f}}$

Let A' be the same edge as in L_{11d} and $B' = \overline{[1,0,-\phi^2][-1,0,-\phi^2]}$, then $L_{11f} = L_{A'B'}$. By shifting, $L_{A'B'} = L_{OA}$, where $O = [0,0,0]$ and $A = A' - B'$ is a polygon with vertices $[-1,\phi^2,\sqrt{5}\phi], [1,\phi^2,\sqrt{5}\phi], [1,\phi^2,\phi], [-1,\phi^2,\phi]$ having area $\operatorname{vol} A = 4$. Projecting O onto $\mathcal{A}(A)$, we obtain $\operatorname{proj}_A O = [0,\phi^2,0]$ and separation $h = \phi^2$. Point-Polygon formula yields

$$L_{11f} = \tfrac{2h^3}{\operatorname{vol} A}\Bigg(I_{00}^{(1)}\left(\tfrac{1}{\phi^2}, \arctan\sqrt{5}\phi\right) + I_{00}^{(1)}\left(\tfrac{\sqrt{5}}{\phi}, \arctan\tfrac{1}{\sqrt{5}\phi}\right) - I_{00}^{(1)}\left(\tfrac{1}{\phi^2}, \arctan\phi\right) - I_{00}^{(1)}\left(\tfrac{1}{\phi}, \arctan\tfrac{1}{\phi}\right)\Bigg) \approx 3.770095521642. \tag{1.266}$$

Explicitly, after series of simplifications,

$$\begin{aligned} L_{11f} = {} & \tfrac{5}{3\sqrt{2}} - \tfrac{1}{2} + \sqrt{\tfrac{5}{2}} - \tfrac{\sqrt{5}}{6} + \left(\tfrac{3}{8} + \tfrac{\sqrt{5}}{6}\right)\pi + \tfrac{1}{48}\left(125 + 53\sqrt{5}\right)\arccos\tfrac{9}{\sqrt{41}} \\ & + \left(\tfrac{3}{8} + \tfrac{\sqrt{5}}{6}\right)\left(2\arccos\tfrac{13}{3\sqrt{41}} + 2\operatorname{arccot} 2 - \arccos\tfrac{1}{9} - 2\arccos\tfrac{3}{5} - 2\arccos\tfrac{1}{\sqrt{41}}\right) \\ & + \tfrac{1}{48}\left(23 + 9\sqrt{5}\right)\operatorname{argcosh}\tfrac{13}{3} - \tfrac{1}{48}\left(125 + 53\sqrt{5}\right)\operatorname{argcosh}\tfrac{7}{\sqrt{41}} \\ & - \tfrac{1}{48}\left(37 + 17\sqrt{5}\right)\operatorname{argsinh} 2 - \tfrac{1}{48}\left(23 + 9\sqrt{5}\right)\ln 3 + \tfrac{1}{96}\left(37 + 17\sqrt{5}\right)\ln 5. \end{aligned} \tag{1.267}$$

$\mathbf{L_{11t}}$

Again, let A' be the same edge as in L_{11d} and $B' = \overline{[\phi,-\phi,-\phi][0,-\phi^2,-1]}$, then $L_{11t} = L_{A'B'}$. By shifting, $L_{A'B'} = L_{OA}$, where $O = [0,0,0]$ and $A = A' - B'$ is a polygon with vertices $[-\phi,\phi^3,\phi^2], [0,2\phi^2,2], [0,2\phi^2,0], [-\phi,\phi^3,1/\phi]$ having area $\operatorname{vol} A = \sqrt{2\left(5+\sqrt{5}\right)}$. Projecting O onto $\mathcal{A}(A)$, we obtain $\operatorname{proj}_A O = [-1-\tfrac{3}{\sqrt{5}}, 2+\tfrac{4}{\sqrt{5}}, 0]$ and separation $h = \sqrt{10 + \tfrac{22}{\sqrt{5}}}$. Point-Polygon formula yields

$$L_{11t} = \tfrac{2h^3}{\operatorname{vol} A}\Bigg(I_{00}^{(1)}\left(\tfrac{1}{\phi}, \tfrac{\pi}{5}\right) + I_{00}^{(1)}\left(\tfrac{1}{2\phi^2}, \tfrac{\pi}{5}\right) - I_{00}^{(1)}\left(\tfrac{1}{2\phi^2}, \tfrac{2\pi}{5}\right)\Bigg) \approx 5.04162416571318. \tag{1.268}$$

Explicitly, after series of simplifications,

$$\begin{aligned} L_{11t} = {} & \tfrac{\sqrt{\frac{2}{5}}}{3} - \tfrac{1}{2} + 2\sqrt{\tfrac{3}{5}} + \tfrac{2}{\sqrt{3}} - \tfrac{7}{6\sqrt{5}} - \tfrac{\pi}{45}\left(29 + 13\sqrt{5}\right) \\ & + \tfrac{2}{15}\left(29 + 13\sqrt{5}\right)\left(2\operatorname{arccot}\sqrt{2} - \arccos\tfrac{2}{3}\right) + \left(\tfrac{61}{15} + \tfrac{9}{\sqrt{5}}\right)\left(\operatorname{argcosh} 4 - \operatorname{argcosh} 2\right) \\ & + \tfrac{1}{120}\left(133 + 61\sqrt{5}\right)\left(\operatorname{argcosh} 3 - \operatorname{argcosh}\tfrac{7}{3} - \ln 3\right). \end{aligned} \tag{1.269}$$

$\mathbf{L_{20e}}$

Let A be the face of K with vertices $[1,0,-\phi^2]$, $[\phi,\phi,-\phi]$, $[0,\phi^2,-1]$, $[-\phi,\phi,-\phi]$, $[-1,0,-\phi^2]$ (a regular pentagon) and let B be vertex $[0,\phi^2,1]$, then $L_{20e} = L_{AB}$.

Projecting B onto $\mathcal{A}(A)$, we obtain $\operatorname{proj}_A B = [0, \frac{3}{\phi^2\sqrt{5}}, -\frac{1}{\sqrt{5}}]$ and separation $h = \sqrt{2+\frac{2}{\sqrt{5}}}$. By Point-Polygon formula,

$$L_{20e} = \frac{2h^3}{\nu}\Bigg(I_{00}^{(1)}\left(\tfrac{1}{2},\tfrac{\pi}{5}\right) - I_{00}^{(1)}\left(\tfrac{1}{2},\tfrac{2\pi}{5}\right) - I_{00}^{(1)}\left(\tfrac{\phi^2}{2},\tfrac{\pi}{5}\right) + I_{00}^{(1)}\left(\tfrac{3\phi}{4},\arctan\tfrac{\sqrt{5-2\sqrt{5}}}{3}\right)$$
$$+ I_{00}^{(1)}\left(\tfrac{\phi^2}{2},\arctan\sqrt{5\left(5-2\sqrt{5}\right)}\right)\Bigg) \approx 3.346942678627. \tag{1.270}$$

Explicitly, after series of simplifications,

$$L_{20e} = \frac{7\sqrt{\frac{2}{5}}}{3} + \frac{13\sqrt{2}}{15} - \frac{4}{15} - \frac{4}{3\sqrt{5}} + \frac{4\pi}{15\sqrt{5}} - \left(\frac{1}{3}+\frac{9}{10\sqrt{5}}\right)\ln 3 - \frac{13\ln 5}{30\sqrt{5}} + \frac{8\arccos\frac{2}{3}}{15\sqrt{5}}$$
$$+ \left(\frac{1}{3}+\frac{9}{10\sqrt{5}}\right)\operatorname{argcosh}\frac{13}{3} + \frac{3}{50}\left(25+8\sqrt{5}\right)\left(\operatorname{argcosh}\frac{9}{\sqrt{41}} - \operatorname{argcosh}\frac{7}{\sqrt{41}}\right)$$
$$- \frac{16\operatorname{arccot}2}{15\sqrt{5}} + \frac{8\arctan\frac{9\sqrt{2}}{17}}{15\sqrt{5}} - \frac{8\arctan\frac{5\sqrt{2}}{7}}{15\sqrt{5}} - \frac{8\arctan\frac{3\sqrt{10}}{19}}{15\sqrt{5}} - \frac{8\arctan\sqrt{10}}{15\sqrt{5}}. \tag{1.271}$$

$\mathbf{L_{20r}}$

Let A be the same face of K as in the section on L_{20e} and let B be vertex $[0, -\phi^2, 1]$, then $L_{20r} = L_{AB}$. Projecting B onto $\mathcal{A}(A)$, we obtain $\operatorname{proj}_A B = [0, \frac{1}{10}\left(\sqrt{5}-5\right), -1-\frac{4}{\sqrt{5}}]$ and separation $h = \sqrt{10+\frac{22}{\sqrt{5}}}$. By Point-Polygon formula,

$$L_{20r} = \frac{2h^3}{\nu}\left(I_{00}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right) + I_{00}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{2\pi}{5}\right) - I_{00}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{5}\right) - I_{00}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{2\pi}{5}\right)\right)$$
$$\approx 4.87605984948. \tag{1.272}$$

Explicitly, after series of simplifications,

$$L_{20r} = \frac{4}{15} - \frac{2\sqrt{2}}{15} + \frac{4}{5\sqrt{3}} + \frac{2}{3\sqrt{5}} + \frac{4}{\sqrt{15}} + \frac{16\pi}{9} + \frac{4\pi}{\sqrt{5}} - \frac{16}{75}\left(20+9\sqrt{5}\right)\operatorname{arccot}\sqrt{2}$$
$$+ \left(\frac{3}{5}+\frac{43}{30\sqrt{5}}\right)\ln 3 + \frac{8}{75}\left(20+9\sqrt{5}\right)\left(\arccos\frac{2}{3} - \arccos\frac{1}{\sqrt{41}} - \arccos\frac{3}{\sqrt{41}}\right)$$
$$- \frac{2}{75}\left(85+37\sqrt{5}\right)\operatorname{argcosh}2 + \left(\frac{3}{5}+\frac{43}{30\sqrt{5}}\right)\left(\operatorname{argcosh}\frac{7}{3} - \operatorname{argcosh}3\right)$$
$$+ \frac{2}{75}\left(85+37\sqrt{5}\right)\operatorname{argcosh}4 + \left(\frac{1}{15}+\frac{9}{10\sqrt{5}}\right)\left(\operatorname{argcosh}\frac{7}{\sqrt{41}} - \operatorname{argcosh}\frac{9}{\sqrt{41}}\right) \tag{1.273}$$

$\mathbf{L_{20f}}$

Let A be the same face of K as in the section on L_{20e} and let B be vertex $[0, -\phi^2, -1]$, then $L_{20f} = L_{AB}$. Projecting B onto $\mathcal{A}(A)$, we obtain $\operatorname{proj}_A B = [0, -\frac{5+3\sqrt{5}}{10}, -2-\frac{3}{\sqrt{5}}]$ and separation $h = 2\sqrt{1+\frac{2}{\sqrt{5}}}$. By Point-Polygon formula,

$$L_{20f} = \frac{2h^3}{\nu}\Bigg(I_{00}^{(1)}\left(\tfrac{\phi^2}{2},\tfrac{\pi}{5}\right) - I_{00}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{2\pi}{5}\right) + I_{00}^{(1)}\left(\tfrac{1}{2\phi^2},\arctan\sqrt{5\left(5+2\sqrt{5}\right)}\right)$$
$$- I_{00}^{(1)}\left(\tfrac{1}{2},\tfrac{\pi}{5}\right) - I_{00}^{(1)}\left(\tfrac{\phi^2}{2},\arctan\sqrt{85-38\sqrt{5}}\right)\Bigg) \approx 4.000363965317. \tag{1.274}$$

Explicitly, after series of simplifications,

$$L_{20f} = 1 + \frac{\sqrt{\frac{2}{5}}}{3} - \frac{\sqrt{2}}{5} + \frac{7}{3\sqrt{5}} + \frac{4\pi}{15} + \frac{8\pi}{15\sqrt{5}} + \frac{2}{3}\left(2+\sqrt{5}\right)\ln 3 + \frac{13}{300}\left(5+2\sqrt{5}\right)\ln 5$$
$$- \frac{16}{75}\left(5+2\sqrt{5}\right)\arccos\frac{2}{3} + \frac{8}{75}\left(5+2\sqrt{5}\right)\left(\operatorname{arccot} 2 + \arctan\frac{5\sqrt{2}}{7} - \arctan\left(7\sqrt{2}\right)\right)$$
$$+ \left(\frac{47}{30} + \frac{52}{15\sqrt{5}}\right)\operatorname{argcosh}\frac{7}{3} - \left(\frac{47}{30} + \frac{52}{15\sqrt{5}}\right)\operatorname{argcosh} 3 + \frac{1}{150}\left(35+4\sqrt{5}\right)\operatorname{argcosh}\frac{13}{3}$$
$$- \frac{13}{150}\left(5+2\sqrt{5}\right)\operatorname{argsinh} 2. \tag{1.275}$$

$\mathbf{L_{22r}}$

Finally, let us take a closer look on parallel configurations L_{21r} and L_{22r}. We start with the latter. Let A and B be opposite faces of dodecahedron K with separation $h = \sqrt{10 + \frac{22}{\sqrt{5}}}$ then $L_{22} = L_{AB}$ with overlap diagram as seen in Figure 1.18. Note that, due to symmetry, only one tenth of the diagram is sufficient to be considered. The subdomains where $\operatorname{vol} A \cap \operatorname{proj} B + k$ can be written as a single polynomial are shown in the diagram. Again, they are labeled by number of sides of polygon of intersection $A \cap (\operatorname{proj} B + k)$, sliding $\operatorname{proj} B + k$ across $\operatorname{proj} A$ by letting k to vary (vector k is shown by a black dot). Let us denote D as the union of the labeled subdomains. Then, by Overlap formula,

$$L_{22r} = \frac{10}{\operatorname{vol} A \operatorname{vol} B} \int_D \sqrt{h^2 + k^2} \operatorname{vol} A \cap \operatorname{proj} B + k \, \mathrm{d}k, \tag{1.276}$$

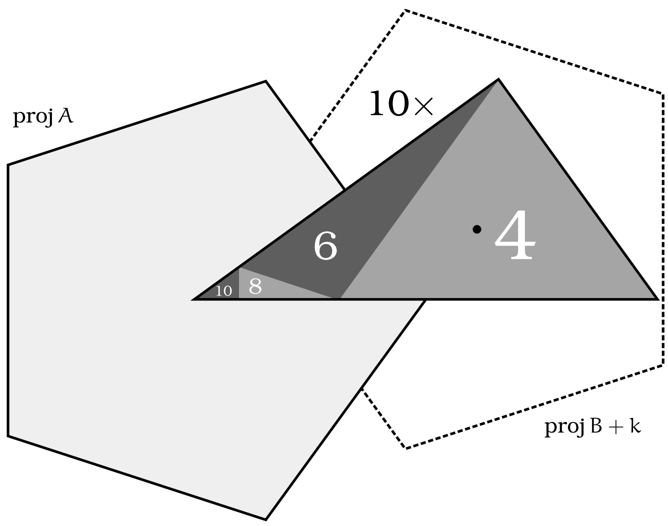


Figure 1.18: Overlap diagram for opposite-faces configuration in dodecahedron

Let us express $\operatorname{vol} A \cap \operatorname{proj} B + k$ in the aforementioned subdomains. We denote $v_j = \operatorname{vol} A \cap \operatorname{proj} B + k$ for all $k \in D_j$. Let us restrict ourselves to the plane $\mathcal{A}(A)$, in which we put $k = (x, y)$ and in which $\operatorname{proj} A$ is a regular pentagon with vertices $\left[\sqrt{2 + \frac{2}{\sqrt{5}}}\cos\frac{2\pi i}{5}, \sqrt{2 + \frac{2}{\sqrt{5}}}\sin\frac{2\pi i}{5}\right]$, $i \in \{0, 1, 2, 3, 4\}$ and area $\operatorname{vol} A = \sqrt{5\left(5 + 2\sqrt{5}\right)}$. Similarly, $\operatorname{proj} B$ is another pentagon with vertices $\left[\sqrt{2 + \frac{2}{\sqrt{5}}}\cos\frac{2\pi(i+1/2)}{5}, \sqrt{2 + \frac{2}{\sqrt{5}}}\sin\frac{2\pi(i+1/2)}{5}\right]$ and area $\operatorname{vol} B = \operatorname{vol} A = \sqrt{5\left(5 + 2\sqrt{5}\right)}$. Under this projection, the labeled subdomains D_j are triangles with vertices

- (subdomain D_4) $\left[\sqrt{\frac{5}{2}+\frac{11}{2\sqrt{5}}}, \frac{1}{2}\left(1+\sqrt{5}\right)\right], \left[\sqrt{2-\frac{2}{\sqrt{5}}}, 0\right], \left[\sqrt{8+\frac{8}{\sqrt{5}}}, 0\right]$, in which

$$v_4=\tfrac{1}{10}\left(8\sqrt{50+22\sqrt{5}}-4\left(5+3\sqrt{5}\right)x+\sqrt{5\left(5+2\sqrt{5}\right)}x^2-5\sqrt{5-2\sqrt{5}}y^2\right), \tag{1.277}$$

- (subdomain D_6) $\left[\sqrt{\frac{5}{2}+\frac{11}{2\sqrt{5}}}, \frac{1}{2}\left(1+\sqrt{5}\right)\right], \left[\sqrt{1-\frac{2}{\sqrt{5}}}, \sqrt{5}-2\right], \left[\sqrt{2-\frac{2}{\sqrt{5}}}, 0\right]$, in which

$$\begin{aligned} v_6 = \frac{1}{20}\Bigg(&8\sqrt{145+62\sqrt{5}}-4\left(5+3\sqrt{5}\right)x-\sqrt{10\left(5+\sqrt{5}\right)}x^2 \\ &-4\sqrt{10\left(5+\sqrt{5}\right)}y+10\left(1+\sqrt{5}\right)xy-5\sqrt{2\left(5+\sqrt{5}\right)}y^2\Bigg), \end{aligned} \tag{1.278}$$

- (subdomain D_8) $\left[\sqrt{1-\frac{2}{\sqrt{5}}}, \sqrt{5}-2\right], \left[\sqrt{1-\frac{2}{\sqrt{5}}}, 0\right], \left[\sqrt{2-\frac{2}{\sqrt{5}}}, 0\right]$, in which

$$v_8=\tfrac{1}{10}\left(4\sqrt{130+58\sqrt{5}}-8\sqrt{5}x-5\sqrt{1+\tfrac{2}{\sqrt{5}}}x^2-5\sqrt{5\left(5+2\sqrt{5}\right)}y^2\right), \tag{1.279}$$

- (subdomain D_{10}) $\left[\sqrt{1-\frac{2}{\sqrt{5}}}, \sqrt{5}-2\right], [0,0], \left[\sqrt{1-\frac{2}{\sqrt{5}}}, 0\right]$, in which

$$v_{10}=\frac{1}{2}\sqrt{5+2\sqrt{5}}\left(4-\sqrt{5}x^2-\sqrt{5}y^2\right). \tag{1.280}$$

In order to use the Overlap formula effectively, that is, to integrate $v_j = \operatorname{vol} A \cap \operatorname{proj} B + k$, $k \in D_j$ over all subdomains D_j, it is convenient to first perform appropriate rotation transformations and inclusion/exclusions. First, by inclusion/exclusion,

$$\begin{aligned} L_{22r} = \frac{10}{\operatorname{vol} A \operatorname{vol} B}\Bigg(&\int_{D_{10}} u_{10}\sqrt{h^2+x^2+y^2}\,\mathrm{d}x\mathrm{d}y+\int_{D_{10}\cup D_8} u_8\sqrt{h^2+x^2+y^2}\,\mathrm{d}x\mathrm{d}y \\ &+\int_{D_{10}\cup D_8\cup D_6} u_6\sqrt{h^2+x^2+y^2}\,\mathrm{d}x\mathrm{d}y+\int_{D_{10}\cup D_8\cup D_6\cup D_4} u_4\sqrt{h^2+x^2+y^2}\,\mathrm{d}x\mathrm{d}y\Bigg), \end{aligned} \tag{1.281}$$

where $u_4 = v_4$, $u_6 = v_6 - v_4$, $u_8 = v_8 - v_6$ and $u_{10} = v_{10} - v_8$. Explicitly

$$u_4 =\tfrac{4\sqrt{2}}{5}\sqrt{25+11\sqrt{5}}-\tfrac{2x}{5}\left(5+3\sqrt{5}\right)+\tfrac{x^2}{10}\sqrt{5\left(5+2\sqrt{5}\right)}-\tfrac{y^2}{2}\sqrt{5-2\sqrt{5}}, \tag{1.282}$$

$$\begin{aligned} u_6 =& -\tfrac{2}{5}\sqrt{5+2\sqrt{5}}+x\left(1+\tfrac{3}{\sqrt{5}}\right)-y\sqrt{2+\tfrac{2}{\sqrt{5}}}+\tfrac{xy}{2}\left(1+\sqrt{5}\right) \\ &-\tfrac{x^2\sqrt{2}}{4}\sqrt{5+\tfrac{11}{\sqrt{5}}}-\tfrac{y^2\sqrt{2}}{4}\sqrt{5-\sqrt{5}}, \end{aligned} \tag{1.283}$$

$$\begin{aligned} u_8 =& -\tfrac{2}{5}\sqrt{5-2\sqrt{5}}+x\left(1-\tfrac{1}{\sqrt{5}}\right)+y\sqrt{2+\tfrac{2}{\sqrt{5}}}-\tfrac{xy}{2}\left(1+\sqrt{5}\right) \\ &-\tfrac{x^2\sqrt{2}}{4}\sqrt{1-\tfrac{1}{\sqrt{5}}}-\tfrac{y^2\sqrt{2}}{4}\sqrt{25+11\sqrt{5}}, \end{aligned} \tag{1.284}$$

$$u_{10} = -\tfrac{2}{5}\sqrt{5-2\sqrt{5}}+\tfrac{4x}{\sqrt{5}}-2x^2\sqrt{1+\tfrac{2}{\sqrt{5}}}. \tag{1.285}$$

Note that domain D_{10} is already in the form of the fundamental triangle domain $D\left(\zeta, \frac{\pi}{5}\right)$ with $\zeta = \sqrt{1-\frac{2}{\sqrt{5}}}$. Since $\zeta/h = 1/(2\phi^4)$, we immediately get in terms of auxiliary integrals,

$$\int_{D_{10}} u_{10}\sqrt{h^2+x^2+y^2}\mathrm{d}x\mathrm{d}y = h^3\Bigg(-\tfrac{2}{5}\sqrt{5-2\sqrt{5}}I_{00}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{\pi}{5}\right) + \tfrac{4h}{\sqrt{5}}I_{10}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{\pi}{5}\right) - 2h^2\sqrt{1+\tfrac{2}{\sqrt{5}}}I_{20}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{\pi}{5}\right)\Bigg). \tag{1.286}$$

Domain $D = D_{10}\cup D_8\cup D_6\cup D_4$ in (x,y) is transformed to the fundamental domain $D(\zeta,\frac{\pi}{5})$ with $\zeta = 2\sqrt{1+\frac{2}{\sqrt{5}}}$ in (x',y') via polar angle substitution $\varphi = \pi/5-\varphi'$, that is $x = r\cos(\frac{\pi}{5}-\varphi')$ and $y = r\sin(\frac{\pi}{5}-\varphi')$. Expanding out the trigonometric functions and writing $x' = r\cos\varphi'$ and $y' = r\cos\varphi'$, we get the following transformation relations

$$x = \frac{1}{4}\left(1+\sqrt{5}\right)x' + \sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}y', \qquad y = \sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}x' - \frac{1}{4}\left(1+\sqrt{5}\right)y' \tag{1.287}$$

and so

$$u_4 = \frac{4}{5}\sqrt{50+22\sqrt{5}} - x'\left(2+\frac{4}{\sqrt{5}}\right) - 2y'\sqrt{1+\frac{2}{\sqrt{5}}} + x'y' + x'^2\sqrt{1-\frac{2}{\sqrt{5}}}. \tag{1.288}$$

Since $\zeta/h = 1/\phi$, we immediately get

$$\int_D u_4\sqrt{h^2+x^2+y^2}\,\mathrm{d}x\mathrm{d}y = h^3\Bigg(\tfrac{4}{5}\sqrt{50+22\sqrt{5}}I_{00}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right) - h\left(2+\tfrac{4}{\sqrt{5}}\right)I_{10}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right) - 2h\sqrt{1+\tfrac{2}{\sqrt{5}}}I_{01}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right) + h^2I_{11}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right) + h^2\sqrt{1-\tfrac{2}{\sqrt{5}}}I_{20}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right)\Bigg). \tag{1.289}$$

In order to express the remaining integrals, we write $D_{10}\cup D_8\cup D_6 = E_4\setminus E_6$ and $D_{10}\cup D_8 = E_8\setminus E_{10}$, where

- E_4 is a triangle $\left[\sqrt{\frac{5}{2}+\frac{11}{2\sqrt{5}}},\frac{1}{2}\left(1+\sqrt{5}\right)\right]$, $[0,0]$, $\left[\frac{1}{2}\sqrt{1+\frac{2}{\sqrt{5}}},-\frac{1}{2}\right]$,
- E_6 is a triangle $\left[\sqrt{2-\frac{2}{\sqrt{5}}},0\right]$, $[0,0]$, $\left[\frac{1}{2}\sqrt{1+\frac{2}{\sqrt{5}}},-\frac{1}{2}\right]$,
- E_8 is a triangle $\left[\frac{1}{2}\sqrt{\frac{5}{2}-\frac{11}{2\sqrt{5}}},\frac{1}{4}\left(\sqrt{5}-1\right)\right]$, $[0,0]$, $\left[\frac{1}{2}\sqrt{\frac{5}{2}-\frac{11}{2\sqrt{5}}},\frac{1}{4}\left(\sqrt{5}-1\right)\right]$,
- E_{10} is a triangle $\left[\frac{1}{2}\sqrt{\frac{5}{2}-\frac{11}{2\sqrt{5}}},\frac{1}{4}\left(\sqrt{5}-1\right)\right]$, $[0,0]$, $\left[\sqrt{1-\frac{2}{\sqrt{5}}},\sqrt{5}-2\right]$.

Note that $E_6\subset E_4$ and $E_{10}\subset E_8$ and thus

$$\begin{aligned}\int_{D_{10}\cup D_8\cup D_6} u_6\sqrt{h^2+x^2+y^2}\mathrm{d}x\mathrm{d}y &= \int_{E_4} u_6\sqrt{h^2+x^2+y^2}\mathrm{d}x\mathrm{d}y - \int_{E_6} u_6\sqrt{h^2+x^2+y^2}\mathrm{d}x\mathrm{d}y,\\ \int_{D_{10}\cup D_8} u_8\sqrt{h^2+x^2+y^2}\mathrm{d}x\mathrm{d}y &= \int_{E_8} u_8\sqrt{h^2+x^2+y^2}\mathrm{d}x\mathrm{d}y - \int_{E_{10}} u_8\sqrt{h^2+x^2+y^2}\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{1.290}$$

Domains E_4, E_6, E_8 and E_{10} can be rotated to fundamental triangle domains after appropriate rotations. First, let $\varphi = \varphi' - \pi/5$, so

$$x = \frac{1}{4}\left(1+\sqrt{5}\right)x' + \sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}y', \qquad y = -\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}x' + \frac{1}{4}\left(1+\sqrt{5}\right)y' \tag{1.291}$$

and thus, after simplifications,

$$u_6 = -\frac{2}{5}\sqrt{5+2\sqrt{5}} + 2x'\left(1+\frac{1}{\sqrt{5}}\right) - x'^2\sqrt{2+\frac{2}{\sqrt{5}}}. \tag{1.292}$$

Suddenly in (x',y'), we have $E_4 = D(\zeta,\frac{2\pi}{5})$ and $E_6 = D(\zeta,\frac{\pi}{5})$ with $\zeta=\sqrt{\frac{5+\sqrt{5}}{10}}$, hence $\zeta/h = 1/(2\phi^2)$ and immediately in terms of auxiliary integrals,

$$\int_{D_{10}\cup D_8\cup D_6} u_6\sqrt{h^2+x^2+y^2}\mathrm{d}x\mathrm{d}y = h^3\Big(-\tfrac{2}{5}\sqrt{5+2\sqrt{5}}\left(I_{00}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{2\pi}{5}\right)-I_{00}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{5}\right)\right)$$
$$+2h\Big(1+\tfrac{1}{\sqrt{5}}\Big)\left(I_{10}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{2\pi}{5}\right)-I_{10}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{5}\right)\right)-h^2\sqrt{2+\tfrac{2}{\sqrt{5}}}\left(I_{20}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{2\pi}{5}\right)-I_{20}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{5}\right)\right)\Big). \tag{1.293}$$

Next, let $\varphi = \frac{2\pi}{5}-\varphi'$, from which we obtain transformation relations

$$x = \frac{1}{4}\left(\sqrt{5}-1\right)x' + \sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}y', \qquad y = \sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}x' - \frac{1}{4}\left(\sqrt{5}-1\right)y', \tag{1.294}$$

so

$$u_8 = -\frac{2}{5}\sqrt{5-2\sqrt{5}} + \frac{4x'}{\sqrt{5}} - 2x'^2\sqrt{1+\frac{2}{\sqrt{5}}}. \tag{1.295}$$

In (x',y'), we have $E_8 = D(\zeta,\frac{2\pi}{5})$ and $E_{10} = D(\zeta,\frac{\pi}{5})$ with $\zeta = \sqrt{1-\frac{2}{\sqrt{5}}}$, hence $\zeta/h = 1/(2\phi^4)$ and immediately in terms of auxiliary integrals,

$$\int_{D_{10}\cup D_8} u_8\sqrt{h^2+x^2+y^2}\,\mathrm{d}x\mathrm{d}y = h^3\Bigg(-\tfrac{2}{5}\sqrt{5-2\sqrt{5}}\left(I_{00}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{2\pi}{5}\right)-I_{00}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{\pi}{5}\right)\right)$$
$$+\tfrac{4h}{\sqrt{5}}\left(I_{10}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{2\pi}{5}\right)-I_{10}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{\pi}{5}\right)\right)-2h^2\sqrt{1+\tfrac{2}{\sqrt{5}}}\left(I_{20}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{2\pi}{5}\right)-I_{20}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{5}\right)\right)\Bigg). \tag{1.296}$$

Therefore, in total,

$$L_{22r} = \tfrac{10h^3}{\operatorname{vol}A\operatorname{vol}B}\Bigg(\tfrac{2}{5}\sqrt{5+2\sqrt{5}}I_{00}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{5}\right)-\tfrac{2}{5}\sqrt{5+2\sqrt{5}}I_{00}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{2\pi}{5}\right)$$
$$-\tfrac{2}{5}\sqrt{5-2\sqrt{5}}I_{00}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{2\pi}{5}\right)+\tfrac{4}{5}\sqrt{50+22\sqrt{5}}I_{00}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right)-2\sqrt{\tfrac{1}{5}\left(5+2\sqrt{5}\right)}hI_{01}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right)$$
$$+\frac{4hI_{10}^{(1)}\left(\frac{1}{2\phi^4},\frac{2\pi}{5}\right)}{\sqrt{5}}-\frac{2hI_{10}^{(1)}\left(\frac{1}{2\phi^2},\frac{\pi}{5}\right)}{\sqrt{5}}-2hI_{10}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{5}\right)+\frac{2hI_{10}^{(1)}\left(\frac{1}{2\phi^2},\frac{2\pi}{5}\right)}{\sqrt{5}}+2hI_{10}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{2\pi}{5}\right)$$
$$-\frac{4hI_{10}^{(1)}\left(\frac{1}{\phi},\frac{\pi}{5}\right)}{\sqrt{5}}-2hI_{10}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right)+h^2I_{11}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right)-2\sqrt{1+\tfrac{2}{\sqrt{5}}}h^2I_{20}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{2\pi}{5}\right)$$
$$+\sqrt{2+\tfrac{2}{\sqrt{5}}}h^2I_{20}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{5}\right)-\sqrt{2+\tfrac{2}{\sqrt{5}}}h^2I_{20}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{2\pi}{5}\right)+\sqrt{1-\tfrac{2}{\sqrt{5}}}h^2I_{20}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right)\Bigg)$$
$$\approx 4.69357209587.$$

Or explicitly, after a lot of simplifications,

$$L_{22r} = \tfrac{2\sqrt{\frac{2}{5}}}{15}-\tfrac{38}{75}-\tfrac{4\sqrt{2}}{75}+\tfrac{44}{25\sqrt{3}}-\tfrac{88}{75\sqrt{5}}+\tfrac{116}{25\sqrt{15}}-\tfrac{8\left(1839+820\sqrt{5}\right)\pi}{1125}$$
$$+\tfrac{16}{125}\left(67+30\sqrt{5}\right)\arccos\tfrac{2}{3}+\tfrac{16}{375}\left(388+173\sqrt{5}\right)\left(\arccos\tfrac{1}{\sqrt{41}}+\arccos\tfrac{3}{\sqrt{41}}\right)$$
$$+\tfrac{2}{375}\left(817+371\sqrt{5}\right)(\operatorname{argcosh}2-\operatorname{argcosh}4)+\tfrac{1}{250}\left(1833+820\sqrt{5}\right)\ln 3$$
$$+\tfrac{1}{750}\left(3538+1523\sqrt{5}\right)\left(\operatorname{argcosh}\tfrac{9}{\sqrt{41}}-\operatorname{argcosh}\tfrac{7}{\sqrt{41}}\right)-\tfrac{32}{125}\left(67+30\sqrt{5}\right)\operatorname{arccot}\sqrt{2}$$
$$+\tfrac{1}{250}\left(1833+820\sqrt{5}\right)\left(\operatorname{argcosh}\tfrac{7}{3}-\operatorname{argcosh}3\right) \tag{1.297}$$

$\mathbf{L_{21r}}$

By definition, $L_{21r} = L_{AB}$, where A is a face of K and B is the perimeter of its corresponding opposite face. Again, we use the Overlap formula to deduce the value of L_{21r}, that is, by symmetry,

$$L_{21r} = \frac{10}{\operatorname{vol} A \operatorname{vol} B} \int_D \sqrt{h^2 + k^2} \operatorname{vol} A \cap \operatorname{proj} B + k \, \mathrm{d}k, \tag{1.298}$$

where $\operatorname{vol} A = \sqrt{5\left(5 + 2\sqrt{5}\right)}$ is the area of A and $\operatorname{vol} B = 10$ is the length of B. The overlap diagram is the same as in the case of L_{22r}, although the value $\operatorname{vol} A \cap \operatorname{proj} B + k$ now corresponds to the total length of polyline $A \cap (\operatorname{proj} B + k)$ of intersection. In order to keep the naming of the subdomains D_j and functions $v_j = \operatorname{vol} A \cap \operatorname{proj} B + k$, $k \in D_j$ the same as in the case of L_{22r}, we let j, exceptionally, to denote *twice* the number line segments of $A \cap (\operatorname{proj} B + k)$ in this section. That way, we get $D = D_{10} \cup D_8 \cup D_6 \cup D_4$ and

$$\begin{aligned} v_4 &= 4 + \tfrac{4}{\sqrt{5}} - x\sqrt{2 + \tfrac{2}{\sqrt{5}}}, & v_6 &= 4 + \frac{2}{\sqrt{5}} - \frac{x}{2}\sqrt{2 + \frac{2}{\sqrt{5}}} + \frac{y}{2}\left(1 - \sqrt{5}\right), \\ v_8 &= 2 + \tfrac{6}{\sqrt{5}} - 2x\sqrt{1 - \tfrac{2}{\sqrt{5}}}, & v_{10} &= 2\sqrt{5}. \end{aligned}$$

Let $u_4 = v_4$, $u_6 = v_6 - v_4$, $u_8 = v_8 - v_6$, $u_{10} = v_{10} - v_8$, that is

$$\begin{aligned} u_4 &= 4 + \tfrac{4}{\sqrt{5}} - x\sqrt{2 + \tfrac{2}{\sqrt{5}}}, & u_6 &= -\tfrac{2}{\sqrt{5}} + \tfrac{x}{2}\sqrt{2 + \tfrac{2}{\sqrt{5}}} + \tfrac{y}{2}\left(1 - \sqrt{5}\right), \\ u_{10} &= -2 + \tfrac{4}{\sqrt{5}} + 2\sqrt{1 - \tfrac{2}{\sqrt{5}}}x, & u_8 &= -2 + \tfrac{4}{\sqrt{5}} + x\sqrt{\tfrac{5}{2} - \tfrac{11}{2\sqrt{5}}} - \tfrac{y}{2}\left(1 - \sqrt{5}\right). \end{aligned}$$

Overall, by inclusion/exclusion,

$$\begin{aligned} L_{21r} = \tfrac{10}{\operatorname{vol} A \operatorname{vol} B}\Big(& \textstyle\int_{D_{10}} u_{10}\sqrt{h^2 + x^2 + y^2} \, \mathrm{d}x\mathrm{d}y + \int_D u_4\sqrt{h^2 + x^2 + y^2} \, \mathrm{d}x\mathrm{d}y \\ & + \textstyle\int_{E_4} u_6\sqrt{h^2 + x^2 + y^2} \, \mathrm{d}x\mathrm{d}y - \int_{E_6} u_6\sqrt{h^2 + x^2 + y^2} \, \mathrm{d}x\mathrm{d}y \\ & + \textstyle\int_{E_8} u_8\sqrt{h^2 + x^2 + y^2} \, \mathrm{d}x\mathrm{d}y - \int_{E_{10}} u_8\sqrt{h^2 + x^2 + y^2} \, \mathrm{d}x\mathrm{d}y\Big), \end{aligned} \tag{1.299}$$

The first integral can be immediately expressed in terms of auxiliary integrals

$$\textstyle\int_{D_{10}} u_{10}\sqrt{h^2+x^2+y^2} \, \mathrm{d}x\mathrm{d}y = h^3\left(\left(\tfrac{4}{\sqrt{5}} - 2\right) I_{00}^{(1)}\left(\tfrac{1}{2\phi^4}, \tfrac{\pi}{5}\right) + 2h\sqrt{1 - \tfrac{2}{\sqrt{5}}} I_{10}^{(1)}\left(\tfrac{1}{2\phi^4}, \tfrac{\pi}{5}\right)\right). \tag{1.300}$$

Performing the same set of transformations as in the previous case of L_{22r}, that is

- $\varphi = \pi/5 - \varphi'$, we get $u_4 = 4 + \frac{4}{\sqrt{5}} - x'\sqrt{1 + \frac{2}{\sqrt{5}}} - y'$,
- $\varphi = \varphi' - \pi/5$, we get $u_6 = -\frac{2}{\sqrt{5}} + x'\sqrt{2 - \frac{2}{\sqrt{5}}}$,
- $\varphi = \frac{2\pi}{5} - \varphi'$, we get $u_8 = -2 + \frac{4}{\sqrt{5}} + 2x'\sqrt{1 - \frac{2}{\sqrt{5}}}$

and as a result, since all the subdomains are now expressed as fundamental triangle domains, we get

$$\int_D u_4\sqrt{h^2+x^2+y^2}\mathrm{d}x\mathrm{d}y = h^3\left(\left(4+\tfrac{4}{\sqrt{5}}\right) I_{00}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right) - h\sqrt{1+\tfrac{2}{\sqrt{5}}} I_{10}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right) - hI_{01}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right)\right),$$

$$\int_{D_{10}\cup D_8\cup D_6} u_6\sqrt{h^2+x^2+y^2}\,\mathrm{d}x\mathrm{d}y = h^3\Bigg(-\tfrac{2}{\sqrt{5}}\left(I_{00}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{2\pi}{5}\right) - I_{00}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{5}\right)\right)$$
$$+ h\sqrt{2-\tfrac{2}{\sqrt{5}}}\left(1+\tfrac{1}{\sqrt{5}}\right)\left(I_{10}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{2\pi}{5}\right) - I_{10}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{5}\right)\right)\Bigg),$$

$$\int_{D_{10}\cup D_8} u_8\sqrt{h^2+x^2+y^2}\,\mathrm{d}x\mathrm{d}y = h^3\Bigg(\left(\tfrac{4}{\sqrt{5}}-2\right)\left(I_{00}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{2\pi}{5}\right) - I_{00}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{\pi}{5}\right)\right)$$
$$+ 2h\sqrt{1-\tfrac{2}{\sqrt{5}}}\left(I_{10}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{2\pi}{5}\right) - I_{10}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{\pi}{5}\right)\right)\Bigg).$$

Therefore, in total Therefore, in total,

$$L_{21r} = \tfrac{10h^3}{\operatorname{vol}A\operatorname{vol}B}\Bigg(\left(\tfrac{4}{\sqrt{5}}-2\right)I_{00}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{2\pi}{5}\right) + \frac{2I_{00}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{5}\right)}{\sqrt{5}} - \frac{2I_{00}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{2\pi}{5}\right)}{\sqrt{5}}$$
$$+ \tfrac{4}{5}\left(5+\sqrt{5}\right)I_{00}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right) - hI_{01}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right) + 2\sqrt{1-\tfrac{2}{\sqrt{5}}}hI_{10}^{(1)}\left(\tfrac{1}{2\phi^4},\tfrac{2\pi}{5}\right)$$
$$- \sqrt{2-\tfrac{2}{\sqrt{5}}}hI_{10}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{\pi}{5}\right) + \sqrt{2-\tfrac{2}{\sqrt{5}}}hI_{10}^{(1)}\left(\tfrac{1}{2\phi^2},\tfrac{2\pi}{5}\right) - \sqrt{1+\tfrac{2}{\sqrt{5}}}hI_{10}^{(1)}\left(\tfrac{1}{\phi},\tfrac{\pi}{5}\right)\Bigg)$$
$$\approx 4.808558828667.$$

Or explicitly,

$$L_{21r} = \tfrac{149}{30} - \tfrac{29\sqrt{\frac{3}{5}}}{5} - \tfrac{\sqrt{2}}{15} - \tfrac{41}{5\sqrt{3}} + \tfrac{166}{15\sqrt{5}} + \tfrac{1}{3\sqrt{10}} - \tfrac{4\pi}{225}\left(19+8\sqrt{5}\right)$$
$$- \tfrac{8}{75}\left(2+\sqrt{5}\right)\left(\arccos\tfrac{1}{\sqrt{41}} + \arccos\tfrac{3}{\sqrt{41}}\right) + \tfrac{1043+468\sqrt{5}}{600}\left(\operatorname{argcosh}\tfrac{7}{3} - \operatorname{argcosh}3\right)$$
$$+ \tfrac{271+117\sqrt{5}}{150}\left(\operatorname{argcosh}4 - \operatorname{argcosh}2\right) + \tfrac{746+283\sqrt{5}}{600}\left(\operatorname{argcosh}\tfrac{9}{\sqrt{41}} - \operatorname{argcosh}\tfrac{7}{\sqrt{41}}\right)$$
$$- \tfrac{8}{75}\left(9+4\sqrt{5}\right)\arccos\tfrac{2}{3} + \tfrac{16}{75}\left(9+4\sqrt{5}\right)\operatorname{arccot}\sqrt{2} + \tfrac{1}{600}\left(1043+468\sqrt{5}\right)\ln 3. \tag{1.301}$$

$\mathbf{L_{33}}$

Putting everything together by using (1.261), we finally arrive, after another series of simplifications and inverse trigonometric and hyperbolic identities, at

$$L_{33} = \tfrac{1516}{1575} + \tfrac{2\sqrt{\frac{2}{5}}}{45} - \tfrac{124\sqrt{\frac{3}{5}}}{175} - \tfrac{71\sqrt{2}}{1575} - \tfrac{12\sqrt{3}}{35} + \tfrac{342}{175\sqrt{5}} + \tfrac{493\pi}{23625} + \tfrac{67\pi}{945\sqrt{5}}$$
$$+ \frac{\left(397-244\sqrt{5}\right)\operatorname{arccot}2}{18900} + \frac{\left(24023+11788\sqrt{5}\right)\left(\arccos\frac{2}{3}-\arccos\frac{1}{3}\right)}{94500}$$
$$- \frac{\left(461+212\sqrt{5}\right)\left(\arccos\frac{23}{41}+\arccos\frac{39}{41}\right)}{1000} - \frac{\left(1031+521\sqrt{5}\right)\operatorname{argcosh}\frac{13}{3}}{75600}$$
$$+ \frac{\left(367+163\sqrt{5}\right)\operatorname{argcosh}9}{16800} + \frac{\left(22197+8149\sqrt{5}\right)\left(\operatorname{argcosh}\frac{121}{41}-\operatorname{argcosh}\frac{57}{41}\right)}{84000}$$
$$+ \frac{\left(15763+7063\sqrt{5}\right)\left(\operatorname{argcosh}\frac{7}{3}-\operatorname{argcosh}3\right)}{21000} + \frac{2\left(423+187\sqrt{5}\right)\left(\operatorname{argcosh}4-\operatorname{argcosh}2\right)}{875}$$
$$+ \frac{\left(288889+129739\sqrt{5}\right)\ln 3}{378000} + \frac{\left(109-3143\sqrt{5}\right)\ln 5}{151200} \approx 2.533488631644.$$

Rescaling, we get our mean distance in a regular dodecahedron having unit volume

$$v_1^{(1)}(\text{dodecahedron}) = \frac{L_{33}}{\sqrt[3]{30+14\sqrt{5}}} \approx 0.65853073. \tag{1.302}$$

1.4.8 Unsolved problems

Weights

We believe that the equation for weights (1.183) possesses a closed form solution in terms of geometrical properties of convex non-parallel polyhedra. However, we were unable to deduce that.

General convex polyhedra

Let $K \subset \mathbb{R}^d$, then for any fixed $p > -d$, $L_{KK}^{(p)}$ is *continuous* with respect to continuous transformations of K. Hence, in principle, we could obtain the formula for convex parallel polyhedra by a continuous limit from some convex non-parallel polyhedron. However, were not able to perform this limit.

Bounds on moments

Also, we believe, since the value $p = 1$ is not special, there could be a bound on $L_{KK}^{(p)}$ similar that of Bonnet, Gusakova, Thäle and Zaporozhets [12].

1.5 Bivariate functionals in higher dimensions

1.5.1 d-Ball

Consider a bivariate symmetric homogeneous functional P of order p dependent on two random points picked uniformly from the unit ball $\mathbb{B}_d = \{\mathbf{x} \in \mathbb{R}^d \mid \|\mathbf{x}\| \le 1\}$ with volume $\mathrm{vol}_d\,\mathbb{B}_d = \kappa_d = \omega_d/d$. Additionally, we require P to be *rotationally* symmetric with respect to the origin. That is, for any $\mathbf{x}, \mathbf{y} \in \mathbb{B}_d$ and any orthogonal matrix R we have $P(R\mathbf{x}, R\mathbf{y}) = P(\mathbf{x}, \mathbf{y})$. This assumption is satisfied by the choice $P = L^p$ (which is implicitly assumed in this section).

Reduction system

According to our convention, let $P_{ab} = \mathbb{E}\left[P(\mathbf{X}, \mathbf{Y}) \mid \mathbf{X} \sim \mathsf{Unif}(A), \mathbf{Y} \sim \mathsf{Unif}(B)\right]$, where $a = \dim A$, $b = \dim B$ and the concrete selection of A, B is deduced from the reduction diagram in Figure 1.19 below. In this diagram, we also included the position of the scaling point $\mathbf{C}$ in cases reduction is possible. The arrows indicate which configurations reduce to which. Each arrow is labeled by a roman numeral corresponding to a given reduction equation in the system of reduction equations.

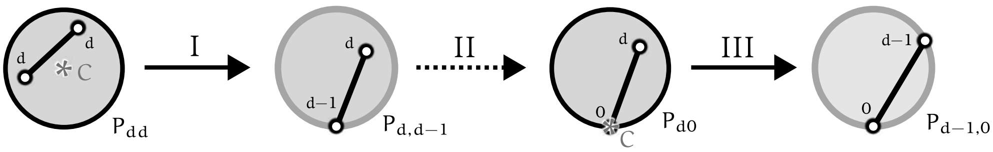


Figure 1.19: All different P_{ab} sub-configurations in $\mathbb{B}_d$

The full system obtained by CRT is

$$\begin{aligned} \mathbf{I}:\ pP_{dd} &= 2d(P_{d,d-1} - P_{dd}) \\ \mathbf{II}:\ P_{d,d-1} &= P_{d0}. \\ \mathbf{III}:\ pP_{d0} &= d(P_{d-1,0} - P_{d0}), \end{aligned}$$

where the equation **II** follows from the rotational symmetry of P. The solution of our system is

$$P_{dd} = \frac{2d^2 P_{20}}{(2d+p)(d+p)}. \tag{1.303}$$

$\mathbf{P_{d\text{-}1,0}}$

In configuration $(d-1, 0)$, one point $\mathbf{X}$ is drawn uniformly from the boundary $\partial\mathbb{B}_d$ while the other $\mathbf{Y}$ is fixed at the boundary. Keep in mind that $P_{d-1,0}$ is defined via generalization of Remark 9 as a mean weighted by the support function

$$P_{d-1,0} = \frac{1}{d\,\mathrm{vol}_d\,\mathbb{B}_d} \int_{\partial\mathbb{B}_d} P(\mathbf{x}, \mathbf{y}) h_{\mathbf{y}}(\mathbf{x}) \lambda_{d-1}(\mathrm{d}\mathbf{x}), \tag{1.304}$$

where the support function $h_{\mathbf{y}}(\mathbf{x})$ of $\mathbb{B}_d$ evaluated in $\mathbf{x}$ and centered at $\mathbf{y} \in \partial\mathbb{B}_d$ (arbitrary fixed point) is given explicitly as $h_{\mathbf{y}}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|^2$. Analogously to the 3-ball case, we parametrise the integral using hyper-spherical coordinates with the axial angle of $\mathbf{x}-\mathbf{y}$ being $\theta \in [0, \pi/2)$. We have $\|\mathbf{x}-\mathbf{y}\| = 2\cos\theta$ and thus $h_{\mathbf{y}}(\mathbf{x}) = 2\cos^2\theta$. Furthermore, since $P = L^p$, we get $P(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|^p = (2\cos\theta)^p$. Integrating out the axial symmetry from the uniform measure on $\partial\mathbb{B}_d$,

$$\lambda_{d-1}(\mathrm{d}\mathbf{x}) = 2\omega_{d-1}(2\cos\theta\sin\theta)^{d-2}\mathrm{d}\theta \tag{1.305}$$

Overall, calculating the following Beta integral and by using *Legendre duplication identity*

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = \tfrac{2\sqrt{\pi}}{2^{2z}}\Gamma(2z), \tag{1.306}$$

we obtain

$$L^{(p)}_{d-1,0} = \frac{\omega_{d-1}}{\omega_d}\int_0^{\pi/2}(2\cos\theta)^{d+p}\sin^{d-2}\theta\ \mathrm{d}\theta = \frac{2^p\,\Gamma(d)\,\Gamma\Big(\frac{1}{2}(d+p+1)\Big)}{\Gamma\Big(\frac{1+d}{2}\Big)\,\Gamma\Big(d+\frac{p}{2}\Big)}. \tag{1.307}$$

$\mathbf{P_{dd}}$

Substituting $P_{d-1,0}$ into Equation (1.303) with $P = L^p$, we get for general $p > -d$ (not necessarily an integer),

$$L^{(p)}_{dd} = \frac{2^{1+p}d\,(d!)\int_0^{\frac{\pi}{2}}\sin^d\theta\cos^{d+p}\theta\ \mathrm{d}\theta}{(d+p)\,\Gamma\Big(\frac{1+d}{2}\Big)^2} = \frac{2^p d\,(d!)\Gamma\Big(\frac{1}{2}(d+p+1)\Big)}{(d+p)\,\Gamma\Big(\frac{1+d}{2}\Big)\,\Gamma\Big(1+d+\frac{p}{2}\Big)}. \tag{1.308}$$

Distance density

The density $f_{dd}(\lambda)$ of the random distance L between two interior points in $\mathbb{B}_d$ can be recovered from moments using inverse Mellin transform (see appendix A.5). By Equation (1.308), we have

$$\mathcal{M}[f_{dd}] = L^{(p-1)}_{dd} = \frac{2^p d\,(d!)}{(d+p-1)\,\Gamma\Big(\frac{1+d}{2}\Big)^2}\int_0^{\frac{\pi}{2}}\sin^d\theta\cos^{d+p-1}\theta\ \mathrm{d}\theta. \tag{1.309}$$

Taking the inverse Mellin transform, we get, formally,

$$f_{dd}(\lambda) = \frac{2d\,(d!)}{\Gamma\Big(\frac{1+d}{2}\Big)^2}\mathcal{I}_{d-1}\left[\int_0^{\pi/2}(\cos\theta\sin\theta)^d\,\delta(\lambda-2\cos\theta)\ \mathrm{d}\theta\right]. \tag{1.310}$$

By Equation (A.41) (see Appendix A), we immediately get

$$f_{dd}(\lambda) = \frac{2^{1-d}d\,(d!)}{\Gamma\Big(\frac{1+d}{2}\Big)^2}\lambda^{d-1}\int_0^{\arccos\frac{\lambda}{2}}\sin^d\theta\ \mathrm{d}\theta. \tag{1.311}$$

This result is not new, see Tu and Fischbach [72].

1.6 Trivariate functionals in two dimensions

1.6.1 Equilateral triangle

To demonstrate the approach of CRT for trivariate functionals, we solve the Silvester problem (c.f. [76]). The objective of the problem is to determine the expected value of the area S of a triangle whose vertices are three points chosen randomly from the intetrior of a given triangle T_2. In our CRT notation, the exact result is expressed as

$$S_{222} = v_2^{(1)}(T_2)\,\mathrm{vol}_2\,T_2 = \frac{\mathrm{vol}_2\,T_2}{12}.$$

Although this result holds for any triangle T_2, thanks to the scale affinity of areas in two-dimensions, we can indeed assume that our triangle is *equilateral* with unit area.

Beyond the Silvester problem, we also find higher area moments $S_{222}^{(k)}$ for any non-negative integer k. Table 1.13 below shows various explicit $S_{222}^{(k)}$ area moments for selected k's (from Equation (1.333) or (1.334)).

$S_{222}^{(1)}$	$S_{222}^{(2)}$	$S_{222}^{(3)}$	$S_{222}^{(4)}$	$S_{222}^{(5)}$	$S_{222}^{(6)}$	$S_{222}^{(7)}$	$S_{222}^{(8)}$
$\frac{1}{12}$	$\frac{1}{72}$	$\frac{31}{9000}$	$\frac{1}{900}$	$\frac{1063}{2469600}$	$\frac{403}{2116800}$	$\frac{211}{2268000}$	$\frac{13}{2646000}$

Table 1.13: Values of volumetric moments $v_2^{(k)}(T_2) = S_{222}^{(k)}$ for selected k's

Consequentially, from the knowledge of all moments, we deduce the probability density function $f_{222}(s)$ of S using the inverse Mellin transform, a method due to Mathai (see [45]).

Furthermore, we show how we can deduce the obtusity probability

$$\eta(T_2^*) = \frac{25}{4} + \frac{\pi}{12\sqrt{3}} + \frac{393}{10}\ln\frac{\sqrt{3}}{2} \approx 0.748197 \qquad (1.312)$$

in (the standard) equilateral triangle T_2^*. Our approach can be generalised to obtain obtusity probability $\eta(T_2)$ in any other triangle T_2 though.

Configurations

In general, let P be a trivariate homogeneous symmetric functional of order p (in case of random triangle area, we have $P = S$ and $p = 2$). In agreement with our convention,

$$P_{abc} = \mathbb{E}\left[P(\mathbf{X},\mathbf{Y},\mathbf{Z}) \mid \mathbf{X}\sim \mathsf{Unif}(A), \mathbf{Y}\sim \mathsf{Unif}(B), \mathbf{Z}\sim \mathsf{Unif}(C)\right], \qquad (1.313)$$

where a, b, c are dimensions of domains A, B, C, respectively, from which the points $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in T_2$ are selected. In the case of ambiguity, the specific meaning of each P_{abc} is deduced from the reduction technique itself or it is shown in Figure 1.20. In there, we also included the position of the scaling point $\mathbf{C}$ in cases reduction is possible. The arrows indicate which configurations reduce to which. Each arrow is labeled by a roman numeral corresponding to a given reduction equation in the system of reduction equations. Note that the assumption the triangle being equilateral gives us a lot of symmetries.

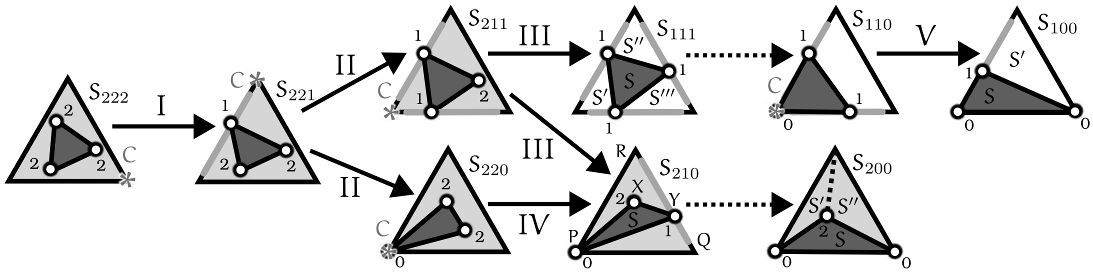


Figure 1.20: All different (abc) configurations for K being a triangle

Reduction system

The full system obtained by the Multivariate Crofton Reduction Technique is

$$\begin{aligned}
\mathbf{I}: &\ pP_{222} = 3 \cdot 2(P_{221} - P_{222}) \\
\mathbf{II}: &\ pP_{221} = 2 \cdot 2(P_{211} - P_{221}) + 1(P_{220} - P_{221}), \\
\mathbf{III}: &\ pP_{211} = 2(P_{111} - P_{211}) + 2 \cdot 1(P_{210} - P_{211}) \\
\mathbf{IV}: &\ pP_{220} = 2 \cdot 2(P_{210} - P_{220}) \\
\mathbf{V}: &\ pP_{110} = 2 \cdot 1(P_{100} - P_{110})
\end{aligned}$$

Solving the system for P_{222}, we get for any functional P the following result which already appeared in Ruben and Reed [61], namely

$$P_{222} = \frac{24(2P_{111} + 3P_{210})}{(4+p)(5+p)(6+p)}. \tag{1.314}$$

Irreducible terms

Using CRT, we have expressed P_{222} as a linear combination of P_{111} and P_{210}. Those terms cannot be reduced since the configurations (111) and (210) are irreducible (no scaling point available). However, for specific functionals, we can use some important symmetries to get us to even more reduced configurations anyway (dashed arrows in Figure 1.20).

First moment of area

$\mathbf{S_{111}}$

Although the configuration (111) is irreducible (no scaling point), note that

$$S = 1 - S' - S'' - S''', \tag{1.315}$$

where S', S'', S''' are areas of triangles formed by fixing one vertex and picking the other two from the adjacent sides (see Figure 1.20). Taking the expectation,

$$S_{111} = 1 - 3S_{110}$$

since by symmetry, $\mathbb{E}[S'] = \mathbb{E}[S''] = \mathbb{E}[S'''] = S_{110}$ which is, however, *reducible*.

$\mathbf{P_{110}}$

One of the unreachable configurations is (110). This configuration is reducible for general functional P by equation **V** in the reduction system. We get

$$P_{110} = \frac{2P_{100}}{(2+p)} \tag{1.316}$$

and therefore with $P = S$ (and thus $p = 2$), we get $S_{110} = \frac{1}{2}S_{100}$.

$\mathbf{S_{100}}$

In configuration (100), the point selected from one of the sides divides T_2 into two triangles with areas S and S' (see Figure 1.20). Therefore, we have $1 = S + S'$. Taking expectation and by symmetry, we get

$$S_{100} = \mathbb{E}[S] = \tfrac{1}{2}. \tag{1.317}$$

$\mathbf{S_{210}}$

This configuration can be solved using conditional expectations. Let $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ be vertices of T_2 and we denote $\mathbf{X}$ as the point selected from the interior and $\mathbf{Y}$ as the point selected from the side $\mathbf{RQ}$ (see Figure 1.20). Let us denote S' as the area of the triangle $\mathbf{PRY}$. The area S of the random triangle with vertices $\mathbf{P}, \mathbf{X}, \mathbf{Y}$ can be conditioned with respect to S' (or $\mathbf{Y}$). Since the point $\mathbf{Y}$ is fixed, we split the problem into two cases by the location of point $\mathbf{X}$. Either $\mathbf{X}$ is above or below the line $\mathbf{PY}$. However, each of those separate cases is equivalent to configuration (200) (apart of scaling so the area of triangle $\mathbf{PRY}$ or $\mathbf{PQY}$ is one). Writing down the correct scaling factors and by $S' \sim \mathsf{Unif}(0,1)$, we obtain

$$S_{210} = \mathbb{E}[S] = \mathbb{E}[\mathbb{E}[S \mid S']] = \mathbb{E}\left[S_{200}S'^2 + S_{200}^{(p)}(1-S')^2\right] = \tfrac{2}{3}S_{200}. \tag{1.318}$$

$\mathbf{S_{200}}$

Note that the only point selected from the interior of T_2 divides T_2 into three triangles with areas S, S' and S'' (see Figure 1.20), for which

$$1 = S + S' + S'', \tag{1.319}$$

Taking the expectation and by symmetry, we immediatelly get $S_{200} = \frac{1}{3}$.

S_{222}

Therefore, we find by backtracking and by Equation (1.314) with $P = S$ ($p = 2$),

$$v_2^{(1)}(T_2) = S_{222} = \frac{1}{14}\left(S_{111} + 3S_{210}\right) = \frac{1}{12}, \tag{1.320}$$

which is the resolution of the Silvester problem. To conclude, the following Table 1.14 shows the mean area of a random triangle in all configurations found along the way.

S_{222}	S_{221}	S_{211}	S_{220}	S_{111}	S_{210}	S_{110}	S_{200}	S_{100}
$\frac{1}{12}$	$\frac{1}{9}$	$\frac{17}{108}$	$\frac{4}{27}$	$\frac{1}{4}$	$\frac{2}{9}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$

Table 1.14: Mean triangle area in T_2 in various configurations

Higher moments of area

$S^{(k)}{}_{100}$

For general $k > -1$, we have in configuration (100) that $S \sim \mathsf{Unif}(0,1)$ and thus

$$S_{100}^{(k)} = \mathbb{E}\left[S^k\right] = \frac{1}{1+k}. \tag{1.321}$$

$S^{(k)}{}_{200}$

In configuration (200), since the area S is proportional with the distance of the base of T_2, we have for its density $f(s) = 2(1-s)$ on $s \in (0,1)$. Therefore

$$S_{200}^{(k)} = \mathbb{E}\left[S^k\right] = 2\int_0^1 s^k(1-s)\,\mathrm{d}s = \frac{2}{(1+k)(2+k)}. \tag{1.322}$$

$S^{(k)}{}_{210}$

By the same approach as in the S_{210} case, writing down the correct scaling factors,

$$S_{210}^{(k)} = \mathbb{E}\left[S_{200}^{(k)}S'^{1+k} + S_{200}^{(k)}(1-S')^{1+k}\right] = \tfrac{2}{2+k}S_{200}^{(k)} = \tfrac{4}{(2+k)^2(1+k)}, \tag{1.323}$$

since $S' \sim \mathsf{Unif}(0,1)$ and thus $\mathbb{E}\left[S'^{1+k}\right] = \mathbb{E}\left[(1-S')^{1+k}\right] = 1/(2+k)$.

$S^{(k)}{}_{111}$

Let $X, Y, Z \sim \mathsf{Unif}(0,1)$ be the (independent) ratios to which the vertices of the random triangle divide each corresponding side of T_2. We can write

$$S' = ZX. \quad S'' = (1-X)(1-Y), \quad S''' = Y(1-Z), \tag{1.324}$$

so

$$S = 1 - S' - S'' - S''' = X - XY - XZ + YZ. \tag{1.325}$$

Taking the expectation by integrating over X, Y, Z, we get

$$S_{111}^{(k)} = \mathbb{E}\left[S^k\right] = \int_0^1 \int_0^1 \int_0^1 (x - xy - xz + yz)^k \,\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{1.326}$$

Integrating out z,

$$S_{111}^{(k)} = \frac{1}{1+k} \int_0^1 \int_0^1 \frac{(x-xy)^{1+k} - (y-yx)^{1+k}}{x-y} \,\mathrm{d}x\mathrm{d}y. \tag{1.327}$$

We may use the formula $a^{1+k} - b^{1+k} = (a-b)(a^k + a^{k-1}b + \ldots + ab^{k-1} + b^k)$, so

$$S_{111}^{(k)} = \frac{1}{1+k} \sum_{l=0}^{k} \int_0^1 \int_0^1 (x-xy)^l (y-yx)^{k-l} \mathrm{d}x\mathrm{d}y = \frac{1}{1+k} \sum_{l=0}^{k} \left(\int_0^1 x^l (1-x)^{k-l} \mathrm{d}x \right)^2. \tag{1.328}$$

The remaining integral is a Beta integral. Straightforwardly, we finally arrive at the explicit result which also appeared in a recent paper by Maesumi [43],

$$S_{111}^{(k)} = \frac{1}{1+k} \sum_{l=0}^{k} \frac{(k-l)!^2 l!^2}{(1+k)!^2}. \tag{1.329}$$

Alternatively, note that the integral

$$I_k = \frac{1}{1+k} \int_0^1 \int_0^1 \frac{x^{1+k}(1-x)^{1+k} - y^{1+k}(1-y)^{1+k}}{x-y} \,\mathrm{d}x\mathrm{d}y \tag{1.330}$$

vanishes, since by substitution $x \to 1-x$ and $y \to 1-y$, we get $-I_k$. Hence, adding I_k to Equation (1.327) and by symmetry,

$$\begin{aligned} S_{111}^{(k)} &= \frac{2}{1+k} \int_0^1 \int_0^1 \frac{(x-xy)^{1+k} - y^{1+k}(1-y)^k}{x-y} \,\mathrm{d}x\mathrm{d}y \\ &= \frac{2}{1+k} \int_0^1 \int_0^1 (1-y)^{1+k} \frac{x^{1+k} - y^{1+k}}{x-y} \,\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{1.331}$$

The formula $x^{1+k} - y^{1+k} = (x-y)(x^k + x^{k-1}y + \ldots + xy^{k-1} + y^k)$ leads to another Beta integral, but only raised to the first power, we get

$$S_{111}^{(k)} = \frac{2}{1+k} \sum_{l=0}^{k} \frac{1}{1+l} \int_0^1 (1-y)^{k+1} y^{k-l} \mathrm{d}y = 2 \sum_{l=0}^{k} \frac{k!\,(k-l)!}{(1+l)(2k-l+2)!}. \tag{1.332}$$

We do not know whether there is some simple combinatorial explanation why those sums are equivalent.

$\mathbf{S^{(2)}}_{\mathbf{222}}$

For the second moment of area, we simply put $P = S^2$, for which $p = 4$. By Equation (1.314), we get

$$v_2^{(2)}(T_2) = S_{222}^{(2)} = \frac{1}{30}\left(2S_{111}^{(2)} + 3S_{210}^{(2)}\right) = \frac{1}{72}.$$

Table 1.15 below again summarises mean square areas of a random triangle in various configurations.

$S^{(2)}_{222}$	$S^{(2)}_{221}$	$S^{(2)}_{211}$	$S^{(2)}_{220}$	$S^{(2)}_{111}$	$S^{(2)}_{210}$	$S^{(2)}_{110}$	$S^{(2)}_{200}$	$S^{(2)}_{100}$
$\frac{1}{72}$	$\frac{5}{216}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$

Table 1.15: Mean square triangle area in T_2 in various configurations

$\mathbf{S^{(k)}}_{\mathbf{222}}$

For a general k integer, we get by Equation (1.314) with $P = S^k$ (so $p = 2k$),

$$v_2^{(k)}(T_2) = S_{222}^{(k)} = 6\frac{2S_{111}^{(k)} + 3S_{210}^{(k)}}{(k+2)(2k+5)(k+3)}.$$

By the results for $S_{111}^{(k)}$ and $S_{210}^{(k)}$, we get explicitly, after some simplifications,

$$S_{222}^{(k)} = 12\frac{6 + \frac{(2+k)^2}{(1+k)!^2}\sum_{j=0}^{k} j!^2(k-j)!^2}{(1+k)(2+k)^3(3+k)(5+2k)} = \frac{24\sum_{j=0}^{k+1}\frac{j!^2(k+1-j)!^2}{(2+k)!^2}}{(1+k)(2+k)^2(3+k)}. \quad (1.333)$$

in agreement with Mathai [46, p. 391], Reed [59] and Alagar [2]. Or equivalently, by rearrangement of the sum as discussed earlier,

$$S_{222}^{(k)} = 24\frac{3 + (2+k)\sum_{j=0}^{k}\frac{(k+2)!(k-j)!}{(1+j)(2k-j+2)!}}{(1+k)(2+k)^3(3+k)(5+2k)} = \frac{48\sum_{j=0}^{k+1}\frac{k!\,(k+1-j)!}{(j+1)(2k-j+4)!}}{(2+k)(3+k)}. \quad (1.334)$$

Either way, we get a very interesting relation between $S_{111}^{(k)}$ and $S_{222}^{(k)}$,

$$S_{222}^{(k)} = \frac{24\,S_{111}^{(k+1)}}{(1+k)(2+k)(3+k)}. \quad (1.335)$$

Area density

$\mathbf{f_{111}}$

For the density $f_{111}(s)$ of the random variable S in configuration (111), we have by Equation (1.327),

$$\mathcal{M}[f_{111}] = S_{111}^{(k-1)} = \frac{1}{k}\int_0^1\int_0^1 \frac{(x-xy)^k - (y-yx)^k}{x-y}\,\mathrm{d}x\,\mathrm{d}y, \quad (1.336)$$

so formally,

$$\begin{aligned} f_{111}(s) &= \mathcal{I}_0\mathcal{M}^{-1}\left[\int_0^1\int_0^1 \frac{(x-xy)^k - (y-yx)^k}{x-y}\,\mathrm{d}x\,\mathrm{d}y\right] \\ &= \mathcal{I}_0\int_0^1\int_0^1 \frac{x(1-y)\delta(s-x(1-y)) - y(1-x)\delta(s-y(1-x))}{x-y}\mathrm{d}x\mathrm{d}y. \end{aligned} \quad (1.337)$$

by Equation (A.41) (in Appendix A) with $r = 1$,

$$\mathcal{I}_0\delta(s-\alpha) = \frac{1}{\alpha}\mathbb{1}_{s<\alpha}. \quad (1.338)$$

via which we can deduce, with $\alpha = x(1-y)$ and $\alpha = y(1-x)$,

$$f_{111}(s) = \int_0^1 \int_0^1 \frac{\mathbb{1}_{s<x(1-y)} - \mathbb{1}_{s<y(1-x)}}{x-y}\, \mathrm{d}x\mathrm{d}y. \tag{1.339}$$

We can deduce that $f_{111}(s)$ is nonzero only when $s \in (0,1)$. Evaluating this integral is not complicated. By using Mathematica, we arrive at

$$f_{111}(s) = \begin{cases} -6\sqrt{1-4s}\,\operatorname{argtanh}\frac{1}{\sqrt{1-4s}} - 3\ln s, & 0 < s < 1/4, \\ \sqrt{4s-1}\left(\pi - 6\arctan\frac{1}{\sqrt{4s-1}}\right) - 3\ln s, & 1/4 \le s < 1. \end{cases} \tag{1.340}$$

Althought the derivation of f_{111} is already part of Alagar's work [2], it is worth to mention that the result was later independently rediscovered by Maesumi [43].

$\mathbf{f_{222}}$

For the density $f_{222}(s)$ of the random variable S in configuration (222), we have by Equation (1.335),

$$\mathcal{M}[f_{222}] = S_{222}^{(k-1)} = \frac{24\, S_{111}^{(k)}}{k(1+k)(2+k)}. \tag{1.341}$$

Taking the inverse Mellin transform, we get

$$f_{222}(s) = 24\, \mathcal{I}_0\mathcal{I}_1\mathcal{I}_2\, \mathcal{M}^{-1}[S_{111}^{(k)}] = 24\, \mathcal{I}_0\mathcal{I}_1\mathcal{I}_2\, [s f_{111}(s)]\,. \tag{1.342}$$

We will write $s f_{111}(s) = \int_0^1 \alpha f_{111}(\alpha)\delta(s-\alpha)\mathrm{d}\alpha$. From Table A.5 (see Appendix A),

$$\mathcal{I}_0\mathcal{I}_1\mathcal{I}_2\delta(s-\alpha) = \frac{(\alpha-s)^2}{2\alpha^3}\mathbb{1}_{s<\alpha}, \tag{1.343}$$

using which we obtain

$$f_{222}(s) = 12\int_s^1 \left(1 - \frac{s}{\alpha}\right)^2 f_{111}(\alpha)\, \mathrm{d}\alpha. \tag{1.344}$$

It is very easy to carry out this integration, we get

$$f_{222}(s) = \begin{cases} \left\{ \begin{array}{c} 12(1-s) - 6(1+24s+6s\ln s)\ln s \\ -12(1+26s)\sqrt{1-4s}\,\operatorname{argtanh}\sqrt{1-4s} \\ -144s(1+s)(\frac{\pi^2}{9} - \operatorname{argtanh}^2\sqrt{1-4s}) \end{array} \right\}, & 0 < s < 1/4, \\ \left\{ \begin{array}{c} 12(1-s) - 6(1+24s+6s\ln s)\ln s \\ -12(1+26s)\sqrt{4s-1}(\frac{\pi}{3} - \arctan\sqrt{4s-1}) \\ -144s(1+s)(\frac{\pi}{3} - \arctan\sqrt{4s-1})^2 \end{array} \right\}, & 1/4 \le s < 1. \end{cases} \tag{1.345}$$

The computation of the area probablity distribution function $f_{222}(s)$ was first carried out by Alagar [2], while the form shown above is due to Philip [51]. A remarkable feature of Philip's paper is that he also found the cumulative density function $F_{222}(s)$ explicitly. Note that the methods of both authors relied crucially on the knowledge of the relation between (111) and (210) configurations. Philip also found the area probability density function of a random triangle for a regular pentagon [56] and hexagon [54].

Obtusity probability

The probability a random triangle is obtuse in a given triangle T_2 is no longer affine (only scale invariant). Hence, the results of this section holds only for T_2 being equilateral. We can use the following standard parametrization

$$T_2^* = \operatorname{conv}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \operatorname{conv}([1,0,0],[0,1,0],[0,0,1]) \tag{1.346}$$

of an equilateral triangle embedded in $\mathbb{R}^3$ with area $\operatorname{vol}_2 T_2^* = \sqrt{3}/2$ and side length $\sqrt{2}$.

η_{111}

In configuration (111), the random points $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are selected from (different) sides of T_2^*. We may parametrise the points as

$$\mathbf{X} = \mathbf{e}_1 + U(\mathbf{e}_2 - \mathbf{e}_1), \quad \mathbf{Y} = \mathbf{e}_1 + V(\mathbf{e}_3 - \mathbf{e}_1), \quad \mathbf{Z} = \mathbf{e}_2 + W(\mathbf{e}_3 - \mathbf{e}_2). \tag{1.347}$$

where we introduced random variables $U, V, W \sim \mathsf{Unif}(0,1)$. Note that

$$(\mathbf{Y} - \mathbf{X})^\top (\mathbf{Z} - \mathbf{X}) = V - 2U + 2U^2 - UV + UW + VW. \tag{1.348}$$

The probability that the triangle $\mathbf{XYZ}$ is obtuse is obtained by integrating the obtusity indicators (Equation (1.31)). Moreover, by symmetry,

$$\eta_{111} = 3\eta_{1^*11} = 3\mathbb{E}\left[\mathbb{1}_{(\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X})<0}\right] = 3\mathbb{E}\left[\mathbb{1}_{V-2U+2U^2-UV+UW+VW<0}\right], \tag{1.349}$$

which can be written as the following integral,

$$\eta_{111} = 3\int_0^1\int_0^1\int_0^1 \mathbb{1}_{v-2u+2u^2-uv+uw+vw<0}\,\mathrm{d}u\mathrm{d}v\mathrm{d}w. \tag{1.350}$$

The integral was evaluated by *Mathematica* (see Code 1.1 below). We obtained

$$\eta_{111} = \frac{9}{2} + 27\ln\frac{\sqrt{3}}{2} \approx 0.616292. \tag{1.351}$$

Code 1.1: Simple code to evaluate η_{111} in T_2^*

```
eta111 = 3*Integrate[Boole[v-2u+2u^2-uv+uw+vw<0],
         {u, 0, 1}, {v, 0, 1}, {w, 0, 1}];
```

η_{210}

In configuration (210), the first vertex $\mathbf{X}$ of the inscribed random triangle $\mathbf{XYZ}$ is selected from the interior of T_2^*, the second vertex $\mathbf{Y}$ is selected from its side and the last vertex $\mathbf{Z}$ is fixed at the vertex of T_2^* opposite to $\mathbf{Y}$. We may parametrise the points as

$$\mathbf{X} = \mathbf{e}_1 + U(\mathbf{e}_2 - \mathbf{e}_1) + V(\mathbf{e}_3 - \mathbf{e}_1), \quad \mathbf{Y} = \mathbf{e}_1 + W(\mathbf{e}_2 - \mathbf{e}_1), \quad \mathbf{Z} = \mathbf{e}_3. \tag{1.352}$$

where we introduced random variables U, V, W such that $(U,V)^\top \sim \mathsf{Unif}(\mathbb{T}_2)$, where $\mathbb{T}_2 = \operatorname{conv}([0,0],[1,0],[0,1])$ is the canonical triangle, and $W \sim \mathsf{Unif}(0,1)$.

Additionally, in order to obtain the probability that the triangle $\mathbf{XYZ}$ is obtuse, we recognize three sub-configurations $(2^*10), (21^*0), (210^*)$ based on the exact location of the obtuse angle (the corresponding vertex domain is indicated by *). We can express the dot products in the decomposition of the obtusity indicator (Equation (1.31)) as follows

$$\begin{aligned}
(2^*10) &: (\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) = 2U^2+2UV-2UW-U+2V^2-VW-2V+W, \\
(21^*0) &: (\mathbf{Z}-\mathbf{Y})^\top(\mathbf{X}-\mathbf{Y}) = U+2V-W-2UW-VW+2W^2 \\
(210^*) &: (\mathbf{X}-\mathbf{Z})^\top(\mathbf{Y}-\mathbf{Z}) = 2-U-2V-W+2UW+VW.
\end{aligned} \tag{1.353}$$

The probability η_{222} that the triangle $\mathbf{XYZ}$ is obtuse is obtained as a sum of probabilities that the random triangle is obtuse at a specific vertex. Via the same technique as in the previous case, we obtained for those probabilities

$$\eta_{2^*10} = \frac{1}{2} + \frac{5\pi}{36\sqrt{3}} - \frac{1}{2}\ln\frac{\sqrt{3}}{2}, \qquad \eta_{21^*0} = \frac{83}{12} + 48\ln\frac{\sqrt{3}}{2}, \qquad \eta_{210^*} = 0. \tag{1.354}$$

Summing those up, we finally get

$$\eta_{210} = \eta_{2^*10} + \eta_{21^*0} + \eta_{210^*} = \frac{89}{12} + \frac{5\pi}{36\sqrt{3}} + \frac{95}{2}\ln\frac{\sqrt{3}}{2} \approx 0.836134. \tag{1.355}$$

η_{222}

Inserting η_{111} and η_{210} into Equation (1.314) with $P = \eta$, for which $p = 0$,

$$\eta_{222} = \frac{1}{5}\left(2\eta_{111} + 3\eta_{210}\right) = \frac{25}{4} + \frac{\pi}{12\sqrt{3}} + \frac{393}{10}\ln\frac{\sqrt{3}}{2} \approx 0.748197. \tag{1.356}$$

1.6.2 Square

Next, by using CRT, we rederive the result of Henze 5.1, that is the volumetric moments $v_2^{(k)}(C_2)$ of a random triangle area. In our convention, we write $v_2^{(k)}(C_2) = S_{222}^{(k)}$ (they are indeed equal, since we already have proper normalization since $\operatorname{vol}_2 C_2 = 1$). Table 1.16 below shows various explicit $S_{222}^{(k)}$ area moments for selected k's (from Equation (1.369)).

$S_{222}^{(1)}$	$S_{222}^{(2)}$	$S_{222}^{(3)}$	$S_{222}^{(4)}$	$S_{222}^{(5)}$	$S_{222}^{(6)}$	$S_{222}^{(7)}$	$S_{222}^{(8)}$
$\frac{11}{144}$	$\frac{1}{96}$	$\frac{137}{72000}$	$\frac{1}{2400}$	$\frac{363}{3512320}$	$\frac{761}{27095040}$	$\frac{7129}{870912000}$	$\frac{61}{24192000}$

Table 1.16: Values of volumetric moments $v_2^{(k)}(C_2) = S_{222}^{(k)}$ for selected k's

Consequentially, from the knowledge of all moments, we again deduce the probability density function $f_{222}(s)$ of S using the inverse Mellin transform.

Furthermore, we deduce the obtusity probability

$$\eta(C_2) = \frac{97}{150} + \frac{\pi}{40} \approx 0.725206, \tag{1.357}$$

which is a result due to Langford [42]. In fact, Langford obtained the obtusity probability in a rectangle of any side-ratio.

Configurations

As usual, let $P_{abc} = \mathbb{E}\left[P(\mathbf{X},\mathbf{Y},\mathbf{Z}) \mid \mathbf{X} \sim \mathsf{Unif}(A), \mathbf{Y} \sim \mathsf{Unif}(B), \mathbf{Z} \sim \mathsf{Unif}(C)\right]$, where $a = \dim A$, $b = \dim B$, $c = \dim C$ and the concrete selection of A, B, C is deduced from the reduction diagram in Figure 1.21 below. In this diagram, we also included the position of the scaling point **C** in cases reduction is possible. The arrows indicate which configurations reduce to which. Each arrow is labeled by a roman numeral corresponding to a given reduction equation in the system of reduction equations.

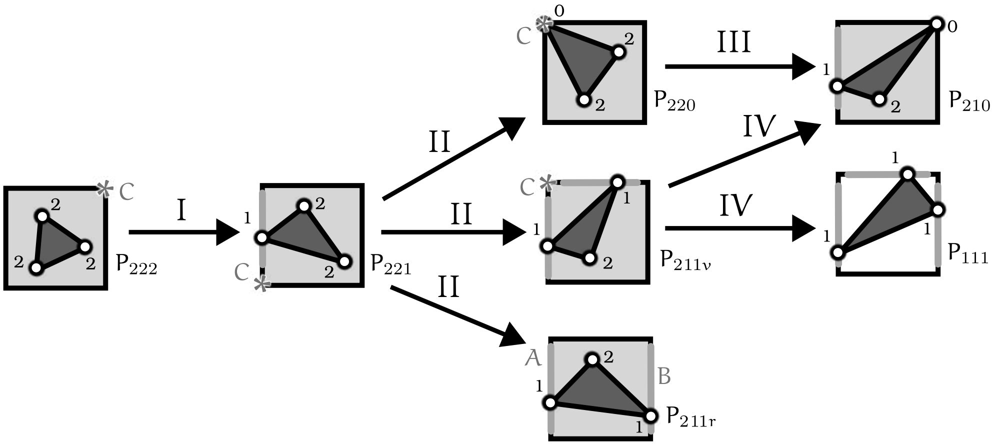


Figure 1.21: All different P_{abc} sub-configurations in C_2

Reduction system

The full system obtained by the Multivariate Crofton Reduction Technique is

$$\begin{aligned}
\mathbf{I}: \; & pP_{222} = 3 \cdot 2(P_{221} - P_{222}) \\
\mathbf{II}: \; & pP_{221} = 2 \cdot 2(P_{211} - P_{221}) + 1(P_{220} - P_{221}), \\
\mathbf{III}: \; & pP_{211v} = 2(P_{111} - P_{211}) + 2 \cdot 1(P_{210} - P_{211}) \\
\mathbf{IV}: \; & pP_{220} = 2 \cdot 2(P_{210} - P_{220})
\end{aligned}$$

with

$$P_{221} = \tfrac{1}{2} P_{221v} + \tfrac{1}{2} P_{221r}.$$

Solving the system for P_{222}, we get for any functional P,

$$P_{222} = \frac{24(P_{111} + 2P_{210})}{(4+p)(5+p)(6+p)} + \frac{12P_{211r}}{(5+p)(6+p)}. \tag{1.358}$$

Moments of area

$\mathbf{S^{(k)}}_{\mathbf{111}}$

Let us parametrise the location of the random points in (111) configuration,

$$\mathbf{X} = U\mathbf{e}_2, \qquad \mathbf{Y} = \mathbf{e}_2 + V\mathbf{e}_1, \qquad \mathbf{Z} = \mathbf{e}_1 + W\mathbf{e}_2, \tag{1.359}$$

where $U, V, W \sim \mathsf{Unif}(0,1)$. Hence, the area is given by

$$S = \frac{1}{2}(1 - U + UV - VW). \tag{1.360}$$

Note that the expression on the right is always positive. Taking the expectation,

$$S_{111}^{(k)} = \int_0^1 \int_0^1 \int_0^1 \left(\frac{u - v + w - uw}{2}\right)^k \, \mathrm{d}u\mathrm{d}w\mathrm{d}v.$$

This integral is straightforward, we obtain

$$S_{111}^{(k)} = \frac{2^{1-k} H_{k+1}}{(1+k)(2+k)}, \tag{1.361}$$

where $H_k = \sum_{j=1}^k 1/j$ is the k-th *harmonic number*.

$\mathbf{S^{(k)}}_{\mathbf{210}}$

In configuration (210), vertices $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ of the random triangle selected from C_2 can be parametrised (see Figure 1.21) as

$$\mathbf{X} = U\mathbf{e}_1 + V\mathbf{e}_2, \qquad \mathbf{Y} = W\mathbf{e}_2, \qquad \mathbf{Z} = \mathbf{e}_1 + \mathbf{e}_2, \tag{1.362}$$

where $U, V, W \sim \mathsf{Unif}(0,1)$. Hence, the area is given by

$$S = \frac{1}{2}|U - V + W - UW|. \tag{1.363}$$

Taking the expectation and splitting the integral into two cases based on $\mathbf{X}$ being located above or below the line segment $\mathbf{YZ}$, we obtain

$$S_{210}^{(k)} = \int_0^1 \int_0^1 \left[\int_0^{u+w-uw} \left(\tfrac{u-v+w-uw}{2}\right)^k \mathrm{d}v + \int_{u-v+w-uw}^1 \left(\tfrac{u-v+w-uw}{2}\right)^k \mathrm{d}v\right] \mathrm{d}u\mathrm{d}w.$$

Integrating out v and u (and also w in case of the second integral), we get

$$S_{210}^{(k)} = \frac{2^{-k}}{(1+k)(2+k)^2} + \frac{2^{-k}\int_0^1 \frac{1-w^{2+k}}{1-w}\mathrm{d}w}{(1+k)(2+k)} = \frac{2^{-k}}{(1+k)(2+k)^2} + \frac{2^{-k}H_{2+k}}{(1+k)(2+k)}. \tag{1.364}$$

$\mathbf{S^{(k)}}_{\mathbf{211r}}$

The last irreducible configuration ($211r$) is the hardest since it has an extra degree of freedom. One point $\mathbf{X}$ is being selected from the interior of C_2, while the remaining two $\mathbf{Y}$ and $\mathbf{Z}$ are taken from the opposite sides. Parametrizing the location of the points,

$$\mathbf{X} = U\mathbf{e}_1 + V\mathbf{e}_2, \qquad \mathbf{Y} = W\mathbf{e}_2, \qquad \mathbf{Z} = \mathbf{e}_1 + R\mathbf{e}_2, \tag{1.365}$$

where $U, V, W, R \sim \mathsf{Unif}(0,1)$. Hence, the area is given by

$$S = \frac{1}{2}|RU - V + W - UW|. \tag{1.366}$$

Taking the expectation and realizing that, by (two-fold) symmetry, we can assume that the point $\mathbf{X}$ is always located *above* the line $\mathbf{YZ}$,

$$S_{211r}^{(k)} = 2\int_0^1\int_0^1\int_0^1\int_0^{ru+w-uw}\left(\frac{ru - v + w - uw}{2}\right)^k \mathrm{d}v\mathrm{d}u\mathrm{d}r\mathrm{d}w.$$

Integrating out v and u is straightforward, we get

$$S_{211r}^{(k)} = \frac{2^{1-k}}{(1+k)(2+k)}\int_0^1\int_0^1 \frac{r^{2+k} - w^{2+k}}{r - w}\mathrm{d}r\mathrm{d}w \tag{1.367}$$

Luckily, the remaining integral is trivial, we get

$$S_{211r}^{(k)} = \frac{2^{2-k}H_{k+2}}{(1+k)(2+k)(3+k)}. \tag{1.368}$$

$\mathbf{S^{(k)}}_{\mathbf{222}}$

For a general k integer, we get by Equation (1.358) with $P = S^k$ (so $p = 2k$) and after some simplifications, we get Henze's [35] result

$$v_2^{(k)}(T_2) = S_{222}^{(k)} = \frac{3\cdot 2^{3-k}H_{2+k}}{(1+k)(2+k)^2(3+k)^2}. \tag{1.369}$$

Note that we can deduce this result independently by the Canonical section integral introduced later in this thesis (see Section 4.3.2 in Chapter 4).

Area density

See Section 4.3.2 in Chapter 4.

Obtusity probability

$\boldsymbol{\eta}_{\mathbf{111}}$

In configuration (111), the vertices of the random triangle $\mathbf{XYZ}$ are selected from three (different) sides. Let $\mathbf{X}$ and $\mathbf{Z}$ be picked from the opposite sides and $\mathbf{Y}$ be picked from (one of) the remaining sides. Again, we parametrise the points as

$$\mathbf{X} = \mathbf{e}_1 + U(\mathbf{e}_2 - \mathbf{e}_1) + V(\mathbf{e}_3 - \mathbf{e}_1), \quad \mathbf{Y} = \mathbf{e}_1 + W(\mathbf{e}_2 - \mathbf{e}_1), \quad \mathbf{Z} = \mathbf{e}_3. \tag{1.370}$$

where $U, V, W \sim \mathsf{Unif}(0,1)$. We recognize three sub-configurations (1^*11), (11^*1), (111^*) based on the exact location of the obtuse angle (the corresponding vertex domain is indicated by *). By symmetry, configurations (1^*11) and (111^*) give the same contribution. We can express the dot products in the decomposition of the obtusity indicator (Equation (1.31)) as follows

$$\begin{aligned} (1^*11):\ (\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) &= U^2 - UW - U + V + W, \\ (11^*1):\ (\mathbf{Z}-\mathbf{Y})^\top(\mathbf{X}-\mathbf{Y}) &= UW - U + V^2 - V - W + 1. \end{aligned} \tag{1.371}$$

The probability η_{222} that the triangle $\mathbf{XYZ}$ is obtuse is obtained as a sum of probabilities that the random triangle is obtuse at a specific vertex. Taking expectations and by *Mathematica* (`Integrate` with `Boole` as an argument),

$$\eta_{1^*11} = \frac{1}{24}, \qquad \eta_{11^*1} = \frac{4}{9}, \qquad \eta_{111^*} = \eta_{1^*11} = \frac{1}{24}. \tag{1.372}$$

Summing those up, we get

$$\eta_{111} = \eta_{1^*11} + \eta_{11^*1} + \eta_{111^*} = \frac{19}{36} \approx 0.527778. \tag{1.373}$$

η_{210}

Recall that in the (210) configuration, the first vertex $\mathbf{X}$ of the inscribed random triangle $\mathbf{XYZ}$ is selected from the interior of C_2, the second vertex $\mathbf{Y}$ is selected from its side and the last vertex $\mathbf{Z}$ is fixed at the vertex of C_2 opposite to $\mathbf{Y}$. We may parametrise the points as

$$\mathbf{X} = U\mathbf{e}_1 + V\mathbf{e}_2, \qquad \mathbf{Y} = W\mathbf{e}_2, \qquad \mathbf{Z} = \mathbf{e}_1 + \mathbf{e}_2, \tag{1.374}$$

where $U, V, W \sim \mathsf{Unif}(0,1)$. We recognize three sub-configurations (2^*10), (21^*0), (210^*) based on the exact location of the obtuse angle. We can express the dot products in the decomposition of the obtusity indicator (Equation (1.31)) as follows

$$\begin{aligned} (2^*10):\ & (\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) = U^2 - U + V^2 - VW - V + W, \\ (21^*0):\ & (\mathbf{Z}-\mathbf{Y})^\top(\mathbf{X}-\mathbf{Y}) = U - VW + V + W^2 - W \\ (210^*):\ & (\mathbf{X}-\mathbf{Z})^\top(\mathbf{Y}-\mathbf{Z}) = 2 - U + VW - V - W. \end{aligned} \tag{1.375}$$

Taking expectations of the corresponding indicators and evaluating the integrals by *Mathematica*, we get

$$\eta_{2^*10} = \frac{13}{18}, \qquad \eta_{21^*0} = \frac{1}{24}, \qquad \eta_{210^*} = 0. \tag{1.376}$$

Summing those up, we get

$$\eta_{210} = \eta_{2^*10} + \eta_{21^*0} + \eta_{210^*} = \frac{55}{72} \approx 0.763889. \tag{1.377}$$

η_{211r}

In the last irreducible configuration $(211r)$, $\mathbf{X}$ is being selected from the interior of C_2, while the remaining two $\mathbf{Y}$ and $\mathbf{Z}$ are taken from the opposite sides. Parametrizing the location of the points,

$$\mathbf{X} = U\mathbf{e}_1 + V\mathbf{e}_2, \qquad \mathbf{Y} = W\mathbf{e}_2, \qquad \mathbf{Z} = \mathbf{e}_1 + R\mathbf{e}_2, \tag{1.378}$$

where $U, V, W, R \sim \mathsf{Unif}(0,1)$. We get three corresponding sub-configurations (2^*11r), (21^*1r), (211^*r), out of which (21^*1r) and (211^*r) give the same contribution by symmetry. For the dot products, we have

$$\begin{aligned} (2^*11r):\ & (\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) = (R-V)(W-V) - U(1-U), \\ (21^*1r):\ & (\mathbf{Z}-\mathbf{Y})^\top(\mathbf{X}-\mathbf{Y}) = (R-W)(V-W) + U. \end{aligned} \tag{1.379}$$

The probabilities that those dot products are negative can be computed by first grouping the random variables, this is the method of Langford [42]. Let

$$\Lambda = (R-V)(W-V), \qquad \Omega = U(1-U), \tag{1.380}$$

then $\Lambda \sim \mathsf{Lang}$ with CDF (see Appendix A.1)

$$F_\Lambda(\lambda) = \begin{cases} 0, & \lambda < -\frac{1}{4} \\ \frac{1}{3}(1-8\lambda)\sqrt{1+4\lambda} + 4\lambda \operatorname{argtanh}\sqrt{1+4\lambda}, & -\frac{1}{4} \le \lambda < 0 \\ \frac{1}{3}\left(1-6\lambda+8\lambda^{3/2}\right) - 2\lambda\ln\lambda, & 0 \le \lambda < 1, \\ 1, & \lambda \ge 1. \end{cases} \tag{1.381}$$

and, trivially, with PDF of Ω being $f_\Omega(\omega) = \frac{2}{\sqrt{1-4\omega}}\mathbb{1}_{\omega\in(0,1/4)}$ (see Example 278). We can write the obtusity probability in (2^*11r) sub-configuration as

$$\begin{aligned} \eta_{2^*11r} &= \mathbb{P}\left[(\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) < 0\right] = \mathbb{P}\left[\Lambda < \Omega\right] = \int_0^{1/4}\int_{-1/4}^{\omega} f_\Lambda(\lambda) f_\Omega(\omega)\, \mathrm{d}\lambda \mathrm{d}\omega \\ &= \int_0^{1/4} F_\Lambda(\omega) f_\Omega(\omega)\, \mathrm{d}\omega = \int_0^{1/4} \frac{\frac{2}{3}\left(1-6\omega+8\omega^{3/2}\right) - 4\omega\ln\omega}{\sqrt{1-4\omega}}\, \mathrm{d}\omega = \frac{5}{9} + \frac{\pi}{16}. \end{aligned} \tag{1.382}$$

Similarly in (21^*1r) configuration, we have $f_U(u) = \mathbb{1}_{u\in(0,1)}$ and thus

$$\begin{aligned} \eta_{21^*1r} &= \mathbb{P}\left[(\mathbf{X}-\mathbf{Y})^\top(\mathbf{Z}-\mathbf{Y}) < 0\right] = \mathbb{P}\left[\Lambda + U < 0\right] = \int_0^{1/4}\int_{-1/4}^{-u} f_\Lambda(\lambda)\, \mathrm{d}\lambda \mathrm{d}u \\ &= \int_0^{1/4} F_\Lambda(-u)\, \mathrm{d}u = \int_0^{1/4} \frac{1+8u}{3}\sqrt{1-4u} - 4u \operatorname{argtanh}\sqrt{1-4u}\, \mathrm{d}u = \frac{1}{60}. \end{aligned} \tag{1.383}$$

Lastly, by symmetry, $\eta_{211^*r} = \eta_{21^*1r} = 1/60$. Summing up three obtusity probabilities we have found so far, we get

$$\eta_{211r} = \eta_{2^*11r} + \eta_{21^*1r} + \eta_{211^*r} = \frac{53}{90} + \frac{\pi}{16} \approx 0.785238. \tag{1.384}$$

η_{222}

Inserting η_{111}, η_{210} and η_{211r} into Equation (1.358) with $P = \eta$, for which $p = 0$,

$$\eta_{222} = \frac{1}{5}(\eta_{111} + 2\eta_{210} + 2\eta_{211r}) = \frac{97}{150} + \frac{\pi}{40} \approx 0.725206 \tag{1.385}$$

as obtained by Langford [42].

1.6.3 Disk

Consider a trivariate symmetric homogeneous functional P of order p dependent on three random points picked uniformly from the unit disk $\mathbb{B}_2 = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \le 1\}$ with area $\mathrm{vol}_2\, \mathbb{B}_2 = \pi$. Additionally, we require P to be *rotationally* symmetric with respect to the origin. That is, for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{B}_2$ and any orthogonal matrix R we have $P(R\mathbf{x}, R\mathbf{y}, R\mathbf{z}) = P(\mathbf{x}, \mathbf{y}, \mathbf{z})$. Table 1.17 below shows various explicit $S_{222}^{(k)}$ area moments for selected k's (from Equation (1.426)).

$S_{222}^{(1)}$	$S_{222}^{(2)}$	$S_{222}^{(3)}$	$S_{222}^{(4)}$	$S_{222}^{(5)}$	$S_{222}^{(6)}$	$S_{222}^{(7)}$	$S_{222}^{(8)}$	$S_{222}^{(9)}$	$S_{222}^{(10)}$
$\frac{35}{48\pi}$	$\frac{3}{32}$	$\frac{1001}{6400\pi}$	$\frac{1}{32}$	$\frac{138567}{2007040\pi}$	$\frac{275}{16384}$	$\frac{1062347}{24772608\pi}$	$\frac{1911}{163840}$	$\frac{86822723}{2664431616\pi}$	$\frac{2499}{262144}$

Table 1.17: Mean triangle area moments $S_{222}^{(k)}$ in $\mathbb{B}_2$

So far, the only known (higher) perimeter moments $\Pi_{222}^{(k)}$ are available in the unit disk (our result in this thesis, see Table 1.18, in which $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ is the *Apéry's constant*). Apparently, to our knowledge, the second and also any higher perimeter moments are still unknown and yet to be determined in any other K_d.

$\Pi_{222}^{(-1)}$	$\Pi_{222}^{(1)}$	$\Pi_{222}^{(2)}$	$\Pi_{222}^{(3)}$	$\Pi_{222}^{(4)}$
$\frac{64}{15\pi} - \frac{64\ln 2}{15\pi}$	$\frac{128}{15\pi}$	$3 + \frac{3383}{72\pi^2} + \frac{35\zeta(3)}{16\pi^2}$	$\frac{93584}{1225\pi} + \frac{1024\ln 2}{245\pi}$	$\frac{49}{2} + \frac{1029\zeta(3)}{32\pi^2} + \frac{9745549}{18000\pi^2}$

Table 1.18: Random triangle perimeter moments $\Pi_{222}^{(k)}$ in the unit disk $\mathbb{B}_2$

Also, we are able to deduce statistics for the smallest and the largest internal angle. As a consequence, we get for the probability a random triangle in a disk is obtuse (a famous result of Woolhouse [77]),

$$\eta(\mathbb{B}_2) = \frac{9}{8} - \frac{4}{\pi^2} \approx 0.719715. \tag{1.386}$$

Configurations

As usual, let $P_{abc} = \mathbb{E}\left[P(\mathbf{X},\mathbf{Y},\mathbf{Z}) \mid \mathbf{X} \sim \mathsf{Unif}(A), \mathbf{Y} \sim \mathsf{Unif}(B), \mathbf{Z} \sim \mathsf{Unif}(C)\right]$, where $a = \dim A$, $b = \dim B$, $c = \dim C$ and the concrete selection of A, B, C is deduced from the reduction diagram in Figure 1.22 below. In this diagram, we also included the position of the scaling point $\mathbf{C}$ in cases reduction is possible. The arrows indicate which configurations reduce to which. Each arrow is labeled by a roman numeral corresponding to a given reduction equation in the system of reduction equations. Recall that the vertices $\mathbf{X} \sim \mathsf{Unif}(A), \mathbf{Y} \sim \mathsf{Unif}(B), \mathbf{Z} \sim \mathsf{Unif}(C)$ of triangle $\mathbf{XYZ}$ are selected independently and we denote $L = |\mathbf{XY}|$, $L' = |\mathbf{XZ}|$ and $L'' = |\mathbf{YZ}|$ its (random) side-lengths and $\Theta = |\angle\mathbf{XZY}|$, $\Theta' = |\angle\mathbf{XYZ}|$ and $\Theta'' = |\angle\mathbf{YXZ}|$ the corresponding (random) sizes of its internal angles.

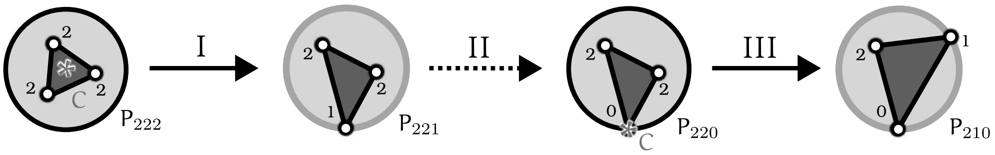


Figure 1.22: All different P_{abc} sub-configurations in $\mathbb{B}_2$

Reduction system

The full system obtained by CRT is

$$\begin{aligned} \mathbf{I}: \ pP_{222} &= 3 \cdot 2(P_{221} - P_{222}) \\ \mathbf{II}: \ P_{221} &= P_{220}. \\ \mathbf{III}: \ pP_{220} &= 2 \cdot 2(P_{210} - P_{220}), \end{aligned}$$

where the equation **II** follows from the rotational symmetry of P. The solution of our system is

$$P_{222} = \frac{24P_{210}}{(6+p)(4+p)}. \tag{1.387}$$

Note that when $p = 0$, we get $P_{222} = P_{210}$, which is essentially Proposition 8.1 of Sullivan [69, p. 65].

$\mathbf{P_{210}}$

In configuration (210), one point $\mathbf{X}$ is drawn uniformly from the interior of $\mathbb{B}_2$, the second point $\mathbf{Y}$ is drawn uniformly from the boundary $\partial\mathbb{B}_2$ and the last one $\mathbf{Z}$ is fixed at the boundary. Keep in mind that P_{210} is defined via generalization of Remark 9 as a mean weighted by the support function

$$P_{210} = \frac{1}{2(\operatorname{vol}_2 \mathbb{B}_2)^2} \int_{\mathbb{B}_2} \int_{\partial\mathbb{B}_2} P(\mathbf{x},\mathbf{y},\mathbf{z}) h_{\mathbf{z}}(\mathbf{y}) \lambda_2(\mathrm{d}\mathbf{x}) \lambda_1(\mathrm{d}\mathbf{y}), \tag{1.388}$$

where the support function $h_{\mathbf{z}}(\mathbf{y})$ of $\mathbb{B}_2$ evaluated in $\mathbf{y}$ and centered at $\mathbf{z} \in \partial\mathbb{B}_2$ (arbitrary fixed point) is given explicitly as

$$h_{\mathbf{z}}(\mathbf{y}) = \tfrac{1}{2}\|\mathbf{y} - \mathbf{z}\|^2. \tag{1.389}$$

Let us parametrise our integral using angular coordinates $(\alpha, \beta, \varepsilon)$, where α, β are internal angles of the random triangle $\mathbf{XYZ}$ located at vertex $\mathbf{Z}$ and $\mathbf{Y}$, respectively. We denote γ as the remaining internal angle at vertex $\mathbf{X}$, but keep in mind that γ implicitly depends on α, β since $\alpha + \beta + \gamma = \pi$. The angle ε is the angle between the chord $\mathbf{ZY}$ and the tangent lines (see Figure 1.23). Also, we define local perpendicular unit vectors $\mathbf{e}_1, \mathbf{e}_2$ as shown in the figure.

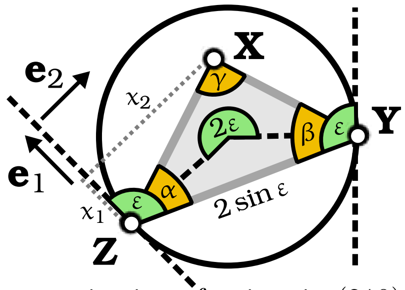


Figure 1.23: Parametrisation of points in (210) disk configuration

By (twofold) symmetry, we can only consider the case where the point $\mathbf{X}$ is located above the chord $\mathbf{ZY}$ (as shown in the figure above). Hence, we get the following set of inequalities for our angular variables

$$0 < \alpha < \varepsilon, \qquad 0 < \beta < \varepsilon, \qquad \alpha + \beta < \varepsilon < \pi. \tag{1.390}$$

The (half-)domain of integration in $(\alpha, \beta, \varepsilon)$ is therefore a tetrahedron

$$\operatorname{conv}([0,0,0],[\pi,0,0],[\pi,0,\pi],[0,\pi,\pi],[0,0,\pi]). \tag{1.391}$$

The parametrization of our points $\mathbf{x}, \mathbf{y}, \mathbf{z}$ (Figure 1.23) is given by

$$\mathbf{x} - \mathbf{z} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2, \qquad \mathbf{y} - \mathbf{z} = 2\sin\varepsilon\,(\cos\varepsilon\,\mathbf{e}_1 + \sin\varepsilon\,\mathbf{e}_2) \tag{1.392}$$

with

$$x_1 = \frac{2\sin\beta\sin\varepsilon\cos(\varepsilon-\alpha)}{\sin\gamma}, \qquad x_2 = \frac{2\sin\beta\sin\varepsilon\sin(\varepsilon-\alpha)}{\sin\gamma}, \tag{1.393}$$

from which

$$\|\mathbf{x}-\mathbf{z}\| = \frac{2\sin\beta\sin\varepsilon}{\sin\gamma}, \qquad \|\mathbf{y}-\mathbf{z}\| = 2\sin\varepsilon, \qquad h_{\mathbf{z}}(\mathbf{y}) = 2\sin^2\varepsilon. \tag{1.394}$$

Calculating the Jacobian, we get the transformation of measures

$$\lambda_2(\mathrm{d}\mathbf{x}) = \mathrm{d}x_1\mathrm{d}x_2 = \frac{4\sin\alpha\sin\beta\sin^2\varepsilon}{\sin^3\gamma}\,\mathrm{d}\alpha\ \mathrm{d}\beta, \qquad \lambda_1(\mathrm{d}\mathbf{y}) = 2\mathrm{d}\varepsilon. \tag{1.395}$$

Therefore, we may write for our integral (including the twofold symmetry factor),

$$P_{210} = \int_0^{\pi}\int_0^{\pi-\alpha}\int_{\alpha+\beta}^{\pi} P(\mathbf{x},\mathbf{y},\mathbf{z})\,\rho_{210}(\alpha,\beta\mid\varepsilon)\ \mathrm{d}\varepsilon\mathrm{d}\beta\mathrm{d}\alpha, \tag{1.396}$$

where we introduced the *internal angle trivariate density*

$$\rho_{210}(\alpha,\beta\mid\varepsilon) = \frac{2\lambda_2(\mathrm{d}\mathbf{x})\lambda_1(\mathrm{d}\mathbf{y})h_{\mathbf{z}}(\mathbf{y})}{2(\operatorname{vol}_2\mathbb{B}_2)^2\ \mathrm{d}\varepsilon\mathrm{d}\beta\mathrm{d}\alpha} = \frac{16\sin\alpha\sin\beta\sin^4\varepsilon}{\pi^2\sin^3\gamma}. \tag{1.397}$$

Angle-only dependent functionals

When the original functional P depends on integral angles α, β only, we can integrate out ε to obtain

$$P_{210} = \int_0^{\pi} \int_0^{\pi-\alpha} P(\mathbf{x}, \mathbf{y}, \mathbf{z})\, \rho_{210}(\alpha, \beta)\; \mathrm{d}\beta \mathrm{d}\alpha, \tag{1.398}$$

where we introduced the (210) configuration *internal angle bivariate density*

$$\rho_{210}(\alpha, \beta) = \int_{\alpha+\beta}^{\pi} \rho_{210}(\alpha, \beta \mid \varepsilon)\; \mathrm{d}\varepsilon = \frac{\sin\alpha \sin\beta\, (12\gamma - 8\sin(2\gamma) + \sin(4\gamma))}{2\pi^2 \sin^3\gamma}, \tag{1.399}$$

which is the PDF of two internal angles Θ (at $\mathbf{Z}$) and Θ' (at $\mathbf{Y}$). Moreover, such functional P must have $p = 0$ (it cannot depend on the scale of the random triangle) and thus, by the solution of the reduction equations,

$$P_{222} = P_{210}. \tag{1.400}$$

Bivariate internal angle distribution

As a simple consequence, we can identify the probability denity function of internal angles in configuration (222) with that of configuration (210). This is essentially Proposition 8.1 of Sullivan [69, p. 65]. However, keep in mind that while α, β and γ are permutable, this is not the case in (210) configuration. In order to obtain the correct function for the distribution of internal angles in (222) configuration, we must first select two vertices whose corresponding internal angles would play the role of α and β in (210) configuration. Symbolically, this corresponds to the following symmetrization construction of the (222) configuration *internal angle bivariate density*

$$\begin{aligned} \rho_{222}(\alpha, \beta) &= \tfrac{1}{3}\left[\rho_{210}(\alpha, \beta) + \rho_{210}(\alpha, \pi-\alpha-\beta) + \rho_{210}(\pi-\alpha-\beta, \beta)\right] \\ &= \frac{\sin\alpha \sin\beta \sin\gamma}{6\pi^2}\Bigg[\frac{12\gamma - 8\sin(2\gamma) + \sin(4\gamma)}{\sin^4\gamma} \\ &\quad + \frac{12\beta - 8\sin(2\beta) + \sin(4\beta)}{\sin^4\beta} + \frac{12\alpha - 8\sin(2\alpha) + \sin(4\alpha)}{\sin^4\alpha}\Bigg], \end{aligned} \tag{1.401}$$

where $\gamma = \pi - \alpha - \beta$.

Univariate internal angle distribution

Finally, integrating out β from $\rho_{222}(\alpha, \beta)$, we get the (222) configuration *internal angle univariate density* (PDF of a random internal angle)

$$\begin{aligned} \rho_{222}(\alpha) = \int_0^{\pi-\alpha} \rho_{222}(\alpha, \beta)\; \mathrm{d}\beta = \frac{\csc^3\alpha}{24\pi^2}\Big[(24(\pi-\alpha)\alpha - 2)\cos(\alpha) \\ + 2\cos(5\alpha) + 8(\pi + 2\alpha)\sin(\alpha) - (\pi-\alpha)(9\sin(3\alpha) + \sin(5\alpha))\Big]. \end{aligned} \tag{1.402}$$

from which we get the CDF of the (222) *random internal angle* Θ of $\mathbf{XYZ}$ triangle

$$\begin{aligned} R_{222}(\alpha) = \mathbb{P}\left[\Theta \le \alpha\right] = \int_0^{\alpha} \rho_{222}(\alpha')\; \mathrm{d}\alpha' = \frac{1}{12\pi^2}\Big[11 + 24\pi\alpha - 12\alpha^2 \\ - 5\cos(2\alpha) + 6(\pi - 2\alpha)\cot\alpha - 2(\pi-\alpha)\left(3\alpha\csc^2\alpha - \sin(2\alpha)\right)\Big] \end{aligned} \tag{1.403}$$

Internal angle order statistics

Our goal is to determine CDF of the largest internal angle $\Omega = \max\{\Theta, \Theta', \Theta''\}$ of a random triangle **XYZ** picked from the unit disk, that is the function $G_{222}(\omega) = \mathbb{P}[\Omega \leq \omega]$ and its corresponding probability density function $g_{222}(\omega) = G'_{222}(\omega)$. Clearly, the PDF is non-zero only when $\pi/3 < \omega < \pi$. Moreover, trivially [29],

$$g_{222}(\omega) = 3\rho_{222}(\omega) \quad \text{and} \quad G_{222}(\omega) = 1 - 3(1 - R_{222}(\omega)), \quad \omega \in [\pi/2, \pi). \tag{1.404}$$

This fact alone enables us to deduce the probability a random triangle is obtuse

$$\eta_{222} = 1 - G_{222}(\pi/2) = 3(1 - R_{222}(\pi/2)) = \frac{9}{8} - \frac{4}{\pi^2} \approx 0.719715 \tag{1.405}$$

as derived by Woolhouse [77]. Finch [29] wrote that the probability density $g_{222}(\omega)$ when $\omega < \pi/2$ is not known. However, Sullivan already found some partial results with CRT [69, Lemma 8.2]. In Eisenberg & Sullivan [27, p. 318], they also derived $g_{222}(\omega)$ when $\omega \geq \pi/2$ but did not give a solution for $\omega < \pi/2$. We finished their calculation and concluded that

$$\begin{aligned} g_{222}(\omega) = \frac{1}{2\pi^2}\Big[&36\omega - 12\pi + 6(\pi - 6\omega - (\pi - 3\omega)\omega\cot\omega)\csc^2\omega \\ &- 2(\pi + 9\omega)\cos(2\omega) + 18\cot\omega + 3(\pi - 2\omega)\sec^2\omega \\ &+ 2\sin(2\omega) + 2\sin(4\omega) - 6\tan\omega\Big], \qquad \omega \in (\pi/3, \pi/2], \end{aligned} \tag{1.406}$$

which matches the numerical result of Small [65, Fig. 1]. In order to derive this result, note that we can write using the (222) bivariate density function

$$G_{222}(\omega) = \int_0^\pi \int_0^{\pi-\alpha} \rho_{222}(\alpha, \beta)\, \mathbb{1}_{\max\{\alpha,\beta,\gamma\} \leq \omega}\, \mathrm{d}\alpha \mathrm{d}\beta \tag{1.407}$$

for all $\omega \in (\pi/3, \pi)$. When $\omega \in (\pi/3, \pi/2)$, this integral becomes

$$G_{222}(\omega) = \int_{\pi-2\omega}^{\omega} \int_{\pi-\omega-\alpha}^{\omega} \rho_{222}(\alpha, \beta)\, \mathrm{d}\beta \mathrm{d}\alpha. \tag{1.408}$$

Differentiating this double integral with respect to ω, we get

$$g_{222}(\omega) = 3\int_{\pi-2\omega}^{\omega} \rho_{222}(\alpha, \omega)\, \mathrm{d}\alpha, \tag{1.409}$$

which is straightforward. Moreover, integrating back, we got for the CDF,

$$\begin{aligned} G_{222}(\omega) = \frac{\csc^2\omega\sec\omega}{32\pi^2}\Big[&\Big(8\pi^2 - 11 - 72\omega^2\Big)\cos\omega - 8(\pi - 3\omega)^2\cos(3\omega) \\ &+ 10\cos(5\omega) + \cos(7\omega) + (20\pi - 36\omega)\sin\omega - 26\pi\sin(3\omega) \\ &+ 78\omega\sin(3\omega) + 2\pi\sin(5\omega) + 18\omega\sin(5\omega)\Big], \qquad \omega \in [\pi/3, \pi/2). \end{aligned} \tag{1.410}$$

Similarly, the CDF of the smallest internal angle $\Xi = \min\{\Theta, \Theta', \Theta''\}$ is given by

$$H_{222}(\xi) = \mathbb{P}[\Xi \leq \xi] = \int_0^\pi \int_0^{\pi-\alpha} \rho_{222}(\alpha, \beta)\, \mathbb{1}_{\min\{\alpha,\beta,\gamma\} \leq \xi}\, \mathrm{d}\alpha \mathrm{d}\beta, \tag{1.411}$$

hence $\xi \in (0, \pi/3)$. Writing out the complement of this integral,

$$H_{222}(\xi) = 1 - \mathbb{P}\left[\Xi > \xi\right] = 1 - \int_{\xi}^{\pi-2\xi} \int_{\xi}^{\pi-\xi-\alpha} \rho(\alpha, \beta). \tag{1.412}$$

Immediately, recognizing that the integral on the right was already calculated,

$$\begin{aligned} H_{222}(\xi) = 1 - \frac{\csc^2\xi \sec\xi}{32\pi^2}\Big[& \left(8\pi^2 - 11 - 72\xi^2\right)\cos\xi - 8(\pi - 3\xi)^2\cos(3\xi) \\ & + 10\cos(5\xi) + \cos(7\xi) + (20\pi - 36\xi)\sin\xi - 26\pi\sin(3\xi) \\ & + 78\xi\sin(3\xi) + 2\pi\sin(5\xi) + 18\xi\sin(5\xi)\Big], \qquad \xi \in (0, \pi/3) \end{aligned} \tag{1.413}$$

and for the PDF given as $h_{222}(\xi) = H'_{222}(\xi)$, we get

$$\begin{aligned} h_{222}(\xi) = -\frac{1}{2\pi^2}\Big[& 36\xi - 12\pi + 6(\pi - 6\xi - (\pi - 3\xi)\xi\cot\xi)\csc^2\xi \\ & - 2(\pi + 9\xi)\cos(2\xi) + 18\cot\xi + 3(\pi - 2\xi)\sec^2\xi \\ & + 2\sin(2\xi) + 2\sin(4\xi) - 6\tan\xi\Big], \qquad \xi \in (0, \pi/3). \end{aligned} \tag{1.414}$$

Perimeter moments

$\Pi^{(k)}{}_{210}$

Let $P = \Pi^k$. In (210) configuration using $(\alpha, \beta, \varepsilon)$ parametrization (see Figure 1.23), we have for the triangle **XYZ** side lengths

$$L = \frac{2\sin\alpha\sin\varepsilon}{\sin(\alpha+\beta)}, \quad L' = \frac{2\sin\beta\sin\varepsilon}{\sin(\alpha+\beta)}, \quad L'' = 2\sin\varepsilon, \tag{1.415}$$

from which, by using a known formula $\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$,

$$\Pi(\mathbf{x}, \mathbf{y}, \mathbf{z}) = L + L' + L'' = \frac{2\sin\varepsilon\,(\sin\alpha + \sin\beta + \sin\gamma)}{\sin\gamma} = \frac{4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\varepsilon}{\sin\frac{\gamma}{2}}. \tag{1.416}$$

Therefore, by Equation (1.396),

$$\Pi^{(k)}_{210} = \int_0^{\pi}\int_0^{\pi-\alpha}\int_{\alpha+\beta}^{\pi} \left(\frac{4\cos\frac{\alpha}{2}\,\cos\frac{\beta}{2}\,\sin\varepsilon}{\sin\frac{\gamma}{2}}\right)^k \rho_{210}(\alpha, \beta \mid \varepsilon)\,\mathrm{d}\varepsilon\mathrm{d}\beta\mathrm{d}\alpha. \tag{1.417}$$

e It is convenient to change our independent variables from (α, β) to (α, γ). Trivially $\mathrm{d}\alpha\mathrm{d}\beta = \mathrm{d}\alpha\mathrm{d}\gamma$ is the transformation of measure and for the integral, substituting $\rho_{210}(\alpha, \beta \mid \varepsilon)$, we get

$$\Pi^{(k)}_{210} = \int_0^{\pi}\int_0^{\pi-\gamma}\int_{\pi-\gamma}^{\pi} \frac{4^{2+k}\sin\alpha\sin\beta\sin^{4+k}\varepsilon}{\pi^2\sin^3\gamma}\left(\frac{\cos\frac{\alpha}{2}\,\cos\frac{\beta}{2}}{\sin\frac{\gamma}{2}}\right)^k \mathrm{d}\varepsilon\mathrm{d}\alpha\mathrm{d}\gamma, \tag{1.418}$$

where $\beta = \pi - \alpha - \gamma$.

$\Pi^{(k)}{}_{222}$

Substituting $\Pi_{210}^{(k)}$ into Equation (1.387) with $P = \Pi^k$ (and thus $p = k$), we get

$$\Pi_{222}^{(k)} = \frac{24\Pi_{210}^{(k)}}{(6+k)(4+k)}, \tag{1.419}$$

so for general $k > -2$ (not necessarily an integer),

$$\Pi_{222}^{(k)} = 6\int_0^{\pi}\int_0^{\pi-\gamma}\int_{\pi-\gamma}^{\pi} \frac{4^{3+k}\sin\alpha\sin\beta\sin^{4+k}\varepsilon}{(6+k)(4+k)\,\pi^2\sin^3\gamma}\left(\frac{\cos\frac{\alpha}{2}\,\cos\frac{\beta}{2}}{\sin\frac{\gamma}{2}}\right)^k \mathrm{d}\varepsilon\mathrm{d}\alpha\mathrm{d}\gamma, \tag{1.420}$$

where $\beta = \pi-\alpha-\gamma$. We do not know whether there is a way how we can simplify this integral for general k's. However, for any given selected k, the integral can be computed in an exact form.

$\Pi^{(2)}{}_{222}$

For example, when $k = 2$, we can integrate out ε and α in Equation (1.420),

$$\begin{aligned}\Pi_{222}^{(2)} = \int_0^{\pi} \frac{\csc^5\frac{\gamma}{2}\sec^3\frac{\gamma}{2}}{2304\pi^2}\Big[45\sin(2\gamma) - 9\sin(4\gamma) + \sin(6\gamma) - 60\gamma\Big]\times \\ \times\Big[6(\pi-\gamma)\cos(2\gamma) - 56\sin\gamma - \cos\gamma(24(\pi-\gamma) + 26\sin\gamma)\Big]\mathrm{d}\gamma,\end{aligned} \tag{1.421}$$

which can be solved using *Mathemtica* or by using derivatives of the Beta function,

$$\Pi_{222}^{(2)} = 3 + \frac{3383}{72\pi^2} + \frac{35\zeta(3)}{16\pi^2}. \tag{1.422}$$

Area moments

$\mathbf{S}^{(k)}{}_{210}$

Let $P = S^k$. In (210) configuration, we have for the area (see Figure 1.23),

$$S(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{2\sin\alpha\sin\beta\sin^2\varepsilon}{\sin\gamma}. \tag{1.423}$$

Therefore, by Equation (1.396),

$$S_{210}^{(k)} = \int_0^{\pi}\int_0^{\pi-\alpha}\int_{\alpha+\beta}^{\pi}\left(\frac{2\sin\alpha\sin\beta\sin^2\varepsilon}{\sin\gamma}\right)^k \rho_{210}(\alpha,\beta\mid\varepsilon)\;\mathrm{d}\varepsilon\mathrm{d}\beta\mathrm{d}\alpha. \tag{1.424}$$

Integrating out β first, we get the following neat result

$$S_{210}^{(k)} = \frac{4}{2+k}\int_0^{\pi}\int_0^{\varepsilon}\frac{(2\sin(\varepsilon-\alpha)\sin\varepsilon\sin\alpha)^{2+k}}{\pi^2\sin^2\alpha}\;\mathrm{d}\alpha\;\mathrm{d}\varepsilon. \tag{1.425}$$

$\mathbf{S^{(k)}}_{\mathbf{222}}$

Substituting $S_{210}^{(k)}$ into Equation (1.387) with $P = S^k$ (and thus $p = 2k$), we get for general $k > -1$ (not necessarily an integer),

$$S_{222}^{(k)} = \frac{6S_{210}^{(k)}}{(2+k)(3+k)} = 24\int_0^\pi \int_0^\varepsilon \frac{(2\sin(\varepsilon-\alpha)\sin\varepsilon\sin\alpha)^{2+k}}{(2+k)^2(3+k)\,\pi^2\sin^2\alpha}\,\mathrm{d}\alpha\mathrm{d}\varepsilon. \quad (1.426)$$

Remarkably, this integral possesses a closed form solution in terms of Gamma functions. This follows from a result by Miles [48, p. 363, Eq. (29)] (Theorem 220 in this thesis). We get for any $k > -1$,

$$S_{222}^{(k)} = \frac{(3/2)\,\Gamma(3+\frac{3k}{2})}{4^k(1+k)(3+k)\Gamma(2+\frac{k}{2})^3}. \quad (1.427)$$

After appropriate normalization, we get for the k-th volumetric moment,

$$v_2^{(k)}(\mathbb{B}_2) = \frac{S_{222}^{(k)}}{(\mathrm{vol}_2\,\mathbb{B}_2)^k} = \frac{(3/2)\,\Gamma(3+\frac{3k}{2})}{(4\pi)^k(1+k)(3+k)\Gamma(2+\frac{k}{2})^3}. \quad (1.428)$$

Area density

The density $f_{222}(s)$ of the random area S can be recovered from moments using inverse Mellin transform (see appendix A.5). By using the *Gamma function triplication identity*

$$\Gamma(z)\Gamma(z+\tfrac{1}{3})\Gamma(z+\tfrac{2}{3}) = \tfrac{2\pi\sqrt{3}}{3^{3z}}\Gamma(3z) \quad (1.429)$$

with $z = 1 + k/2$, we can rewrite Equation (1.427), in terms of a product of two Beta integrals as follows

$$S_{222}^{(k)} = \frac{81}{4\pi^2(1+k)(2+k)(3+k)} \int_0^1\int_0^1 \frac{\left(\frac{3}{4}\sqrt{3yz}\right)^k y^{1/3}z^{2/3}}{(1-y)^{1/3}(1-z)^{2/3}}\,\mathrm{d}y\mathrm{d}z \quad (1.430)$$

Taking the inverse Mellin transform of $S_{222}^{(k-1)}$, we get, formally,

$$f_{222}(s) = \frac{81}{4\pi^2}\mathcal{I}_0\mathcal{I}_1\mathcal{I}_2\left[\int_0^1\int_0^1 \frac{\delta\left(s-\frac{3}{4}\sqrt{3yz}\right)y^{1/3}z^{2/3}}{(1-y)^{1/3}(1-z)^{2/3}}\,\mathrm{d}y\mathrm{d}z\right]. \quad (1.431)$$

From Table A.5 (see Appendix A),

$$\mathcal{I}_0\mathcal{I}_1\mathcal{I}_2\delta(s-\alpha) = \frac{(\alpha-s)^2}{2\alpha^3}\mathbb{1}_{s<\alpha}, \quad (1.432)$$

via which we can deduce that in the unit disk $s \in (0, 3\sqrt{3}/4)$ and

$$f_{222}(s) = \frac{8}{\sqrt{3}\,\pi^2}\int_0^1\int_0^1 \frac{\left(\frac{3}{4}\sqrt{3yz}-s\right)^2 y^{1/3}z^{2/3}}{(yz)^{3/2}(1-y)^{1/3}(1-z)^{2/3}}\mathbb{1}_{s<\frac{3}{4}\sqrt{3yz}}\,\mathrm{d}y\mathrm{d}z. \quad (1.433)$$

Unfortunately, this integral is nontrivial. There exists a closed form expression in terms of generalised hypergeometric functions due to Mathai [45], but we are not showing it here since it is not particularly illuminating.

1.7 Trivariate functionals in three dimensions

1.7.1 Ball

See Finch [29, p. 694] and refferences therein.

1.7.2 Cube

Consider a trivariate symmetric homogeneous functional P of order p dependent on three random points picked uniformly from the unit cube C_3 with volume $\operatorname{vol}_3 C_3 = 1$. Nothing is known about area moments (Finch [29, p. 691–692]). However, we are able to deduce the obtusity probability,

$$\eta(C_3) = \frac{323338}{385875} - \frac{13G}{35} + \frac{4859\pi}{62720} - \frac{73\pi}{1680\sqrt{2}} - \frac{\pi^2}{105} + \frac{3\pi\ln 2}{224} - \frac{3\pi\ln(1+\sqrt{2})}{224} \qquad (1.434)$$
$$\approx 0.54265928142722907450111187258177267165716732602495,$$

where $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.9159655941$ is the *Catalan's constant*. This result is new as far as we know [29].

Configurations

As usual, let $P_{abc} = \mathbb{E}\left[P(\mathbf{X},\mathbf{Y},\mathbf{Z}) \mid \mathbf{X}\sim\mathsf{Unif}(A), \mathbf{Y}\sim\mathsf{Unif}(B), \mathbf{Z}\sim\mathsf{Unif}(C)\right]$, where $a = \dim A$, $b = \dim B$, $c = \dim C$ and the concrete selection of A, B, C is deduced from the reduction diagram in Figure 1.24 below.

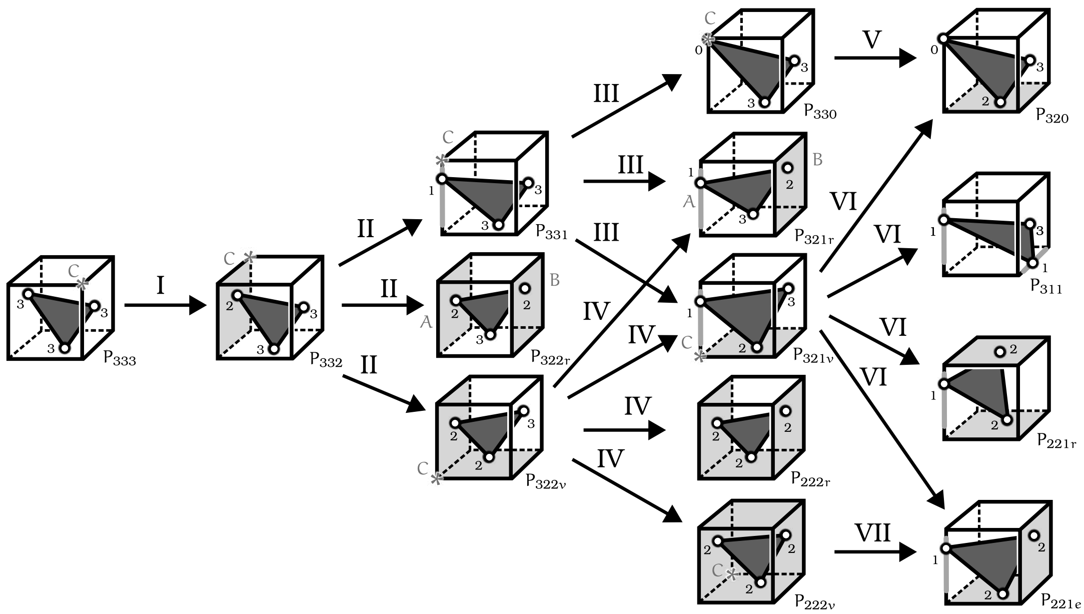


Figure 1.24: All different P_{abc} sub-configurations in C_3

Reduction system

The full system obtained by CRT is

$$\begin{aligned}
\mathbf{I}: \ pP_{333} &= 3\cdot 3(P_{332}-P_{333})\\
\mathbf{II}: \ pP_{332} &= 2(P_{331}-P_{332})+2\cdot 3(P_{322}-P_{332}),\\
\mathbf{III}: \ pP_{331} &= 1(P_{330}-P_{331})+2\cdot 3(P_{321}-P_{331}),\\
\mathbf{IV}: \ pP_{322v} &= 2\cdot 2(P'_{321}-P_{322v})+3(P_{222}-P_{322v}),\\
\mathbf{V}: \ pP_{330} &= 2\cdot 3(P_{320}-P_{330}),\\
\mathbf{VI}: \ pP_{321v} &= 1(P_{320}-P_{321v})+2(P_{311}-P_{321v})+3(P_{221}-P_{321v}),\\
\mathbf{VII}: \ pP_{222v} &= 3\cdot 2(P_{221e}-P_{222v})
\end{aligned}$$

with

$$\begin{aligned}
P_{322} &= \tfrac{1}{3}P_{322r}+\tfrac{2}{3}P_{322v},\\
P_{321} &= \tfrac{2}{3}P_{321r}+\tfrac{1}{3}P_{321v},\\
P'_{321} &= \tfrac{1}{2}P_{321r}+\tfrac{1}{2}P_{321v},\\
P_{222} &= \tfrac{2}{3}P_{222r}+\tfrac{1}{3}P_{222v},\\
P_{221} &= \tfrac{1}{3}P_{221r}+\tfrac{2}{3}P_{221e}.
\end{aligned}$$

The solution of our system is

$$\begin{aligned}
P_{333} = &\frac{108(4P_{221e}+P_{221r}+2P_{311}+2P_{320})}{(6+p)(7+p)(8+p)(9+p)}\\
&+\frac{72(P_{222r}+2P_{321r})}{(7+p)(8+p)(9+p)}+\frac{18P_{322r}}{(8+p)(9+p)}.
\end{aligned}\tag{1.435}$$

Obtusity probability

In order to decuce $\eta(C_3)$, it is convenient to introduce the *auxiliary Langford random variables* (see Appendix A.2). Let $U, U', U'' \sim \mathsf{Unif}(0,1)$ (independent), we define those random variables as having the same distribution as the functions of U, U', U'' on the right of the following equalities:

$$\Lambda = (U'-U)(U''-U), \quad \Sigma = (U-U')U, \quad \Xi = UU', \quad \Omega = U(1-U). \tag{1.436}$$

Moreover, we write $\Lambda \sim \mathsf{Lang}$ (Langford distribution). Probability and cumulative density functions of Λ, Σ, Ξ and Ω are shown in Table A.2. By symmetry, we get for the obtusity probability in C_3

$$\eta(C_3) = 3\eta_{3*33} = 3\,\mathbb{P}\left[(\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X})<0 \mid \mathbf{X},\mathbf{Y},\mathbf{Z}\sim\mathsf{Unif}(C_3)\right]. \tag{1.437}$$

We can rewrite $\eta(C_3)$ in terms auxiliary variables introduced above. This is the method used by Langford [42] to deduce $\eta(C_2)$. In configuration (333), we may parametrise the random points $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ as

$$\mathbf{X} = \sum_{i=1}^{3} X_i\mathbf{e}_i, \quad \mathbf{Y} = \sum_{i=1}^{3} Y_i\mathbf{e}_i, \quad \mathbf{Z} = \sum_{i=1}^{3} Z_i\mathbf{e}_i, \tag{1.438}$$

where $X_i, Y_i, Z_i \sim \mathsf{Unif}(0,1), i = 1,2,3$. Hence, for the scalar product of the (3*33) configuration (obtuse vertex at $\mathbf{X}$),

$$(3^*33): \ (\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) = \sum_{i=1}^{3}(Y_i - X_i)(Z_i - X_i) \tag{1.439}$$

and thus, using our auxiliary random variables,

$$\eta(C_3) = 3\mathbb{P}\left[\Lambda+\Lambda'+\Lambda''<0\right] = 3\int_{\lambda_1+\lambda_2+\lambda_3<0} f_\Lambda(\lambda) f_\Lambda(\lambda') f_\Lambda(\lambda') \ \mathrm{d}\lambda \mathrm{d}\lambda' \mathrm{d}\lambda'', \tag{1.440}$$

where $\Lambda, \Lambda', \Lambda'' \sim \mathsf{Lang}$ are independent random variables following the Langford distribution. Unfortunately, we were not able to find the closed form expression of the integral in Equation (1.440) with $d = 3$ straightaway. The intermediate result involves dilogarithms with intricate arguments. However, there is a workaround – CRT.

η_{322r}

In configuration ($332r$), the first vertex $\mathbf{X}$ of the inscribed random triangle $\mathbf{XYZ}$ is selected from the interior of C_3, while the other two $\mathbf{Y}$ and $\mathbf{Z}$ are picked from (any fixed) opposite faces. We may parametrise the points as

$$\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3, \quad \mathbf{Y} = Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2, \quad \mathbf{Z} = Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2 + \mathbf{e}_3, \tag{1.441}$$

where $X_1, X_2, X_3, Y_1, Y_2, Z_1, Z_2 \sim \mathsf{Unif}(0,1)$. Based on the exact location of the obtuse angle, we recognize three sub-configurations (3^*22r), (32^*2r) and (322^*r), out of which the last two give the same contribution by symmetry. Expressing the dot products in the decomposition of the obtusity indicator (Equation (1.31)), we get

$$\begin{aligned}
(3^*22r): (\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) &= (Y_1-X_1)(Z_1-X_1)+(Y_2-X_2)(Z_2-X_2)-X_3(1-X_3), \\
(32^*2r): (\mathbf{Z}-\mathbf{Y})^\top(\mathbf{X}-\mathbf{Y}) &= (X_1-Y_1)(Z_1-Y_1)+(X_2-Y_2)(Z_2-Y_2)+X_3.
\end{aligned} \tag{1.442}$$

The probabilities that those dot products are negative can be computed by the method of Langford [42]. The method relies on noticing that the dot product can be written as linear combinations of auxiliary Langford random variables $\Lambda, \Lambda' \sim \mathsf{Lang}$ (two independent copies). We can write the obtusity probability in (3^*22r) sub-configuration as

$$\begin{aligned}
\eta_{3^*22r} &= \mathbb{P}\left[(\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X})<0\right] = \mathbb{P}\left[\Lambda+\Lambda'-\Omega<0\right] \\
&= \int_0^{1/4}\int_{-1/4}^{\omega+1/4}\int_{-1/4}^{\omega-\lambda} f_\Lambda(\lambda)f_\Lambda(\lambda')f_\Omega(\omega)\ \mathrm{d}\lambda'\mathrm{d}\lambda\mathrm{d}\omega \\
&= \int_0^{1/4}\int_{-1/4}^{\omega+1/4} f_\Lambda(\lambda)F_\Lambda(\omega-\lambda)f_\Omega(\omega)\ \mathrm{d}\lambda\mathrm{d}\omega.
\end{aligned} \tag{1.443}$$

Unfortunately, the leftover integral is far from trivial and even *Mathematica* is unable to find its closed form solution straightaway. Nevertheless, via simple

Weierstrass substitution, the integral can be decomposed into linear combination of special integrals recently discussed on *MSE* website [47, 74], via which

$$\eta_{3^*22r} = \frac{6739}{6750} - \frac{8G}{15} + \frac{211\pi}{1440} - \frac{17\pi}{252\sqrt{2}} - \frac{\pi^2}{45} - \frac{\pi\ln(1+\sqrt{2})}{24} + \frac{\pi\ln 2}{24} \approx 0.576363509, \tag{1.444}$$

where $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.9159655941$ is the *Catalan's constant*. Somehow, the situation is much more elementary in (32^*2r) configuration. We have

$$\begin{aligned}\eta_{32^*2r} &= \mathbb{P}\left[(\mathbf{X}-\mathbf{Y})^\top(\mathbf{Z}-\mathbf{Y}) < 0\right] = \mathbb{P}\left[\Lambda + \Lambda' + U < 0\right] \\ &= \int_{-1/4}^{1/4}\int_{-1/4}^{-\lambda}\int_{0}^{-\lambda-\lambda'} f_\Lambda(\lambda) f_\Lambda(\lambda') f_U(u)\ \mathrm{d}u\mathrm{d}\lambda'\mathrm{d}\lambda \\ &= \int_{-1/4}^{1/4} F_\Lambda(-\lambda) F_\Lambda(\lambda)\ \mathrm{d}\lambda = \frac{121}{7350} + \frac{\pi}{2688} \approx 0.0176313323.\end{aligned} \tag{1.445}$$

Lastly, by symmetry, $\eta_{322^*r} = \eta_{32^*2r}$. Summing up the three obtusity probabilities,

$$\begin{aligned}\eta_{322r} &= \eta_{3^*22r} + \eta_{32^*2r} + \eta_{322^*r} = \eta_{3^*22r} + 2\eta_{32^*2r} = \frac{341101}{330750} - \frac{8G}{15} + \frac{2969\pi}{20160} \\ &- \frac{17\pi}{252\sqrt{2}} - \frac{\pi^2}{45} + \frac{\pi\ln 2}{24} - \frac{\pi\ln(1+\sqrt{2})}{24} \approx 0.61162617366523535 6686.\end{aligned} \tag{1.446}$$

η_{321r}

In configuration ($321r$), vertex $\mathbf{X}$ is selected from the interior of C_3 and $\mathbf{Y}$ and $\mathbf{Z}$ are picked from one face and its opposite edge, respectively. We may parametrise the points as

$$\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3, \quad \mathbf{Y} = Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2, \quad \mathbf{Z} = Z_1\mathbf{e}_1 + \mathbf{e}_3, \tag{1.447}$$

where $X_1, X_2, X_3, Y_1, Y_2, Z_1 \sim \mathsf{Unif}(0,1)$. Based on the exact location of the obtuse angle, we recognize three sub-configurations (3^*21r), (32^*1r) and (321^*r). Expressing the dot products in the decomposition of the obtusity indicator (Equation (1.31)), we get

$$\begin{aligned}(3^*21r) &: (\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) = (Y_1-X_1)(Z_1-X_1) + (X_2-Y_2)X_2 - X_3(1-X_3), \\ (32^*1r) &: (\mathbf{Z}-\mathbf{Y})^\top(\mathbf{X}-\mathbf{Y}) = (X_1-Y_1)(Z_1-Y_1) + (Y_2-X_2)Y_2 + X_3, \\ (321^*r) &: (\mathbf{X}-\mathbf{Z})^\top(\mathbf{Y}-\mathbf{Z}) = (X_1-Z_1)(Y_1-Z_1) + X_2Y_2 + 1 - X_3.\end{aligned} \tag{1.448}$$

Using auxiliary Langford random variables, we can write the obtusity probability in (3^*21r) sub-configuration as

$$\begin{aligned}\eta_{3^*21r} &= \mathbb{P}\left[(\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) < 0\right] = \mathbb{P}\left[\Lambda + \Sigma - \Omega < 0\right] \\ &= \int_{0}^{1/4}\int_{-1/4}^{\omega+1/4}\int_{-1/4}^{\omega-\lambda} f_\Lambda(\lambda) f_\Sigma(\sigma) f_\Omega(\omega)\ \mathrm{d}\sigma\mathrm{d}\lambda\mathrm{d}\omega \\ &= \int_{0}^{1/4}\int_{-1/4}^{\omega+1/4} f_\Lambda(\lambda) F_\Sigma(\omega-\lambda) f_\Omega(\omega)\ \mathrm{d}\lambda\mathrm{d}\omega.\end{aligned} \tag{1.449}$$

By using the *MSE* integrals, we get

$$\eta_{3^*21r} = \frac{49043}{54000} - \frac{8G}{15} + \frac{1567\pi}{11520} - \frac{67\pi}{720\sqrt{2}} - \frac{\pi^2}{240} + \frac{\pi\ln 2}{192} - \frac{\pi\ln(1+\sqrt{2})}{96} \approx 0.5816795685, \tag{1.450}$$

Next, in (32*1r) configuration,

$$\begin{aligned} \eta_{32^*1r} &= \mathbb{P}\left[(\mathbf{X}-\mathbf{Y})^\top(\mathbf{Z}-\mathbf{Y})<0\right] = \mathbb{P}\left[\Lambda+\Sigma+U<0\right] \\ &= \int_{-1/4}^{1/4}\int_{-1/4}^{-\lambda}\int_0^{-\lambda-\sigma} f_\Lambda(\lambda)f_\Sigma(\sigma)f_U(u)\ \mathrm{d}u\mathrm{d}\sigma\mathrm{d}\lambda \\ &= \int_{-1/4}^{1/4} F_\Lambda(-\lambda)F_\Sigma(\lambda)\ \mathrm{d}\lambda = \frac{37}{1176}+\frac{\pi}{1344} \approx 0.03380008. \end{aligned} \tag{1.451}$$

At last, in (321*r), configuration, since $1-X_3 \sim \mathsf{Unif}(0,1)$, we get

$$\begin{aligned} \eta_{321^*r} &= \mathbb{P}\left[(\mathbf{X}-\mathbf{Z})^\top(\mathbf{Y}-\mathbf{Z})<0\right] = \mathbb{P}\left[\Lambda+\Xi+U<0\right] \\ &= \int_{-1/4}^{0}\int_{0}^{-\lambda}\int_0^{-\lambda-\xi} f_\Lambda(\lambda)f_\Xi(\xi)f_U(u)\ \mathrm{d}u\mathrm{d}\xi\mathrm{d}\lambda \\ &= \int_{0}^{1/4} F_\Lambda(-\lambda)F_\Xi(\lambda)\ \mathrm{d}\lambda = \frac{43}{14700} \approx 0.00292517. \end{aligned} \tag{1.452}$$

Summing up the three obtusity probabilities we obtained in all sub-configurations,

$$\begin{aligned} \eta_{321r} = \eta_{3^*21r}+\eta_{32^*1r}+\eta_{321^*r} &= \frac{2494097}{2646000} - \frac{8G}{15} + \frac{11029\pi}{80640} - \frac{67\pi}{720\sqrt{2}} \\ &- \frac{\pi^2}{240} + \frac{\pi\ln 2}{192} - \frac{\pi\ln(1+\sqrt{2})}{96} \approx 0.61840481814327429018. \end{aligned} \tag{1.453}$$

$\boldsymbol{\eta_{222r}}$

In configuration (222r), vertices $\mathbf{Y}$ and $\mathbf{Z}$ are selected from opposite faces of C_3 and $\mathbf{X}$ is selected from another face in between the two. We may parametrise the points as

$$\mathbf{X} = X_2\mathbf{e}_2 + X_3\mathbf{e}_3, \quad \mathbf{Y} = Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2, \quad \mathbf{Z} = Z_1\mathbf{e}_1 + Z_2\mathbf{e}_2 + \mathbf{e}_3, \tag{1.454}$$

where $X_2, X_3, Y_1, Y_2, Z_1, Z_2 \sim \mathsf{Unif}(0,1)$. Based on the exact location of the obtuse angle, we recognize three sub-configurations (2*22r), (22*2r) and (222*r), the last last two of which give the same contribution by symmetry. Expressing the dot products in the decomposition of the obtusity indicator, we get

$$\begin{aligned} &(2^*22r) : (\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) = Y_1Z_1 + (Y_2-X_2)(Z_2-X_2) - X_3(1-X_3), \\ &(22^*2r) : (\mathbf{Z}-\mathbf{Y})^\top(\mathbf{X}-\mathbf{Y}) = (Y_1-Z_1)Y_1 + (X_2-Y_2)(Z_2-Y_2) + X_3. \end{aligned} \tag{1.455}$$

Using auxiliary Langford random variables, we can write the obtusity probability in (2*22r) sub-configuration as

$$\begin{aligned} \eta_{2^*22r} &= \mathbb{P}\left[(\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X})<0\right] = \mathbb{P}\left[\Xi+\Lambda-\Omega<0\right] \\ &= \int_{0}^{1/4}\int_{-1/4}^{\omega}\int_0^{\omega-\lambda} f_\Xi(\xi)f_\Lambda(\lambda)f_\Omega(\omega)\ \mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}\omega \\ &= \int_{0}^{1/4}\int_{-1/4}^{\omega} F_\Xi(\omega-\lambda)f_\Lambda(\lambda)f_\Omega(\omega)\ \mathrm{d}\lambda\mathrm{d}\omega. \end{aligned} \tag{1.456}$$

By using the *MSE* integrals, we get

$$\eta_{2^*22r} = \frac{14393}{27000} - \frac{2G}{15} + \frac{11\pi}{1152} - \frac{\pi^2}{72} + \frac{\pi \ln 2}{96} \approx 0.326548524. \tag{1.457}$$

Next, in (22^*2r) configuration,

$$\begin{aligned} \eta_{22^*2r} &= \mathbb{P}\left[(\mathbf{X}-\mathbf{Y})^\top(\mathbf{Z}-\mathbf{Y}) < 0\right] = \mathbb{P}\left[\Sigma + \Lambda + U < 0\right] = \eta_{32^*1r} \\ &= \frac{37}{1176} + \frac{\pi}{1344} \approx 0.03380008. \end{aligned} \tag{1.458}$$

At last, $\eta_{222^*r} = \eta_{22^*2r}$ by symmetry. Summing up the three obtusity probabilities,

$$\begin{aligned} \eta_{222r} &= \eta_{2^*22r} + \eta_{22^*2r} + \eta_{222^*r} = \eta_{2^*22r} + 2\eta_{22^*2r} \\ &= \frac{788507}{1323000} - \frac{2G}{15} + \frac{89\pi}{8064} - \frac{\pi^2}{72} + \frac{\pi \ln 2}{96} \approx 0.39414868337494. \end{aligned} \tag{1.459}$$

η_{320}

In configuration (320), $\mathbf{X}$ is selected from the interior of C_3, $\mathbf{Y}$ is selected from a face and $\mathbf{Z}$ in one of the vertices opposite to the selected face. We may parametrise the points as

$$\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3, \quad \mathbf{Y} = Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2, \quad \mathbf{Z} = \mathbf{e}_3, \tag{1.460}$$

where $X_1, X_2, X_3, Y_1, Y_2 \sim \mathsf{Unif}(0,1)$. Based on the exact location of the obtuse angle, we recognize three sub-configurations (3^*20), (32^*0) and (320^*). Expressing the dot products in the decomposition of the obtusity indicator, we get

$$\begin{aligned} (3^*20) &: (\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) = (X_1 - Y_1)X_1 + (X_2 - Y_2)X_2 - X_3(1 - X_3), \\ (32^*0) &: (\mathbf{Z}-\mathbf{Y})^\top(\mathbf{X}-\mathbf{Y}) = (Y_1 - X_1)Y_1 + (Y_2 - X_2)Y_2 + X_3, \\ (320^*) &: (\mathbf{X}-\mathbf{Z})^\top(\mathbf{Y}-\mathbf{Z}) = X_1Y_1 + X_2Y_2 + 1 - X_3, \end{aligned} \tag{1.461}$$

Using auxiliary Langford random variables, we can write the obtusity probability in (3^*20) sub-configuration as

$$\begin{aligned} \eta_{3^*20} &= \mathbb{P}\left[(\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) < 0\right] = \mathbb{P}\left[\Sigma + \Sigma' - \Omega < 0\right] \\ &= \int_0^{1/4}\int_{-1/4}^{\omega+1/4}\int_{-1/4}^{\omega-\sigma} f_\Sigma(\sigma) f_\Sigma(\sigma') f_\Omega(\omega)\ \mathrm{d}\sigma'\mathrm{d}\sigma\mathrm{d}\omega \\ &= \int_0^{1/4}\int_{-1/4}^{\omega+1/4} f_\Sigma(\sigma) F_\Sigma(\omega-\sigma) f_\Omega(\omega)\ \mathrm{d}\sigma\mathrm{d}\omega. \end{aligned} \tag{1.462}$$

By using the *MSE* integrals, we get

$$\eta_{3^*20} = \frac{42977}{54000} - \frac{7G}{30} - \frac{\pi^2}{1440} \approx 0.575291173117. \tag{1.463}$$

Next, in (32^*0) configuration,

$$\begin{aligned} \eta_{32^*0} &= \mathbb{P}\left[(\mathbf{X}-\mathbf{Y})^\top(\mathbf{Z}-\mathbf{Y}) < 0\right] = \mathbb{P}\left[\Sigma + \Sigma' + U < 0\right] \\ &= \int_{-1/4}^{1/4}\int_{-1/4}^{-\sigma}\int_0^{-\sigma-\sigma'} f_\Sigma(\sigma) f_\Sigma(\sigma') f_U(u)\ \mathrm{d}u\mathrm{d}\sigma'\mathrm{d}\sigma \\ &= \int_{-1/4}^{1/4} F_\Sigma(-\sigma) F_\Sigma(\sigma)\ \mathrm{d}\sigma = \frac{23}{450} \approx 0.0511111. \end{aligned} \tag{1.464}$$

At last, in (320*), configuration, since $1 - X_3 \sim \mathsf{Unif}(0,1)$ and both independent copies Ξ and Ξ' are positive, we get trivially

$$\eta_{320^*} = \mathbb{P}\left[(\mathbf{X}-\mathbf{Z})^\top(\mathbf{Y}-\mathbf{Z}) < 0\right] = \mathbb{P}\left[\Xi + \Xi' + U < 0\right] = 0. \tag{1.465}$$

Summing up the three obtusity probabilities we got,

$$\eta_{320} = \eta_{3^*20} + \eta_{32^*0} + \eta_{320^*} = \frac{45737}{54000} - \frac{7G}{30} - \frac{\pi^2}{1440} \approx 0.6264022842. \tag{1.466}$$

η_{311}

In configuration (311), $\mathbf{X}$ is selected from the interior of C_3 and $\mathbf{Y}$ and $\mathbf{Z}$ are selected from perpendicular edges which do not share a common vertex. We may parametrise the points as

$$\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3, \quad \mathbf{Y} = Y_3\mathbf{e}_3, \quad \mathbf{Z} = \mathbf{e}_1 + Z_2\mathbf{e}_2, \tag{1.467}$$

where $X_1, X_2, X_3, Y_3, Z_2 \sim \mathsf{Unif}(0,1)$. Based on the exact location of the obtuse angle, we recognize three sub-configurations (3*11), (31*1) and (311*), out of which the last two give the same contribution. Expressing the dot products in the decomposition of the obtusity indicator, we get

$$\begin{aligned} (3^*11) &: (\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) = -(1-X_1)X_1 + (X_2-Z_2)X_2 - (X_3-Y_3)X_3, \\ (31^*1) &: (\mathbf{Z}-\mathbf{Y})^\top(\mathbf{X}-\mathbf{Y}) = X_1 + X_2Z_2 + (Y_3 - X_3)Y_3, \end{aligned} \tag{1.468}$$

Using auxiliary Langford random variables, we can write the obtusity probability in (3*20) sub-configuration as

$$\begin{aligned} \eta_{3^*11} &= \mathbb{P}\left[(\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) < 0\right] = \mathbb{P}\left[-\Omega + \Sigma + \Sigma' < 0\right] = \eta_{3^*20} \\ &= \frac{42977}{54000} - \frac{7G}{30} - \frac{\pi^2}{1440} \approx 0.575291173117. \end{aligned} \tag{1.469}$$

Next, in (31*1) configuration,

$$\begin{aligned} \eta_{31^*1} &= \mathbb{P}\left[(\mathbf{X}-\mathbf{Y})^\top(\mathbf{Z}-\mathbf{Y}) < 0\right] = \mathbb{P}\left[U + \Xi + \Sigma < 0\right] \\ &= \int_{-1/4}^{0}\int_0^{-\sigma}\int_0^{-\sigma-\xi} f_U(u) f_\Xi(\xi) f_\Sigma(\sigma)\,\mathrm{d}u\mathrm{d}\xi\mathrm{d}\sigma \\ &= \int_0^{1/4} F_\Sigma(-\sigma)F_\Xi(\sigma)\,\mathrm{d}\sigma = \frac{17}{1800} \approx 0.00944444. \end{aligned} \tag{1.470}$$

At last, by symmetry, $\eta_{311^*} = \eta_{31^*1}$. Summing up the three obtusity probabilities,

$$\eta_{320} = \eta_{3^*20} + \eta_{32^*0} + \eta_{320^*} = \frac{43997}{54000} - \frac{7G}{30} - \frac{\pi^2}{1440} \approx 0.5941800620. \tag{1.471}$$

η_{221r}

In configuration ($221r$), $\mathbf{X}$ and $\mathbf{Y}$ are selected from opposite faces of C_3 while $\mathbf{Z}$ is selected from one of the edges connecting them. We may parametrise the points as

$$\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2, \quad \mathbf{Y} = Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2 + \mathbf{e}_3, \quad \mathbf{Z} = Z_3\mathbf{e}_3, \tag{1.472}$$

where $X_1, X_2, Y_1, Y_2, Z_3 \sim \mathsf{Unif}(0,1)$. Based on the exact location of the obtuse angle, we recognize three sub-configurations (2^*21r), (22^*1r) and (221^*r), out of which the first two give the same contribution by symmetry. Expressing the dot products in the decomposition of the obtusity indicator, we get

$$\begin{aligned}(2^*21r) &: (\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) = (X_1-Y_1)X_1 + (X_2-Y_2)X_2 + Z_3,\\ (221^*r) &: (\mathbf{X}-\mathbf{Z})^\top(\mathbf{Y}-\mathbf{Z}) = X_1Y_1 + X_2Y_2 + Z_3(1-Z_3),\end{aligned} \tag{1.473}$$

Using auxiliary Langford random variables, we can write the obtusity probability in (2^*21r) sub-configuration as

$$\eta_{2^*21r} = \mathbb{P}\left[(\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) < 0\right] = \mathbb{P}\left[\Sigma + \Sigma' - U < 0\right] = \eta_{32^*0} = \frac{23}{450}. \tag{1.474}$$

By symmetry, $\eta_{22^*1r} = \eta_{2^*21r}$. Finally, in ($221^*r$) configuration,

$$\begin{aligned}\eta_{221^*r} &= \mathbb{P}\left[(\mathbf{X}-\mathbf{Z})^\top(\mathbf{Y}-\mathbf{Z}) < 0\right] = \mathbb{P}\left[\Xi + \Xi' - \Omega < 0\right]\\ &= \int_0^{\frac14}\int_0^{\frac14-\xi}\int_{\xi+\xi'}^{\frac14} f_\Xi(\xi) f_\Xi(\xi') f_\Sigma(\sigma)\,\mathrm{d}\sigma\mathrm{d}\xi'\mathrm{d}\xi = \int_0^{\frac14}\int_0^{\frac14-\xi} \ln\xi \ln\xi' \sqrt{1-4(\xi+\xi')}\,\mathrm{d}\xi'\mathrm{d}\xi\\ &= \int_0^{\frac14} \frac{(1-4\xi)^{3/2}}{18}(3\ln(1-4\xi)-8)\ln\xi\,\mathrm{d}\xi = \frac{788}{3375} - \frac{\pi^2}{120} \approx 0.151234778.\end{aligned} \tag{1.475}$$

Summing up the three obtusity probabilities,

$$\eta_{221r} = \eta_{2^*21r} + \eta_{22^*1r} + \eta_{221^*r} = \frac{1133}{3375} - \frac{\pi^2}{120} \approx 0.2534570004. \tag{1.476}$$

$\boldsymbol{\eta_{221e}}$

In the last irreducible configuration ($221e$), $\mathbf{X}$ and $\mathbf{Y}$ are selected from adjacent faces of C_3 while $\mathbf{Z}$ is selected from an edge opposite to the face on which $\mathbf{Y}$ reside. We may parametrise the points as

$$\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2, \quad \mathbf{Y} = \mathbf{e}_1 + Y_2\mathbf{e}_2 + Y_3\mathbf{e}_3, \quad \mathbf{Z} = Z_3\mathbf{e}_3, \tag{1.477}$$

where $X_1, X_2, Y_2, Y_3, Z_3 \sim \mathsf{Unif}(0,1)$. Based on the exact location of the obtuse angle, we recognize three sub-configurations (2^*21e), (22^*1e) and (221^*e). Expressing the dot products in the decomposition of the obtusity indicator, we get

$$\begin{aligned}(2^*21e) &: (\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) = -X_1(1-X_1) + (X_2-Y_2)X_2 + Y_3Z_3,\\ (22^*1e) &: (\mathbf{X}-\mathbf{Y})^\top(\mathbf{Z}-\mathbf{Y}) = 1 - X_1 + (Y_2-X_2)Y_2 + (Y_3-Z_3)Y_3,\\ (221^*e) &: (\mathbf{X}-\mathbf{Z})^\top(\mathbf{Y}-\mathbf{Z}) = X_1 + X_2Y_2 + (Z_3-Y_3)Z_3,\end{aligned} \tag{1.478}$$

Using auxiliary Langford random variables, we can write the obtusity probability in (2^*21e) sub-configuration as

$$\begin{aligned}\eta_{2^*21e} &= \mathbb{P}\left[(\mathbf{Y}-\mathbf{X})^\top(\mathbf{Z}-\mathbf{X}) < 0\right] = \mathbb{P}\left[-\Omega + \Sigma + \Xi < 0\right]\\ &= \int_0^{1/4}\int_{-1/4}^{\omega}\int_0^{\omega-\sigma} f_\Omega(\omega) f_\Sigma(\sigma) f_\Xi(\xi)\,\mathrm{d}\xi\mathrm{d}\sigma\mathrm{d}\omega\\ &= \int_0^{1/4}\int_{-1/4}^{\omega} f_\Omega(\omega) f_\Sigma(\sigma) F_\Xi(\omega-\sigma)\,\mathrm{d}\sigma\mathrm{d}\omega.\end{aligned} \tag{1.479}$$

By using the *MSE* integrals, we get

$$\eta_{2^*21e} = \frac{32629}{54000} - \frac{7G}{30} - \frac{\pi^2}{360} \approx 0.3630998677. \tag{1.480}$$

Next, in (22^*1e) configuration, we have since $1 - X_1 \sim \mathsf{Unif}(0,1)$,

$$\eta_{22^*1e} = \mathbb{P}\left[(\mathbf{X}-\mathbf{Y})^\top(\mathbf{Z}-\mathbf{Y}) < 0\right] = \mathbb{P}\left[U+\Sigma+\Sigma'<0\right] = \eta_{32^*0} = \frac{23}{450}. \tag{1.481}$$

Finally, in (221^*e) configuration,

$$\eta_{221^*e} = \mathbb{P}\left[(\mathbf{X}-\mathbf{Z})^\top(\mathbf{Y}-\mathbf{Z}) < 0\right] = \mathbb{P}\left[U+\Xi+\Sigma<0\right] = \eta_{31^*1} = \frac{17}{1800}. \tag{1.482}$$

Summing up the three obtusity probabilities,

$$\eta_{221r} = \eta_{2^*21r} + \eta_{22^*1r} + \eta_{221^*r} = \frac{35899}{54000} - \frac{7G}{30} - \frac{\pi^2}{360} \approx 0.4236554232. \tag{1.483}$$

η_{333}

Inserting η_{221e}, η_{221r}, η_{222r}, η_{311}, η_{320}, η_{321r} and η_{322r} into Equation (1.435) with $P = \eta$, for which $p = 0$, we finally obtain

$$\begin{aligned}\eta(C_3) = \eta_{333} &= \tfrac{1}{28}(4\eta_{221e}+\eta_{221r}+4\eta_{222r}+2\eta_{311}+2\eta_{320}+8\eta_{321r}+7\eta_{322r})\\ &= \frac{323338}{385875} - \frac{13G}{35} + \frac{4859\pi}{62720} - \frac{73\pi}{1680\sqrt{2}} - \frac{\pi^2}{105} + \frac{3\pi\ln 2}{224} - \frac{3\pi\ln(1+\sqrt{2})}{224} \\ &\approx 0.542659281427229074501111872581772671657167326024 95\ldots,\end{aligned} \tag{1.484}$$

which is a natural generalization of Langford's $\eta(C_2)$ [42].

2. Even Moments of Random Determinants

In this chapter, we study the moments of random determinants. It turns out that they are closely related with moments of volumes of random simplices. We will see this correspondence later in Chapter 3. For now, we shall study the moments of random matrices on their own, the usefulness of which will be apparent later on.

2.1 Preliminaries

2.1.1 Definitions

Definition 39. Let X_{ij}'s be independent and identically distributed (i.i.d.) random variables with (non-central) moments $m_r = \mathbb{E}\, X_{ij}^r$, from which we construct two (random) matrices $A = (X_{ij})_{n\times n}$ and $U = (X_{ij})_{n\times p}$. Let $f_k(n) = \mathbb{E}\,(\det A)^k$ and $f_k(n,p) = \mathbb{E}\,(\det U^\top U)^{k/2}$ be their $k-$th determinant moment and k-th Gram moment, respectively. By definition, we set $f_k(0) = 1$ and $f_k(n,0) = 1$ (we put $\det(U^\top U) = 1$ when $p = 0$). Also, we define their corresponding generating functions

$$F_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} f_k(n), \qquad F_k(t,\omega) = \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{(n-p)!}{n!p!} t^p \omega^{n-p} f_k(n,p). \tag{2.1}$$

Remark 40. This definition of generating functions makes sense only for $k \le 5$, otherwise it does not in general define an analytic function of t on any interval. Although, we can still treat them formally.

Remark 41. Notice that, when $n = p$, that is when $U = A$, we get by the multiplicative property of determinant, $\det(U^\top U) = (\det A)^2$. Therefore, $f_k(n,n) = f_k(n)$ and thus $F_k(t,0) = F_k(t)$.

Example 42. When $n = 2$ and $k = 4$, we have

$$\begin{aligned} f_4(2) = \mathbb{E}\,(\det A)^4 = \mathbb{E}\begin{vmatrix} X_{11} & X_{12} \\ X_{21} & X_{22}\end{vmatrix}^4 &= \mathbb{E}\,(X_{11}X_{22} - X_{12}X_{21})^4 = \mathbb{E}\,(X_{11}^4X_{22}^4 \\ &- 4X_{11}^3X_{22}^3X_{12}X_{21} + 6X_{11}^2X_{22}^2X_{12}^2X_{21}^2 - 4X_{11}X_{22}X_{12}^3X_{21}^3 + X_{12}^4X_{21}^4) \\ &= m_4^2 - 4m_3^2m_1^2 + 6m_2^4 - 4m_1^2m_3^2 + m_4^2 = 2m_4^2 - 8m_3^2m_1^2 + 6m_2^4. \end{aligned} \tag{2.2}$$

Definition 43. Sometimes, we restrict the distribution of X_{ij}'s:

- We say X_{ij}'s follow a **symmetrical** distribution, if the odd moments are equal to zero up to the order k (that is, $m_{2l+1} = 0$ for $2l+1 \le k$). We denote $f_k^{\text{sym}}(n)$ and $F_k^{\text{sym}}(t)$ the corresponding k-th moment of the random determinant formed by those random variables, and its generating function, respectively. Similarly, $f_k(n,p) = f_k^{\text{sym}}(n,p)$ and $F_k(t,\omega) = F_k^{\text{sym}}(t,\omega)$ if $m_1 = m_3 = m_5 = \ldots = m_{2\lceil k/2\rceil - 1} = 0$.
- We say X_{ij}'s follow a **centered** distribution (or equivalently, we say X_{ij}'s

are centered random variables) if $m_1 = 0$. For those variables, we consider $f_k^{\text{cen}}(n)$ and $F_k^{\text{cen}}(t)$ in the same way. Similarly, $f_k(n,p) = f_k^{\text{cen}}(n,p)$ and $F_k(t,\omega) = F_k^{\text{cen}}(t,\omega)$ if $m_1 = 0$.

2.1.2 Polynomial nature and scalability

We present some general statements about the random determinant moments.

Proposition 44. *Let k be even, then*

$$f_k(n) = \varphi_k(n, m_1, m_2, m_3, \ldots, m_{k-1}, m_k), \tag{2.3}$$

where φ_k is a polynomial in $m_1, \ldots, m_k$. Equivalently, there exists a function Φ_k whose expansion coefficients are polynomials in $m_1, \ldots, m_k$, we can write

$$F_k(t) = \Phi_k(t, m_1, \ldots, m_k). \tag{2.4}$$

Similarly, $f_k(n,p)$ is also some polynomial $\varphi_k(n,p,m_1,\ldots,m_k)$ and $F_k(t,\omega) = \Phi_k(t,\omega,m_1,\ldots,m_k)$ for some functions Φ_k with polynomial expansion coefficients.

Corollary 44.1.

$$f_k^{\text{sym}}(n) = \varphi_k(n, 0, m_2, 0, m_4, 0, m_6, \ldots), \tag{2.5}$$
$$F_k^{\text{sym}}(t) = \Phi_k(t, 0, m_2, 0, m_4, 0, m_6, \ldots), \tag{2.6}$$
$$f_k^{\text{cen}}(n) = \varphi_k(n, 0, m_2, m_3, m_4, m_5, m_6, \ldots), \tag{2.7}$$
$$F_k^{\text{cen}}(t) = \Phi_k(t, 0, m_2, m_3, m_4, m_5, m_6, \ldots), \tag{2.8}$$

similarly for $f_k(n,p)$ and $F_k(t,\omega)$.

The following proposition allow us to fix one m_r and still retain the full generality:

Proposition 45. *Let φ_k and Φ_k be defined as in Proposition 44, then for any $\beta \in \mathbb{R}$ and k even,*

$$\varphi_k(n, \beta m_1, \beta^2 m_2, \beta^3 m_3, \ldots, \beta^k m_k) = \beta^{nk} \varphi_k(n, m_1, m_2, m_3, \ldots, m_k) \tag{2.9}$$

and as a consequence,

$$\Phi_k(t, \beta m_1, \beta^2 m_2, \ldots, \beta^k m_k) = \Phi_k(\beta^k t, m_1, m_2, \ldots, m_k). \tag{2.10}$$

Similarly for the non-symmetric case,

$$\varphi_k(n, p, \beta m_1, \beta^2 m_2, \beta^3 m_3, \ldots, \beta^k m_k) = \beta^{pk} \varphi_k(n, m_1, m_2, m_3, \ldots, m_k) \tag{2.11}$$

and

$$\Phi_k(t, \beta m_1, \beta^2 m_2, \ldots, \beta^k m_k) = \Phi_k(\beta^k t, m_1, m_2, \ldots, m_k). \tag{2.12}$$

Proof. Let $X_{ij}^* = \beta X_{ij}$, $m_r^* = \mathbb{E}\,(X_{ij}^*)^r = \beta^r m_r$, $A = (X_{ij}^*)_{n\times n}$, $U = (X_{ij}^*)_{n\times p}$. On one hand, by definition,

$$\mathbb{E}\,(\det A^*)^k = \varphi_k(n, m_1^*, \ldots, m_k^*) = \varphi_k(n, \beta m_1, \ldots, \beta^k m_k) \tag{2.13}$$
$$\mathbb{E}\,(\det U^{*\top} U^*)^{k/2} = \varphi_k(n, p, m_1^*, \ldots, m_k^*) = \varphi_k(n, p, \beta m_1, \ldots, \beta^k m_k). \tag{2.14}$$

On the other hand, by linearity of determinants,

$$\mathbb{E}\,(\det A^*)^k = \beta^{nk}\mathbb{E}\,(\det A)^k = \beta^{nk}\varphi_k(n, m_1, \ldots, m_k) \tag{2.15}$$
$$\mathbb{E}\,(\det U^{*\top}U^*)^{k/2} = \beta^{pk}\mathbb{E}\,(\det U^\top U)^{k/2} = \beta^{pk}\varphi_k(n, p, m_1, \ldots, m_k). \tag{2.16}$$

The assertion for the generating functions follows simply by plugging those results into Definition 39. ■

Corollary 45.1. *Assume we know $f_k(n)$ and $F_k(t)$ with $m_2 = 1$, that is*

$$f_k(n)|_{m_2=1} = \varphi_k(n, m_1, 1, m_3, m_4, \ldots, m_k), \tag{2.17}$$
$$F_k(t)|_{m_2=1} = \Phi_k(t, m_1, 1, m_3, m_4, \ldots, m_k), \tag{2.18}$$

then

$$f_k(n) = m_2^{nk/2}\varphi_k\left(n, \frac{m_1}{m_2^{1/2}}, 1, \frac{m_3}{m_2^{3/2}}, \frac{m_4}{m_2^{4/2}}, \ldots, \frac{m_k}{m_2^{k/2}}\right), \tag{2.19}$$
$$F_k(t) = \Phi_k\left(m_2^{k/2}t, \frac{m_1}{m_2^{1/2}}, 1, \frac{m_3}{m_2^{3/2}}, \frac{m_4}{m_2^{4/2}}, \ldots, \frac{m_k}{m_2^{k/2}}\right). \tag{2.20}$$

Similarly

$$f_k(n,p) = m_2^{pk/2}\varphi_k\left(n, \frac{m_1}{m_2^{1/2}}, 1, \frac{m_3}{m_2^{3/2}}, \frac{m_4}{m_2^{4/2}}, \ldots, \frac{m_k}{m_2^{k/2}}\right), \tag{2.21}$$
$$F_k(t,\omega) = \Phi_k\left(m_2^{k/2}t, \omega, \frac{m_1}{m_2^{1/2}}, 1, \frac{m_3}{m_2^{3/2}}, \frac{m_4}{m_2^{4/2}}, \ldots, \frac{m_k}{m_2^{k/2}}\right). \tag{2.22}$$

2.1.3 Permutations and derangements

Definition 46. Let P_n be the set of all permutations (that is, bijections) of order n on $[n] = \{1, 2, 3, \ldots, n\}$. An inversion is a permutation which only switches two elements. We define the sign $\operatorname{sgn}\pi$ of a permutation π to be the number of inversion necessary to get π from the identity. This definition is unambiguous.

A permutation can be represented in the *Cauchy notation*. It is well known that the permutation can be decomposed in cycles. This is becomes obvious by showing the same π in the previous example in its *cycle representation* of π. Both representations are shown on Figure 2.1.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & 1 & 9 & 4 & 7 & 2 & 5 & 8 \end{pmatrix}$$

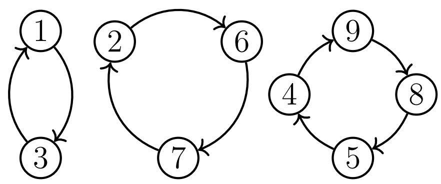


Figure 2.1: A permutation $\pi \in P_9$ being represented in (tabular) Cauchy notation (left panel) and the cycle notation (right panel)

Let $C(\pi)$ be the number of cycles which π decomposes to, then $\operatorname{sgn}\pi = (-1)^{n-C(\pi)}$, where n is the order of the permutation π. We can write this formula as the product over cycles. Let $\pi = \pi_1 \sqcup \pi_2 \sqcup \cdots \sqcup \pi_m$ be (disjoint) decomposition into cycles

$\pi_r, r = 1, \ldots, m$, where we denoted $m = C(\pi)$. Then

$$\operatorname{sgn} \pi = (-1)^n \prod_{r=1}^{m} (-1) = \prod_{r=1}^{m} (-1)^{|\pi_r|-1}, \tag{2.23}$$

where $|\pi_r|$ is the order of π_r (that is, the *length* of the cycle π_r). Note that, technically, π_r are not permutations since they are not bijections on $\{1, \ldots, |\pi_r|\}$, but rather on a subset of n. In fact, our permutation π from Figure 2.1 is also a special special case of another set of permutations called derangements.

Lemma 47. *Let D_n be the set of all* derangements *of order n on $[n]$. That is, D_n are permutations in P_n which have no fixed points (cycles of leghth one). If we let $C(\pi)$ denote the number of cycles in a permutation π and take $C_n(u) = \sum_{\pi \in D_n} u^{C(\pi)}$, then*

$$C_n(u) = (n-1)(C_{n-1}(u) + u\, C_{n-2}(u))$$

and

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} C_n(u) = \frac{e^{-ux}}{(1-x)^u}.$$

Proof. See the chapter on Bivariate generating functions in the textbook "Analytic Combinatorics" by Flajolet and Sedgewick [30]. For completeness, we present our own derivation. We proceed recursively based on the position of the node n in the cycle representation of π. We can create a derangement $\pi \in D_n$ by either:

1. Adding the node n to one of the cycles of a derangement $\pi' \in D_{n-1}$. That is, if $i \to \pi(i)$, then we insert n as $i \to n \to \pi(i)$. Since there are $n-1$ nodes in π', there are $n-1$ different $\pi \in D_n$ we can create. In this case, the number of cycles is unchanged, i.e., $C(\pi) = C(\pi')$.
2. Adding a cycle $(n, n-1)$ of length two to $\pi'' \in D_{n-2}$. We can then replace $n-1$ by any $i \in \pi''$. This gives $n-1$ new derangements $\pi \in D_n$ created from π'', all of them having $C(\pi) = C(\pi'') + 1$.

We can obtain all derangements D_n in this way. These two possibilities are shown in the figures below.

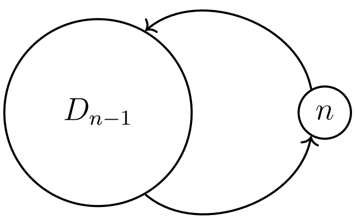


Figure 2.2: $D_{n-1} \to D_n$.

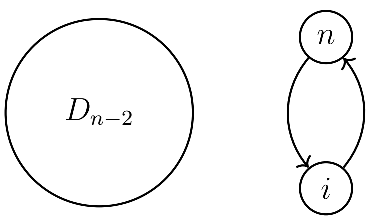


Figure 2.3: $D_{n-2} \to D_n$.

In terms of $C_n(u)$, we get the desired recurrence relation

$$\begin{aligned} C_n(u) &= \sum_{\pi \in D_n} u^{C(\pi)} = (n-1) \sum_{\pi' \in D_n} u^{C(\pi')} + (n-1) \sum_{\pi'' \in D_{n-2}} u^{C(\pi'')+1} \\ &= (n-1)(C_{n-1}(u) + u\, C_{n-2}(u)), \end{aligned}$$

from which one can deduce its generating function easily. ■

2.1.4 Analytic combinatorics

We follow the notation from the textbook *Analytic combinatorics* [30] by Flajolet and Sedgewick. Let $\mathcal{A}$ be a combinatorial structure with weight $w_{\mathcal{A}} : \mathcal{A} \to \mathbb{N}_0$. The structure is said to be labeled if any of its members $\alpha \in \mathcal{A}$ (an *object* of structure $\mathcal{A}$) is composed of atoms numbered by $[w_{\mathcal{A}}(\alpha)] = \{1, 2, 3, 4, \ldots, w_{\mathcal{A}}(\alpha)\}$. Moreover, we assume that $\mathcal{A}_n = \{\alpha \in \mathcal{A} \mid w_{\mathcal{A}}(\alpha) = n\}$ is finite for any natural $n \geq 0$. We also define $\hat{a}_n = |\mathcal{A}_n|$ as the number of objects with weight equal to n. Combinatorial structures can be composed together. If a structure $\mathcal{C}$ is composed out of structures $\mathcal{A}$ and $\mathcal{B}$, we can depict this dependency in a form of a *structural relation* (or *structural equation*)

$$\mathcal{C} = \Phi(\mathcal{A}, \mathcal{B}) \tag{2.24}$$

One common composition of labeled structures is the *star product*. Note that a tuple (α, β) cannot represent a labeled object of any structure, since the atoms of α and β are labeled by $[w_{\mathcal{A}}(\alpha)]$ and $[w_{\mathcal{B}}(\beta)]$, respectively. Relabeling our α, β into α', β', so every number from 1 to $w_{\mathcal{A}}(\alpha) + w_{\mathcal{B}}(\beta)$ appears once, we get a correctly labeled object. There are of course many ways how to re-label the objects. The canonical way is to use the *star product*. We say $(\alpha', \beta') \in \alpha \star \beta$ if the new labels in both α' and β' increase in the same order as in α and β separately. An example is illustrated below in Figure 2.4.

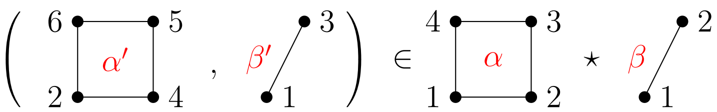


Figure 2.4: Star product

The key concept in labeled structures is their *generating function* (EGF for short) defined as

$$\hat{A}(t) = \sum_{\alpha \in \mathcal{A}} \frac{t^{w(\alpha)}}{w_{\mathcal{A}}(\alpha)!} = \sum_{n=0}^{\infty} \hat{a}_n \frac{t^n}{n!}. \tag{2.25}$$

Generating functions encode the relation between combinatorial structures (i.e. how are they composed). In general, there is often a relation in the form

$$\hat{C}(t) = \phi(\hat{A}(t), \hat{B}(t)) \tag{2.26}$$

for some function (or an operator) ϕ. The following Table 2.1 enlists the most common constructions.

Let us make some further comments of the constructions in the table above.

- $\textsc{Seq}_k(\mathcal{A})$ is a shorthand for a *sequence* and indeed it can be represented as (re-labeled) k-tuples of objects taken from $\mathcal{A}$. Note that since everything is re-labeled, even though α_i, α_j might be the same for different i, j, the corresponding α'_i, α'_j are always distinct. Formally, $\textsc{Seq}_k(\mathcal{A}) = \{(\alpha'_1, \ldots, \alpha'_k) \mid \alpha_i \in \mathcal{A}, i \in [k]\}$, where $(\alpha'_1, \ldots, \alpha'_k) \in \alpha_1 \star \cdots \star \alpha_k$.

$\mathcal{C}$	representation of $\mathcal{C}$	$w_{\mathcal{C}}(\gamma), \gamma \in \mathcal{C}$	$\hat{C}(t)$
$\mathcal{A}+\mathcal{B}$	$\mathcal{A} \cup \mathcal{B}$	$\begin{cases} w_{\mathcal{A}}(\gamma) & \text{if } \gamma \in \mathcal{A}, \\ w_{\mathcal{B}}(\gamma) & \text{if } \gamma \in \mathcal{B} \end{cases}$	$\hat{A}(t)+\hat{B}(t)$
$\mathcal{A} \star \mathcal{B}$	$\{\gamma \mid \gamma \in \alpha \star \beta, \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$	$w_{\mathcal{A}}(\alpha) w_{\mathcal{B}}(\beta)$	$\hat{A}(t)\hat{B}(t)$
$\textsc{Seq}_k(\mathcal{A})$	$\mathcal{A}^k \stackrel{\text{def.}}{=} \underbrace{\mathcal{A} \star \mathcal{A} \star \cdots \star \mathcal{A}}_{k}$		$\hat{A}^k(t)$
$\textsc{Set}_k(\mathcal{A})$	$\frac{1}{k!}\mathcal{A}^k$		$\frac{1}{k!}\hat{A}^k(t)$
$\textsc{Cyc}_k(\mathcal{A})$	$\frac{1}{k}\mathcal{A}^k$		$\frac{1}{k}\hat{A}^k(t)$
$\textsc{Seq}(\mathcal{A})$	$\sum_{k=0}^{\infty} \textsc{Seq}_k(\mathcal{A})$		$\frac{1}{1-\hat{A}(t)}$
$\textsc{Set}(\mathcal{A})$	$\sum_{k=0}^{\infty} \textsc{Set}_k(\mathcal{A})$		$\exp \hat{A}(t)$
$\textsc{Cyc}(\mathcal{A})$	$\sum_{k=1}^{\infty} \textsc{Cyc}_k(\mathcal{A})$		$\ln \frac{1}{1-\hat{A}(t)}$

Table 2.1: Composition of combinatorial structures and the corresponding exponential generating functions

- $\textsc{Set}_k(\mathcal{A})$ is a structure of *sets* of k relabeled elements, that is, the order of objects α_i' is irrelevant. Formally, $\textsc{Set}_k(\mathcal{A}) = \{\{\alpha_1', \ldots, \alpha_k'\} \mid \alpha_i \in \mathcal{A}, i \in [k]\}$. Alternatively, $\textsc{Set}_k(\mathcal{A})$ can be represented as the structure of classes of k-tuples in $\textsc{Seq}_k(\mathcal{A})$ which differ up to some permutation.
- $\textsc{Cyc}_k(\mathcal{A})$ represents the structure of classes of k-tuples in $\textsc{Seq}_k(\mathcal{A})$ which differ up to some cyclical permutation.

For completeness, we briefly explain these results. To see that the exponential generating function for $\mathcal{A} \star \mathcal{B}$ is $\hat{A}(t)\hat{B}(t)$, let $\hat{a}_n$, $\hat{b}_n$, and $\hat{c}_n$ be the number of elements of weight n in $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A} \star \mathcal{B}$. We have that

$$\hat{c}_n = \sum_{j=0}^{n} \binom{n}{j} \hat{a}_j \hat{b}_{n-j}$$

so

$$\hat{C}(t) = \sum_{n=0}^{\infty} \frac{\hat{c}_n t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{\hat{a}_j t^j}{j!} \cdot \frac{\hat{b}_{n-j} t^{n-j}}{(n-j)!} = \hat{A}(t)\hat{B}(t)$$

The generating functions for $\textsc{Seq}(\mathcal{A})$, $\textsc{Set}(\mathcal{A})$, and $\textsc{Cyc}(\mathcal{A})$ come from the Taylor series $\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k$, $e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$ and $\ln\left(\frac{1}{1-t}\right) = \sum_{k=1}^{\infty} \frac{x^k}{k}$.

Tagging

Often, we asign a parameter (real or complex) when a given combinatorial substructure appears in a more general construction. Let us say that each time a substructure $\alpha \in \mathcal{A}$ appears, we multiply the weight by $\mu_{\mathcal{A}}$. The generating function for a labeled combinatorial structure $\mathcal{A}_\mu = \mu_{\mathcal{A}} \times \mathcal{A}$ is then $\hat{A}_\mu(t) = \mu_{\mathcal{A}} \hat{A}(t)$. Similarly, let us attach a parameter $\mu_{\mathcal{B}}$ to a substructure $\mathcal{B}$ and let us create the following construction

$$\mathcal{C}_\mu = \Phi(\mu_\mathcal{A} \times \mathcal{A}, \mu_\mathcal{B} \times \mathcal{B}). \tag{2.27}$$

In terms of EGF's, this translates to

$$\hat{C}(t) = \phi(\mu_\mathcal{A} \hat{A}(t), \mu_\mathcal{B} \hat{B}(t)). \tag{2.28}$$

Example 48. Let us denote $\mathcal{D}$ as the combinatorial structure of all derangements. Any derangement can be decomposed into cycles of length at least two. Attaching a tag u to each cycle, we get a structure $\mathcal{D}_u$, which can be also constructed as follows

$$\mathcal{D}_u = \textsc{Set}\left(u\,\textsc{Cyc}_{\geq 2}\left(\textcircled{1}\right)\right) \tag{2.29}$$

and thus immediately in terms of generating functions,

$$\hat{D}_u(t) = \exp(-ut - u\ln(1-t)) = \frac{e^{-ut}}{(1-t)^u}. \tag{2.30}$$

This is an alternative proof of Lemma 47.

2.2 Permutation tables

We can express the value of $f_k(n) = \mathbb{E}\,(\det A)^k$ as a sum of terms over *permutation tables*. Permutation tables are well known and have been used to find older results [50][24]. However, they have never been used as a tool, but merely as a visualisation. The main ingredient how to obtain random determinant moments thus still remained a plain recursion. Although recursions are versatile (the author of this thesis found $F_4(t)$ and $F_4(t,\omega)$ in his original work [8] by recursions only), they have a major disadvantage – they hide the underlining structure. Simply because there are so many of them and they are connected nontrivialy (see Figure 1 in [8]). After finding $F_4(t)$ and $F_4(t,\omega)$, the author begun a collaboration with Zelin Lv and Aaron Potechin, who found $f_6^{\mathrm{sym}}(n)$ earlier the same year. Together, we are able to deduce a slight generalisation, namely $f_6^{\mathrm{cen}}(n)$, using a clever handling of generation functions. In our paper [5], the bijection between random determinants and permutation tables is used extensively. For the first time, the overall structure of permutation tables played a crucial role in obtaining the moment of a random determinant. Nevertheless, the paper still relied only on recursions (coupled with exclusion/inclusion technique), which again made the derivation incredibly technical and hided some crucial insights. It was only later after publishing our work that we realised that each block of our final generating functions $F_4(t)$ and $F_6^{\mathrm{cen}}(t)$ have a concrete combinatorial meaning. Rather then top to bottom, the natural question is how we can build our permutation tables from bottom up from generating functions. The new approach is thus to view permutation tables as a standalone combinatorial structure on which we can perform analytic calculations in the spirit of Flajolet and Sedgewick [30]. This is the method of permutation tables in its present form as it arose from the collaboration with the aforementioned authors. We will show how the method works in the remaining sections.

Definition 49. We say τ is a k by n permutation table, if its rows are permutations $\pi_j, j = 1, \ldots, k$ of order n. We denote $F_{k,n}$ the set of all such tables.

Definition 50. We define the sign of a table $\tau \in F_{k,n}$ as the product of signs of permutations which form its rows.

Definition 51. We define the weight of the i-th column of $\tau \in F_{k,n}$ as the expectation $\mathbb{E}\prod_{j=1}^{k} X_{i\pi_j(i)}$ Then, we define the weight $w(\tau)$ of the whole table τ as the product of weights of its individual columns.

Definition 52. Finally, we denote F_k as the set of all permutation tables with k rows with the above weights and signs. That is, structurally$\mathcal{F}_k$,

$$F_k = \bigcup_{n=0}^{\infty} F_{k,n}. \tag{2.31}$$

Example 53. The following example in Figure 2.5 shows a permutation table $\tau \in F_{4,9}$ with weight $w(\tau) = m_1^{12}m_2^7m_3^2m_4$. Weight of each individual column is shown below each column. For instance, the second column corresponds to term $X_{26}X_{22}X_{26}X_{23}$, whose expectation is obviously $m_1^2m_2$ since $\mathbb{E}X_{26}^2 = m_2$ and $\mathbb{E}X_{22} = \mathbb{E}X_{23} = m_1$.

1	6	3	9	5	2	7	8	4	+
3	2	1	9	4	6	7	5	8	+
4	6	1	9	3	2	7	5	8	+
2	3	1	5	4	6	7	8	9	−
m_1^4	$m_1^2m_2$	m_1m_3	m_1m_3	$m_1^2m_2$	m_2^2	m_4	m_2^2	$m_1^2m_2$	

Figure 2.5: A permutation table $\tau \in F_{4,9}$ with $w(\tau) = m_1^{12}m_2^7m_3^2m_4$ and $\operatorname{sgn}\tau = -1$.

Proposition 54. *For any distribution X_{ij},*

$$f_k(n) = \mathbb{E}\,(\det A)^k = \sum_{\tau\in F_{k,n}} w(\tau)\operatorname{sgn}(\tau). \tag{2.32}$$

Proof. Follows directly from the expansion $\det A = \sum_{\pi\in P_n}\operatorname{sgn}\pi\prod_{i\in[n]}X_{i\pi(i)}$ raised to k-th power and by taking expectation. ∎

Example 55. The correspondence between $f_k(n)$ and permutation tables is shown below in Figure 2.6 for $n = 2$ and $k = 2$ showing $f_2(2) = 2(m_2^2 - m_1^4) = 2!(m_2 + m_1^2)(m_2 - m_1^2)$ by summing the contributions from all permutation tables.

$(\det A)^2$	$= X_{11}^2X_{22}^2$	$- X_{11}X_{22}X_{12}X_{21}$	$- X_{12}X_{21}X_{11}X_{22}$	$+ X_{12}^2X_{21}^2$
$F_{2,2}$:	**1 2** / **1 2**	**1 2** / 2 1	2 1 / **1 2**	2 1 / 2 1
Weight:	m_2m_2	$m_1^2m_1^2$	$m_1^2m_1^2$	m_2m_2
Sign:	+	−	−	+

Figure 2.6: Correspondence between determinant moment $f_2(2)$ and permutation tables $F_{2,2}$

2.2.1 Exponential generating function and analytic combinatorial nature of permutation tables

Note that our generating functions (see Definition 39) are *double exponential*, meaning they have a factor $n!$ squared in the denominator, where the usual exponential generating function (EGF) has only first power of $n!$ in the denominator. It turns out there is a simple way how the tables so their generating functions are EGF. In order to achieve this, we relax the assumptions of knowing the order of the columns.

Definition 56. Let $A_{k,n}$ be some set of $k \times n$ nontrivial tables with usual weight defined as a product of weights of its columns. We denote $\mathcal{A}_{k,n}$ as tables formed from $A_{k,n}$ in which the order of the columns is irrelevant. That is, $A_{k,n}$ is split into equivalent classes of tables which differ only by some permutation of columns. The set $\mathcal{A}_{k,n}$ can be then viewed as the set of representats (one per each class). Or equivalently, in $\mathcal{A}_{k,n}$, tables which differ up to permutation of columns are treated as the same table. Accordingly, we define

$$a_k(n) = \sum_{\tau \in A_{k,n}} w(\tau) \operatorname{sgn} \tau, \qquad \hat{a}_k(n) = \sum_{\tau \in \mathcal{A}_{k,n}} w(\tau) \operatorname{sgn} \tau$$

and their corresponding generating functions

$$A_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} a_k(n), \qquad \hat{A}_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \hat{a}_k(n). \tag{2.33}$$

Lemma 57. *Let $a_k(n)$, $\hat{a}_k(n)$, $A_k(t)$ and $\hat{A}_k(t)$ be defined as above, then*

$$a_k(n) = n!\hat{a}_k(n) \qquad \text{and} \qquad \hat{A}_k(t) = A_k(t). \tag{2.34}$$

Proof. Let $\tau \in A_{k,n}$. Since k is even (otherwise $a_k(n)$ is zero), permutation of columns of τ does not change the sign nor weight of τ. Select one representant τ' from each class of tables whose columns differ only by permutation of columns. Since there are $n!$ ways how we can arrange the columns,

$$a_k(n) = \sum_{\tau \in A_{k,n}} w(\tau) \operatorname{sgn} \tau = \sum_{\tau' \in \mathcal{A}_{k,n}} n! w(\tau) \operatorname{sgn} \tau = n!\hat{a}_k(n), \tag{2.35}$$

from which immediately

$$A_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} a_k(n) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \hat{a}_k(n) = \hat{A}_k(t). \tag{2.36}$$

■

Definition 58. Finally, we denote $\mathcal{A}_k$ as the combinatorial structure of all permutation tables with k rows with the above weights and signs (whose column order is irrelevant). That is, structurally,

$$\mathcal{A}_k = \sum_{n=0}^{\infty} \mathcal{A}_{k,n}. \tag{2.37}$$

Definition 59 (Tables $\mathcal{F}_k$)**.** According to the above definition, we write $\mathcal{F}_{k,n}$ for the set of all permutation tables with k rows and n columns with irrelevant order. Similarly, $\mathcal{F}_k = \sum_{n=0}^{\infty} \mathcal{F}_{k,n}$ is the combinatorial structure of all such tables regardless of the number of columns (including zero columns).

Example 60. Let us compute $f_2(3)$. We may write $f_2(3) = 3!\hat{f}_2(3)$, where $\hat{f}_2(3) = \sum_{\tau \in \mathcal{F}_{2,3}} w(\tau) \operatorname{sgn} \tau$. Figure 2.7 enlists all elements of $\mathcal{F}_{2,3}$ and shows their weights and signs. Summing the contribution, we get $\hat{f}_2(3) = m_2^3 - 3m_2m_1^4 + 2m_1^6 = (m_2 + 2m_1^2)(m_2 - m_1^2)^2$ and thus $f_2(3) = 3!(m_2 + 2m_1^2)(m_2 - m_1^2)^2$.

$\mathcal{F}_{2,3}$:	1 2 3 / 1 2 3	1 2 3 / 1 3 2	1 2 3 / 3 2 1	1 2 3 / 2 1 3	1 2 3 / 3 1 2	1 2 3 / 2 3 1
Weight:	$m_2m_2m_2$	$m_2m_1^2m_1^2$	$m_1^2m_2m_1^2$	$m_1^2m_1^2m_2$	$m_1^2m_1^2m_1^2$	$m_1^2m_1^2m_1^2$
Sign:	$+$	$-$	$-$	$-$	$+$	$+$

Figure 2.7: Correspondence between $\hat{f}_2(3)$ and permutation tables $\mathcal{F}_{2,3}$

Remark 61. Since the order of the columns in any $\tau \in \mathcal{F}_k$ is irrelevant by definition, we often sort them by the first permutation (the first row in the given table τ).

Sub-table factorization

Let $n(\tau)$ denote the number of columns (which is the same as the number of elements) of a table $\tau \in \mathcal{A}_k$. Using this definition, we can write the exponential generating function from Equation (2.33) more compactly as

$$A_k(t) = \sum_{\tau \in \mathcal{A}_k} \frac{t^{n(\tau)}}{n(\tau)!} w(\tau) \operatorname{sgn} \tau, \tag{2.38}$$

Any table τ can be viewed as being build up by smaller constituents. Those constituents are *sub-tables*, which we define as the smallest subsets of columns not sharing any element which cannot be further divided. The following proposition underlines the property of tables with irrelevant column order $\mathcal{A}_k$ being a combinatorial structure with the usual property of the star product, namely that the (exponential) generating functions *factorise over sub-tables*.

Definition 62. Let τ be a $\mathcal{A}_k$ table. We denote $EGF[\tau]$ as the contribution of τ to $A_k(t)$ (Equation (2.38)). By definition,

$$EGF[\tau] = \frac{t^{n(\tau)}}{n(\tau)!} w(\tau) \operatorname{sgn} \tau. \tag{2.39}$$

Similarly, if $\mathcal{B} \subset \mathcal{A}_k$ is a subset of tables from $\mathcal{A}_k$, then $EGF[\mathcal{B}] = \sum_{\tau \in \mathcal{B}} EGF[\tau]$.

Proposition 63 (sub-table factorization)**.** *Let τ be a $\mathcal{A}_k$ table build up from exactly two disjoint sub-tables τ_1 and τ_2, then*

$$EGF[\tau_1 \star \tau_2] = EGF[\tau_1]EGF[\tau_2]. \tag{2.40}$$

Proof. Let n_1 and n_2 be the number of columns of τ_1 and τ_2, respectively. We also denote $n = n_1 + n_2$ as the total number of columns of τ (See Figure 2.8).

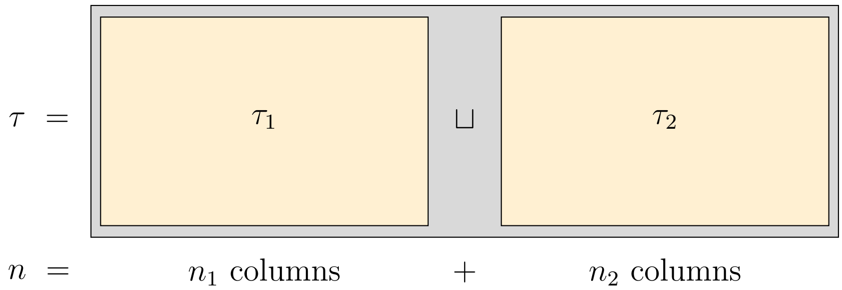


Figure 2.8: Table τ consisted of two dis-joint sub-tables τ_1 and τ_2.

Any τ from the set $\tau_1 \star \tau_2$ (with elements shuffled) gives the same contribution to $A_k(t)$. Since there are $\binom{n}{n_1}$ ways how can we select elements for τ_1 and τ_2,

$$EGF[\tau_1 \star \tau_2] = \binom{n}{n_1} EGF[\tau] = \binom{n}{n_1} \frac{t^n}{n!} w(\tau) \operatorname{sgn} \tau = \frac{t^n}{n_1! n_2!} w(\tau) \operatorname{sgn} \tau. \tag{2.41}$$

On the other hand, we already know that both weight and sign factorises over sub-tables, that is $w(\tau) \operatorname{sgn} \tau = w(\tau_1) w(\tau_2) \operatorname{sgn} \tau_1 \operatorname{sgn} \tau_2$. Hence,

$$EGF[\tau_1]EGF[\tau_2] = \frac{t^{n_1}}{n_1!} w(\tau_1) \operatorname{sgn} \tau_1 \frac{t^{n_2}}{n_2!} w(\tau_2) \operatorname{sgn} \tau_2 = \frac{t^{n_1+n_2}}{n_1! n_2!} w(\tau) \operatorname{sgn} \tau, \tag{2.42}$$

which concludes the proof. ■

2.2.2 Highest moment recursion formulae

The following statement and its proof due to Prékopa [57] enables us to replace m_k with any arbitrary value and still not loose any generality:

Proposition 64.

$$\frac{\partial F_k(t)}{\partial m_k} = t F_k(t), \tag{2.43}$$

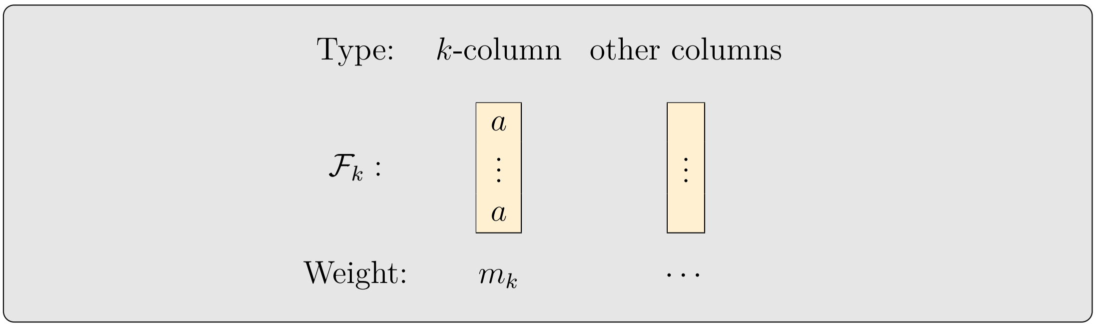
from which, for $m_k^* \in \mathbb{R}$ arbitrary,

$$F_k(t) = e^{(m_k - m_k^*)t} (F_k(t)|_{m_k \to m_k^*}), \tag{2.44}$$

or equivalently

$$\Phi_k(t, m_1, \ldots, m_{k-1}, m_k) = e^{(m_k - m_k^*)t} \Phi_k(t, m_1, \ldots, m_{k-1}, m_k^*) \tag{2.45}$$

Proof. For each factor of m_k, there must be one column filled with the same elements in a given permutation table (a k-column). The columns of $\mathcal{F}_k$ can be depicted in the diagram below.



Crucially, these k-columns are disjoint from the rest of the table. If we denote $\tilde{\mathcal{F}}_k$ as the structure of tables not containing k-columns, we can write the following structural equation

$$\mathcal{F}_k = \textsc{Set}\left(\boxed{\begin{matrix} 1 \\ \vdots \\ 1 \end{matrix}} \right) \star \tilde{\mathcal{F}}_k. \tag{2.46}$$

Note that the star product arises naturally, since it handles relabeling. The first term in the star product is precisely the structure of k-columns with EGF equal to $\exp(m_k t)$, where $m_k t$ is the EGF of a single k-column. For the second term, we have for its EGF that $\tilde{F}_k(t) = F_k(t)|_{m_k \to 0}$, since we can erase the k-columns by setting $m_k = 0$. Combining those generating functions together,

$$F_k(t) = e^{m_k t}(F_k(t)|_{m_k \to 0}), \tag{2.47}$$

which is equivalent to the assertion of the proposition. ■

Dembo (Lemma 2 in [24]) showed that a similar result holds also for $F_k(t, \omega)$, namely

Proposition 65.

$$\frac{\partial F_k(t, \omega)}{\partial m_k} = t F_k(t, \omega), \tag{2.48}$$

from which, for $m_k^* \in \mathbb{R}$ arbitrary,

$$F_k(t, \omega) = e^{(m_k - m_k^*)t}(F_k(t, \omega)|_{m_k \to m_k^*}), \tag{2.49}$$

or equivalently

$$\Phi_k(t, \omega, m_1, \ldots, m_{k-1}, m_k) = e^{(m_k - m_k^*)t}\Phi_k(t, \omega, m_1, \ldots, m_{k-1}, m_k^*) \tag{2.50}$$

> *Remark* 66. There is a general pattern found throughout the thesis. Namely, if a table is composed of two disjoint sub-tables, its EGF is a product of those EGF's for the two sub-tables. This is because not only weights decompose into product over sub-tables, but also the signs (as shown in Proposition 63).

2.2.3 Second moment general

The following formula for $f_2(n)$ was derived by Fortet [32] as a special case of a more general setting by recursions, although it could be derived in a much more elementary way [68]. In this thesis, however, we shall prove this formula using permutation tables and their corresponding combinatorial constructions.

Proposition 67 (Fortet [32])**.** *For any distribution of* X_{ij}*,*

$$f_2(n) = n!(m_2 + m_1^2(n-1))(m_2 - m_1^2)^{n-1}, \tag{2.51}$$

$$F_2(t) = (1 + m_1^2 t)e^{(m_2 - m_1^2)t}. \tag{2.52}$$

Proof. In order to deduce $f_2(n)$, we can add up weights and signs in tables $\mathcal{F}_2$. Let us examine their structure. Let a, b be different integers, then there are two types of columns in $\mathcal{F}_2$ (see the diagram below).

Type:	2-column	1-column
$\mathcal{F}_2$:	$\begin{matrix} a \\ a \end{matrix}$	$\begin{matrix} a \\ b \end{matrix}$
Weight:	m_2	m_1^2

A crucial observation is as follows: Each table $\tau \in \mathcal{F}_2$ can be decomposed into *sub-tables* (disjoint sets of columns). This should be obvious since any τ can be associated with a corresponding permutation π in the Cauchy notation. The fixed points of π correspond to 2-columns and the cycles of π are created by connecting the 1-columns (first row to second row). An example of $\tau \in \mathcal{F}_2$ with its corresponding permutation is shown in Figures 2.9 and 2.10 – since the order of columns in τ is irrelevant, we grouped the columns into sub-tables right away.

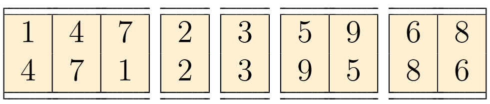


Figure 2.9: $\tau \in \mathcal{F}_{2,9}$ with two 2-columns and seven 1-columns (5 sub-tables)

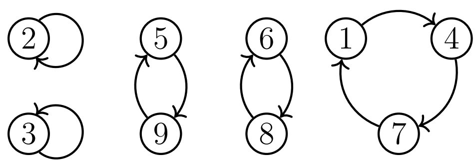


Figure 2.10: The corresponding permutation $\pi \in F_9$ for table τ

As a consequence, we can write down a structural equation for the structure of all 2-tables $\mathcal{F}_2$ as follows

$$\mathcal{F}_2 = \textsc{Set}\left(\boxed{\begin{matrix}1\\1\end{matrix}}\right) \star \textsc{Set}\left(-\textsc{Cyc}_{\geq 2}\left(-m_1^2 \textcircled{1}\right)\right). \tag{2.53}$$

That is, the second term in the star product is exactly the structure $\tilde{\mathcal{F}}_2$. Based on analytical combinatorics, we immediately get in terms of generating functions,

$$F_2(t) = \exp(m_2 t)\exp(-m_1^2 t + \ln(1 + m_1^2 t)) = (1 + m_1^2 t)e^{(m_2 - m_1^2)t}. \tag{2.54}$$

This concludes the proof. By using Taylor expansion, we immediately recover also $f_2(n)$. For completeness, let us discuss how the signs are handled in the EGF of $\tilde{\mathcal{F}}_2$. Those tables are decomposable into sub-tables, each sub-table of τ is identified with a corresponding cycle. Each cycle π of length n has the sign equal to $(-1)^{n-1}$ and this must be the same sign of the sub-table of τ of the same size (n columns). We can therefore write for the EGF of all tables composed of one cycle only

$$-\left(m_1^2(-t) + m_1^4\frac{(-t)^2}{2!} + m_1^6\frac{(-t)^3}{3!} + m_1^8\frac{(-t)^4}{4!} + \cdots\right) = \ln(1 + m_1^2 t). \tag{2.55}$$

Note that the power of the minus sign at t^n is exactly $(-1)^{n-1}$ as it should be. Finally, since the cycle of length one is impossible (a single 1-column can never be disjoint), we have to subtract the first term in the series expansion above.

■

2.2.4 Fourth moment central

Note that when $m_1 = 0$, the number of tables with nontrivial weight is reduced significantly. As a consequence, we can easily derive the result of Nyquist, Rice and Riordan [50], namely $F_4^{\mathrm{sym}}(t)$ and the corresponding $f_4^{\mathrm{sym}}(n)$.

Proposition 68 (Nyquist, Rice and Riordan [50])**.**

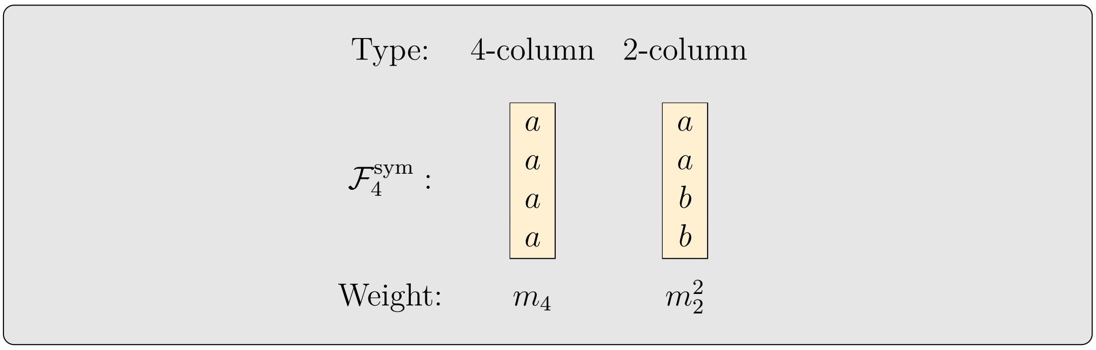
$$F_4^{\mathrm{sym}}(t) = \frac{e^{t(m_4 - 3m_2^2)}}{(1 - m_2^2 t)^3}, \tag{2.56}$$

Corollary 68.1.

$$f_4^{\mathrm{sym}}(n) = (n!)^2 m_2^{2n} \sum_{j=0}^{n} \frac{1}{j!}\left(\frac{m_4}{m_2^2} - 3\right)^j \binom{n-j+2}{2}. \tag{2.57}$$

Remark 69. In fact, those formulae hold even if X_{ij}'s follow just a centered distribution. That is, $f_4^{\mathrm{cen}}(n) = f_4^{\mathrm{sym}}(n)$ and $F_4^{\mathrm{cen}}(t) = F_4^{\mathrm{sym}}(t)$. This is due to the fact that m_3 appears always as a product $m_1 m_3$ in the $f_4(n)$ polynomial.

Proof of Proposition 68. Let a, b be different integers, then there are two types of columns we need to consider which give rise to tables with nontrivial weights (see the diagram below). It is convenient to denote those tables as F_4^{sym} (or $\mathcal{F}_4^{\mathrm{sym}}$ if we do not care about the order of columns).



By definition

$$f_4^{\text{sym}}(n) = n!\hat{f}_4^{\text{sym}}(n) = n! \sum_{\tau \in \mathcal{F}_{4,n}^{\text{sym}}} w(\tau) \operatorname{sgn} \tau. \tag{2.58}$$

Again, we can write down a structural equation for the 4-tables $\mathcal{F}_4^{\text{sym}}$ as follows

$$\mathcal{F}_4^{\text{sym}} = \textsc{Set}\left(\begin{array}{|c|}\hline 1\\ 1\\ 1\\ 1\\ \hline\end{array}\right) \star \mathcal{Q}_4. \tag{2.59}$$

The first part of the star product corresponds to tables with 4-columns only. Its EGF is equal to $\exp(m_4 t)$. The second term $\mathcal{Q}_4$ denote the tables with 2-columns only. Let us further examine how we can construct the latter tables using disjoint sub-tables. Let b be a number in the first row of a given column of table $\tau \in \mathcal{Q}_4$. Since it is a 2-column, we denote the other number in the column as b'. We construct a permutation π to a given table τ as composed from all those pairs $b \to b'$. Note that since b and b' are always different, the set off all admissible permutations corresponds to the set of all **derangements**. On top of that, since the first row of τ can be assumed to be fixed to identity (we simply reorder the columns), there are 3 possibilities how to arrange the leftover numbers in the 2-columns of a given cycle of π as the number in the first row of each 2-column can reappear either in the second, third or in the fourth row. For each possibility, we draw a vertical box with four slots filled with two dots representing in which rows the number in the first row appears (see Figure 2.11).

1	3	2	6	7	4	5	8	9
1	3	6	7	2	9	4	5	8
3	1	2	6	7	9	4	5	8
3	1	6	7	2	4	5	8	9

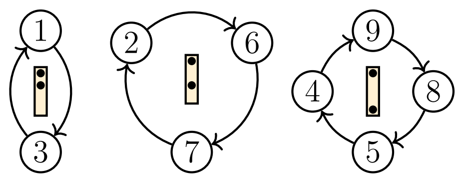


Figure 2.11: One-to-one correspondence between a table τ in $\mathcal{Q}_{4,9}$ with nine 2-columns decomposable into three disjoint sub-tables, and its associated derangement π with cycles labeled according to the repetitions of the number in the first row of τ

Any derangement can be decomposed into cycles of length at least two. Those cycles correspond to disjoint sub-tables of τ. Since each permutation appears twice in any sub-table, the sign of those sub-tables is always positive. Hence,

$$\mathcal{Q}_4 = \textsc{Set}\left(3\,\textsc{Cyc}_{\geq 2}\left(m_2^2 \,\textcircled{1}\right)\right) \tag{2.60}$$

and thus immediately in terms of generating functions,

$$F_4^{\text{sym}}(t) = \exp(m_4 t)\exp(-3m_2^2 t - \ln(1 - m_2^2 t)) = \frac{e^{(m_4 - 3m_2^2)t}}{(1 - 3m_2^2 t)^3}. \tag{2.61}$$

This concludes the proof. By using Taylor expansion, we immediatelly recover also $f_4^{\text{sym}}(n)$. ■

Remark 70. By considering all tables, one could theoretically tackle also the case $m_1 \neq 0$. However, this approach is rather ineffective since it turns out the problem drastically simplifies if we shift the random variables X_{ij} by their first moment m_1, as we will see later on in the section on marked permutation tables (Section 2.4).

2.2.5 Normal moments

In the case of X_{ij} being normally distributed, we know much more. In fact we know all determinant moments (even for the Gram case as we will see later). For now, we focus only on the special case of the standard normal distribution $X_{ij} \sim \mathsf{N}(0,1)$.

Definition 71. If $X_{ij} \sim \mathsf{N}(0,1)$, we denote $f_k(n)$ as $n_k(n)$ and $F_k(t)$ as $N_k(t)$.

Proposition 72 (Prékopa 1967)**.** *For any even* $k = 2m$*,*

$$n_{2m}(n) = \prod_{r=0}^{m-1} \frac{(n+2r)!}{(2r)!}. \tag{2.62}$$

For now, we take this proposition as granted. It was first derived in this form by Nyquist, Rice and Riordan (Equation (3.12) in [50], their treatment even covers the case of arbitrary complex moment), although much more elementary derivation of this result was later given by Prékopa (Section 3 in [57]). Both proofs rely on a deep connection of random determinants with volumetric moments of random polytopes (see Chapter 6). The proposition is also a special case of Lemma 143 with $\mu = 0$.

Fourth normal moment

When $k = 4$, we get $n_4(n) = n!(n+2)!/2$ and thus

$$N_4(t) = \frac{1}{2}\sum_{n=0}^{\infty}(n+2)(n+1)\,t^n = \frac{1}{(1-t)^3}. \tag{2.63}$$

Alternatively, we can deduce $N_4(t)$ independently from Proposition 72 by using the general formula for $F_4^{\mathrm{sym}}(t)$ (Proposition 68), since the standard normal distribution is modeled by plugging its moments $m_2 = 1$ and $m_4 = 3$ into $F_4^{\mathrm{sym}}(t)$.

Sixth normal moment

When $k = 6$, the function

$$N_6(t) = \frac{1}{48}\sum_{n=0}^{\infty}(n+1)(n+2)(n+4)!\,t^n. \tag{2.64}$$

is no longer analytic. In fact, it diverges everywhere except $t = 0$, however, we can still treat it formally. Note that in this case, there also exists a fully combinatorial proof (independent from Proposition 72) due to Potechin and Lv (see Appendix A of [5]).

2.2.6 Sixth moment central

The proof of the following theorem was already established by B., Potechin and Lv [5]. In this section, we provide a more compact version of the proof, based on inclusion/exclusion, analytic combinatorics and the fact we know the EGF for the special case X_{ij} being normally distributed.

Theorem 73 (B., Lv, Potechin 2023)**.** *For any central distribution of X_{ij},*

$$F_6^{\text{cen}}(t) = (1+m_3^2t)^{10}\,\frac{e^{t\left(m_6-15m_4m_2-10m_3^2+30m_2^3\right)}}{\left(1+3m_2^3t-m_4m_2t\right)^{15}}N_6\left(\frac{m_2^3t}{\left(1+3m_2^3t-m_4m_2t\right)^3}\right).$$

Furthermore, via Taylor expansion,

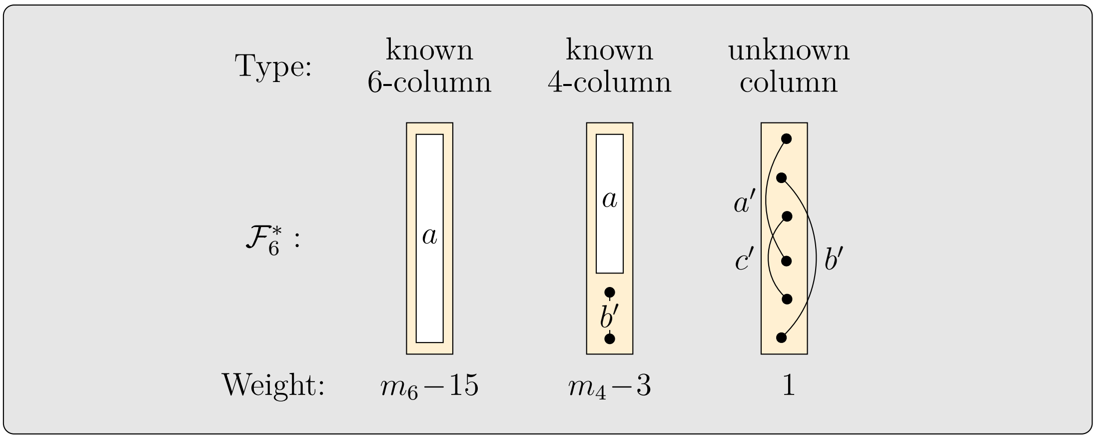
$$f_6^{\text{cen}}(n)=(n!)^2m_2^{3n}\sum_{j=0}^{n}\sum_{i=0}^{j}\sum_{k=0}^{n-j}\frac{(1+i)(2+i)(4+i)!}{48(n-j-k)!}\binom{10}{k}\binom{14+j+2i}{j-i}q_6^{n-j-k}q_4^{j-i}q_3^{k},$$

where $q_6=\dfrac{m_6}{m_2^3}-10\dfrac{m_3^2}{m_2^3}-15\dfrac{m_4}{m_2^2}+30,\qquad q_4=\dfrac{m_4}{m_2^2}-3,\qquad q_3=\dfrac{m_3^2}{m_2^3}.$

Proof. Without the loss of generality, we assume $m_2=1$ throughout the proof. The fact we have $m_1=0$ reduces the number of tables with nontrivial weight. It is convenient to denote $\mathcal{F}_6^{\text{cen}}$ as the set of those tables (irrelevant column order), which in turn contribute to the sum $f_6^{\text{cen}}(n)$. These tables can be constructed out of the following columns (apart from permutation of rows):

Type:	6-column	4-column	2-column	3-column
$\mathcal{F}_6^{\text{cen}}$:	a a a a a a	a a a a b b	a a b b c c	a a a b b b
Weight:	m_6	m_4	1	m_3^2

First, we examine the special case when also $m_3=0$. It is convenient to denote $\mathcal{F}_6^{\text{sym}}\subseteq\mathcal{F}_6^{\text{cen}}$ as the set of tables which contribute only to the sum $f_6^{\text{sym}}(n)$. These are precisely those tables which are composed out of 6-, 4- and 2-columns only. In order to utilise inclusion/exclusion, we further divide the columns into two types *known* and *unknown*. An unknown column is a column where the numbers are, in addition, paired up (only the same ones). Let a,b,c be distinct integers different from integers a',b',c' (which themselves are **not necessarily distinct**, so we might have $a'=b'$). We construct our new structure of tables $\mathcal{F}_6^*$ build up from the following columns (apart from permutation of rows) with carefully designed weights:



Example 74. Below, in Figure 2.12, there is an example of a table $\tau \in \mathcal{F}_{6,9}^*$ with two known four-columns (each having weight $m_4 - 3$) and one known six-column.

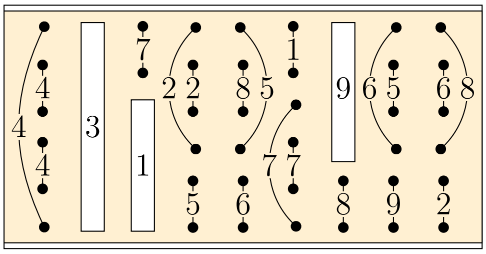


Figure 2.12: A table $\tau \in \mathcal{F}_{6,9}^*$ with weight $w(\tau) = (m_6 - 15)(m_4 - 3)^2$.

Note that since a 4-column in any $\tau \in \mathcal{F}_6^{\text{sym}}$ can either be known (weight $m_4 - 3$) or unknown (there are 3 ways how we can pair up the four identical elements), the total contribution from all $\mathcal{F}_6^{\text{sym}}$ columns is $m_4 - 3 + 3 = m_4$, which is precisely as in plain $\mathcal{F}_6^{\text{sym}}$. This decomposition is shown in Figure 2.13.

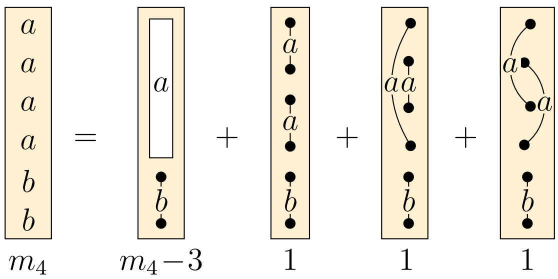
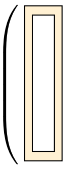


Figure 2.13: Inclusion/Exclusion of 4-columns

Similarly, the weights add up to m_6 from each known and unknown 6-column. To see this, note that there are in total 15 pairings of the six identical elements, this gives us the factor of $m_6 - 15 + 15 = m_6$ again (known 4-columns with $a = b'$ are forbidden!). Overall, we must get the following matching

$$f_6^{\text{sym}}(n) = n! \sum_{\tau \in \mathcal{F}_{6,n}^{\text{sym}}} w(\tau) \operatorname{sgn} \tau = n! \sum_{\tau \in \mathcal{F}_{6,n}^*} w(\tau) \operatorname{sgn} \tau. \tag{2.65}$$

Even though the set $\mathcal{F}_6^*$ is much larger, we will see how it can be constructed out of $\mathcal{N}_6$ we saw earlier (tables constructed out of unknown columns only). First, notice that the the known 6-columns are disjoint from the other columns, we can therefore write

$$\mathcal{F}_6^* = \textsc{Set}\left(\right. $$  $$\left. \right) \star \tilde{\mathcal{F}}_6^*, \tag{2.66}$$

where we denoted $\tilde{\mathcal{F}}_6^*$ as the tables constructed out of known 4-columns and unknown columns only. In terms of EGF's,

$$F_6^*(t) = \exp((m_6 - 15)t)\tilde{F}_6^*(t). \tag{2.67}$$

Let $\tau \in \tilde{\mathcal{F}}_6^*$, we construct its associated oriented graph g using the following rules: Let a be the number which appears four times in a known 4-column of τ, this number must appear elsewhere in table τ as a pair of connected a's, then in g,

- there will be a vertex associated to each known 4-column labeled by the number a.
- Apart of that, our graph will have one special vertex ν.
- If the remaining pair of a's is located in a known 4-column, we draw an oriented edge $a \to b$, where b is the number which appears four times in that known 4-column,
- else if the remaining pair of a's is not located in some known 4-column, then we draw an edge $a \to \nu$.

An example how the graph is constructed is shown in Figure 2.14.

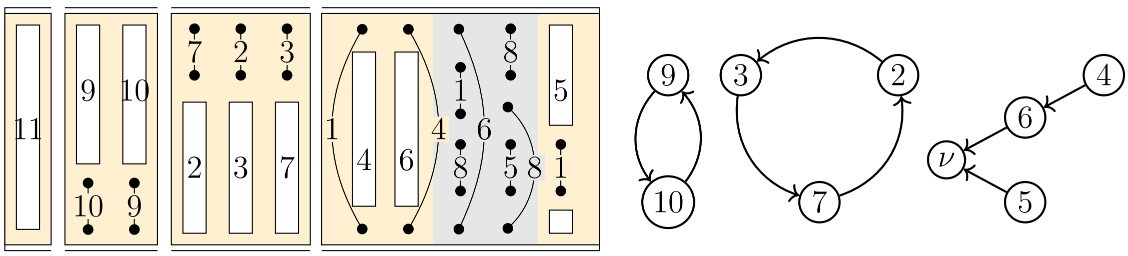


Figure 2.14: A table $\tau \in \tilde{\mathcal{F}}_{6,11}^*$ with its associated graph g of known 4-columns (columns in the core ν are shown in grey)

Upon seeing an example above, we discover the following structure – any graph g must be composed out of

- disjoint cycles of length at least two
- a (single) tree whose root is ν

Disjoint cycles of g directly correspond to disjoint sub-tables found in τ. This correspondence is not a bijection. Luckily, for a given cycle, there are exactly $\binom{6}{2} = 15$ ways how a sub-table corresponding to a given cycle can look like based on the location of the remaining pair in the known 4-columns. These ways can be depicted using the following diagram in Figure 2.15.

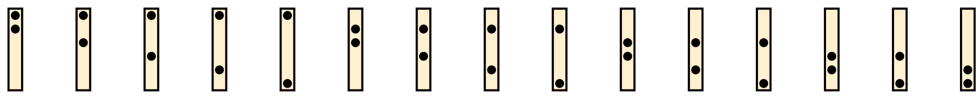
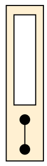


Figure 2.15: Positions of the remaining pair in a given cycle of known 4-columns

We may write the following structural equation

$$\tilde{\mathcal{F}}_6^* = \textsc{Set}\left(15\,\textsc{Cyc}_{\geq 2}\left(\right)\right) \star \mathcal{R}_6, \tag{2.68}$$



where we denoted $\mathcal{R}_6$ as the set of all $\tilde{\mathcal{F}}_6^*$ tables whose graph contains no cycles of known 4-columns. In terms of generating functions, we must have

$$\tilde{F}_6^*(t) = \exp(-15(m_4 - 3)t - 15\ln(1 - (m_4 - 3)t))R_6(t). \tag{2.69}$$

Although the known 4-columns in some $\tau \in \mathcal{R}_6$ do not form sub-tables since they are not disjoint from the remaining unknown columns, they are still very tightly associated with them. We can consider the following operation of **collapse**

- each chain of 4-columns attached to ν in g corresponds to a subset of 4-columns in τ.
- These columns contain an unmatched pair of a's not equal to any number tagging the chain.
- By deleting those columns and inserting the unmatched pairs into the known columns, we get a collapsed table τ' on unknowns columns only.

An example how a table is collapsed is shown in Figure 2.16.

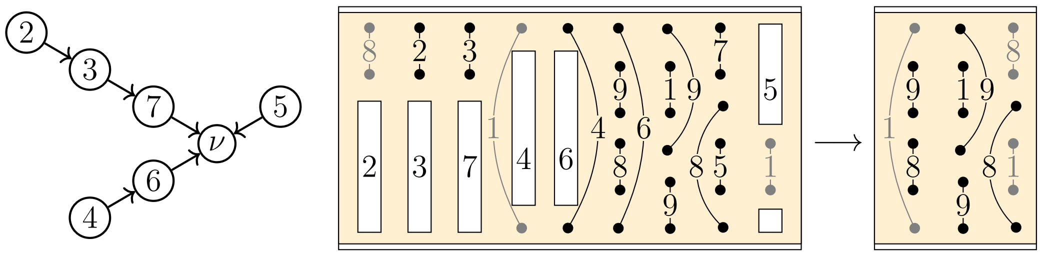
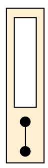


Figure 2.16: A table $\tau \in \mathcal{R}_{6,9}$ with its associated collapsed table $\tau' \in \mathcal{N}_{6,3}$.

In order to create (any) τ from τ', notice that we can mount a chain of 4-columns to a pair in the collapsed τ'. The chains stem out of τ' almost like branches out of a trunk. A chain (with length including zero) of known 4-columns can be described as a structure (we always have positive signs)

$$\textsc{Seq}\left(\right) \tag{2.70}$$



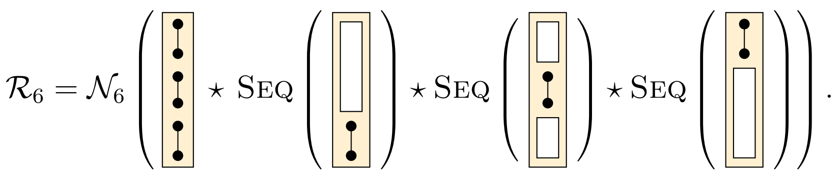
with the corresponding EGF equal to

$$\frac{1}{1 - (m_4 - 3)t}. \tag{2.71}$$

Each unknown column has three pairs to which we can attach a chain of known 4-columns. Since it is also equipped with the factor of t in the generating function, we can simply replace t in $N_6(t)$ with

$$\frac{t}{(1-(m_4-3)t)^3}. \tag{2.72}$$

Structurally,



(2.73)

In terms of generating functions

$$R_6(t) = N_6\left(\frac{t}{(1-(m_4-3)t)^3}\right). \tag{2.74}$$

All together, by Equations (2.65), (2.67), (2.69) and (2.74)

$$F_6^{\mathrm{sym}}(t) = F_6^*(t) = \frac{e^{(m_6-15m_4+30)t}}{(1-(m_4-3)t)^{15}} N_6\left(\frac{t}{(1-(m_4-3)t)^3}\right). \tag{2.75}$$

Finally, we generalise our approach to deduce $F_6^{\mathrm{cen}}(t)$. Relaxing the condition $m_3 = 0$ by making m_3 arbitrary, we get one extra column type which appears in the structure of all tables $\mathcal{F}_6^{\mathrm{cen}}$ with nontrivial weight and that is the 3-column. These 3-columns, however, form a disjoint set of sub-tables. This is because they are the only columns in which a number can appear in a triplet. Denote $\mathcal{Q}_6$ as the set of all tables constructed out of 3-columns only (with column order irrelevant), then

$$\mathcal{F}_6^{\mathrm{cen}} = \mathcal{F}_6^{\mathrm{sym}} \star \mathcal{Q}_6 \tag{2.76}$$

Our goal is to find the construction relation for $\mathcal{Q}_6$. The triplets in 3-columns can be connected in a similar way as the columns in tables $\tilde{\mathcal{F}}_2$. Hence, we can associate a derangement π whose cycles correspond to disjoint sub-tables into which $\tau \in \mathcal{Q}_6$ decomposes. This association, however, is no longer a bijection.

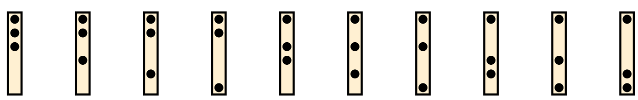


Figure 2.17: Positions of the number which appears in the first row

To make it a bijection, notice that for a given cycle, there are $\binom{5}{2} = 10$ ways (see Figure 2.17) how we can arrange the remaining numbers in 3-columns in the sub-table corresponding to that cycle. The correspondence is shown in Figure 2.18.

1	3	2	6	7	4	5	8	9
1	3	2	6	7	9	4	5	8
1	3	6	7	2	9	4	5	8
3	1	2	6	7	4	5	8	9
3	1	6	7	2	9	4	5	8
3	1	6	7	2	4	5	8	9

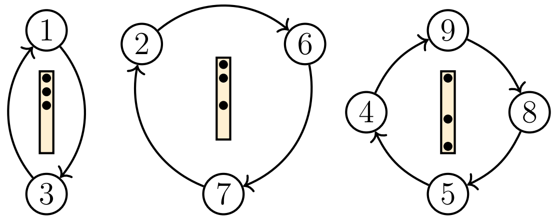


Figure 2.18: One-to-one correspondence between a table τ in $\mathcal{Q}_{6,9}$ with nine 3-columns decomposable into three disjoint sub-tables, and its associated derangement π with cycles labeled according to the repetitions of the number in the first row of τ

For a given sub-table of size n, there are three rows which are themselves cycles of size n and sign $(-1)^{n-1}$, so the overall sign of a sub-table is again $(-1)^{n-1}$. All together, we can write the following construction relation

$$\mathcal{Q}_6 = \text{SET}\left(-10\,\text{CYC}_{\geq 2}\left(-m_3^2 \textcircled{1}\right)\right). \tag{2.77}$$

Hence, the EGF of $\mathcal{Q}_6$ must be

$$\exp(-10m_3^2 t + 10\ln(1 + m_3^2 t)) = \left(1 + m_3^2 t\right)^{10} e^{-10m_3^2 t} \tag{2.78}$$

and therefore, restating Equation (2.76) in terms of generating functions,

$$F_6^{\text{cen}}(t) = \left(1 + m_3^2 t\right)^{10} e^{-10m_3^2 t}\, F_6^{\text{sym}}(t). \tag{2.79}$$

This concludes the proof (see Corollary 45.1 how we can get from $m_2 = 1$ to m_2 arbitrary). By using Taylor expansion, we immediately recover also $f_6^{\text{cen}}(n)$. ∎

2.2.7 Mounting argument for higher moments

Note that the previous approach enables us to generalise the result of Dembo [24] (Note that we present a slightly extended version since in fact, the result is correct for any value of m_{k-1} as it always appears in a product with m_1 which vanishes).

Definition 75. Denote $\mathfrak{m}_q$ the moments of a $\mathsf{N}(0,1)$ variable, that is $\mathfrak{m}_q = (q-1)!!$ when q is even and $\mathfrak{m}_q = 0$ otherwise.

Proposition 76 (Dembo 1989)**.** *Let $k \geq 2$ be even and let X_{ij} have moments m_q which coincide with moments $\mathfrak{m}_q$ of the Normal distribution upto $q \leq k-2$. That is, $m_q = \mathfrak{m}_q$ for $q \leq k-2$ (we thus have two free parameters m_k and m_{k-1}). Then*

$$F_k(t) = e^{(m_k - \mathfrak{m}_k)t} N_k(t), \tag{2.80}$$

where

$$N_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} \prod_{r=0}^{\frac{k}{2}-1} \frac{(n+2r)!}{(2r)!} \tag{2.81}$$

is the generating function $F_k(t)$ for the full normal distribution $X_{ij} \sim \mathsf{N}(0,1)$.

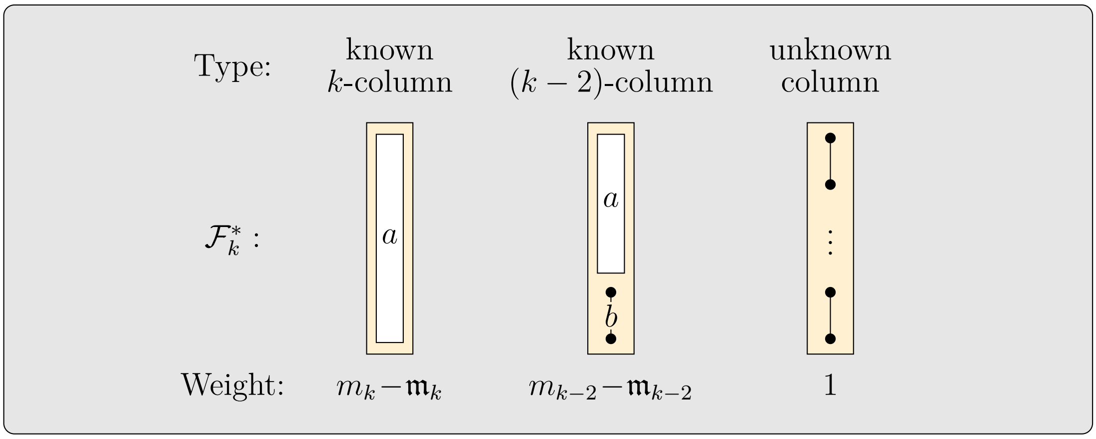
However, by mounting to tables which have normal weights (unknown columns only), we can extend this result so m_{k-2} can attain any value. From a corresponding functional equation, we deduced:

Proposition 77. *Let $k \geq 4$ be even and let X_{ij} have moments m_q which coincide with moments $\mathfrak{m}_q$ of the Normal distribution upto $q \leq k-3$ (we thus have three free parameters m_k, m_{k-1} and m_{k-2}). Then*

$$F_k(t) = \frac{\exp\left[\left(m_k - \binom{k}{2} m_{k-2} - \mathfrak{m}_k + \binom{k}{2}\mathfrak{m}_{k-2}\right)t\right]}{(1+\mathfrak{m}_{k-2}t - m_{k-2}t)^{\binom{k}{2}}} N_k\left(\frac{t}{(1+\mathfrak{m}_{k-2}t - m_{k-2}t)^{k/2}}\right), \tag{2.82}$$

where $N_k(t)$ as before.

Proof. The same argument as in the case of the sixth moment. Even though we assume $m_1 = 0$, the structure of all nontrivial tables $\mathcal{F}_k^{\text{sym}}$ still contains a lot of types of columns. However, when the condition imposed on moments enables us to make the same inclusion/exclusion argument by carefully designing the weights for their known/unknown counterparts. As a result, we replace the structure $\mathcal{F}_k^{\text{sym}}$ with the structure $\mathcal{F}_k^*$ of tables build up from the following columns (a, b distinct):



The matching of columns assures that generating functions $F_k^{\text{sym}}(t)$ and $F_k^*(t)$ coincide. The known k-columns are disjoint (each), so we get the factor $\exp((m_k - \binom{k}{2} m_{k-2} - \mathfrak{m}_k + \binom{k}{2}\mathfrak{m}_{k-2})t)$. There are $\binom{k}{2}$ of known $(k-2)$-columns based on the position of the only pair in this column. These columns can either form cycles of length at least two, from those we get the factor $\exp(-\binom{k}{2}(m_{k-2} - \mathfrak{m}_{k-2})t)/(1 - (m_{k-2} - \mathfrak{m}_{k-2})t)^{\binom{k}{2}}$. Or, they could be mounted to pairs of $\mathcal{N}_k$ tables. In each column of $\mathcal{N}_k$, there are $k/2$ pairs, from which we can grow a chain of known $(k-2)$-columns. This gives us the last factor in EGF of $\mathcal{F}_k^*$. ∎

Remark 78. Note that when $k = 6$ this result has enough freedom to give us the value of $F_6(t)$ for any symmetrical distribution – the condition $m_2 = 1$ can be relaxed by scaling and m_4 and m_6 are already free. For $k \geq 8$, that's not the case though since m_4 is no longer free (it must be equal to 3 in order the argument to work).

2.2.8 Direct mounting without inclusion/exclusion

Notice how the inclusion/exclusion turned out to be crucial in the proof above. Without is, the collapse would not work since we could not guarantee that the collapse is surjective. However, by carefully working with structural compositions, we can avoid the inclusion/exlusion entirely. Although at a cost of getting a functional relation for $F_6^{\mathrm{sym}}(t)$ rather than a direct expression. See the proof below.

Alternative proof of Theorem 73. Let us consider all $\mathcal{F}_6^{\mathrm{sym}}$ tables with their usual 6-, 4- and 2-columns (weights m_6, m_4m_2 and m_2^3, respectively). We denote $\mathcal{F}_6^{\mathrm{tree}}$ as the subset of all tables $\mathcal{F}_6^{\mathrm{sym}}$ which lack 6-columns and whose 4-columns do not form cycles. Those tables are composed out of the following columns:

Type:	4-column	2-column
$\mathcal{F}_6^{\mathrm{tree}}$:	a a a a b b	a a b b c c
Weight:	m_4m_2	1

However, not all compositions of columns are allowed in $\mathcal{F}_6^{\mathrm{tree}}$ since we must ensure there are no cycles of 4-columns. Note that now the cycles of 4-columns in $\mathcal{F}_6^{\mathrm{sym}}$ must be of length at least two (otherwise we would end up with a 6-column). Considering this, we can write down the following structural relation

$$\mathcal{F}_6^{\mathrm{sym}} = \textsc{Set}\left(\begin{array}{|c|}\hline a\\ a\\ a\\ a\\ a\\ a\\ \hline\end{array}\right) \star \textsc{Set}\left(15\,\textsc{Cyc}_{\geq 2}\left(\begin{array}{|c|}\hline a\\ a\\ a\\ a\\ b\\ b\\ \hline\end{array}\right)\right) \star \mathcal{F}_6^{\mathrm{tree}}. \tag{2.83}$$

From the structural equation, we get immediately in terms of EGFs,

$$F_6^{\mathrm{sym}}(t) = e^{m_6 t} e^{-15 m_4 m_2 t - 15 \ln(1 - m_4 m_2 t)} F_6^{\mathrm{tree}}(t) = \frac{e^{(m_6 - 15 m_4 m_2)t}}{(1 - m_2^3 t)^{15}} F_6^{\mathrm{tree}}(t). \tag{2.84}$$

Analogously, we can collapse all of the 4-columns in $\tau \in \mathcal{F}_6^{\mathrm{tree}}$ to get a new table τ'. An example of this procedure is shown in Figure 2.19.

4	3	**7**	2	5	1	9	6	8
2	4	**7**	3	8	1	9	5	6
2	4	1	3	8	7	9	5	6
4	3	1	2	5	7	9	6	8
4	3	1	5	6	9	**7**	8	2
4	3	1	5	6	9	**7**	8	2

$\longrightarrow$

2	5	**7**	6	8
2	8	**7**	5	6
2	8	7	5	6
2	5	7	6	8
5	6	**7**	8	2
5	6	**7**	8	2

Figure 2.19: A correspondence between table $\tau \in \mathcal{F}_{6,9}^{\mathrm{tree}}$ and its collapsed $\tau' \in \mathcal{F}_{6,5}^{\mathrm{sym}}$.

Notice that, although τ does not contain any 6-columns, the corresponding collapsed table τ' can. Similarly, τ' can contain 4-columns which form cycles. Hence, in general, τ' must be a $\mathcal{F}_6^{\text{sym}}$ table. In fact, we can get any $\mathcal{F}_6^{\text{sym}}$ table by collapsing a larger $\mathcal{F}_6^{\text{tree}}$ table so the operation is *surjective*. This observation suggests we can construct $\mathcal{F}_6^{\text{tree}}$ tables out of $\mathcal{F}_6^{\text{sym}}$ tables by mounting a chain of 4-columns to appropriate pairs of identical elements. Unlike in the previous proof, we distinguish between 4-column chains of length at least one and at least zero. We will call the former as improper chain structure and the latter as proper chain structure. The improper chain structure is constructed as follows

$$\textsc{Seq}\left(\boxed{\begin{matrix}a\\a\\a\\a\\b\\b\end{matrix}}\right), \tag{2.85}$$

so its EGF is equal to $1/(1-m_4m_2t)$. The proper chain structure is given by

$$\textsc{Seq}_{\geq 1}\left(\boxed{\begin{matrix}a\\a\\a\\a\\b\\b\end{matrix}}\right) \tag{2.86}$$

and its EGF equals $\sum_{n=1}^{\infty}(m_4m_2t)^n = m_4m_2t/(1-m_4m_2t)$. Now, we are ready to construct any $\mathcal{F}_6^{\text{tree}}$ table by enlarging a specific (collapsed) $\mathcal{F}_6^{\text{sym}}$ table. Note that the generating function $F_6^{\text{sym}}(t)$ is in general a function of parameters (t, m_6, m_4, m_2). In fact, we can write it as a function of only three combinations of those parameters, namely

$$F_6^{\text{sym}}(t) = \Omega(m_6t, m_4m_2t, m_2^3t) \tag{2.87}$$

for some function Ω. This is because we have the following options for the contribution of a column in the overall generating function:

- 6-column, factor m_6t,
- 4-column, factor m_4m_2t,
- 2-column, factor m_2^3t.

Alternatively, assuming $m_2 = 1$, we can write

$$F_6^{\text{sym}}(t) = \Lambda(t, m_6, m_4) \tag{2.88}$$

for some function Λ. Now, we proceed to actual mounting. In order to get a larger $\mathcal{F}_6^{\text{tree}}$ tables from a smaller $\mathcal{F}_6^{\text{sym}}$ table, we must

- turn each 6-column of the $\mathcal{F}_6^{\text{sym}}$ table into 2-column by mounting three proper chains in 15 ways or by mounting two proper chains in $\frac{1}{2!}\binom{6}{2}\binom{4}{2} = 45$ ways. There cannot be just one proper chain as the column would become a 4-column. Overall, we get the following replacement rule

$$m_6t \to \left[15\left(\frac{m_4m_2t}{1-m_4m_2t}\right)^3 + 45\left(\frac{m_4m_2t}{1-m_4m_2t}\right)^2\right]m_2^3t = \frac{15m_4^2m_2^5t^3\,(3-2m_4m_2t)}{(1-m_4m_2t)^3} \tag{2.89}$$

- turn each 4-column of an $\mathcal{F}_6^{\text{sym}}$ table into 2-column by mounting either two or one proper chains to four copies of a single number in the column.

Remember that we can mount an improper chain to the remaining pair in both cases. Overall,

$$\begin{aligned} m_4 m_2 t \to & \left[3\left(\frac{m_4 m_2 t}{1-m_4 m_2 t}\right)^2 + 6\left(\frac{m_4 m_2 t}{1-m_4 m_2 t}\right)\right]\left(\frac{1}{1-m_4 m_2 t}\right) m_3^2 t \\ = & \frac{3 m_4 m_2^4 t^2 \left(2 - m_4 m_2 t\right)}{\left(1-m_4 m_2 t\right)^3} \end{aligned} \tag{2.90}$$

- turn a 2-column into another 2-column, but this time we mount an improper chain to each of the three pairs in the column, so

$$m_2^3 t \to \left(\frac{1}{1-m_4 m_2 t}\right)^3 m_2^3 t = \frac{m_2^3 t}{(1-m_4 m_2 t)^3}. \tag{2.91}$$

Note that, we can rewrite this set of replacement rules as

$$\begin{aligned} \frac{m_6}{m_2^3} m_2^3 t &\to \left[15 m_4^2 m_2^2 t^2 \left(3 - 2 m_4 m_2 t\right)\right] \frac{m_2^3 t}{\left(1-m_4 m_2 t\right)^3}, \\ \frac{m_4}{m_2^2} m_2^3 t &\to \left[3 m_4 m_2 t \left(2 - m_4 m_2 t\right)\right] \frac{m_2^3 t}{\left(1-m_4 m_2 t\right)^3}, \\ m_2^3 t &\to \frac{m_2^3 t}{\left(1-m_4 m_2 t\right)^3}. \end{aligned} \tag{2.92}$$

Thus, by dividing the first two rules by the last $m_2^3 t$ rule and letting $m_2 = 1$, we get the following corresponding set of rules for all valid chain mountings

$$\begin{aligned} m_6 &\to 15 m_4^2 t^2 \left(3 - 2 m_4 m_2 t\right), \\ m_4 &\to 3 m_4 t \left(2 - m_4 t\right), \\ t &\to \frac{t}{\left(1-m_4 t\right)^3}. \end{aligned} \tag{2.93}$$

In terms of generating functions, we get

$$F_6^{\text{tree}}(t) = \Lambda\left(\frac{t}{(1-m_4 t)^3}, 15 m_4^2 t^2 \left(3 - 2 m_4 t\right), 3 m_4 t \left(2 - m_4 t\right)\right). \tag{2.94}$$

Together with Equation (2.84), we get the following function equation

$$\Lambda(t, m_6, m_4) = \frac{e^{(m_6 - 15 m_4) t}}{(1-m_4 t)^{15}} \Lambda\left(\frac{t}{(1-m_4 t)^3}, 15 m_4^2 t^2 \left(3 - 2 m_4 t\right), 3 m_4 t \left(2 - m_4 t\right)\right). \tag{2.95}$$

We can check this functional equation is satisfied by

$$F_6^{\text{sym}}(t) = \Lambda(t, m_6, m_4) = \frac{e^{t(m_6 - 15 m_4 + 30)}}{(1+3t-m_4 t)^{15}} \Lambda\left(\frac{t}{(1+3t-m_4 t)^3}, 15, 3\right). \tag{2.96}$$

However, note that $\Lambda(t, 15, 3) = N_6(t)$ and thus

$$\Lambda(t, m_6, m_4) = F_6^{\text{sym}}(t) = \frac{e^{t(m_6 - 15 m_4 + 30)}}{(1+3t-m_4 t)^{15}} N_6\left(\frac{t}{(1+3t-m_4 t)^3}\right). \tag{2.97}$$

Expressing $F_6^{\text{sym}}(t)$ in terms of $N_6(t) = F_6^{\text{sym}}(t)|_{m_6=15,m_4=3}$ essentially solves the problem as we can now take advantage of the fact we can express $N_6(t)$ exactly (normally distributed entries X_{ij}). The remaining argument how we can obtain $F_6^{\text{cen}}(t)$ from $F_6^{\text{sym}}(t)$ by attaching cycles of 3-columns to $\mathcal{F}_6^{\text{sym}}$ remains the same. ∎

2.2.9 Chain counting generating function

We consider the following generalisation of the problem of finding the sixth moment of a random determiant: Let us select a table τ randomly uniformly out of the set of all $\mathcal{F}_6^{\text{sym}}$ tables. We can then ask the following questions:

- What is the probability of τ having a chain?
- What is the mean number of chains in τ?

Those questions are at heart of analytic combinatorics. In order to answer them, we can attach a tag z (a variable) to each proper chain and then we can just either determine if there is a label to answer the first question or collect the labels in order to answer the second question. We are interested in finding the following quantities which generalise the notion of a moment

Definition 79. Let $\tau \in F_6^{\text{sym}}$ (or $\mathcal{F}_6^{\text{sym}}$), we denote $p(\tau)$ as the number of chains of 4-columns found in it. We define the **chain-tagged moments** as

$$\hat{f}_6^{\text{sym}}(n)_z = \sum_{\tau \in \mathcal{F}_{6,n}^{\text{sym}}} z^{p(\tau)} w(\tau) \operatorname{sgn} \tau, \qquad \hat{f}_6^{\text{tree}}(n)_z = \sum_{\tau \in \mathcal{F}_{6,n}^{\text{tree}}} z^{p(\tau)} w(\tau) \operatorname{sgn} \tau. \tag{2.98}$$

Correspondingly, we write for their generating functions

$$F_6^{\text{sym}}(t)_z = \sum_{n=0}^{\infty} \frac{t^n}{n!} \hat{f}_6^{\text{sym}}(n)_z, \qquad F_6^{\text{tree}}(t)_z = \sum_{n=0}^{\infty} \frac{t^n}{n!} \hat{f}_6^{\text{tree}}(n)_z. \tag{2.99}$$

We can consider another generalisation of the sixth moment problem as follows

Definition 80. We define the *chain moment* $\hat{p}_6(n) = \sum_{\tau \in \mathcal{F}_{6,n}^{\text{sym}}} p(\tau) w(\tau) \operatorname{sgn} \tau$ and $P_6(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \hat{p}_6(n)$ its exponential *chain counting* generating function. As usual, we can sum over tables with distinguishable column position and get $p_6(n) = \sum_{\tau \in F_{6,n}^{\text{sym}}} p(\tau) w(\tau) \operatorname{sgn} \tau = n! \hat{p}_6(n)$.

Lemma 81. *For any symmetric distribution X_{ij},*

$$F_6^{\text{tree}}(t)_z = \frac{e^{30t}(1-m_4 t)^{15}}{(1+(3-m_4)\,t-3(1-z)m_4 t^2)^{15}} N_6\left(\frac{t\,(1-t(1-z)m_4)^3}{(1+(3-m_4)\,t-3(1-z)m_4 t^2)^3}\right)$$

and

$$F_6^{\text{sym}}(t)_z = \frac{e^{(m_6-15m_4+30)t}}{(1+(3-m_4)\,t-3(1-z)m_4 t^2)^{15}} N_6\left(\frac{t\,(1-t(1-z)m_4)^3}{(1+(3-m_4)\,t-3(1-z)m_4 t^2)^3}\right).$$

Proof. Since we are working with generating functions, we can build up the structure of tagged tables $(\mathcal{F}_6^{\text{sym}})_z$ from bottom to top simply by the rules of combinatorial constructions and deduce its corresponding EGF. That is, the corresponding exponential generating function $F_6^{\text{sym}}(t)_z$ is obtained by attaching z to each instance where a chain appeared in the original construction of non-tagged $F_6^{\text{sym}}(t)$. Separating 6-columns and closed loops of 4-columns (they contain no chains), we can write

$$F_6^{\text{sym}}(t)_z = \frac{e^{(m_6-15m_4m_2)t}}{(1-m_2^3t)^{15}} F_6^{\text{tree}}(t)_z, \tag{2.100}$$

where $F_6^{\text{tree}}(t)_z$ is the EGF of chain-tagged $\mathcal{F}_6^{\text{tree}}$ tables. These tagged loop-free tables (we can denote them as $(\mathcal{F}_6^{\text{tree}})_z$) can be composed out of chain-free $\mathcal{F}_6^{\text{sym}}$ tables by mounting chains to their pairs of identical elements. Dependent on the type of a chain (proper or improper), their EGF's get updated accordingly. Attaching a tag z, the *tagged proper chain generating function* will be now

$$z\sum_{s=1}^{\infty}(m_4m_2t)^s = \frac{zm_4m_2t}{1-m_4m_2t}. \tag{2.101}$$

Next, for the *tagged improper chain generating function*, we get

$$1+z\sum_{s=1}^{\infty}(m_4m_2t)^s = \frac{1-(1-z)m_4m_2t}{1-m_4m_2t}. \tag{2.102}$$

Notice that the first term in the sum did not receive the tag z as we did not increase the number of chains (no chain was mounted). By mounting the chains, we then build up $(\mathcal{F}_6^{\text{tree}})_z$ from $\mathcal{F}_6^{\text{sym}}$ by the following updated replacement rules:

$$\begin{aligned}
m_6t &\to \left[15\left(\frac{zm_4m_2t}{1-m_4m_2t}\right)^3 + 45\left(\frac{zm_4m_2t}{1-m_4m_2t}\right)^2\right] m_2^3t \\
m_4m_2t &\to \left[3\left(\frac{zm_4m_2t}{1-m_4m_2t}\right)^2 + 6\left(\frac{zm_4m_2t}{1-m_4m_2t}\right)\right]\left(\frac{1-(1-z)m_4m_2t}{1-m_4m_2t}\right) m_3^2t \\
m_2^3t &\to \left(\frac{1-(1-z)m_4m_2t}{1-m_4m_2t}\right)^3 m_2^3t
\end{aligned} \tag{2.103}$$

By dividing the first two rules by the last m_2^3t rule and setting $m_2=1$, we get, after simplification, the following corresponding set of rules

$$\begin{aligned}
m_6 &\to \frac{15z^2m_4^2t^2\,(3-(3-z)m_4t)}{(1-(1-z)m_4t)^{\,3}} \\
m_4 &\to \frac{3zm_4t\,(2-(2-z)m_4t)}{(1-(1-z)m_4t)^{\,2}} \\
t &\to t\left(\frac{1-(1-z)m_4t}{1-m_4t}\right)^3
\end{aligned} \tag{2.104}$$

Hence, in terms of generating functions

$$F_6^{\text{tree}}(t)_z = \Lambda\!\left(t\left(\frac{1-(1-z)m_4t}{1-m_4t}\right)^3, \frac{15z^2m_4^2t^2\,(3-(3-z)m_4t)}{(1-(1-z)m_4t)^3}, \frac{3zm_4t\,(2-(2-z)m_4t)}{(1-(1-z)m_4t)^2}\right).$$

Since we already know $\Lambda(t, m_6, m_4)$, we can substitute for it from Equation (2.97) and get $F_6^{\text{tree}}(t)_z$ and $F_6^{\text{sym}}(t)_z$ by Equation (2.100). ■

Proposition 82. *For any symmetric distribution of X_{ij} with $m_2 = 1$,*

$$P_6(t)=m_4t\frac{e^{t(m_6-15m_4+30)}}{(1+3t-tm_4)^{16}}\left[(1-m_4t)N_6^2\left(\frac{t}{(1+3t-tm_4)^3}\right)-45tN_6^0\left(\frac{t}{(1+3t-tm_4)^3}\right)\right].$$

where $N_6^2(t)$ is an auxiliary function defined as

$$N_6^2(t) = 3t\,\frac{\mathrm{d}}{\mathrm{d}t}N_6^0(t) = \frac{1}{16}\sum_{n=0}^{\infty} n(n+1)(n+2)(n+4)!\,t^n. \tag{2.105}$$

Proof. Clearly, $\hat{p}_6(n) = \partial\hat{f}_6^{\mathrm{sym}}(n)_z/\partial z\,|_{z\to 1}$ and thus $P_6(t) = \partial F_6^{\mathrm{sym}}(t)_z/\partial z\,|_{z\to 1}$. Differentiating $F_6^{\mathrm{sym}}(t)_z$ from Lemma 81 gives the desired result. ∎

We will establish the precise combinatorial connection of $N_6^2(t)$ to permutation tables later.

2.2.10 Position approach

We will introduce another approach to tables. Instead of treating each column separately (column-approach), we focus on individual numbers (position-approach).

Definition 83. Let $\tau \in F_{6,n}^{\mathrm{sym}}$ (or $\mathcal{F}_{6,n}^{\mathrm{sym}}$), we denote $I_j(t)$ as the set of numbers $i \in [n]$ which appear in j different columns of τ.

Proposition 84. *Let $\tau \in F_{6,n}^{\mathrm{sym}}$ (or $\mathcal{F}_{6,n}^{\mathrm{sym}}$) have c 6-columns and d 4-columns. Then $[n] = I_1(t) \sqcup I_2(t) \sqcup I_3(t)$ and $\#I_1(t) = c$, $\#I_2(t) = d$ and $\#I_3(t) = n-c-d$, where $\#$ denotes the number of elements in a set.*

Proof. In each table τ, there are three types of numbers. Either number i appears in three different columns (and thus belongs to set $I_3(t)$), or it appears in two different columns (and belongs to $I_2(t)$) or it appears alone in a column (and belongs to $I_1(t)$). These sets are disjoint. Obviously, numbers in $\#I_1(t)$ are the ones that form 6-columns, thus $\#I_1(t) = c$. Similarly, numbers in $I_2(t)$ are precisely those which appear in exactly four copies in some column (the remaining pair is displaced in some other column of τ). This forms a bijection between 4-columns and numbers in $I_2(t)$, thus $\#I_2(t) = d$. Finally, from the disjoint union property, $\#I_3(t) = n - c - d$. ∎

Definition 85. Let $\tau \in F_6^{\mathrm{sym}}$ (or $\mathcal{F}_6^{\mathrm{sym}}$). We denote $\nu_i(t)$ the number of 4-columns in τ in which i appears.

Example 86. To see how the definition works, consider the following table $\tau \in F_{6,11}^{\mathrm{sym}}$ in the figure below. In here, we have $I_1(t) = \{10\}$, $I_2(t) = \{1,3,4,7,9,11,12\}$ and $I_3(t) = \{2,5,6,8\}$. Next, $\nu_i(t) = 0$ if $i \in \{5,10\}$, $\nu_i(t) = 1$ if $i \in \{1,2,3,4,7,9,12\}$, $\nu_i(t) = 2$ if $i \in \{6,11\}$ and $\nu_i(t) = 3$ if $i \in \{8\}$.

12	4	8	2	3	5	1	9	6	7	11	10
12	4	8	2	3	5	1	9	6	7	11	10
12	6	3	5	2	4	1	8	9	7	11	10
12	6	3	5	2	4	1	8	9	7	11	10
6	4	3	1	5	7	8	9	12	11	2	10
6	4	3	1	5	7	8	9	12	11	2	10

Proposition 87. *Let* $\tau \in F_{6,n}^{\text{sym}}$ *(or* $\mathcal{F}_{6,n}^{\text{sym}}$*) have* c *6-columns and* d *4-columns, then* $\nu_i(t) = 0$ *if* $i \in I_1(t)$*,* $\nu_i(t) \in \{1,2\}$ *if* $i \in I_2(t)$ *and* $\nu_i(t) \in \{0,1,2,3\}$ *if* $i \in I_3(t)$*. Moreover,*

$$\sum_{i\in[n]} \nu_i(t) = 2d. \tag{2.106}$$

Proof. If $i \in I_1(t)$, $\nu_i(t) = 0$ as this number i automatically forms a 6-column (six copies of i's). If $i \in I_2(t)$, then one of the columns is automatically a 4-column as it has four copies of i's. The other column with displaced pair of i's can be either a 4-column or a 2-column depending on the remaining numbers in the column. If i appears in three different columns, each of this column is either a 4-column or a 2-column. To prove $\sum_{i\in[n]} \nu_i(t) = 2d$, note that each 4-column is counted twice in the sum (there are two different numbers in any 4-column). Alternatively, denote

$$\chi_{ij} = \begin{cases} 1 & j\text{-th column contains } i \\ 0 & \text{otherwise} \end{cases} \tag{2.107}$$

and $f_j = \mathbb{1}_{j\text{-th column is a four-column}}$. On one hand, $\nu_i(t) = \sum_{j\in[n]} \chi_{ij} f_j$, on the other $\sum_{i\in[n]} \chi_{ij} f_j = 2f_j$ since there are two numbers in each 4-column. Immediately, by changing the order of summation $\sum_{i\in[n]} \nu_i(t) = \sum_{i,j\in[n]} \chi_{ij} f_j = 2\sum_{j\in[n]} f_j = 2d$. ∎

Lemma 88. *Let* $\tau \in F_{6,n}^{\text{sym}}$ *(or* $\mathcal{F}_{6,n}^{\text{sym}}$*) with* c *6-columns and* d *4-columns out of which* $p(\tau)$ *form chains of 4-columns (the remaining* $d - p(\tau)$ *of 4-columns form cycles). Then*

$$\sum_{i\in I_2(t)} \nu_i(t) = 2d - p(\tau) \qquad \text{and} \qquad \sum_{i\in I_3(t)} \nu_i(t) = p(\tau).$$

Proof. First, notice the sum $\sum_{i\in I_3(t)} \nu_i(t)$ increases by one if it encounters a 4-column, but when the residual pair forms 4-column elsewhere, it does not contribute to the sum. Adding all occurrences of pairs in 4-columns, we get d (this is precisely the number of 4 columns). This number is then reduced by the number of numbers in those four-columns which form four-columns themselves. Let τ have p total chains of lengths $j_1, j_2, \dots, j_p$ (each $j_q \geq 1$) and denote $j = j_1 + j_2 + \cdots + j_p$. The total number of cycles is then $d - j$. The number of pairs which form 4-columns themselves is then equal to the number of cycles $d - j$ plus $\sum_{q=1}^{p}(j_q - 1) = j - p$, that is $d - p$. Hence,

$$\sum_{i\in I_3(t)} \nu_i(t) = d - (d - p) = p.$$

Finally, using Proposition 87 and since $\nu_i(t) = 0$ if $i \in I_1(t)$, we have immediately $\sum_{i\in I_3(t)} \nu_i(t) + \sum_{i\in I_2(t)} \nu_i(t) = 2d$, from which $\sum_{i\in I_2(t)} \nu_i(t) = 2d - p(\tau)$ follows. ∎

Example 89. In Example 86, we have $d = 7$ of 4-columns, which form $p = 6$ chains, out of which 4 have length one and one has length two. There are no loops of 4-columns. We have $\sum_{i\in I_2(t)} \nu_i(t) = 8 = 2d - p$ and $\sum_{i\in I_3(t)} \nu_i(t) = 6 = p$.

2.3 Gram moment and permutation tables

So far, we only considered determinant moments for square matrices. Recall the generalization we introduced earlier (see Definition 39), restating its relevant part:

Definition 90. Let $U = (X_{ij})_{n\times p}$ be rectangular random matrix and $f_k(n,p) = \mathbb{E}\,(\det U^\top U)^{k/2}$ be its k-th Gram moment. We may rewrite the Gram moment using scalar product of random vectors as follows: Let $\mathbf{X}_j = (X_{1j}, X_{2j}, \dots, X_{nj})^\top$, so $U = (\mathbf{X}_1 \mid \mathbf{X}_2 \mid \cdots \mid \mathbf{X}_p)$ and for the Gram matrix $J = U^\top U = (\mathbf{X}_j^\top \mathbf{X}_{j'})_{p\times p}$, that is $J_{jj'} = \mathbf{X}_j^\top \mathbf{X}_{j'}$. By definition, we set $f_k(n,0) = 1$ (we put $\det(U^\top U) = 1$ when $p = 0$). Note that if $p > n$, $\det(U^\top U)$ vanishes, that means that $f_k(n,p) = 0$ whenever $p > n$. Also, we define the corresponding generating function

$$F_k(t,\omega) = \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{(n-p)!}{n!p!} t^p \omega^{n-p} f_k(n,p). \tag{2.108}$$

Again, restricting the distribution of X_{ij}'s, we write

- *(centered distribution)* $f_k(n,p) = f_k^{\text{cen}}(n,p)$ and $F_k(t,\omega) = F_k^{\text{cen}}(t,\omega)$ if $m_1 = 0$; and similarly
- *(symmetrical distribution)* $f_k(n,p) = f_k^{\text{sym}}(n,p)$ and $F_k(t,\omega) = F_k^{\text{sym}}(t,\omega)$ if $m_1 = m_3 = m_5 = \ldots = 0$.

The fact that $f_k(n,p)$ is a polynomial in m_j leads to the important instant equality

$$f_k^{\text{cen}}(n,p) = f_k^{\text{sym}}(n,p) \qquad \text{valid for} \qquad k = 2,4. \tag{2.109}$$

Therefore, we also have $F_2^{\text{cen}}(t,\omega) = F_2^{\text{sym}}(t,\omega)$ and $F_4^{\text{cen}}(t,\omega) = F_4^{\text{sym}}(t,\omega)$. When $k \geq 6$, $f_k^{\text{cen}}(n,p)$ contains extra products of even powers of odd m_j moments $(m_3^2, m_3^4, m_5^2, \ldots)$.

Example 91. When $n = 3$ and $p = 2$, we have $\mathbf{X}_j = (X_{1j}, X_{2j}, X_{3j})^\top$, $j = 1,2$ and thus $J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$ with $J_{11} = X_{11}^2 + X_{21}^2 + X_{31}^2$, $J_{22} = X_{12}^2 + X_{22}^2 + X_{32}^2$ and $J_{12} = J_{21} = X_{11}X_{12} + X_{21}X_{22} + X_{31}X_{32}$. For example when $k = 6$, we get

$$\begin{aligned} f_6(3,2) = \mathbb{E}\,(\det J)^3 = \mathbb{E} \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix}^3 &= \mathbb{E}\,(J_{11}J_{22} - J_{12}^2)^3 \\ &= \mathbb{E}\,(J_{11}^3 J_{22}^3 - 3J_{11}^2 J_{22}^2 J_{12}^2 + 3J_{11}J_{22}J_{12}^4 - J_{12}^6). \end{aligned} \tag{2.110}$$

Computing the expectations of individual terms is straightforward, but tedious. We only show how the term $\mathbb{E}\,[J_{11}^3 J_{22}^3]$ is obtained: Expanding J_{11}^3, we get

$$\begin{aligned} J_{11}^3 =& X_{11}^6 + X_{21}^6 + X_{31}^6 + 3X_{11}^2X_{21}^4 + 3X_{11}^4X_{21}^2 + 3X_{11}^2X_{31}^4 \\ &+ 3X_{11}^4X_{31}^2 + 3X_{21}^2X_{31}^4 + 3X_{21}^4X_{31}^2 + 6X_{11}^2X_{21}^2X_{31}^2. \end{aligned} \tag{2.111}$$

Taking expectation, we get $\mathbb{E}\,[J_{11}^3] = 3\,(m_6 + 6m_2m_4 + 2m_2^3)$ and by independence, $\mathbb{E}\,[J_{11}^3 J_{22}^3] = \mathbb{E}\,[J_{11}^3]\,\mathbb{E}\,[J_{22}^3] = \mathbb{E}^2\,[J_{11}^3] = 9(m_6 + 6m_2m_4 + 2m_2^3)^2$. Overall,

$$\begin{aligned} f_6(3,2) = 6m_6^2 &+ 36m_2m_4m_6 + 162m_2^2m_4^2 + 216m_2^4m_4 - 36m_2^6 - 60m_3^4 \\ &- 144m_1^2m_2^2m_3^2 - 144m_1m_2m_3^3 + 288m_1^3m_2m_3m_4 - 72m_1^2m_3^2m_4 \\ &- 144m_1^2m_2m_3m_5 - 72m_1^3m_4m_5 - 36m_1^2m_5^2. \end{aligned} \tag{2.112}$$

For centrally distributed X_{ij}'s, we get simply by putting $m_1 = 0$,

$$f_6^{\mathrm{cen}}(3,2) = 6m_6^2 + 36m_2m_4m_6 + 162m_2^2m_4^2 + 216m_2^4m_4 - 36m_2^6 - 60m_3^4. \quad (2.113)$$

Similarly, letting also $m_3 = m_5 = 0$, we get

$$f_6^{\mathrm{sym}}(3,2) = 6m_6^2 + 36m_2m_4m_6 + 162m_2^2m_4^2 + 216m_2^4m_4 - 36m_2^6. \quad (2.114)$$

Notice that $f_6^{\mathrm{cen}}(3,2) \neq f_6^{\mathrm{sym}}(3,2)$ which we indeed expect for $k \geq 6$.

The aim of this section is to present a combinatorial construction associated with the Gram moments defined above. Again, the central role is played by permutation tables. We will see how they naturally arise in the Gram case via *Cauchy-Binet formula*.

Lemma 92 (Cauchy-Binet formula). *Let $L = (l_{ij})_{n\times p}$ and $M = (m_{ij})_{n\times p}$ be real n by p matrices and let $C = \{i_1, i_2, \ldots, i_p\}$ denotes a subset of $[n] = \{1, 2, 3, \ldots, n\}$ with p elements i_j taken from $[n]$ such that $i_1 < i_2 < \cdots < i_p$. Denote L_C and M_C to be square matrices formed from matrices L and M by selecting the rows $i_1, i_2, \ldots i_p$, respectively. Then*

$$\det(L^\top M) = \sum_{\substack{C\subset[n]\\ |C|=p}} \det(L_C)\det(M_C). \quad (2.115)$$

2.3.1 Gram second moment

Cauchy-Binet formula offers an elementary derivation of $f_2(n,p)$, generalizing Fortet's $f_2(n)$ (Proposition 67).

Proposition 93. *For any distribution of X_{ij},*

$$f_2(n,p) = \binom{n}{p} p!(m_2 + m_1^2(p-1))(m_2 - m_1^2)^{p-1}, \quad (2.116)$$

$$F_2(t,\omega) = \frac{1 + m_1^2 t}{1-\omega} e^{(m_2 - m_1^2)t}. \quad (2.117)$$

Proof (Stanley [67]). Choosing $L = M = U$ in Cauchy-Binet formula,

$$\det(U^\top U) = \sum_{\substack{C\subset[n]\\ |C|=p}} \det(U_C)^2. \quad (2.118)$$

Taking the expectation and by linearity, we get $\binom{n}{p}$ identical terms, each attending the value $\mathbb{E}\det(U_C)^2 = f_2(p)$ from which $f_2(n,p)$ follows immediately. ∎

2.3.2 Pair-tables

In general, taking $k/2$-power of Cauchy-Binet formula with $L = M = U$ and taking expectation,

$$f_k(n,p) = \mathbb{E}\det(U^\top U)^{k/2} = \mathbb{E}\left[\sum_{\substack{C\subset[n]\\ |C|=p}} (\det U_C)^2\right]^{k/2}. \quad (2.119)$$

This formula has a nice permutation tables interpretation due to Dembo [24]:

Definition 94. We define $F_{\langle k\rangle,n,p}$ as the set of all k (even number) by p pair-tables on n numbers. We say $\tau \in F_{\langle k\rangle,n,p}$ is a pair-table, if there are subsets $C_1, C_2, \ldots, C_{k/2} \subset [n]$ with $|C_1| = |C_2| = \cdots = |C_{k/2}| = p$, such that τ is a k by p table whose first two rows are permutations of numbers from C_1, next two rows are permutations of numbers from C_2, and so on. As in the case of regular tables, we define weight $w(\tau)$ of table τ as the product of weights of its individual columns (we multiply corresponding X_{ij}'s) and $\operatorname{sgn}(\tau)$ as the product of sign of the corresponding permutations. Correspondingly, we also define $\mathcal{F}_{\langle k\rangle}$ as the pair-tables on n numbers with k rows and p columns whose column order is irrelevant and we denote $\mathcal{F}_{\langle k\rangle} = \bigcup_{k\geq 0} \mathcal{F}_{\langle k\rangle,n,p}$ to be the set of all irrelevant column order pair-tables (with k rows).

We state the following analogue of Proposition 54:

Proposition 95. *For any distribution of X_{ij}, assuming k even,*

$$f_k(n,p) = \mathbb{E}\,\det(U^\top U)^{k/2} = \sum_{\tau\in F_{\langle k\rangle,n,p}} w(\tau)\operatorname{sgn}(\tau) \tag{2.120}$$

or alternatively, defining $\hat f_k(n,p) = f_k(n,p)/p!$

$$\hat f_k(n,p) = \sum_{\tau\in \mathcal{F}_{\langle k\rangle,n,p}} w(\tau)\operatorname{sgn}(\tau) \tag{2.121}$$

Proof. The right hand side of Equation (2.119) can be expanded as

$$f_k(n,p) = \mathbb{E}\prod_{j=1}^{k/2}\sum_{\substack{C_j\subset[n]\\|C_j|=p}} \det(U_{C_j})^2. \tag{2.122}$$

The proof follows simply from realization that the square of $\det(U_{C_j})$ possesses the meaning of two identical rows of (permuted) elements C_j. ∎

Example 96. Let us compute $f_4(3,2)$. We may write $f_4(3,2) = 2!\hat f_4(3,2)$, where $\hat f_4(3,2) = \sum_{\tau\in\mathcal{F}_{\langle 4\rangle,3,2}} w(\tau)\operatorname{sgn}\tau$. Let a,b,c be distinct elements of $\{1,2,3\}$, Figure 2.20 enlists all members of $\mathcal{F}_{\langle 4\rangle,3,2}$ and shows their weights and signs.

C_1:	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
C_2:	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$
$\mathcal{F}_{\langle 4\rangle,3,2}$:	a b	a b	a b	a b	a c	a b	a b
	a b	b a	b a	a b	a c	b a	b a
	a b	b a	b a	a c	b a	c a	a c
	a b	a b	b a	a c	b a	a c	a c
Weight:	$3m_4^2$	$9m_2^4$	$12m_1^2m_3^2$	$6m_4m_2^2$	$6m_2^4$	$12m_1^4m_2^2$	$24m_1^2m_2m_3$
Sign:	$+$	$+$	$-$	$+$	$+$	$+$	$+$

Figure 2.20: Correspondence between $\hat f_4(3,2)$ and permutation tables $\mathcal{F}_{\langle 4\rangle,3,2}$

Each member is displayed apart of (valid) permutation of rows and selections of a,b,c. The first three members build up tables $\mathcal{F}_{4,2}$. Since there are three ways how we can select for a,b from $\{1,2,3\}$, their contribution is $3\hat f_4(2)$. The remaining terms are only found in $\mathcal{F}_{\langle 4\rangle,3,2}$ tables. Summing the contribution up, we get $\hat f_4(3,2) = 3m_4^2 + 15m_2^4 - 12m_1^2m_3^2 + 6m_2^2m_4 + 12m_1^4m_2^2 - 24m_1^3m_2m_3$.

2.3.3 Sub-table factorization

The choice of the EGF for $F_{\langle k\rangle}$ (which is the same as for $\mathcal{F}_{\langle k\rangle}$), namely

$$F_k(t,\omega)=\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{(n-p)!}{n!p!}t^p\omega^{n-p}f_k(n,p)=\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{(n-p)!}{n!}t^p\omega^{n-p}\hat{f}_k(n,p) \quad (2.123)$$

as introduced by Dembo [24] may seem arbitrary. Because of the additional variable ω, it seems we are no longer able to use the factorization property of the star product. However, we will see that the unique feature of $F_k(t,\omega)$ is, after appropriate transformation in ω, that it again *factorises over sub-tables*. First, let us introduce a few definitions.

Definition 97. For a given subset $\mathcal{A}_{\langle k\rangle}\subset\mathcal{F}_{\langle k\rangle}$ of pair-tables with k rows and irrelevant column order, let $\mathcal{A}_{\langle k\rangle,n,p}$ be the subset of $\mathcal{A}_{\langle k\rangle}$ with p columns and elements selected from the set $[n]=\{1,2,3,\dots,n\}$ of the so called *potential elements*. As usual, we define $\hat{a}_k(n,p)=\sum_{\tau\in\mathcal{A}_{\langle k\rangle,n,p}}w(\tau)\operatorname{sgn}\tau$ with the corresponding exponential generating function

$$\begin{aligned}A_k(t,\omega)&=\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{(n-p)!}{n!}t^p\omega^{n-p}\hat{a}_k(n,p)\\&=\sum_{\tau\in\mathcal{A}_{\langle k\rangle}}\frac{(n(\tau)-p(\tau))!\,t^{p(\tau)}\omega^{n(\tau)-p(\tau)}}{n(\tau)!}w(\tau)\operatorname{sgn}\tau,\end{aligned} \quad (2.124)$$

where we denoted $p(\tau)$ as the number of columns of τ and $n(\tau)$ as the number of potential elements.

Note that the number of potential elements $n(\tau)$ one chooses from to fill table τ may be arbitrarily large and it is always at least equal to the *number of actual elements* in τ (the number of distinct elements in τ), which we will denote as $q(\tau)$. With this said, we construct another generating function of $\mathcal{A}_{\langle k\rangle}$ (*modified generating function*) as follows

Definition 98 (Modified generating function)**.** Let $\mathcal{A}_{\langle k\rangle}\subset\mathcal{F}_{\langle k\rangle}$, then we define

$$\tilde{A}_k(t,\omega)=\sum_{\tau\in\mathcal{A}_{\langle\!\langle k\rangle\!\rangle}}\frac{t^{p(\tau)}\omega^{q(\tau)-p(\tau)}}{q(\tau)!}w(\tau)\operatorname{sgn}\tau, \quad (2.125)$$

where $\mathcal{A}_{\langle\!\langle k\rangle\!\rangle}\subset\mathcal{A}_{\langle k\rangle}$ are pair-tables with all of their potential elements being used up (all potential elements are in fact actual elements).

There is a simple connection between $A_k(t,\omega)$ and $\tilde{A}_k(t,\omega)$.

Definition 99. Let τ be a $\mathcal{A}_{\langle k\rangle}$ pair-table. We denote $EGF[\tau]$ as the contribution of τ to $A_k(t,\omega)$ and $MGF[\tau]$ as the contribution of τ to $\tilde{A}_k(t,\omega)$. By definition,

$$\begin{aligned}EGF[\tau]&=\frac{(n(\tau)-p(\tau))!}{n(\tau)!}t^{p(\tau)}\omega^{n(\tau)-p(\tau)}w(\tau)\operatorname{sgn}\tau,\\MGF[\tau]&=\frac{t^{p(\tau)}\omega^{q(\tau)-p(\tau)}}{n(\tau)!}w(\tau)\operatorname{sgn}\tau.\end{aligned} \quad (2.126)$$

Similarly for subsets $\mathcal{B}\subset\mathcal{A}_{\langle k\rangle}$.

Definition 100. Let $\mathbf{\Omega}$ be a linear operator acting on formal power series in ω such that for any integer r, we have $\mathbf{\Omega}[\omega^r] = r!\,\omega^r$.

Proposition 101.

$$A_k(t,\omega) = \mathbf{\Omega}[e^{\omega}\tilde{A}_k(t,\omega)]. \tag{2.127}$$

Proof. Applying $\mathbf{\Omega}^{-1}$, where $\mathbf{\Omega}^{-1}[\omega^r] = \omega^r/r!$, to Equation (2.124), we get

$$\mathbf{\Omega}^{-1}[A_k(t,\omega)] = \sum_{\tau\in\mathcal{A}_{\langle k\rangle}} \frac{t^{p(\tau)}\omega^{n(\tau)-p(\tau)}}{n(\tau)!} w(\tau)\,\mathrm{sgn}\,\tau. \tag{2.128}$$

In a given table $\tau \in \mathcal{A}_{\langle\!\langle k\rangle\!\rangle}$, we can replace its q actual elements $[q] = \{1,2,3,\ldots,q\}$ by another q actual elements from a larger set $[n] = \{1,2,3,\ldots,n\}$ forming a general table $\tau \in \mathcal{A}_{\langle k\rangle}$ with $n \geq q$ potential elements (all of which have the same weight and sign). Hence, $\mathcal{A}_{\langle k\rangle}$ is split into classes of tables which only differ by selection of actual elements. Since there are $\binom{n}{q}$ ways how we can substitute for the actual elements from the potential ones, we get that the total contribution of $\tau \in \mathcal{A}_{\langle\!\langle k\rangle\!\rangle}$ to $\mathbf{\Omega}^{-1}[A_k(t,\omega)]$ is

$$\sum_{n=q}^{\infty}\binom{n}{q}\frac{t^p\omega^{n-p}}{n!}w(\tau)\,\mathrm{sgn}\,\tau = e^{\omega}\frac{t^p\omega^{q-p}}{q!}. \tag{2.129}$$

Summing over all $\tau \in \mathcal{A}_{\langle\!\langle k\rangle\!\rangle}$, we get $\tilde{A}_k(t,\omega) = e^{-\omega}\mathbf{\Omega}^{-1}[A_k(t,\omega)]$, which is equivalent to the statement of the proposition. ∎

Moreover, it is exactly the modified generating function $\tilde{A}_k(t,\omega)$ which satisfies the star-product factorization property (an analogue of Proposition 63), turning $\mathcal{A}_{\langle k\rangle}$ into proper combinatorial structure.

Proposition 102 (sub-table factorization in pair-tables)**.** *Let τ be a $\mathcal{A}_{\langle\!\langle k\rangle\!\rangle}$ pair-table build up from exactly two disjoint sub-tables τ_1 and τ_2, then*

$$MGF[\tau_1 \star \tau_2] = MGF[\tau_1]MGF[\tau_2]. \tag{2.130}$$

Proof. Let p_1 and p_2 be the number of columns and q_1 and q_2 be the number of actual elements of τ_1 and τ_2, respectively. We also denote $p = p_1 + p_2$ as the total number of columns of τ and $q = q_1 + q_2$ as the total number of actual elements of τ (See Figure 2.21).

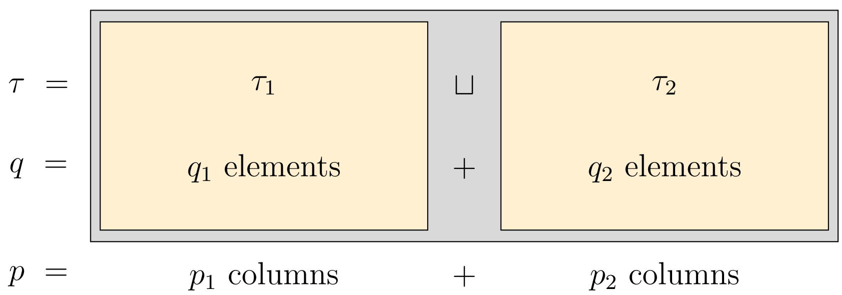


Figure 2.21: Pair-table τ consisted of two dis-joint sub-tables τ_1 and τ_2.

Any τ from the set $\tau_1 \star \tau_2$ (with elements shuffled) gives the same contribution to $\tilde{A}_k(t,\omega)$. Since there are $\binom{q}{q_1}$ ways how can we select elements for τ_1 and τ_2,

$$MGF[\tau_1 \star \tau_2] = \binom{q}{q_1} MGF[\tau] = \binom{q}{q_1} \frac{t^p \omega^{q-p}}{q!} w(\tau) \operatorname{sgn} \tau = \frac{t^p \omega^{q-p}}{q_1! q_2!} w(\tau) \operatorname{sgn} \tau. \tag{2.131}$$

On the other hand, since $w(\tau) \operatorname{sgn} \tau = w(\tau_1) w(\tau_2) \operatorname{sgn} \tau_1 \operatorname{sgn} \tau_2$,

$$\begin{aligned} MGF[\tau_1] MGF[\tau_2] &= \frac{t^{p_1} \omega^{q_1 - p_1}}{q_1!} w(\tau_1) \operatorname{sgn} \tau_1 \frac{t^{p_2} \omega^{q_2 - p_2}}{q_2!} w(\tau_2) \operatorname{sgn} \tau_2 \\ &= \frac{t^{p_1 + p_2} \omega^{q_1 + q_2 - p_1 - p_2}}{q_1! q_2!} w(\tau) \operatorname{sgn} \tau, \end{aligned} \tag{2.132}$$

which concludes the proof. ■

2.3.4 Gram fourth moment central

Note that when $m_1 = 0$, the number of pair-tables with nontrivial weights is reduced significantly. As a consequence, we can easily derive the result of Dembo [24], namely $F_4^{\text{sym}}(t,\omega)$ and the corresponding $f_4^{\text{sym}}(n,p)$.

Proposition 103 (Dembo, 1989)**.** *For any distribution X_{ij},*

$$F_4^{\text{sym}}(t,\omega) = \frac{e^{t(m_4 - 3m_2^2)}}{(1 - m_2^2 t)^2 (1 - \omega - m_2^2 t)}. \tag{2.133}$$

Corollary 103.1.

$$f_4^{\text{sym}}(n,p) = p!^2 \binom{n}{p} m_2^{2p} \sum_{j=0}^{p} \frac{1}{j!} \left(\frac{m_4}{m_2^2} - 3 \right)^j \binom{n - j + 2}{n - p + 2}. \tag{2.134}$$

Remark 104. Note that, letting $\omega = 0$ (or $p = n$), we recover the formulae of Nyquist, Rice and Riordan (Proposition 68) we saw earlier

$$F_4^{\text{sym}}(t) = \frac{e^{t(m_4 - 3m_2^2)}}{(1 - m_2^2 t)^3}, \qquad f_4^{\text{sym}}(n) = (n!)^2 m_2^{2n} \sum_{j=0}^{n} \frac{1}{j!} \left(\frac{m_4}{m_2^2} - 3 \right)^j \binom{n - j + 2}{2}.$$

Proof of Proposition 103. Let a, b be different integers, then there are again only two types of columns in tables with nontrivial weights assuming $m_1 = 0$ (see the diagram below). It is convenient to denote those tables as $F_{\langle 4 \rangle}^{\text{sym}}$ (or $\mathcal{F}_{\langle 4 \rangle}^{\text{sym}}$ if we do not care about the order of columns).

Type:	4-column	2-column
$\mathcal{F}_{\langle 4 \rangle}^{\text{sym}}$:	a a a a	a a b b
Weight:	m_4	m_2^2

By definition

$$f_4^{\mathrm{sym}}(n,p) = p!\hat{f}_4^{\mathrm{sym}}(n,p) = p! \sum_{\tau \in \mathcal{F}^{\mathrm{sym}}_{\langle 4 \rangle, n}} w(\tau)\,\mathrm{sgn}\,\tau. \tag{2.135}$$

Let us consider only tables $\mathcal{F}^{\mathrm{sym}}_{\langle\!\langle 4 \rangle\!\rangle} \subset \mathcal{F}^{\mathrm{sym}}_{\langle 4 \rangle}$ which use up all their potential elements as actual elements and whose modified generating function $\tilde{F}_4^{\mathrm{sym}}(t,\omega)$ factorises over sub-tables. Based on Exmaple 96, it is not hard to see that any τ is composed out of single 4-columns, closed cycles of 2-columns and *open chains* of 2-columns (see Figure 2.22 below).

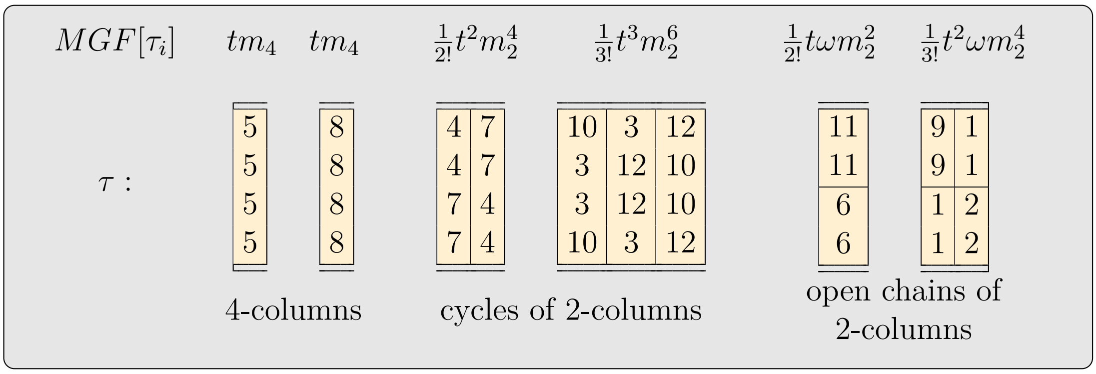


Figure 2.22: Table $\tau \in \mathcal{F}_{\langle\!\langle 4 \rangle\!\rangle}$ with $C_1 = \{1,3,4,5,7,8,9,10,11,12\}$, $C_2 = \{1,2,3,4,5,6,7,8,10,12\}$ and its decomposition into sub-tables τ_i

In general, we can write the following structural relation

$$\mathcal{F}^{\mathrm{sym}}_{\langle\!\langle 4 \rangle\!\rangle} = \mathcal{F}^{\mathrm{sym}}_4 \star \mathcal{L}_{\langle\!\langle 4 \rangle\!\rangle}, \tag{2.136}$$

where we denote $\mathcal{L}_{\langle\!\langle 4 \rangle\!\rangle}$ as the structure of open chains with modified generating function $\tilde{L}_4(t,\omega)$. Since we already know that the generating function of all 4-columns and cycles of 2-columns is $F_4^{\mathrm{sym}}(t)$ this follows immediately from the fact that $q = p$ for those sub-tables and thus $MGF[\,\cdot\,]$ coincide with $EGF[\,\cdot\,]$. The only remaining part is to deduce the modified generating function of open chains of 2-columns. However, there is one to one correspondence between a single open chain and a permutation of its (actual) elements (in Figure 2.22 above, the permutations are $11 \to 6$ and $9 \to 1 \to 2$). Hence, the single open chain modified generating function is given by, summing all $MGF[\tau]$ terms,

$$\sum_{q=2}^{\infty} \frac{t^{q-1}\omega^{q-(q-1)}}{q!} q! m_2^{2(q-1)} = \frac{\omega m_2^2 t}{1 - m_2^2 t} \tag{2.137}$$

Since the structure $\mathcal{L}_{\langle\!\langle 4 \rangle\!\rangle}$ is merely a concatenation of all possible chains (in view of the SET operation), we immediately get

$$\tilde{L}_4(t,\omega) = \exp\left(\frac{\omega m_2^2 t}{1 - m_2^2 t}\right) \tag{2.138}$$

and thus

$$\tilde{F}_4^{\mathrm{sym}}(t,\omega) = F_4^{\mathrm{sym}}(t)\tilde{L}_4^{\mathrm{sym}}(t,\omega) = \frac{e^{t(m_4 - 3m_2^2)}}{(1 - m_2^2 t)^3} \exp\left(\frac{\omega m_2^2 t}{1 - m_2^2 t}\right). \tag{2.139}$$

Finally, by using definition of $\mathbf{\Omega}$,

$$\begin{aligned} F_4^{\text{sym}}(t,\omega) &= \mathbf{\Omega}[e^{\omega}\tilde{F}_4^{\text{sym}}(t,\omega)] = \mathbf{\Omega}\left[\frac{e^{t(m_4-3m_2^2)}}{(1-m_2^2t)^3}\exp\left(\frac{\omega}{1-m_2^2t}\right)\right] \\ &= \mathbf{\Omega}\left[\frac{e^{t(m_4-3m_2^2)}}{(1-m_2^2t)^3}\sum_{r=0}^{\infty}\frac{\omega^r/r!}{(1-m_2^2t)^r}\right] = \frac{e^{t(m_4-3m_2^2)}}{(1-m_2^2t)^3}\sum_{r=0}^{\infty}\frac{\omega^r}{(1-m_2^2t)^r}. \end{aligned} \tag{2.140}$$

This concludes the proof. By using Taylor expansion, we immediatelly recover also $f_4^{\text{sym}}(n,p)$. ■

Remark 105. Somewhat similarly to the $f_2(n,p)$ case, we can derive Dembo's $f_4^{\text{sym}}(n,p) = \mathbb{E}\det(U^\top U)^2$ by taking expectation of Equation (2.119) directly. What we obtain are recursion relations (based on the overlap of C_1 and C_2) which are trivial to solve using Binomial transform (see Section 3.3 of our older work [8]).

2.3.5 Normal Gram moments

We have the following generalization of Prékopa's result for $n_{2m}(n)$ (Proposition 72) due to Dembo [24].

Definition 106. When $X_{ij} \sim \mathsf{N}(m_1,1)$, we denote $f_k(n,p)$ as $n_k(n,p)$ and $F_k(t,\omega)$ as $N_k(t,\omega)$.

Proposition 107 (Dembo 1989)**.** *For any even $k=2m$,*

$$n_{2m}(n,p) = \prod_{r=0}^{m-1}\frac{(n+2r)!}{(n-p+2r)!}. \tag{2.141}$$

Dembo's proof is an adaptation of the simplified proof of $n_{2m}(n)$ by Prékopa [57]. It relies again on a correspondence with Gaussian random polytopes. Hence, the proposition t is also equivalent to the well known result of Miles [48, p. 377, (70)] – see Proposition 254 in Chapter 6 of this thesis and its proof (our own). Proposition 107 is also special case of Theorem 181 with $\mu = 0$.

Fourth normal Gram moment

When $k=4$, we get

$$n_4(n,p) = \frac{n!(n+2)!}{(n-p)!(n-p+2)!} \tag{2.142}$$

and thus

$$N_4(t,\omega) = \sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{n!(n+2)!t^p\omega^{n-p}}{p!(n-p+2)!} = \frac{1}{(1-t)^2(1-\omega-t)}. \tag{2.143}$$

Alternatively, we can deduce $N_4(t,\omega)$ independently from Proposition 107 by using the general formula for $F_4^{\text{sym}}(t,\omega)$ (Proposition 103) with $m_2 = 1$ and $m_4 = 3$.

Sixth normal Gram moment

When $k = 6$, we obtain

$$n_6(n,p) = \frac{n!(n+2)!(n+4)!}{(n-p)!(n-p+2)!(n-p+4)!}. \tag{2.144}$$

However, its generating function

$$N_6(t,\omega) = \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{t^p\omega^{n-p}(n+2)!(n+4)!}{p!(n-p+2)!(n-p+4)!}. \tag{2.145}$$

is no longer analytic.

2.3.6 Gram sixth moment central

Theorem 108. *For any distribution X_{ij} with $m_1 = 0$ and $m_2 = 1$,*

$$F_6(t,\omega) = (1+m_3^2 t)^{10}\,\frac{e^{t(m_6-10m_3^2-15m_4+30)}}{(1+3t-m_4t)^{15}} N_6\left(\frac{t}{1+3t-m_4t}, \frac{\omega}{1+3t-m_4t}\right).$$

Expanding, we get for any distribution X_{ij} with $m_1 = 0, m_2 = 1$, that

$$f_6(n,p) = \frac{n!p!}{(n-p)!}\sum_{j=0}^{p}\sum_{i=0}^{j}\sum_{k=0}^{p-j} \frac{(2+i+n-p)!(4+i+n-p)!}{i!(n-p+2)!(n-p+4)!(p-j-k)!} \times \binom{10}{k}\binom{14+3n-3p+j+2i}{j-i} q_6^{p-j-k} q_4^{j-i} q_3^{k},$$

where $q_6 = m_6 - 10m_3^2 - 15m_4 + 30$, $q_4 = m_4 - 3$, and $q_3 = m_3^2$.

The proof is rather technical and rely on decomposing tables into disjoint sub-tables. Any table $\tau \in \mathcal{F}_{\langle 6\rangle}$ contains sub-tables already present in $\mathcal{F}_6$ (cycles of 3-columns, cycles of 4-columns and a core of 2-columns with attached chains of 4-columns) and also extra sub-tables consisted of open chains of 4-columns similar as in $\mathcal{F}_{\langle 4\rangle}$.

2.4 Marked permutation tables

2.4.1 Shifted basis

Instead of expressing determinant moments in m_r, there exists another basis which turns out to be much more convenient.

Definition 109. Let $Y_{ij} = X_{ij} - m_1$ with moments $\mu_r = \mathbb{E}\,Y_{ij}^r$.

> *Remark* 110. By expanding $Y_{ij}^r = (X_{ij} - m_1)^r$ and taking expectation, we get
>
> $$\begin{aligned} \mu_2 &= m_2 - m_1^2, \\ \mu_3 &= m_3 - 3m_1m_2 + 2m_1^3, \\ \mu_4 &= m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4 \\ \mu_5 &= m_5 - 5m_1m_4 + 10m_1^2m_3 - 10m_1^3m_2 + 4m_1^5, \\ \mu_6 &= m_6 - 6m_1m_5 + 15m_1^2m_4 - 20m_1^3m_3 + 15m_1^4m_2 - 5m_1^6. \end{aligned}$$
>
> Note that $\mu_1 = 0$ always.

Proposition 111. *Let k be even, then*

$$f_k(n) = \psi_k(n, m_1, \mu_2, \mu_3, \ldots, \mu_{k-1}, \mu_k), \tag{2.146}$$

where ψ_k is a polynomial in $m_1, \mu_2, \ldots, \mu_k$. Equivalently, there exists a function Ψ_k whose expansion coefficients are polynomials in $m_1, \mu_2, \ldots, \mu_k$ such that

$$F_k(t) = \Psi_k(t, m_1, \mu_2, \mu_3 \ldots, \mu_{k-1}, \mu_k). \tag{2.147}$$

Similarly, $f_k(n,p)$ is also some polynomial $\psi_k(n, p, m_1, \mu_2, \ldots, \mu_k)$ and $F_k(t, \omega) = \Psi_k(t, \omega, m_1, \mu_2, \ldots, \mu_k)$ for some function Φ_k with polynomial expansion coefficients.

Proposition 112. *Let ψ_k and Ψ_k be defined as in Proposition 111, then for any $\beta \in \mathbb{R}$ and k even,*

$$\psi_k(n, \beta m_1, \beta^2\mu_2, \beta^3\mu_3, \ldots, \beta^k\mu_k) = \beta^{nk}\psi_k(n, m_1, \mu_2, \mu_3, \ldots, \mu_k) \tag{2.148}$$

and as a consequence,

$$\Psi_k(t, \beta m_1, \beta^2 m_2, \ldots, \beta^k m_k) = \Psi_k(\beta^k t, m_1, m_2, \ldots, m_k). \tag{2.149}$$

Similarly for the non-symmetric case,

$$\psi_k(n, p, \beta m_1, \beta^2\mu_2, \beta^3\mu_3, \ldots, \beta^k\mu_k) = \beta^{pk}\psi_k(n, m_1, \mu_2, \mu_3, \ldots, \mu_k) \tag{2.150}$$

and

$$\Psi_k(t, \beta m_1, \beta^2\mu_2, \ldots, \beta^k\mu_k) = \Psi_k(\beta^k t, m_1, \mu_2, \ldots, \mu_k). \tag{2.151}$$

Proof. We only show the statement for the symmetric case. Write $X_{ij}^* = \beta X_{ij}$, $m_r^* = \mathbb{E}\,(X_{ij}^*)^r$, $A^* = (X_{ij}^*)_{n\times n}$. Note that $\mu_r^* = \mathbb{E}\,(X_{ij}^* - m_1^*)^r = \beta^r \mathbb{E}\,(X_{ij} - m_1)^r = \beta^r\mu_r$. Therefore, on one hand,

$$\mathbb{E}\,(\det A^*)^k = \psi_k(n, m_1^*, \mu_2^*, \ldots, \mu_k^*) = \psi_k(n, \beta m_1, \beta^2\mu_2, \ldots, \beta^k\mu_k), \tag{2.152}$$

on the other hand, by linearity of determinants,

$$\mathbb{E}\,(\det A^*)^k = \beta^{nk}\mathbb{E}\,(\det A)^k = \beta^{nk}\psi_k(n, m_1, \mu_2, \ldots, \mu_k). \tag{2.153}$$

■

Corollary 112.1. *Assume we know $f_k(n)$ and $F_k(t)$ with $\mu_2 = 1$, that is*

$$f_k(n)|_{\mu_2=1} = \psi_k(n, m_1, 1, \mu_3, \mu_4, \dots, \mu_k), \tag{2.154}$$
$$F_k(t)|_{\mu_2=1} = \Psi_k(t, m_1, 1, \mu_3, \mu_4, \dots, \mu_k), \tag{2.155}$$

then

$$f_k(n) = \mu_2^{nk/2}\psi_k\left(n, \frac{m_1}{\mu_2^{1/2}}, 1, \frac{\mu_3}{\mu_2^{3/2}}, \frac{\mu_4}{\mu_2^{4/2}}, \dots, \frac{\mu_k}{\mu_2^{k/2}}\right), \tag{2.156}$$
$$F_k(t) = \Psi_k\left(\mu_2^{k/2}t, \frac{m_1}{\mu_2^{1/2}}, 1, \frac{\mu_3}{\mu_2^{3/2}}, \frac{\mu_4}{\mu_2^{4/2}}, \dots, \frac{\mu_k}{\mu_2^{k/2}}\right). \tag{2.157}$$

Similarly

$$f_k(n,p) = \mu_2^{pk/2}\psi_k\left(n, \frac{m_1}{\mu_2^{1/2}}, 1, \frac{\mu_3}{\mu_2^{3/2}}, \frac{\mu_4}{\mu_2^{4/2}}, \dots, \frac{\mu_k}{\mu_2^{k/2}}\right), \tag{2.158}$$
$$F_k(t,\omega) = \Psi_k\left(\mu_2^{k/2}t, \omega, \frac{m_1}{\mu_2^{1/2}}, 1, \frac{\mu_3}{\mu_2^{3/2}}, \frac{\mu_4}{\mu_2^{4/2}}, \dots, \frac{\mu_k}{\mu_2^{k/2}}\right). \tag{2.159}$$

Remark 113. Without loss of generality, we put $\mu_2 = 1$ from now on (if not stated differently).

It is convenient to extend the definition of random matrix moments to Y_{ij} variables:

Definition 114. From $Y_{ij}'s$, we construct two (random) matrices $B = (Y_{ij})_{n\times n}$ and $V = (Y_{ij})_{n\times p}$. Let $g_k(n) = \mathbb{E}\,(\det B)^k$ and $g_k(n,p) = \mathbb{E}\,(\det V^\top V)^{k/2}$ be their $k-$th determinant moment and k-th Gram moment, respectively. By definition, we set $g_k(0) = 1$ and $g_k(n,0) = 1$ (we put $\det(V^\top V) = 1$ when $p = 0$). Also, we define their corresponding (formal) generating functions

$$G_k(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!^2}g_k(n), \qquad G_k(t,\omega) = \sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{(n-p)!}{n!p!}t^p\omega^{n-p}g_k(n,p). \tag{2.160}$$

Proposition 115. *For any distribution of X_{ij},*

$$G_k(t) = F_k(t)|_{m_1=0, m_r\to\mu_r} = F_k^{\text{cen}}(t)|_{m_r\to\mu_r},$$
$$G_k(t,\omega) = F_k(t,\omega)|_{m_1=0, m_r\to\mu_r} = F_k^{\text{cen}}(t,\omega)|_{m_r\to\mu_r}.$$

Proof. We only show the square matrix case, since the Gram case is analogous. Since Y_{ij}'s are independent and identically distributed (i.i.d.), we may replace X_{ij} by Y_{ij} and $m_r = \mathbb{E}\,X_{ij}^r$ by $\mu_r = \mathbb{E}\,Y_{ij}^r$ in the definition of $F_k(t)$. Hence

$$G_k(t) = \Phi_k(t, \mu_1, \mu_2, \dots, \mu_{k-1}, \mu_k), \tag{2.161}$$

which is equal to the right hand side of the proposition as $\mu_1 = 0$. ∎

2.4.2 Finite decomposition

Crucial observation, which appeared in our previous work [8], states

Proposition 116. *Let ψ_k and Ψ_k as in Proposition 111, then ψ_k and Ψ_k are polynomials in m_1 upto order k only. That is, there exist functions $t_k^r(n)$ and $T_k^r(t), r = 0, \dots k$ whose only parameters are $\mu_2, \dots, \mu_k$, such that*

$$F_k(t) = \sum_{r=0}^{k} m_1^r T_k^r(t) \qquad \textit{or equivalently,} \qquad f_k(n) = \sum_{r=0}^{k} m_1^r t_k^r(n). \tag{2.162}$$

The fact that the expansion of Ψ_k in m_1 is finite enabled us to deduce $F_4(t)$ and $F_4(t,\omega)$ in general (see B. [8]). The former was deduced using recursions for $T_k^r(t)$. The latter, however, was much harder to deduce because we thought there is no analogue of Proposition 116 applicable for $F_6(t,\omega)$. Instead, we showed that the Binomial expansion of $F_4(t,\omega)$, that is

$$F_4(t,\omega) = \sum_{j=0}^{\infty} \omega^j \left(\frac{1-\mu_2^2 t}{1-\omega-\mu_2^2 t} \right)^{j+1} \Phi_j(t) \tag{2.163}$$

with expansion functions $\Phi_j(t)$ (not related with Φ_k introduced earlier), has in fact finitely many terms. The proof of this assertion is highly nontrivial and combines the Cauchy-Binet formula with the Binomial transform, which we showed they are in fact closely related, as well as various recursion relations. For an interested reader, we reffer to our original work [8]. In this thesis, we show a much simpler proof based on marked permutation tables. It turned out that the functions $T_k^r(t)$ in the $F_k(t)$ expansion are generating functions associated to some kind of combinatorial construction similar to that of Niquist, Rice and Riordan [50].

In this thesis, we shall interpret the combinatorial meaning of functions $T_k^r(t)$ using permutation tables construction, we first show that $\det A$ is in fact linear in m_1. To see this, we use the following lemma.

Lemma 117.
$$\sum_{\pi \in P_n} \operatorname{sgn} \pi = \begin{cases} 1, & n \in \{0,1\} \\ 0, & n \geq 2 \end{cases}$$

Proposition 118.

$$\det A = \sum_{\pi \in P_n} \operatorname{sgn} \pi \prod_{i \in [n]} Y_{i\pi(i)} \; + \; m_1 \sum_{j \in [n]} \sum_{\pi \in P_n} \prod_{i \in [n] \setminus \{j\}} Y_{i\pi(i)}. \tag{2.164}$$

Proof. Write $X_{ij} = Y_{ij} + m_1$ in the definition of determinant. Multiplying everything out,

$$\begin{aligned} \det A &= \sum_{\pi \in P_n} \operatorname{sgn} \pi \prod_{i \in [n]} X_{i\pi(i)} \\ &= \sum_{\pi \in P_n} \operatorname{sgn} \pi \prod_{i \in [n]} (Y_{i\pi(i)} + m_1) = \sum_{\pi \in P_n} \operatorname{sgn} \pi \sum_{M \subseteq [n]} \prod_{i \in [n] \setminus M} m_1^{\#M} Y_{i\pi(i)}, \end{aligned}$$

where $\#M$ denotes the number of elements in M. However, by Lemma 117, the terms with $\#M \geq 2$ vanish. ■

Note that the proposition is actually a special case of the Matrix Determinant Lemma

Lemma 119 (Matrix Determinant Lemma)**.** *Let $C = (c_{ij})_{n\times n}$ be any real matrix, $u = (u_i)_{n\times 1}$, $v = (v_i)_{n\times 1}$ real (column) vectors and $\lambda \in \mathbb{R}$, then*

$$\det\left(C + \lambda u v^\top\right) = (\det C) + \lambda v^\top C^{\mathrm{adj}} u, \tag{2.165}$$

*where $(C^{\mathrm{adj}})_{ij} = (-1)^{i+j}\det C_{ji}$ is called the **adjugate matrix** of C and C_{ji} denotes a matrix formed from C by deleting its j-th row and i-th column, as usual.*

In fact, we have

Proposition 120.

$$\det A = \det(B) + m_1 S, \qquad \textit{where} \qquad S = \sum_{ij}(-1)^{i+j}\det(B_{ij}). \tag{2.166}$$

Proof. By the definition of Y_{ij}'s and B, we can write

$$A = B + m_1 u u^\top, \tag{2.167}$$

where u is a column vector with n rows having all components equal to one. Hence, by Lemma 117,

$$\begin{aligned}\det A &= \det(B + m_1 u u^\top) = (\det B) + m_1 u^\top B^{\mathrm{adj}} u \\ &= (\det B) + m_1 \sum_{ij} u_i (-1)^{i+j}\det(B_{ji})\, u_j = \det(B) + m_1 S.\end{aligned} \tag{2.168}$$

■

Remark 121. Also note that S can be expressed as $S = \sum_{s=1}^{n}\det B^{[s]}$, where we define

$$B^{[s]} = \begin{pmatrix} Y_{11} & Y_{12} & \dots & Y_{1,n-1} & Y_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_{s-1,1} & Y_{s-1,2} & \dots & Y_{s-1,n-1} & Y_{s-1,n} \\ 1 & 1 & \dots & 1 & 1 \\ Y_{s+1,1} & Y_{s+1,2} & \dots & Y_{s+1,n-1} & Y_{s+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Y_{n1} & Y_{n2} & \dots & Y_{n,n-1} & Y_{nn} \end{pmatrix}. \tag{2.169}$$

Corollary 121.1. *Rising Proposition 120 to the k-th power and taking expectation, we get*

$$f_k(n) = \mathbb{E}\,(\det A)^k = \mathbb{E}\,\left(\det(B) + m_1 S\right)^k = \sum_{r=0}^{k}\binom{k}{r} m_1^r\,\mathbb{E}\,(\det B)^{k-r} S^r. \tag{2.170}$$

This statement already proves Proposition 116, we take

$$t_k^r(n) = \binom{k}{r}\mathbb{E}\,(\det B)^{k-r} S^r, \qquad T_k^r(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!^2} t_k^r(n). \tag{2.171}$$

2.4.3 Marked permutations and tables

In this section, we uncover combinatorial nature of the functions $t_k^r(n)$ and $T_k^r(n)$.

Definition 122. We say σ is a *marked permutation* if it was formed from some $\pi \in P_n$ in which we replaced at most one element by the mark "$\times$". We define $\operatorname{sgn}\sigma = \operatorname{sgn}\pi$ and $Y_{i\sigma(i)}^{\times} = m_1$ if i is marked and $Y_{i\sigma(i)}^{\times} = Y_{i\pi(i)}$ otherwise. We write $P_n^{\times}$ for the set of all marked permutations.

Proposition 123. *Restating Proposition 118 in terms of marked permutations.*

$$\det A = \sum_{\sigma \in P_n^{\times}} \operatorname{sgn}\sigma \prod_{i=1}^{n} Y_{i\sigma(i)}^{\times}.$$

Definition 124. We say t is a k by n marked table, if its rows are marked permutations $\sigma_j, j = 1, \ldots, k$ of order n. We denote $G_{k,n}^{\times}$ the set of all such tables (in the case the order of columns is irrelevant, we write $\mathcal{G}_{k,n}^{\times}$ instead). We define the marked weight $w_\times$ of the i-th column of $\tau \in G_{k,n}^{\times}$ as the expectation $\mathbb{E}\prod_{j=1}^{k} Y_{i\sigma_j(i)}^{\times}$. Similarly, we define the sign $\operatorname{sgn}(\tau)$ of table τ as the product of signs of $\sigma_j, j = 1, \ldots, k$. Also, we define the marked weight $w_\times(t)$ of the whole table τ as the product of weights of its individual columns. Finally, we define another weight $w(\tau)$ as $w_\times(t)$ in which we put $m_1 = 1$.

Example 125. The following figures show two example marked permutation tables.

1	×	3	4	5	2	7	8	9
3	2	1	9	4	6	7	5	8
1	×	3	9	4	2	7	5	8
3	2	1	4	5	6	7	8	9

Figure 2.23: An example of a table $\tau \in G_{4,9}^2$ with weights $w_\times(t) = m_1^2\mu_2^{15}\mu_4$ and $w(\tau) = \mu_2^{15}\mu_4$,

×	2	3	4	5	6	7	8	9
×	2	1	9	4	6	7	5	8
2	×	1	9	4	6	7	5	8
2	×	3	4	5	6	7	8	9

Figure 2.24: An example of a table $\tau \in G_{4,9}^4$ with weights $w_\times(t) = m_1^4\mu_2^{12}\mu_4^2$ and $w(\tau) = \mu_2^{12}\mu_4^2$.

Remark 126. Note that since only one number is marked in any row, we can recover the original permutation in the row uniquely. That is, a column of a table $\tau \in G_{k,n}^{\times}$ is created by marking an (unmarked) original column of a corresponding table in $F_{k,n}$. Alternatively, we can show the numbers covered by marks in a special curly bracket column alongside tables. The same tables as above would look like

{ }									
6	1	×	3	4	5	2	7	8	9
	3	2	1	9	4	6	7	5	8
6	1	×	3	9	4	2	7	5	8
	3	2	1	4	5	6	7	8	9

{ }									
1	×	2	3	4	5	6	7	8	9
3	×	2	1	9	4	6	7	5	8
3	2	×	1	9	4	6	7	5	8
1	2	×	3	4	5	6	7	8	9

Proposition 127. *For any distribution X_{ij},*

$$f_k(n) = \mathbb{E}(\det A)^k = \sum_{\tau \in G_{k,n}^{\times}} w_\times(t)\operatorname{sgn}(\tau). \tag{2.172}$$

2.4.4 Nontrivial marked tables

Since $\mu_1 = 0$, it turns out that most tables in $G^{\times}_{k,n}$ have zero weight.

Definition 128. We say a table $\tau \in G^{\times}_{k,n}$ is trivial if its weight vanishes, otherwise the table is nontrivial. The set all all nontrivial tables form a subset $T^{\times}_{k,n} \subseteq G^{\times}_{k,n}$. Similarly for tables with irrelevant column order, we write $\mathcal{T}^{\times}_{k,n} \subseteq \mathcal{G}^{\times}_{k,n}$.

Proposition 129. *For any distribution X_{ij},*

$$f_k(n) = \mathbb{E}\,(\det A)^k = \sum_{\tau \in T^{\times}_{k,n}} w_{\times}(t)\,\mathrm{sgn}(\tau). \tag{2.173}$$

Example 130. The correspondence between $f_k(n)$ and marked permutation tables is shown below in Figure 2.25 for $n = 2$ and $k = 2$ showing once again $f_2(2) = 2(m_2^2 - m_1^4)$. Note that in the expansion of $\det A$, only terms which give nonzero expectation are listed.

$(\det A)^2 =$	$Y_{11}^2 Y_{22}^2$	$+\ Y_{12}^2 Y_{21}^2$	$+\ m_1^2 Y_{22}^2$	$+\ m_1^2 Y_{21}^2$	$+\ Y_{11}^2 m_1^2$	$+\ Y_{12}^2 m_1^2$
$T^{\times}_{2,2}$:	$\begin{array}{\|c\|c\|}1 & 2 \\ 1 & 2\end{array}$	$\begin{array}{\|c\|c\|}2 & 1 \\ 2 & 1\end{array}$	$\begin{array}{\|c\|c\|}\times & 2 \\ \times & 2\end{array}$	$\begin{array}{\|c\|c\|}\times & 1 \\ \times & 1\end{array}$	$\begin{array}{\|c\|c\|}1 & \times \\ 1 & \times\end{array}$	$\begin{array}{\|c\|c\|}2 & \times \\ 2 & \times\end{array}$
$w_{\times}$:	$\mu_2\mu_2$	$\mu_2\mu_2$	$m_1^2\mu_2$	$m_1^2\mu_2$	$\mu_2 m_1^2$	$\mu_2 m_1^2$
Sign:	$+$	$+$	$+$	$+$	$+$	$+$

Figure 2.25: Correspondence between determinant moment $f_2(2)$ and marked permutation tables $T^{\times}_{2,2}$

By summing the contribution from all marked tables, we get

$$f_2(2) = 2\mu_2(\mu_2 + 2m_1^2) = 2!(m_2 - m_1^2)(m_2 + m_1^2), \tag{2.174}$$

where we have used $\mu_2 = m_2 - m_1^2$.

Remark 131. In terms of tables with irrelevant column order, Proposition 129 can be written alternatively $f_k(n) = n!\hat{f}_k(n)$, where by definition

$$\hat{f}_k(n) = \sum_{\tau \in \mathcal{T}^{\times}_{k,n}} w_{\times}(t)\,\mathrm{sgn}(\tau). \tag{2.175}$$

Example 132. Let us derive $f_4(2)$. We may write $f_4(2) = 2!\hat{f}_4(2)$ and sum the contribution from tables $\mathcal{T}^{\times}_{4,2}$ with irrelevant column order. Figure 2.26 below shows the members of $\mathcal{T}^{\times}_{4,2}$ with the corresponding sign and weight including multiplicity (#) as the members are displayed apart of permutation of rows and substitution of $\{1, 2\}$ for elements $\{a, b\}$.

$\mathcal{T}_{4,2}^{\times}$:											
	$a\ b$	$a\ b$	$\times\ b$	$\times\ b$	$\times\ b$	$\times\ b$	$\times\ b$	$\times\ b$	$\times\ b$	$\times\ b$	$\times\ b$
	$a\ b$	$a\ b$	$a\ b$	$\times\ b$	$\times\ b$	$a\ \times$	$a\ \times$	$\times\ b$	$\times\ b$	$\times\ b$	$\times\ b$
	$a\ b$	$b\ a$	$a\ b$	$a\ b$	$b\ a$	$a\ b$	$a\ \times$	$\times\ b$	$\times\ a$	$b\ \times$	$a\ \times$
	$a\ b$	$b\ a$	$a\ b$	$a\ b$	$b\ a$	$a\ b$	$a\ b$	$\times\ b$	$\times\ a$	$b\ \times$	$a\ \times$
$w_\times$:	μ_4^2	μ_2^4	$m_1\mu_3\mu_4$	$m_1^2\mu_2\mu_4$	$m_1^2\mu_2^3$	$m_1^2\mu_3^2$	$m_1^3\mu_2\mu_3$	$m_1^4\mu_4$	$m_1^4\mu_2^2$	$m_1^4\mu_2^2$	$m_1^4\mu_2^2$
Sign:	+	+	+	+	+	+	+	+	+	+	+
#	1	3	8	12	12	12	24	2	6	6	6

Figure 2.26: Correspondence between determinant moment $f_4(2)$ and marked permutation tables $\mathcal{T}_{4,2}^{\times}$

By summing the contribution from all marked tables, we get

$$\begin{aligned}\hat{f}_4(2) = \mu_4^2 + 3\mu_2^4 + 8m_1\mu_3\mu_4 + 12m_1^2\mu_2\mu_4 + 12m_1^2\mu_2^3 \\ + 12m_1^2\mu_3^2 + 24m_1^3\mu_2\mu_3 + 2m_1^4\mu_4 + 18m_1^4\mu_2^2.\end{aligned} \tag{2.176}$$

Plugging $\mu_2 = m_2 - m_1^2$, $\mu_3 = m_3 - 3m_2m_1 + 2m_1^3$, $\mu_4 = m_4 - 4m_3m_1 + 6m_2m_1^2 - 3m_1^4$ and expanding, we get

$$f_4(2) = 2!\hat{f}_4(2) = 2(m_4^2 - 4m_1^2m_3^2 + 3m_2^4), \tag{2.177}$$

which coincides with the introductory Example 42.

Definition 133. Let $T_{k,n}^r \subseteq T_{k,n}^{\times}$ be the subset of those tables which have exactly r marks.

Definition 134. In accordance with Beck, Lv and Potechin [5], we write $T_{k,n} = T_{k,n}^0$ for nontrivial permutation tables (with no marks). More generally, we often omit r when we have $r = 0$.

Definition 135. We define

$$t_k^r(n) = \sum_{\tau \in T_{k,n}^r} w(\tau)\,\mathrm{sgn}(\tau) \tag{2.178}$$

and its corresponding generation function

$$T_k^r(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} t_k^r(n). \tag{2.179}$$

Proposition 136. *For any distribution X_{ij},*

$$f_k(n) = \sum_{r=0}^{k} m_1^r t_k^r(n) \qquad \textit{and thus} \qquad F_k(t) = \sum_{r=0}^{k} m_1^r T_k^r(t). \tag{2.180}$$

Proof. Write

$$\sum_{\tau \in T_{k,n}^{\times}} w_\times(t)\,\mathrm{sgn}(\tau) = \sum_{r=0}^{k} \sum_{\tau \in T_{k,n}^r} w_\times(t)\,\mathrm{sgn}(\tau) = \sum_{r=0}^{k} m_1^r \sum_{\tau \in T_{k,n}^r} w(\tau)\,\mathrm{sgn}(\tau). \tag{2.181}$$

■

2.4.5 Second moment general (alternative proof)

We are ready to rederive Fortet's formula (Proposition 67) in by means of marked permutations tables. Restating the proposition,

Proposition 137 (Fortet [32]). *For any distribution of X_{ij},*

$$f_2(n) = n!(m_2 + m_1^2(n-1))(m_2 - m_1^2)^{n-1}, \tag{2.182}$$

$$F_2(t) = (1 + m_1^2 t)e^{(m_2 - m_1^2)t}. \tag{2.183}$$

Proof. We consider writing $f_2(n)$ as a sum over all (nontrivial) marked permutations tables $T_{2,n}^{\times}$ with two rows and n columns. Note that $T_{2,n}^{\times} = T_{2,n}^0 \sqcup T_{2,n}^1 \sqcup T_{2,n}^2$ and thus

$$f_2(n) = t_2^0(n) + m_1 t_2^1(n) + m_1^2 t_2^2(n). \tag{2.184}$$

However, as $\mu_1 = 0$, we get $t_2^1(n) = 0$ since $T_{2,n}^1$ is empty. In fact, there are only two types of columns which give rise to nontrivial weights, namely the 2-column (the same two numbers a) and the $\times^2$-column (the same two numbers marked).

Type:	2-column	$\times^2$-column
$T_2^{\times}$:	a a	$\times$ $\times$
$w_{\times}$:	μ_2	m_1^2

There are $n!$ possibilities how the permutation in the first row of a table can look like. Since tables $T_{2,n}^0$ only contain 2-columns, the second row must be filled with the same permutation (thus the sign is always positive). This gives us the factor

$$t_2^0(n) = n!\mu_2^n \tag{2.185}$$

to the overall sum $f_2(n) = \sum_{\tau \in T_{2,n}^{\times}} w_{\times}(t) \operatorname{sgn} \tau$. As a consequence,

$$T_2^0(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} t_2^0(n) = e^{\mu_2 t}. \tag{2.186}$$

Let $\tau' \in T_{2,n}^0$ have c two-columns. The weight of this table is given as $w(\tau') = \mu_2^c$. Note that for the weights $w(\tau)$ of a marked table $\tau \in T_{2,n}^2$ created from τ' by marking one of its two-columns by two marks, we have $w(\tau) = \mu_2^{c-1}$. Thus, from τ', we get the following contribution to $t_2^2(n) = \sum_{\tau \in T_{2,n}^2} w(\tau) \operatorname{sgn} \tau$,

$$c\mu_2^{c-1}. \tag{2.187}$$

Summing up this contribution over all tables $\tau' \in T_{2,n}^0$, we get

$$t_2^2(n) = \frac{\partial t_2^0(n)}{\partial \mu_2} = n! n \mu_2^{n-1} \tag{2.188}$$

or in terms of generating functions,

$$T_2^2(t) = \frac{\partial T_2^0(t)}{\partial \mu_2} = te^{\mu_2 t}, \tag{2.189}$$

from which

$$F_2(t) = T_2^0(t) + m_1 T_2^1(t) + m_1^2 T_2^2(t) = (1 + m_1^2 t)e^{\mu_2 t} = (1 + m_1^2 t)e^{(m_2 - m_1^2)t}. \tag{2.190}$$

Alternatively, since there can be only one mark per row, there can be only one $\times^2$-column. For a given permutation of the first row, there are n positions for this marked column and thus we get the factor

$$n! n \mu_2^{n-1} m_1^2. \tag{2.191}$$

In other words, $t_2^2(n) = n! n \mu_2^{n-1}$. Summing the two factors together, we get

$$f_2(n) = n! \mu_2^n + n! n \mu_2^{n-1} m_1^2 = n!(\mu_2 + n m_1^2)\mu_2^{n-1}, \tag{2.192}$$

from which follows the first assertion by putting $\mu_2 = m_2 - m_1^2$. Note that we can also get $F_2(t)$ by directly inserting $f_2(n)$ into the definition of generating function $F_2(t)$.

Yet another, the shortest proof is given by analytic combinatorics. Note that we can formally write the following construction relation for the set $\mathcal{T}_2^\times$ of all marked tables (with column order irrelevant)

$$\mathcal{T}_2^\times = \mathcal{T}_2^0 + \mathcal{T}_2^0 \star \boxed{\begin{matrix}\times \\ \times\end{matrix}} = \textsc{Set}\left(\boxed{\begin{matrix}1 \\ 1\end{matrix}}\right) \star \left(\emptyset + \boxed{\begin{matrix}\times \\ \times\end{matrix}}\right) \tag{2.193}$$

from which immediatelly, in terms of generating functions,

$$F_2(t) = \exp(\mu_2 t)(1 + m_1^2 t). \tag{2.194}$$

■

2.4.6 Even marked tables

Definition 138. We denote $S_{k,n}^\times$ as the subset of tables $T_{k,n}^\times$ which have still nonzero weight when $\mu_3 = \mu_5 = \cdots = 0$. As a consequence, their columns have only even number of marks (since k is even). We call these tables *even (marked)* tables or simply S tables.

Definition 139. For any distribution of X_{ij}, we write

$$s_k^r(n) = \sum_{\tau \in S_{k,n}^r} w(\tau)\,\mathrm{sgn}(\tau) \tag{2.195}$$

and its corresponding generation function

$$S_k^r(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} s_k^r(n). \tag{2.196}$$

Remark 140. Summing over $S^{\times}_{k,n}$ is equivalent of summing over $T^{\times}_{k,n}$ and then putting $\mu_3 = \mu_5 = \cdots = 0$. That is $s^r_k(n) = t^r_k(n)|_{\mu_3=\mu_5=\cdots=0}$ and $S^r_k(t) = T^r_k(t)|_{\mu_3=\mu_5=\cdots=0}$.

Remark 141. In the case of the sixth moment, even marked tables are the notrivial tables under intermedial distribution of entries (with $\mu_3 = 0$).

2.4.7 Shifted normal moments

In the case $Y_{ij} \sim \mathsf{N}(0,1)$, or equivalently $X_{ij} \sim \mathsf{N}(m_1,1)$, we can find $f_k(n)$ and thus $t^r_k(n)$ and $T^r_k(t)$ explicitely for all r and k.

Definition 142. When $Y_{ij} \sim \mathsf{N}(0,1)$, we denote $t^r_k(n)$ as $n^r_k(n)$ and $T^r_k(t)$ as $N^r_k(t)$.

Lemma 143 (Square matrix Wishart expansion)**.** *Let $X_{ij} \sim \mathsf{N}(\mu,\sigma^2)$ and $k = 2m$ be an even integer, then*

$$f_{2m}(n) = \sigma^{2mn}\left(\prod_{j=0}^{m-1}\frac{(n+2j)!}{(2j)!}\right)\sum_{s=0}^{m}\binom{m}{s}\frac{(n-2)!!}{(n+2s-2)!!}\left(\frac{n\mu}{\sigma}\right)^{2s}. \tag{2.197}$$

The lemma is a special case of Theorem 181 with $n = p$ and follows from known properties of the non-central Wishart distribution (see Theorem 10.3.7 in [49]).

Corollary 143.1. *Selecting $\mu = m_1$ and $\sigma = 1$, and comparing m_1 powers, we get that when k or r is odd, then $n^r_k(n)$ and $N^r_k(t)$ vanish. Otherwise, if $k = 2l$ and $r = 2s$ for some integers m, s, then*

$$n^{2s}_{2l}(n) = \binom{l}{s}\frac{n^{2s}(n-2)!!}{(n+2s-2)!!}\prod_{j=0}^{l-1}\frac{(n+2j)!}{(2j)!}. \tag{2.198}$$

Proposition 144. *When $Y_{ij} \sim \mathsf{N}(0,1)$, then $s^r_k(n) = n^r_k(n)$. Or in terms of generating functions, $S^r_k(t) = N^r_k(t)$.*

Proof. We already know that by definition for any distribution of X_{ij},

$$t^r_k(n) = \sum_{\tau\in T^r_n} w(\tau)\,\mathrm{sgn}(\tau). \tag{2.199}$$

However, since $\mu_3 = \mu_5 = \cdots = 0$ in the case of the normal distribution, we have that all tables in $T^r_{k,n}/S^r_{k,n}$ are trivial (their weight equals zero), as the weight $w(\tau)$ of each column with odd number of marks vanishes. ■

Fourth shifted normal moment

We will examine two special cases. First, when, $k = 4$, we have the following:

Proposition 145.

$$n^0_4(n) = \frac{n!(n+2)!}{2}, \qquad n^2_4(n) = n!(n+2)!\,n, \qquad n^4_4(n) = \frac{n^3 n!(n+1)!}{2}.$$
$$n^1_4(n) = 0, \qquad n^3_4(n) = 0,$$

Corollary 145.1. *By summing the series,*

$$N_4^0(t) = \frac{1}{(1-t)^3}, \qquad N_4^2(t) = \frac{6t}{(1-t)^4}, \qquad N_4^4(t) = \frac{t\,(1+7t+4t^2)}{(1-t)^5}.$$
$$N_4^1(t) = 0, \qquad N_4^3(t) = 0,$$

Sixth shifted normal moment

Next, when $k = 6$,

Proposition 146.

$$n_6^0(n) = \frac{n!(n+2)!(n+4)!}{48}, \qquad n_6^2(n) = \frac{n!(n+2)!(n+4)!\,n}{16},$$
$$n_6^4(n) = \frac{n^3 n!(n+1)!(n+4)!}{16}, \qquad n_6^6(n) = \frac{n^5 n!(n+1)!(n+3)!}{48},$$
$$n_6^1(n) = n_6^3(n) = n_6^5(n) = 0.$$

Proof. Directly from Corollary 143.1 or by Lemma 143 with $\mu = m_1$ and $\sigma = 1$,

$$f_6(n) = \tfrac{n!(n+1)!(n+3)!}{48}\left((n+2)(n+4) + 3m_1^2 n(n+2)(n+4) + 3m_1^4 n^3(n+4) + m_1^6 n^5\right).$$

Comparing m_1 powers, we get $n_6^r(n)$ for $r = 0, \ldots, 6$. ■

Corollary 146.1. *By summing the series*

$$N_6^0(t) = \frac{1}{48}\sum_{n=0}^{\infty}(n+1)(n+2)(n+4)!\,t^n,$$
$$N_6^2(t) = \frac{1}{16}\sum_{n=0}^{\infty} n(n+1)(n+2)(n+4)!\,t^n,$$
$$N_6^4(t) = \frac{1}{16}\sum_{n=0}^{\infty} n^3(n+1)(n+4)!\,t^n,$$
$$N_6^6(t) = \frac{1}{48}\sum_{n=0}^{\infty} n^5(n+1)(n+3)!\,t^n,$$
$$N_6^1(t) = N_6^3(t) = N_6^5(t) = 0.$$

Remark 147. Note that those series (with even r) have zero radius of convergence, so they have to be treated formally.

Remark 148. Note that those auxiliary series are not independent. For example, it holds

$$N_6^2(t) = 3t\,\frac{\mathrm{d}}{\mathrm{d}t}N_6^0(t), \tag{2.200}$$
$$t^2\frac{\mathrm{d}}{\mathrm{d}t}N_6^2(t) = (1-8t)\,N_6^2(t) - 45t\,N_6^0(t), \tag{2.201}$$
$$t\,N_6^4(t) = \left(4t^2 - 10t + 1\right)N_6^2(t) + 15t(4t-3)N_6^0(t), \tag{2.202}$$
$$3t^2 N_6^6(t) = \left(1 - 23t + 125t^2 - 120t^3\right)N_6^2(t) - 3t\left(15 - 210t + 344t^2\right)N_6^0(t). \tag{2.203}$$

Note that these are not the only relations between $N_6^r(t)$'s. For example, $N_6^2(t)$ can be also expressed using $N_6^0(t)$ but the relation involves rather complicated rational functions.

2.4.8 Odd-end marked tables

Definition 149. We say a marked column is $\times_{k-1}^1$, when it is consisted of one mark and $k-1$ copies of some identical number.

Definition 150. We denote $O_{k,n}^\times$ as the subset of *nontrivial* tables $T_{k,n}^\times$ which lack $\times_{k-1}^1$ columns. Equivalently, $O_{k,n}^\times$ are the tables which stays nontrivial when $\mu_{k-1} = 0$. We call these tables as *odd-end* marked tables or simply O tables.

Definition 151. For any distribution of X_{ij}, we write

$$o_k^r(n) = \sum_{\tau \in O_{k,n}^r} w(\tau)\,\mathrm{sgn}(\tau) \tag{2.204}$$

and its corresponding generation function

$$O_k^r(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} o_k^r(n). \tag{2.205}$$

Remark 152. Summing over $O_{k,n}^\times$ is equivalent of summing over $T_{k,n}^\times$ if we put $\mu_{k-1} = 0$, that is, $o_k^r(n) = t_k^r(n)|_{\mu_{k-1}=0}$ and $O_k^r(t) = T_k^r(t)|_{\mu_{k-1}=0}$. Note that, however, as μ_{k-1} always appear alongside with μ_1 in unmarked tables, $o_k^0(n) = t_k^0(n)$ and $O_k^0(t) = T_k^r(t)$.

Remark 153. In any nontrivial table, every $\times_{k-1}^1$ column must be originally a column with k identical copies of the same number a that gets covered by a mark. The other option would be that the covered number is different. That would mean, however, these is a single displaced number a elsewhere in the table. But since $\mu_1 = 0$, this would turn the table to be trivial (zero weight).

2.4.9 Decomposition into odd-end marked tables

It turns out that instead of all marked tables $T_{k,n}^\times$, we can only consider the tables $O_{k,n}^\times$ as there exists a natural decomposition.

Proposition 154. *For any distribution of X_{ij} with $\mu_2 = 1$,*

$$t_k^r(n) = \sum_{s=0}^{r} \binom{k-r+s}{s} \frac{n!^2 \mu_{k-1}^s}{(n-s)!^2} o_k^{r-s}(n-s). \tag{2.206}$$

Proof. We collect the terms according to the number of $\times_{k-1}^1$ columns. Let there be r marks out of which s form $\times_{k-1}^1$ columns. According to Remark 153, they are disjoint from other columns (they do not share any numbers). We can select $\binom{n}{s}$ positions for those columns. Also, $\binom{n}{s}$ is the number of selections of concrete numbers from $[n] = \{1, 2, 3, \ldots, n\}$ to fill in those $\times_{k-1}^1$ columns, there are then $s!$ permutations of those numbers. Erasing $\times_{k-1}^1$ columns, what we are left with is a table $\tau' \in O^{r-s}_{k,n-s}$ (lacking $\times^1_{k-1}$ columns). To construct a table $\tau \in T^r_{k,n}$ from τ', we select rows where the marks of $\times^1_{k-1}$ columns will be. Since $r-s$ marks are already placed, there are $\binom{k-(r-s)}{s}$ ways how to place our s marks. Thus

$$\sum_{\tau \in T^r_{k,n}} w(\tau)\operatorname{sgn}(\tau) = \sum_{s=0}^{r} \binom{n}{s}^2 \binom{k-r+s}{s} s!^2 \mu^s_{k-1} \sum_{\tau' \in O^{r-s}_{n-s}} w(\tau')\operatorname{sgn}(\tau'). \tag{2.207}$$

$$\left\{\begin{array}{c}6\\7\\6\\3\end{array}\right\}\left\{\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \mathbf{3} & \times & 1 & 4 & 5 & 2 & \mathbf{7} & 8 & 9\\ \hline \mathbf{3} & 2 & 1 & 9 & 4 & 6 & \times & 5 & 8\\ \hline \mathbf{3} & \times & 1 & 9 & 4 & 2 & \mathbf{7} & 5 & 8\\ \hline \times & 2 & 1 & 4 & 5 & 6 & \mathbf{7} & 8 & 9\\ \hline\end{array}\right. \leftarrow \left\{\begin{array}{c}6\\ \\6\\ \\ \end{array}\right\}\begin{array}{|c|c|c|c|c|c|c|}\hline \times & 1 & 4 & 5 & 2 & 8 & 9\\ \hline 2 & 1 & 9 & 4 & 6 & 5 & 8\\ \hline \times & 1 & 9 & 4 & 2 & 5 & 8\\ \hline 2 & 1 & 4 & 5 & 6 & 8 & 9\\ \hline\end{array}$$

Example: $n = 9$, $r = 4$, $s = 2$

■

Corollary 154.1. *In terms of generating functions, we have for any distribution X_{ij} with $\mu_2 = 1$,*

$$T^r_k(t) = \sum_{s=0}^{r} \binom{k-r+s}{s} t^s \mu^s_{k-1} O^{r-s}_k(t). \tag{2.208}$$

Proof. Changing the order of summation, we get from the definition of $T^r_k(t)$,

$$\begin{aligned} T^r_k(t) &= \sum_{n=0}^{\infty} \frac{t^n}{n!^2} t^r_k(n) = \sum_{n=0}^{\infty}\sum_{s=0}^{r} \binom{k-r+s}{s} \frac{t^n \mu^s_{k-1}}{(n-s)!^2} o^{r-s}_k(n-s) \\ &= \sum_{s=0}^{r} \binom{k-r+s}{s} t^s \mu^s_{k-1} O^{r-s}_k(t). \end{aligned} \tag{2.209}$$

■

Corollary 154.2. *In terms of generating functions, we have for any distribution X_{ij} with $\mu_2 = 1$,*

$$F_k(t) = \sum_{r=0}^{k} m^r_1 (1 + m_1 \mu_{k-1} t)^{k-r} O^r_k(t), \tag{2.210}$$

Proof. Using Proposition 136 and by substitution $r = s + j$ in the previous corollary,

$$\begin{aligned} F_k(t) &= \sum_{r=0}^{k} m^r_1 T^r_k(t) = \sum_{r=0}^{k}\sum_{s=0}^{r} \binom{k-r+s}{s} m^r_1 t^s \mu^s_{k-1} O^{r-s}_k(t) \\ &= \sum_{j=0}^{k}\sum_{s=0}^{k-j} \binom{k-j}{s} m^{s+j}_1 t^s \mu^s_{k-1} O^j_k(t). \end{aligned} \tag{2.211}$$

■

To finish this section, let us state an inverse relations between tables $T^r_{k,n}$ and $O^r_{k,n}$.

Proposition 155. *In terms of generating functions, we have for any distribution X_{ij} with $\mu_2 = 1$,*

$$O_k^s(t) = \sum_{r=0}^{s} (-1)^r \binom{k-s+r}{r} t^r \mu_{k-1}^r T_k^{s-r}(t). \tag{2.212}$$

One can deduce a generalisation of Corollary 154.2, connecting generating functions for distributions with two different μ_{k-1}.

Corollary 155.1. *In terms of generating functions, we have for any distribution X_{ij},*

$$[m_1^s]F_k(t)|_{\mu_{k-1}=\mu''_{k-1}} = \sum_{r=0}^{k} m_1^r (1 + m_1(\mu''_{k-1} - \mu'_{k-1})t)^{k-r} [m_1^r]F_k(t)|_{\mu_{k-1}=\mu'_{k-1}}. \tag{2.213}$$

2.5 Fourth moment general

In this section, we derive the general fourth determinant moment:

Theorem 156 (B. 2022)**.** *For any distribution X_{ij},*

$$F_4(t) = \frac{e^{t(\mu_4 - 3\mu_2^2)}}{(1-\mu_2^2 t)^3} \left((1+m_1\mu_3 t)^4 + 6m_1^2 \mu_2 t \frac{(1+m_1\mu_3 t)^2}{1-\mu_2^2 t} + m_1^4 t \frac{1+7\mu_2^2 t + 4\mu_2^4 t^2}{(1-\mu_2^2 t)^2} \right).$$

Corollary 156.1. *For any distribution X_{ij},*

$$f_4(n) = (n!)^2 \sum_{w=0}^{2} \sum_{s=0}^{4-2w} \sum_{c=0}^{n-s} \binom{4-2w}{s} \frac{(1+c) m_1^{s+2w} \mu_2^{2c-w} \mu_3^s \left(\mu_4 - 3\mu_2^2\right)^{n-c-s}}{(n-c-s)!(2-w)!w!} d_w(c),$$

where

$$d_0(c) = (2+c), \qquad d_1(c) = c(2+c), \qquad d_2(c) = c^3.$$

When $m_1 = 0$, we recover the original special case $F_4^{\text{sym}}(t)$ derived by Nyquist, Rice and Riordan (Proposition 68).

Example 157 (General Gaussian distribution)*.* If $X_{ij} \sim N(\mu, \sigma^2)$, we have $m_1 = \mu$, $(\mu_2, \mu_3, \mu_4) = (\sigma^2, 0, 3\sigma^4)$, from which we get

$$f_4(n) = \frac{1}{2}(n!)^2 (1+n) \sigma^{4(n-1)} \left(n^3 \mu^4 + (2+n)\sigma^2 \left(2n\mu^2 + \sigma^2 \right) \right). \tag{2.214}$$

Example 158*.* ((0, 2) matrices)*.* Let $X_{ij} = 0, 2$ with equal probability, thus $(m_1, m_2, m_3, m_4) = (1, 2, 4, 8)$ and $(\mu_2, \mu_3, \mu_4) = (1, 0, 1)$. As pointed out by Terence Tao [70], the determinant of a random $n \times n$ $(-1, +1)$ matrix is equal to the determinant of a random $n-1 \times n-1$ $(0, 2)$ matrix for which $(m_1, m_2, m_3, m_4) = (0, 1, 0, 1)$. In terms of generating functions, that means

$$F_4(t) = \frac{\partial}{\partial t} \left(t \frac{\partial F_4^{\text{sym}}(t)}{\partial t} \right) = \frac{\partial}{\partial t} \left(t \frac{\partial}{\partial t} \frac{e^{-2t}}{(1-t)^3} \right) = \frac{e^{-2t}\left(1 + 5t + 2t^2 + 4t^3\right)}{(1-t)^5}, \tag{2.215}$$

where in $F_4^{\text{sym}}(t)$ we put $(m_1, m_2, m_3, m_4) = (0, 1, 0, 1)$. This result coincides exactly with our general formula for $F_4(t)$ with $(m_1, m_2, m_3, m_4) = (1, 2, 4, 8)$.

Example 159 (Exponential distribution). If $X_{ij} \sim \text{Exp}(1)$, that is if $m_j = j!$, we have $(\mu_2, \mu_3, \mu_4) = (1, 2, 9)$. For n large, we get an asymptotic behavior

$$f_4(n) = \frac{1}{2} e^6 (n!)^2 \left(n^4 - 5n^3 - 27n^2 + 141n + 450 + O(1/n) \right). \tag{2.216}$$

The first seven exact moments are shown in Table 2.2 below.

n	1	2	3	4	5	6	7
$f_4(n)$	24	960	51840	3511872	287953920	27988001280	3181325414400

Table 2.2: Fourth moment of a random determinant with entries exponentially distributed

2.5.1 Structure of marked tables

Let a, b denote different numbers selected from $[n] = \{1, 2, 3, \ldots, n\}$. Up to permutation of rows, the only way how the columns of 4 by n tables with nonzero weight could look like is the following:

Type:	4-column	2-column	$\times^1$-column	$\times^2$-column	$\times^4$-column
$\mathcal{T}_4^\times$:	a a a a	a a b b	$\times$ a a a	$\times$ $\times$ a a	$\times$ $\times$ $\times$ $\times$
Weight $w_\times$:	μ_4	1	$m_1\mu_3$	m_1^2	m_1^4

Figure 2.27: Structure of $\mathcal{T}_4^\times$ tables

See Example 132 showing the full $\mathcal{T}_{4,2}^\times$ for better illustration.

2.5.2 Odd-end tables decomposition

Since the odd-end and even marked tables coincide when $k = 4$, we can only consider the sums over even marked tables $S_4^\times \subset T_4^\times$. By Corollary 154.2,

$$F_4(t) = (1 + m_1\mu_3 t)^4 S_4^0(t) + (1 + m_1\mu_3 t)^2 m_1^2 S_4^2(t) + m_1^4 S_4^4(t). \tag{2.217}$$

As the only nonzero terms are $S_4^0(t), S_4^2(t)$ and $S_4^4(t)$ in the expansion of $F_4(t)$ (see Equation (2.217)), we have the following options (upto permutations of rows) how to nontrivially place marks in a table $\tau \in S_{4,n}^\times$.

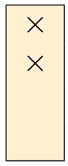


Figure 2.28: Marked columns $S^2_{4,n}$

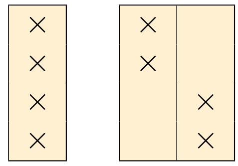


Figure 2.29: Marked columns $S^4_{4,n}$

Definition 160. We define tables $S^{r/s}_{4,n} \subseteq S^r_{4,n}$ such that their r marks occupy s columns. Accordingly, we define

$$s^{r/s}_4(n) = \sum_{\tau \in S^{r/s}_{4,n}} w(\tau)\,\mathrm{sgn}(\tau) \qquad \text{and} \qquad S^{r/s}_4(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} s^{r/s}_4(n). \tag{2.218}$$

Remark 161. Note that $S^2_{4,n} = S^{2/1}_{4,n}$ and $S^4_{4,n} = S^{4/1}_{4,n} \sqcup S^{4/2}_{4,n}$ disjoint union as shown below in Figure 2.30.

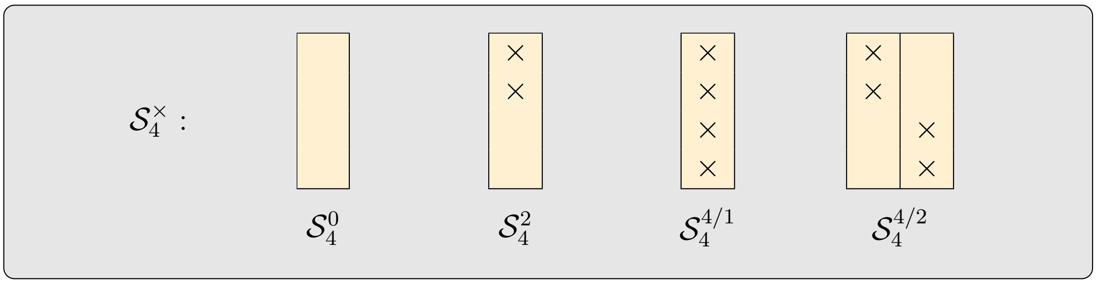


Figure 2.30: Structure of $\mathcal{S}^{\times}_4$ tables

2.5.3 Covering technique

Zero marks

The generating function $S^0_4(t)$ coincides with the already obtained $F^{\mathrm{sym}}_4(t)$ of Nyquist, Rice and Riordan (see Proposition 68) replacing m_k with μ_k, that is

$$S^0_4(t) = G_4(t) = \frac{e^{t(\mu_4 - 3)}}{(1-t)^3}. \tag{2.219}$$

Two marks

Proposition 162. *Tables $S^2_{4,n}$ are formed by marking one pair of numbers $S_{4,n}$ in a given column.*

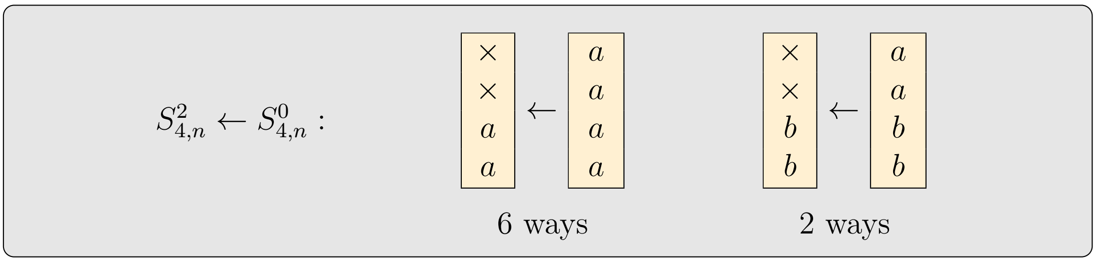
Proof. Let $\tau \in S^2_{4,n}$, then the numbers which are covered by one pair of marks are the same numbers. If they were different, say a, b there would have been another a elsewhere in the table, making the table trivial in $S^2_{4,n}$ (since we would have odd number of a's uncovered). ■

Corollary 162.1. *For any distribution X_{ij} with $\mu_2 = 1$,*

$$S^2_4(t) = (6 - 2\mu_4)\frac{\partial S^0_4(t)}{\partial \mu_4} + 2t\,\frac{\partial S^0_4(t)}{\partial t} = \frac{6t}{(1-t)^4} e^{t(\mu_4 - 3)}. \tag{2.220}$$

Proof. Let $\tau' \in S_{4,n}$ have c four-columns. Thus, there are $n-c$ two-columns. The weight of this table is given as $w(\tau') = \mu_4^c$. Let us find the weights $w(\tau)$ of all marked tables $\tau \in S_{4,n}^2$ created from τ' by marking. There are the following possibilities where we can put those two marks:

- in 4-column of τ' in 6 ways, creating a table τ with weight μ_4^{c-1},
- in 2-column of τ' in 2 ways, creating a table τ with weight μ_4^c,



Thus, from τ', we get the following contribution to $s_4^2(n) = \sum_{\tau \in S_{4,n}^2} w(\tau)\,\mathrm{sgn}(\tau)$,

$$6c\mu_4^{c-1} + 2(n-c)\mu_4^c. \tag{2.221}$$

Grouping the terms, this is equal to

$$c\mu_4^{c-1}(6-2\mu_4) + 2n\mu_4^c. \tag{2.222}$$

Summing up this contribution over all tables $\tau' \in S_{4,n}$, we get,

$$s_4^2(n) = (6-2\mu_4)\frac{\partial s_4^0(n)}{\partial \mu_4} + 2ns_4^0(n) \tag{2.223}$$

or in terms of generating functions,

$$S_4^2(t) = (6-2\mu_4)\frac{\partial S_4^0(t)}{\partial \mu_4} + 2t\,\frac{\partial S_4^0(t)}{\partial t}. \tag{2.224}$$

By substituting Equation (2.219), we get, by computing the derivatives, the statement of the corollary. ∎

Four marks

Proposition 163. *Similarly, by marking, tables $S_{4,n}^{4/1}$ are formed from $S_{4,n}$ by marking one of its columns with four marks.*

Corollary 163.1. *For any distribution X_{ij} with $\mu_2 = 1$,*

$$S_4^{4/1}(t) = (1-\mu_4)\frac{\partial S_4^0(t)}{\partial \mu_4} + t\,\frac{\partial S_4^0(t)}{\partial t} = \frac{t(1+2t)}{(1-t)^4}e^{t(\mu_4-3)}. \tag{2.225}$$

Proof. Again let $\tau' \in S_{4,n}$ have c four-columns and thus $n-c$ two-columns. Its weight is then $w(\tau') = \mu_4^c$. To create a table $\tau \in S_{4,n}^{4/1}$, we can put four marks

- in 4-column of τ' in 1 way, creating a table τ with weight μ_4^{c-1},
- in 2-column of τ' in 1 way, creating a table τ with weight μ_4^c,

$$S_{4,n}^{4/1} \leftarrow S_{4,n}^{0}: \quad \begin{array}{|c|}\hline \times \\ \times \\ \times \\ \times \\ \hline\end{array} \leftarrow \begin{array}{|c|}\hline a \\ a \\ a \\ a \\ \hline\end{array} \qquad \begin{array}{|c|}\hline \times \\ \times \\ \times \\ \times \\ \hline\end{array} \leftarrow \begin{array}{|c|}\hline a \\ a \\ b \\ b \\ \hline\end{array}$$

Thus, from τ', we get the following contribution to $s_4^{4/1}(n) = \sum_{\tau \in S_{4,n}^{4/1}} w(\tau)\,\mathrm{sgn}(\tau)$,

$$c\mu_4^{c-1} + (n-c)\mu_4^c. \tag{2.226}$$

Grouping the terms, this is equal to

$$c\mu_4^{c-1}(1-\mu_4) + n\mu_4^c. \tag{2.227}$$

Summing up this contribution over all tables $\tau' \in S_{4,n}$, we get,

$$s_4^{4/1}(n) = (1-\mu_4)\frac{\partial s_4^0(n)}{\partial \mu_4} + n s_4^0(n) \tag{2.228}$$

or in terms of generating functions,

$$S_4^{4/1}(t) = (1-\mu_4)\frac{\partial S_4^0(t)}{\partial \mu_4} + t\,\frac{\partial S_4^0(t)}{\partial t}. \tag{2.229}$$

■

Proposition 164. *Tables $S_{4,n}^{4/2}$ are formed from $S_{4,n}^{4/1}$ by swapping two marks in $\times^4$ column with a pair of numbers in some other column. Via this swapping, each table from $S_{4,n}^{4/2}$ is counted twice.*

Proof. Let $\tau \in S_{4,n}^{4/2}$. There are two options how the table can look like based on the uncovered numbers in $\times^2$ columns. Either they are the same (a) or they are different (a,b). In the first case, by swapping two marks with two a's, we get a corresponding table $\tau' \in S_{4,n}^{4/1}$ with a four-column filled with a's. In the second option, by swapping, we get a two-column with numbers a and b (see figures below).

$$\begin{array}{|c|c|}\hline \times & a \\ \times & a \\ a & \times \\ a & \times \\ \hline\end{array} \leftarrow \begin{array}{|c|c|}\hline \times & a \\ \times & a \\ \times & a \\ \times & a \\ \hline\end{array}$$

Figure 2.31: First option for $S_{4,n}^{4/2}$

$$\begin{array}{|c|c|}\hline \times & a \\ \times & a \\ b & \times \\ b & \times \\ \hline\end{array} \leftarrow \begin{array}{|c|c|}\hline \times & a \\ \times & a \\ \times & b \\ \times & b \\ \hline\end{array}$$

Figure 2.32: Second option for $S_{4,n}^{4/2}$

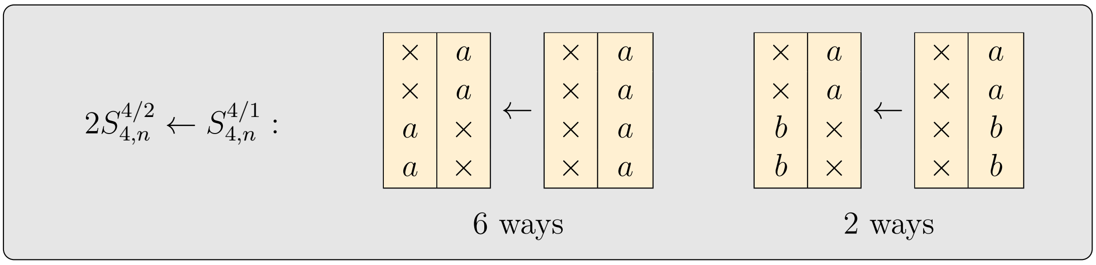
■

Corollary 164.1. *For any distribution X_{ij} with $\mu_2 = 1$,*

$$S_4^{4/2}(t) = (3-\mu_4)\frac{\partial S_4^{4/1}(t)}{\partial \mu_4} + t\,\frac{\partial S_4^{4/1}(t)}{\partial t} - S_4^{4/1}(t) = \frac{6t^2(1+t)}{(1-t)^5}e^{t(\mu_4-3)}. \tag{2.230}$$

Proof. Let $\tau' \in S_{4,n}^{4/1}$ have c four-columns, its weight is then $w(\tau') = \mu_4^c$. However, now there are only $n-c-1$ two-columns as one column is covered with four marks. To create a table $\tau \in S_{4,n}^{4/2}$, we can swap two marks of $\times^4$-column with

- a 4-column of τ' in 6 ways, creating a table τ with weight μ_4^{c-1},
- a 2-column of τ' in 2 ways, creating a table τ with weight μ_4^c,



Thus, from τ', we get the following contribution to $s_4^{4/2}(n) = \sum_{\tau \in S_{4,n}^{4/2}} w(\tau)\operatorname{sgn}(\tau)$,

$$6c\mu_4^{c-1} + 2(n-c-1)\mu_4^c. \tag{2.231}$$

Grouping the terms, this is equal to

$$c\mu_4^{c-1}(6-2\mu_4) + 2n\mu_4^c - 2\mu_4^c. \tag{2.232}$$

Summing up this contribution over all tables $\tau' \in S_{4,n}^{4/1}$ (note that as each table in $S_{4,n}^{4/2}$ is counted twice, we get twice the sum),

$$2s_4^{4/2}(n) = (6-2\mu_4)\frac{\partial s_4^{4/1}(n)}{\partial \mu_4} + 2ns_4^{4/1}(n) - 2s_4^{4/1}(n) \tag{2.233}$$

or in terms of generating functions,

$$2S_4^{4/2}(t) = (6-2\mu_4)\frac{\partial S_4^{4/1}(t)}{\partial \mu_4} + 2t\,\frac{\partial S_4^{4/1}(t)}{\partial t} - 2S_4^{4/1}(t). \tag{2.234}$$

∎

Corollary 164.2. *For any distribution X_{ij} with $\mu_2 = 1$,*

$$S_4^4(t) = \frac{t(1+7t+4t^2)}{(1-t)^5}e^{t(\mu_4-3)}. \tag{2.235}$$

Proof. As $S_{4,n}^4 = S_{4,n}^{4/1} \sqcup S_{4,n}^{4/2}$ disjoint union, we have that $S_4^4(t) = S_4^{4/1}(t) + S_4^{4/2}(t)$. ∎

Corollary 164.3. *For any distribution of X_{ij}, we get the statement of Theorem 156 with $\mu_2 = 1$, that is*

$$F_4(t) = \tfrac{e^{t(\mu_4-3)}}{(1-t)^3}\left((1+m_1\mu_3 t)^{\,4} + 6m_1^2 t\tfrac{(1+m_1\mu_3 t)^2}{1-t} + m_1^4 t\tfrac{1+7t+4t^2}{(1-t)^2}\right). \tag{2.236}$$

Proof. As all $S_4^0(t), S_4^2(t)$ and $S_4^4(t)$ have been found, we use Equation (2.217). ∎

Corollary 164.4. *By Corollary 154.1 or by $T_4^r(t) = [m_1^r]F_4(t)$, we get explicitly*

$$T_4^0(t) = S_4^0(t) = \frac{e^{t(\mu_4-3)}}{(1-t)^3},$$

$$T_4^1(t) = 4\mu_3 t S_4^0(t) = 4\mu_3 t \frac{e^{t(\mu_4-3)}}{(1-t)^3},$$

$$T_4^2(t) = 6\mu_3^2 t^2 S_4^0(t) + S_4^2(t) = \frac{e^{t(\mu_4-3)}}{(1-t)^3}\left(6\mu_3^2 t^2 + \frac{6t}{1-t}\right),$$

$$T_4^3(t) = 6\mu_3^3 t^3 S_4^0(t) + 2\mu_3 t S_4^2(t) = \frac{e^{t(\mu_4-3)}}{(1-t)^3}\left(6\mu_3^3 t^3 + \frac{12\mu_3 t^2}{1-t}\right),$$

$$T_4^4(t) = \mu_3^4 t^4 S_4^0(t) + \mu_3^2 t^2 S_4^2(t) + S_4^4(t) = \frac{e^{t(\mu_4-3)}}{(1-t)^3}\left(\mu_3^4 t^4 + \frac{6\mu_2^2 t^3}{1-t} + t\frac{1+7t+4t^2}{(1-t)^2}\right).$$

Remark 165. Note that we get $F_4(t)$ in its full generality (Theorem 156) by simply using the scalability property given by Corollary 112.1.

2.5.4 Addition technique

We present an alternative technique how to obtain $S_4^2(t)$ and $S_4^4(t)$ by finding a correspondence between marked tables with n columns and unmarked tables with one extra added column.

Let $\tau \in S_{4,n}^2$. Then, we construct $\tau' \in S_{4,n+1}^0$ in such a way we replace two $\times$'s by the number "$n+1$" and add an extra column filled with "$n+1$"'s and the covered numbers (see Figure 2.33 below). The crucial observation is that these two covered numbers must be the same, so the added column is always nontrivial.

6	4	×	5	2	7	3	1
	1	2	4	6	7	5	3
6	1	×	4	2	7	5	3
	4	2	5	6	7	3	1

$\longleftrightarrow$

4	**8**	5	2	7	6	3	1
1	2	4	6	7	**8**	5	3
1	**8**	4	2	7	6	5	3
4	2	5	6	7	**8**	3	1

Figure 2.33: A correspondence between table $\tau \in S_{4,7}^2$ and table $\tau' \in S_{4,8}^0$

Given a table $\tau' \in S_{4,n+1}^0$ with c 4-columns and thus weight μ_4^c, there are $2(n+1-c)$ ways how we can select one 2-column and one pair of numbers in this 2-column. We then erase this column and turn the other pair found in τ' into two marks. As there are $n+1$ ways how we can place back the erased column and $n+1$ ways how we can select n numbers from $n+1$ numbers, each table $\tau \in S_{4,n}^2$ is counted $(n+1)^2$ times, thus

$$\begin{aligned} s_4^2(n) &= \sum_{\tau \in S_{4,n}^2} w(\tau)\operatorname{sgn}\tau = \sum_{\tau' \in S_{4,n+1}^0} \frac{2(n+1-c)}{(n+1)^2}\mu_4^c \operatorname{sgn}\tau' \\ &= 2\frac{s_4^0(n+1)}{n+1} - \frac{2\mu_4}{(n+1)^2}\frac{\partial s_4^0(n+1)}{\partial \mu_4}. \end{aligned}$$

Or in terms of generating functions,

$$S_4^2(t) = 2\frac{\partial S_4^0(t)}{\partial t} - \frac{2\mu_4}{t}\frac{\partial S_4^0(t)}{\partial \mu_4} = \frac{6t}{(1-t)^4}e^{t(\mu_4-3)}.$$

Similarly, there is a correspondence between $S_{4,n}^{4/1}$ and $S_{4,n+1}^{0}$ as shown in Figure 2.34 below.

6	4	$\times$	5	2	7	3	1		4	**8**	5	2	7	6	3	1
2	1	$\times$	4	6	7	5	3	$\longleftrightarrow$	1	**8**	4	6	7	2	5	3
6	1	$\times$	4	2	7	5	3		1	**8**	4	2	7	6	5	3
2	4	$\times$	5	6	7	3	1		4	**8**	5	6	7	2	3	1

Figure 2.34: A correspondence between table $\tau \in S_{4,7}^{4/1}$ and table $\tau' \in S_{4,8}^{0}$

Now it depends whether previously covered numbers (in the gray column above) form a 4-column or a 2-column. Let $\tau' \in S_{4,n+1}^{0}$ have c 4-columns. To count the number of tables $\tau \in S_{4,n}^{4/1}$, first, we select one 4-column in c ways and mark all of its entries, next, either we select

- one 2-column in $n+1-c$ ways and erase it, creating table τ with $w(\tau) = w(\tau')/\mu_4$
- or one 4-column in $c-1$ ways and erase it, creating table τ with $w(\tau) = w(\tau')/\mu_4^2$

In total,

$$s_4^{4/1}(n) = \sum_{\tau \in S_{4,n}^{4/1}} w(\tau) \operatorname{sgn} \tau = \sum_{\tau' \in S_{4,n+1}^{0}} \frac{\frac{c(n+1-c)}{\mu_4} + \frac{c(c-1)}{\mu_4^2}}{(n+1)^2} \mu_4^c \operatorname{sgn} \tau'$$
$$= \frac{n}{(n+1)^2} \frac{\partial s_4^0(n+1)}{\partial \mu_4} - \frac{1-\mu_4}{(n+1)^2} \frac{\partial^2 s_4^0(n+1)}{\partial \mu_4^2}.$$

In terms of generating functions,

$$S_4^{4/1}(t) = \frac{\partial^2 S_4^0(t)}{\partial \mu_4 \partial t} - \frac{1}{t} \frac{\partial S_4^0(t)}{\partial \mu_4} + \frac{1-\mu_4}{t} \frac{\partial^2 S_4^0(t)}{\partial \mu_4^2} = \frac{t(1+2t)}{(1-t)^4} e^{t(\mu_4 - 3)}.$$

And finally, there is a correspondence between $S_{4,n}^{4/2}$ and $S_{4,n+1}^{0}$ as shown in Figure 2.35 below. Again, a column formed by covered numbers must have nonzero weight.

6	4	$\times$	5	2	7	3	1		4	**8**	5	2	7	6	3	1
6	1	2	4	$\times$	7	5	3	$\longleftrightarrow$	1	2	4	**8**	7	6	5	3
6	1	$\times$	4	2	7	5	3		1	**8**	4	2	7	6	5	3
6	4	2	5	$\times$	7	3	1		4	2	5	**8**	7	6	3	1

Figure 2.35: A correspondence between table $\tau \in S_{4,7}^{4/2}$ and table $\tau' \in S_{4,8}^{0}$

Again it depends whether covered numbers form a 4-column or a 2-column. Let $\tau' \in S_{4,n+1}^{0}$ have c 4-columns. To count the number of tables $\tau \in S_{4,n}^{4/1}$, first, we select one 2-column in $n+1-c$ ways and one pair of numbers in it. We then mark those numbers and the other pair found elsewhere. By symmetry, however, there is only $n+1-c$ pairs which can be marked. Next, either we select

- one 2-column (other than the two with marked numbers) in $n-1-c$ ways and then erase it, creating table τ with $w(\tau) = w(\tau')$

- or one 4-column in c ways and erase it, creating table τ with $w(\tau) = w(\tau')/\mu_4$

In total,

$$s_4^{4/2}(n) = \sum_{\tau \in S_{4,n}^{4/2}} w(\tau)\,\mathrm{sgn}\,\tau = \sum_{\tau' \in S_{6,n+1}^{0}} \frac{(n+1-c)(n-1-c) + \frac{(n+1-c)c}{\mu_4}}{(n+1)^2} \mu_4^c\,\mathrm{sgn}\,\tau'$$
$$= \frac{(n+1)(n-1)}{(n+1)^2} s_4^0(n+1) + \frac{\mu_4 - 2n\mu_4 + n}{(n+1)^2} \frac{\partial s_4^0(n+1)}{\partial \mu_4} + \frac{\mu_4(\mu_4-1)}{(n+1)^2} \frac{\partial^2 s_4^0(n+1)}{\partial \mu_4^2}.$$

In terms of generating functions,

$$S_4^{4/2}(t) = t\frac{\partial^2 S_4^0(t)}{\partial t^2} - \frac{\partial S_4^0(t)}{\partial t} + (1-2\mu_4)\frac{\partial^2 S_4^0(t)}{\partial \mu_4 \partial t} + \frac{3\mu_4 - 1}{t}\frac{\partial S_4^0(t)}{\partial \mu_4}$$
$$+ \frac{\mu_4(\mu_4-1)}{t}\frac{\partial^2 S_4^0(t)}{\partial \mu_4^2} = \frac{6t^2(1+t)}{(1-t)^5} e^{t(\mu_4 - 3)}.$$

2.5.5 Inclusion/Exclusion

Finally, we show yet another derivation whose only ingredient to deduce $F_4(t)$ is the Wishart expansion (Lemma 143). The method of inclusion/exlusion shown here was introduced in the context of permutation tables by Lv and Potechin in [5].

Definition 166. Denote $S_{4,n,C}^r$ the subset of tables $S_{4,n}^r$ such that if $\tau \in S_{4,n,C}^r$ then $C \subseteq [n]$ are numbers which only appear in 4 columns of τ with no marks. Those 4-columns are reffered to as *known* 4-column. The other columns of τ can be also 4 columns or other columns, marked or unmarked. In contrast, we denote $\bar{S}_{4,n,C'}^r$ a subset of $S_{4,n,C'}^r$, such that if $\tau \in \bar{S}_{4,n,C'}^r$, then C' is the set of **all** numbers appearing in unmarked 4 columns of τ.

Remark 167. Clearly, we can write the following disjoint union representation

$$S_{4,n,C}^r = \bigcup_{C' \supseteq C} \bar{S}_{4,n,C'}^r. \tag{2.237}$$

Remark 168. As all columns other than C' have weight $w(\cdot)$ equal to one, we have for any $\tau \in \bar{S}_{4,n,C'}^r$ that $w(\tau) = \mu_4^{\#C'}$, where $\#C'$ denotes the number of elements of set C'.

Definition 169. We define the *residual normal weight* $w_C(t)$ of a table $\tau \in S_{4,n,C}^r$ as the product of weights $w(\cdot)$ in columns other than C in which we treat $Y_{ij} \sim \mathsf{N}(0,1)$.

Proposition 170 (Inclusion/exclusion)**.** *For any distribution of X_{ij} with $\mu_2 = 1$,*

$$s_4^r(n) = \sum_{\tau \in S_{4,n}^r} w(\tau)\,\mathrm{sgn}(\tau) = \sum_{C \subseteq [n]} \sum_{\tau \in S_{4,n,C}^r} (\mu_4 - 3)^{\#C} w_C(t)\,\mathrm{sgn}(\tau). \tag{2.238}$$

Proof. Directly, via a chain of equalities and by using Remark 168,

$$\sum_{\tau\in S^r_{4,n}} w(\tau)\,\mathrm{sgn}(\tau) = \sum_{C'\subseteq[n]}\sum_{\tau\in\bar{S}^r_{4,n,C'}} \mu_4^{\#C'}\,\mathrm{sgn}(\tau)$$
$$= \sum_{C'\subseteq[n]}\sum_{\tau\in\bar{S}^r_{4,n,C'}}\sum_{C\subseteq C'} (\mu_4-3)^{\#C}3^{\#C'/C}\,\mathrm{sgn}(\tau)$$
$$= \sum_{C'\subseteq[n]}\sum_{C\subseteq C'}\sum_{\tau\in\bar{S}^r_{4,n,C'}} (\mu_4-3)^{\#C}w_C(t)\,\mathrm{sgn}(\tau)$$
$$= \sum_{C\subseteq[n]}\sum_{C'\supseteq C}\sum_{\tau\in\bar{S}^r_{4,n,C'}} (\mu_4-3)^{\#C}w_C(t)\,\mathrm{sgn}(\tau)$$
$$= \sum_{C\subseteq[n]}\sum_{\tau\in S^r_{4,n,C}} (\mu_4-3)^{\#C}w_C(t)\,\mathrm{sgn}(\tau).$$

■

Corollary 170.1. *For any distribution of X_{ij} with $\mu_2 = 1$,*

$$s^r_4(n) = \sum_{c=0}^{n} \frac{n!^2(\mu_4-3)^c}{(n-c)!^2c!} n^r_4(n-c). \tag{2.239}$$

Proof. There are $\binom{n}{c}$ ways how we can select $C \in [n]$. As the sum depends only on $\#C$,

$$s^r_4(n) = \sum_{C\subseteq[n]}\sum_{\tau\in S^r_{4,n,C}} (\mu_4-3)^{\#C}w_C(t)\,\mathrm{sgn}(\tau) = \sum_{c=0}^{n}\binom{n}{c}(\mu_4-3)^c \sum_{\tau\in S^r_{4,n,C}} w_C(t)\,\mathrm{sgn}(\tau) \tag{2.240}$$

for $C \in [n]$ arbitrary. Next, since the columns C in $\tau \in S^r_{4,n,C}$ are disjoint from other columns, we can write, as there are $\binom{n}{c}c!$ ways how we can arrange the corresponding unmarked 4 columns in a table,

$$\sum_{\tau\in S^r_{4,n,C}} w_C(t)\,\mathrm{sgn}(\tau) = \binom{n}{c}c! \sum_{\tau\in S^r_{4,n-c}} w_\emptyset(t)\,\mathrm{sgn}(\tau) = \binom{n}{c}c!n^r_4(n-c). \tag{2.241}$$

■

Corollary 170.2. *In terms of generating functions, for any distribution of X_{ij} with $\mu_2 = 1$,*

$$S^r_4(t) = e^{t(\mu_4-3)}N^r_4(t). \tag{2.242}$$

Proof. By definition and using the previous corollary,

$$S^r_4(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!^2}s^r_4(n) = \sum_{n=0}^{\infty}\sum_{c=0}^{n}\frac{t^n(\mu_4-3)^c}{(n-c)!^2c!}n^r_4(n-c) = \sum_{c=0}^{\infty}\frac{t^c}{c!}(\mu_4-3)^cN^r_4(t). \tag{2.243}$$

■

Corollary 170.3. *For any distribution of X_{ij}, we get the statement of Theorem 156 with $\mu_2 = 1$ (Corollary 164.3).*

Proof. Directly from Equation (2.217),

$$\begin{aligned}F_4(t) &= \sum_{r=0}^{4} m_1^r (1+m_1\mu_3 t)^{4-r} S_4^r(t) = e^{t(\mu_4-3)} \sum_{r=0}^{4} m_1^r (1+m_1\mu_3 t)^{4-r} N_4^r(t) \\ &= e^{t(\mu_4-3)} \left((1+m_1\mu_3 t)^4 N_4^0(t) + m_1^2 (1+m_1\mu_3 t)^2 N_4^2(t) + m_1^4 N_4^4(t) \right).\end{aligned}$$

■

Remark 171. Note that, in this new derivation of the general fourth moment, we do not require the knowledge of the formula for $F_4(t)|_{m_1=0}$ of Nyquist, Rice and Riordan [50].

Remark 172. Tracing back the definitions of auxiliary variables, we can write the following expression for any distribution X_{ij} with $\mu_2 = 1$,

$$f_4(n) = \sum_{r=0}^{4} \sum_{s=0}^{r} \sum_{c=0}^{n-s} \frac{(4-r+s)! m_1^r n!^2 \mu_3^s (\mu_4-3)^c}{(4-r)!(n-s-c)!^2 c! s!} n_4^{r-s}(n-s-c), \tag{2.244}$$

which is equivalent to Corollary 156.1 using scalability property again.

2.6 Paired Marked Permutation Tables

2.6.1 Marked pair-tables

Shifting the random entries by their first moment, we can again find a simplification in terms of marked permutation pair-tables.

Definition 173. We define $G^{\times}_{\langle k\rangle,n,p}$ as the set of all marked pair-tables with at most one mark per row. For them, we define marked weight accordingly as in the previous non-Gram case (expectation over products of $Y_{ij}^{\times}$'s). Also, we denote $T^{\times}_{\langle k\rangle,n,p}$ as the subset of all tables $G^{\times}_{\langle k\rangle,n,p}$ which are nontrivial (having nonzero marked weight). Finally, we define $T^{r}_{\langle k\rangle,n,p}$ as the subset of tables $T^{\times}_{\langle k\rangle,n,p}$ having r marks.

Proposition 174. *For any distribution of X_{ij}, assuming k even,*

$$f_k(n,p) = \mathbb{E}\,\det(U^\top U)^{k/2} = \sum_{\tau\in T^{\times}_{\langle k\rangle,n,p}} w_\times(t)\,\mathrm{sgn}(\tau). \tag{2.245}$$

Remark 175. In the case of marked pair-tables, as the selection in pairs of rows is a subset of $[n]$, there could be ambiguity if what numbers are covered if we would depicted the tables using just marks. Thus, instead of just using "×" for marks, we always append a column showing the numbers hidden under marks alongside our tables (in curly brackets).

$\left\{\begin{matrix}6\\6\\ \\ \end{matrix}\right\}$	1	×	4	5	7	8	9
	1	×	9	4	7	5	8
	3	2	9	4	7	5	8
	3	2	4	5	7	8	9

Figure 2.36: An example of nontrivial table $\tau \in T^2_{\langle 4\rangle,9,7}$ with weights $w_\times(t) = m_1^2\mu_4$ and $w(\tau) = \mu_4$, $C_1 = \{1,4,5,6,7,8,9\}$, $C_2 = \{2,3,4,5,7,8,9\}$.

$\left\{\begin{matrix}4\\4\\9\\9\end{matrix}\right\}$	2	×	3	6	7	8	9
	×	2	3	6	7	8	9
	×	2	1	6	7	5	8
	2	×	1	6	7	5	8

Figure 2.37: An example of nontrivial table $\tau \in T^4_{\langle 4\rangle,9,7}$ with weights $w_\times(t) = m_1^4\mu_4^2$ and $w(\tau) = \mu_4^2$, $C_1 = \{2,3,4,6,7,8,9\}$, $C_2 = \{1,2,5,6,7,8,9\}$.

Definition 176. We define

$$t^r_k(n,p) = \sum_{\tau\in T^r_{\langle k\rangle,n,p}} w(\tau)\,\mathrm{sgn}(\tau) \tag{2.246}$$

and its corresponding generation function

$$T^r_k(t,\omega) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} t^r_k(n,p). \tag{2.247}$$

Proposition 177. *For any distribution X_{ij} with $\mu_2 = 1$,*

$$f_k(n,p) = \sum_{r=0}^{k} m_1^r t^r_k(n,p) \qquad \textit{and thus} \qquad F_k(t,\omega) = \sum_{r=0}^{k} m_1^r T^r_k(t,\omega). \tag{2.248}$$

2.6.2 Even marked pair-tables

Definition 178. We denote $S^\times_{\langle k\rangle,n,p}$ as the subset of nontrivial tables $T^\times_{\langle k\rangle,n,p}$ whose weight does not vanish when $\mu_3 = \mu_5 = \cdots = 0$. As a consequence, the columns of those tables must have only even number of marks. We write $S^r_{\langle k\rangle,n,p}$ as the subset of tables $S^\times_{\langle k\rangle,n,p}$ having r marks.

Definition 179. For any distribution of X_{ij}, we write

$$s^r_k(n,p) = \sum_{\tau\in S^r_{\langle k\rangle,n,p}} w(\tau)\,\mathrm{sgn}(\tau) \tag{2.249}$$

and its corresponding generation function

$$S^r_k(t,\omega) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} s^r_k(n,\omega). \tag{2.250}$$

2.6.3 Shifted normal Gram moments

The case of Normal distribution is the only one for which we know $f_k(n,p)$ exactly for any k, n and p. We have the following generalization of Dembo's result for $n_{2m}(n,p)$ (Proposition 107).

Definition 180. When $X_{ij} \sim \mathsf{N}(m_1, 1)$, we denote $t^r_4(n,p)$ as $n^r_4(n,p)$ and $T^r_4(t,\omega)$ as $N^r_4(t,\omega)$.

Theorem 181 (Wishart expansion)**.** *Let $X_{ij} \sim \mathsf{N}(\mu, \sigma^2)$ and $k = 2m$ be an even integer, then*

$$f_{2m}(n,p)=\sigma^{2mp}\left(\prod_{r=0}^{m-1}\frac{(n+2r)!}{(n-p+2r)!}\right)\sum_{s=0}^{m}\binom{m}{s}\frac{(n-2)!!}{(n+2s-2)!!}\left(\frac{np\mu^2}{\sigma^2}\right)^s. \quad (2.251)$$

The assertion follows from the properties of the Wishart distribution, see Theorem 10.3.7 in [49]. For completeness, we will show our own derivation later in Chapter 6 on Random simplices (see Proposition 256 and Remark 257).

Proposition 182. *When $X_{ij} \sim \mathsf{N}(m_1, 1)$, then $s_4^r(n,p) = n_4^r(n,p)$. Also, in terms of generating functions, $S_4^r(t,\omega) = N_4^r(t,\omega)$.*

Proof. Since any table $\tau \in T^r_{\langle 4\rangle,n,p}/S^r_{\langle 4\rangle,n,p}$ has weight zero when $X_{ij} \sim \mathsf{N}(m_1, 1)$, we can replace $T^r_{\langle 4\rangle,n,p}$ by $S^r_{\langle 4\rangle,n,p}$ in Proposition 177. ■

Fourth shifted normal Gram moment

Proposition 183. *Selecting $\mu = m_1$, $\sigma = 1$ and $k = 4$, we get when $X_{ij} \sim \mathsf{N}(m_1, 1)$.*

$$f_4(n,p) = \frac{n!(n+1)!\,(np^2m_1^4 + (2+n)\,(2pm_1^2+1))}{(n-p)!(n-p+2)!}. \quad (2.252)$$

Corollary 183.1.

$$n_4^1(n,p) = n_4^3(n,p) = 0, \qquad n_4^2(n,p) = \frac{2n!(n+2)!\,p}{(n-p)!(n-p+2)!},$$

$$n_4^0(n,p) = \frac{n!(n+2)!}{(n-p)!(n-p+2)!}, \qquad n_4^4(n,p) = \frac{np^2n!(n+1)!}{(n-p)!(n-p+2)!}.$$

Proof. Comparing m_1 powers in Proposition 177 with Proposition 183. ■

Proposition 184. *By summing the series,*

$$N_4^1(t,\omega) = N_4^3(t,\omega) = 0,$$

$$N_4^0(t,\omega) = \frac{1}{(1-t)^2(1-\omega-t)},$$

$$N_4^2(t,\omega) = \frac{1}{(1-t)^3}\left(\frac{6t}{1-\omega-t} + \frac{2t\omega}{(1-\omega-t)^2}\right),$$

$$N_4^4(t,\omega) = \frac{1}{(1-t)^4}\left(\frac{t\,(1+7t+4t^2)}{1-\omega-t} + \frac{t\,(1+5t+2t^2)\,\omega}{(1-\omega-t)^2} + \frac{2t^2\omega^2}{(1-\omega-t)^3}\right).$$

2.7 Gram fourth moment (general)

Surprisingly, using the method of marked tables and inclusion/exclusion, we can derive the full $F_4(t,\omega)$ in an elementary way, thus generalizing both Nyquist's, Rice's and Riordan's $F_4^{\mathrm{sym}}(t)$ (Proposition 68) and Dembo's $F_4^{\mathrm{sym}}(t,\omega)$ (Proposition 103). We show that:

Theorem 185 (B. 2022)**.** *For any distribution X_{ij},*

$$F_4(t,\omega) = \frac{e^{t(\mu_4-3\mu_2^2)}}{(1-\mu_2^2t)^2\left(1-\omega-\mu_2^2t\right)}\Bigg[(1+m_1\mu_3t)^4 + \frac{6m_1^2\mu_2t(1+m_1\mu_3t)^2}{1-\mu_2^2t} + \frac{m_1^4t\left(1+7\mu_2^2t+4\mu_2^4t^2\right)}{\left(1-\mu_2^2t\right)^2}$$
$$+\frac{\omega m_1^2t}{1-\omega-\mu_2^2t}\left(\frac{2\mu_2(1+m_1\mu_3t)^2}{1-\mu_2^2t} + \frac{m_1^2\left(1+5t\mu_2^2+2t^2\mu_2^4\right)}{\left(1-\mu_2^2t\right)^2}\right) + \frac{2t^2\omega^2m_1^4\mu_2^2}{\left(1-\omega-\mu_2^2t\right)^2\left(1-\mu_2^2t\right)^2}\Bigg].$$

Remark 186. Letting $\omega = 0$, we recover $F_4(t)$. On the other hand, letting $m_1 = 0$, we get $F_4^{\text{sym}}(t,\omega)$.

Corollary 186.1. *Defining q_i and μ_j as above, we get, by Taylor expansion,*

$$f_4(n,p) = p!^2\binom{n}{p}\mu_2^{2p}\sum_{j=0}^{p}\frac{1}{j!}\left(\frac{\mu_4}{\mu_2^2}-3\right)^j\sum_{i=-2}^{4}(q_i+\tilde{q}_i(n-p)+\tilde{\tilde{q}}_i(n-p)(n-p+7))\binom{n-j+i}{n-p+i},$$

where

$$\tilde{q}_0 = -\frac{2m_1^4\mu_3^2}{\mu_2^5},\quad \tilde{q}_1 = \frac{2m_1^3\left(2\mu_2^2\mu_3+3m_1\mu_3^2-m_1\mu_2^3\right)}{\mu_2^5},\quad \tilde{q}_2 = \frac{m_1^2\left(3m_1^2\mu_2^3-2\mu_2^4-8m_1\mu_2^2\mu_3-6m_1^2\mu_3^2\right)}{\mu_2^5},$$
$$\tilde{q}_3 = \frac{m_1^2\left(2\mu_2^4+4m_1\mu_2^2\mu_3+2m_1^2\mu_3^2-m_1^2\mu_2^3\right)}{\mu_2^5},\qquad \tilde{\tilde{q}}_2 = \frac{m_1^4}{\mu_2^2},\qquad \tilde{\tilde{q}}_3 = -\frac{2m_1^4}{\mu_2^2},\qquad \tilde{\tilde{q}}_4 = \frac{m_1^4}{\mu_2^2}$$

and $\tilde{q}_i, \tilde{\tilde{q}}_i$ otherwise zero.

Example 187 (General Gaussian distribution). If $X_{ij} \sim N(\mu,\sigma^2)$, we have $m_1 = \mu$, $(\mu_2,\mu_3,\mu_4) = (\sigma^2, 0, 3\sigma^4)$, which gives, after series of simplifications,

$$f_4(n,p) = \frac{n!(n+1)!\sigma^{4(p-1)}}{(n-p)!(n-p+2)!}\left(np^2\mu^4 + (n+2)\left(2p\mu^2\sigma^2+\sigma^4\right)\right). \tag{2.253}$$

This formula agrees with the general case given by Theorem 181.

Example 188 (Exponential distribution). If $X_{ij} \sim \text{Exp}(1)$, that is if $m_j = j!$, we have $(\mu_2,\mu_3,\mu_4) = (1,2,9)$ and $(q_{-2},q_{-1},q_0,q_1,q_2,q_3,q_4,\tilde{q}_0,\tilde{q}_1,\tilde{q}_2,\tilde{q}_3,\tilde{\tilde{q}}_2,\tilde{\tilde{q}}_3,\tilde{\tilde{q}}_4) =$ (16,−96,192,−124,−26,27,12,−8,30,−39,17,1,−2,1). The exact moments $f_4(n,p)$ for low n and p are shown in Table 2.3 below.

$f_4(n,p)$		p				
		1	2	3	4	5
$n-p$	0	24	960	51840	3511872	287953920
	1	56	3744	297216	27708480	3004024320
	2	96	9432	1022400	124675200	17182609920
	3	144	19320	2724480	419207040	71341240320
	4	200	34920	6189120	1169602560	240336875520
	5	264	57960	12579840	2858913792	696776048640
	6	336	90384	23538816	6325119360	1801876285440
	7	416	134352	41299200	12939696000	4256462960640

Table 2.3: Fourth moment of a random Gram determinant with entries exponentially distributed

From now on, we assume $\mu_2 = 1$ in this section. What follows is the proof of Theorem 185.

2.7.1 Structure of marked pair-tables

From now on, we put $k = 4$. As columns of $\tau \in T^r_{\langle 4\rangle,n,p}$ do not see what resides in other columns, the column types there must be the same as in $\tau \in T^r_{4,n}$. The structure of all nontrivial (nonzero weight) tables $T^r_{\langle 4\rangle,n,p}$ is however cumbersome, as there might be many nontrivial marked pair-tables formed by marking trivial unmarked pair-tables.

> *Remark* 189. If τ is nontrivial, there is again only one possibility for a column to have odd number of marks, and that is it is a $\times^1$ column (all number were the same before marking).

2.7.2 Decomposition over even marked columns

Proposition 190. *For any distribution of X_{ij} with $\mu_2 = 1$,*

$$t^r_4(n,p) = \sum_{s=0}^{r} \binom{4-r+s}{s} \frac{n!p!\mu_3^s}{(n-s)!(p-s)!} s_4^{r-s}(n-s,p-s). \tag{2.254}$$

Proof. The proof is a modification of Proof of Proposition 154. This time, however, if we assume the number of $\times^1$ columns is s, we can select for them $\binom{p}{s}$ numbers (with $s!$ permutations), but only $\binom{n}{s}$ column positions. To create a table $\tau \in T^r_{\langle 4\rangle,n,p}$, we start with a table $\tau' \in S^{r-s}_{\langle 4\rangle,n-s,p-s}$ as we decreased the number of possible positions and numbers by s. Thus

$$\sum_{\tau\in T^r_{\langle 4\rangle,n,p}} w(\tau)\operatorname{sgn}(\tau) = \sum_{s=0}^{r} \sum_{\tau'\in S^{r-s}_{\langle 4\rangle,n-s,p-s}} \binom{4-r+s}{s}\binom{n}{s}\binom{p}{s} s!^2\mu_3^s w(\tau')\operatorname{sgn}(\tau'). \tag{2.255}$$

■

Corollary 190.1. *In terms of generating functions, for any distribution X_{ij} with $\mu_2 = 1$,*

$$T^r_4(t,\omega) = \sum_{s=0}^{r} \binom{4-r+s}{s} t^s\mu_3^s S_4^{r-s}(t,\omega) \qquad \textit{and thus} \tag{2.256}$$

$$F_4(t,\omega) = \sum_{r=0}^{4} m_1^r(1+m_1\mu_3 t)^{4-r} S^r_4(t,\omega). \tag{2.257}$$

2.7.3 Inclusion/Exclusion

Definition 191. Similarly as in Section 2.5.5, we define $S^r_{\langle 4\rangle,n,p,C}$ the subset of tables $S^r_{\langle 4\rangle,n,p}$ with numbers which are in $C \subseteq [n]$ are in 4 columns of τ with no marks. The other columns of τ can be also 4 columns or other columns, marked or unmarked. In contrast, we denote $\bar{S}^r_{\langle 4\rangle,n,p,C'}$ a subset of $S^r_{\langle 4\rangle,n,p,C'}$ so that C' contains all numbers of unmarked 4 columns.

Remark 192. As all other columns have weight one, we have for any $\tau \in \bar{S}^r_{\langle 4\rangle,n,p,C'}$,

$$w(\tau) = \mu_4^{\#C'}. \tag{2.258}$$

Definition 193. We define the *residual normal weight* $w_C(t)$ of a table $\tau \in S^r_{\langle 4\rangle,n,p,C}$ as the product of weights in columns other than C in which we assume normal distribution.

Proposition 194 (Inclusion/exclusion)**.**

$$\sum_{\tau\in S^r_{\langle 4\rangle,n,p}} w(\tau)\,\mathrm{sgn}(\tau) = \sum_{C\subseteq[n]}\sum_{\tau\in S^r_{\langle 4\rangle,n,p,C}} (\mu_4-3)^{\#C} w_C(t)\,\mathrm{sgn}(\tau). \tag{2.259}$$

Corollary 194.1. *For any distribution X_{ij} with $\mu_2 = 1$,*

$$s_4^r(n,p) = \sum_{\tau\in S^r_{\langle 4\rangle,n,p}} w(\tau)\,\mathrm{sgn}(\tau) = \sum_{c=0}^{p} \frac{n!p!(\mu_4-3)^c}{(n-c)!(p-c)!c!} n_4^r(n-c,p-c). \tag{2.260}$$

Proof. There are $\binom{n}{c}$ ways how we can select $C \in [n]$. As the sum depends only on $\#C$,

$$\sum_{C\subseteq[n]}\sum_{\tau\in S^r_{\langle 4\rangle,n,p,C}} (\mu_4-3)^{\#C} w_C(t)\,\mathrm{sgn}(\tau) = \sum_{c=0}^{n}\binom{n}{c}(\mu_4-3)^c \sum_{\tau\in S^r_{\langle 4\rangle,n,p,C}} w_C(t)\,\mathrm{sgn}(\tau) \tag{2.261}$$

for $C \in [n]$ arbitrary. Next, since the columns C in $\tau \in S^r_{\langle 4\rangle,n,p,C}$ are disjoint from other columns, we can write, as there are $\binom{p}{c}c!$ ways how we can arrange the corresponding unmarked 4 columns in a table,

$$\sum_{\tau\in S^r_{\langle 4\rangle,n,p,C}} w_C(t)\,\mathrm{sgn}(\tau) = \binom{p}{c}c! \sum_{\tau\in S^r_{\langle 4\rangle,n-c,p-c,C}} w_\emptyset(t)\,\mathrm{sgn}(\tau) = \binom{p}{c}c!\, n_4^r(n-c,p-c). \tag{2.262}$$

■

Corollary 194.2. *In terms of generating functions, for any distribution X_{ij} with $\mu_2 = 1$,*

$$S_4^r(t,\omega) = e^{t(\mu_4-3)} N_4^r(t,\omega). \tag{2.263}$$

Proof. By definition, and then by consecutive summation (we first extend the c summation to ∞, as negative factorials in the denominator force the terms to vanish after finitely many c's),

$$\begin{aligned} S_4^r(t,\omega) &= \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{(n-p)!}{n!p!} t^p\omega^{n-p} s_4^r(n,p) \\ &= \sum_{n=0}^{\infty}\sum_{p=0}^{n}\sum_{c=0}^{p} \frac{(n-p)!t^p\omega^{n-p}(\mu_4-3)^c}{(n-c)!(p-c)!c!} n_4^r(n-c,p-c) \\ &= \sum_{c=0}^{\infty} \frac{t^c}{c!}(\mu_4-3)^c N_4^r(t,\omega) = e^{t(\mu_4-3)} N_4^r(t,\omega). \end{aligned} \tag{2.264}$$

■

Corollary 194.3. *For any distribution of X_{ij}, we get the statement of Theorem 185 with $\mu_2 = 1$, that is*

$$F_4(t,\omega)=\frac{e^{t(\mu_4-3)}}{(1-t)^2(1-\omega-t)}\Bigg[(1+m_1\mu_3 t)^4+\frac{6m_1^2 t\,(1+m_1\mu_3 t)^2}{1-t}+\frac{m_1^4 t\,(1+7t+4t^2)}{(1-t)^2}$$
$$+\frac{\omega m_1^2 t}{1-\omega-t}\left(\frac{2\,(1+m_1\mu_3 t)^2}{1-t}+\frac{m_1^2\,(1+5t+2t^2)}{(1-t)^2}\right)+\frac{2t^2\omega^2 m_1^4}{(1-\omega-t)^2\,(1-t)^2}\Bigg].$$

Proof. Directly from Corollary 190.1,

$$F_4(t,\omega)=e^{t(\mu_4-3)}\left[(1+m_1\mu_3 t)^4 N_0(t,\omega)+m_1^2(1+m_1\mu_3 t)^2 N_2(t,\omega)+m_1^4 N_4(t,\omega)\right].$$

∎

By scaling, we get for any distribution X_{ij} the statement of Theorem 185 for any μ_2.

2.7.4 Covering technique

The fact the column types of tables $S^r_{\langle 4\rangle,n,p}$ are the same as in $S^r_{4,n}$ tables enables us to find $F_4(t,\omega)$ in elementary way. By Dembo [24], we already know $S^0_4(t,\omega)$,

$$S_4^0(t,\omega)=\frac{e^{t(\mu_4-3)}}{(1-t)^2(1-\omega-t)}. \tag{2.265}$$

Next, per analogy, we must have

Proposition 195. *For any distribution X_{ij} with $\mu_2 = 1$,*

$$S_4^2(t,\omega)=(6-2\mu_4)\frac{\partial S_4^0(t,\omega)}{\partial\mu_4}+2t\,\frac{\partial S_4^0(t,\omega)}{\partial t}, \tag{2.266}$$
$$S_4^{4/1}(t,\omega)=(1-\mu_4)\frac{\partial S_4^0(t,\omega)}{\partial\mu_4}+t\,\frac{\partial S_4^0(t,\omega)}{\partial t}, \tag{2.267}$$
$$S_4^{4/2}(t,\omega)=(3-\mu_4)\frac{\partial S_4^{4/1}(t,\omega)}{\partial\mu_4}+t\,\frac{\partial S_4^{4/1}(t,\omega)}{\partial t}-S_4^{4/1}(t,\omega), \tag{2.268}$$
$$S_4^4(t,\omega)=S_4^{4/1}(t,\omega)+S_4^{4/2}(t,\omega). \tag{2.269}$$

Corollary 195.1. *For any distribution X_{ij} with $\mu_2 = 1$,*

$$S_4^2(t,\omega)=\frac{e^{t(\mu_4-3)}}{(1-t)^3}\left(\frac{6t}{1-\omega-t}+\frac{2t\omega}{(1-\omega-t)^2}\right), \tag{2.270}$$
$$S_4^{4/1}(t,\omega)=\frac{e^{t(\mu_4-3)}}{(1-t)^3}\left(\frac{t(1+2t)}{1-\omega-t}+\frac{t\omega}{(1-\omega-t)^2}\right), \tag{2.271}$$
$$S_4^{4/2}(t,\omega)=\frac{e^{t(\mu_4-3)}}{(1-t)^4}\left(\frac{6t^2(1+t)}{1-\omega-t}+\frac{2t^2(3+t)\omega}{(1-\omega-t)^2}+\frac{2t^2\omega^2}{(1-\omega-t)^3}\right), \tag{2.272}$$
$$S_4^4(t,\omega)=\frac{e^{t(\mu_4-3)}}{(1-t)^4}\left(\frac{t\,(1+7t+4t^2)}{1-\omega-t}+\frac{t\,(1+5t+2t^2)\,\omega}{(1-\omega-t)^2}+\frac{2t^2\omega^2}{(1-\omega-t)^3}\right). \tag{2.273}$$

Corollary 195.2. *For any distribution of X_{ij}, we get the statement of Theorem 185 with $\mu_2 = 1$.*

Proof. By combining this result with Corollary 190.1, we get

$$F_4(t, \omega) = (1 + m_1\mu_3 t)^4 S_4^0(t, \omega) + m_1^2(1 + m_1\mu_3 t)^2 S_4^2(t, \omega) + m_1^4 S_4^4(t, \omega). \quad (2.274)$$

■

2.8 Ordinary sixth moment (intermedial)

In the last section of this chapter, we show how to obtain the sixth determinant moment $F_6(t)$ when $\mu_3 = 0$ and m_1 arbitrary.

Theorem 196 (B., 2023). *For X_{ij} with $m_1 = 0, \mu_2 = 1$ and $\mu_3 = 0$, we have*

$$F_6(t) = \frac{e^{t(30-15\mu_4+\mu_6)}}{3t^2(1+3t-t\mu_4)^{17}}\Big(N_6^2\left(\tfrac{t}{(1+3t-t\mu_4)^3}\right)\left(m_1^6\left(1-8t-4t^2+72t^3-216t^4\right.\right.$$
$$+243t^5 - t\left(5-13t-23t^2-129t^3+81t^4\right)\mu_4 + t^2\left(10+t-37t^2-54t^3\right)\mu_4^2$$
$$\left.-t^3\left(10+9t-18t^2\right)\mu_4^3 + t^4(5+3t)\mu_4^4 - t^5\mu_4^5\right) + 3tm_1^4\left(1+3t-t\mu_4\right)\left(1-4t\right.$$
$$\left.\left.t^2-2t(1+t)\mu_4+t^2\mu_4^2\right)\left(1+tm_1\mu_5\right)^2 + 3t^2m_1^2\left(1+3t-t\mu_4\right)\left(1+tm_1\mu_5\right)^4\right)$$
$$+\,3tN_6^0\left(\tfrac{t}{(1+3t-t\mu_4)^3}\right)\left(m_1^6\left(-15+120t+106t^2-1659t^3+2304t^4\right.\right.$$
$$\left.+t\left(30-105t+598t^2-1536t^3\right)\mu_4 + t^2\left(-15-15t+256t^2\right)\mu_4^2\right)$$
$$-15tm_1^4\left(1+3t-t\mu_4\right)\left(3-13t+3t\mu_4\right)\left(1+tm_1\mu_5\right)^2$$
$$\left.+15t^2m_1^2\left(\mu_4-3\right)\left(1+3t-t\mu_4\right)\left(1+tm_1\mu_5\right)^4 + t\left(1+3t-t\mu_4\right)^2\left(1+tm_1\mu_5\right)^6\right)\Big)$$

What follows is the proof of this theorem.

2.8.1 Structure of marked tables

In $T_{6,n}$, the only nontrivial tables are the ones consisted of four types of columns:

- 6 column: six identical copies of the same number, weight μ_6
- 4 column: four identical copies of the same number and a pair of another number, weight μ_4
- 3 column: two triplets of two distinct numbers, weight μ_3
- 2 column: three pairs of three distinct numbers, weight 1

Additional columns with nonzero weight in marked tables, based on number of marks, are

- $\times^1$ column: two marks, we distinguish two sub-types:
  - $\times^1_5$ column: one mark and five identical copies of the same number, weight μ_5
  - $\times^1_3$ column: two marks, three same numbers and one pair of distinct numbers, weight μ_3
- $\times^2$ column: two marks, we distinguish two sub-types:
  - $\times^2_4$ column: two marks and four identical numbers, weight μ_4
  - $\times^2_2$ column: two marks and two pairs of two distinct numbers, weight 1
- $\times^3$ column: three marks and one triplet of the same number, weight μ_3
- $\times^4$ column: four marks and one pair of the same number, weight 1
- $\times^6$ column: six marks, weight 1

In general case, $S^\times_{6,n}$ is a proper subset of $O^\times_{6,n}$. However, assuming $\mu_3 = 0$, we get that $\times^1_3$ and 3-columns vanish. In that case, corresponding nontrivial tables from $O^\times_{6,n}$ coincide with $S^\times_{6,n}$, which only contain the following columns (with weight $w_\times$):

6-column	4-column	2-column	$\times_4^2$-column	$\times_2^2$-column	$\times^4$-column	$\times^6$-column
a	a	a	$\times$	$\times$	$\times$	$\times$
a	a	a	$\times$	$\times$	$\times$	$\times$
a	a	b	a	a	$\times$	$\times$
a	a	b	a	a	$\times$	$\times$
a	b	c	a	b	a	$\times$
a	b	c	a	b	a	$\times$
μ_6	μ_4	1	$\mu_4 m_1^2$	m_1^2	m_1^4	m_1^6

Note that, based on our earlier result (Theorem 73), we can determine the following

Proposition 197. *For any distribution X_{ij} with $\mu_2 = 1$,*

$$O_6^0(t) = G_6(t) = \left(1+\mu_3^2 t\right)^{10} \frac{e^{t\left(\mu_6 - 10\mu_3^2 - 15\mu_4 + 30\right)}}{(1+3t-\mu_4 t)^{15}} \sum_{i=0}^{\infty} \frac{(1+i)(2+i)(4+i)!t^i}{(1+3t-\mu_4 t)^{3i}}.$$
$$= \left(1+\mu_3^2 t\right)^{10} \frac{e^{t\left(\mu_6 - 10\mu_3^2 - 15\mu_4 + 30\right)}}{(1+3t-\mu_4 t)^{15}} N_6^0\left(\frac{t}{(1+3t-\mu_4 t)^3}\right). \tag{2.275}$$

From now on, assume that $\mu_3 = 0$. In this case, $o_6^r(n) = s_6^r(n)$ and $O_6^r(t) = S_6^r(t)$ as nontrivial tables of $O_{6,n}^{\times}$ contain marked columns with even number of marks only (marked column types with odd number of marks disappear). That is, nontrivial $O_{6,n}^{\times}|_{\mu_3=0} = S_{6,n}^{\times}$ (which additionally do not contain 3-columns).

Corollary 197.1. *For any distribution X_{ij} with $\mu_2 = 1, \mu_3 = 0$,*

$$S_6^0(t) = \frac{e^{t(\mu_6 - 15\mu_4 + 30)}}{(1+3t-\mu_4 t)^{15}} N_6^0\left(\frac{t}{(1+3t-\mu_4 t)^3}\right). \tag{2.276}$$

Proposition 198. *In terms of generating functions, we have for any distribution X_{ij} with $\mu_2 = 1$ and $\mu_3 = 0$,*

$$F_6(t) = \sum_{r=0}^{6} m_1^r (1+m_1\mu_5 t)^{6-r} S_6^r(t), \tag{2.277}$$

2.8.2 Displacement of marks in S tables

We use the covering technique described in the fourth moment scenario. As the only nonzero terms are $S_6^0(t), S_6^2(t), S_6^4(t), S_6^6(t)$ in the expansion of $F_6(t)$ when $\mu_3 = 0$, we have the following options (upto permutations of rows) how to nontrivially place marks in a table $\tau \in S_{6,n}^{\times}$:

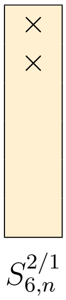


Figure 2.38: Marked columns $S^2_{6,n}$

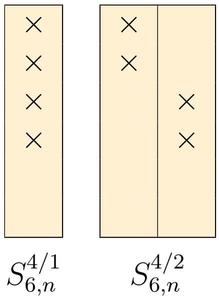


Figure 2.39: Marked columns $S^4_{6,n}$

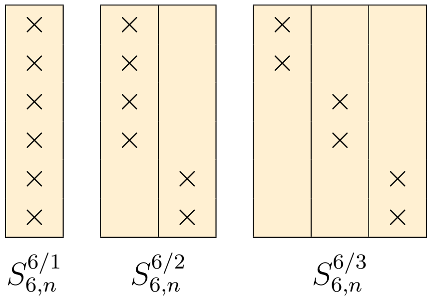


Figure 2.40: Marked columns $S^6_{6,n}$

Definition 199. We define tables $S^{r/s}_{6,n} \subseteq S^r_{6,n}$ such that their r marks occupy s columns. Accordingly, we define

$$s_6^{r/s}(n) = \sum_{\tau \in S^{r/s}_{6,n}} w(\tau)\,\mathrm{sgn}(\tau) \qquad \text{and} \qquad S_6^{r/s}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} s_6^{r/s}(n). \tag{2.278}$$

Remark 200. Note that $S^2_{6,n} = S^{2/1}_{6,n}$, and $S^4_{6,n} = S^{4/1}_{6,n} \sqcup S^{4/2}_{6,n}$ disjoint union and $S^6_{6,n} = S^{6/1}_{6,n} \sqcup S^{6/2}_{6,n} \sqcup S^{6/3}_{6,n}$ disjoint union (see figures above and Figure 2.41).

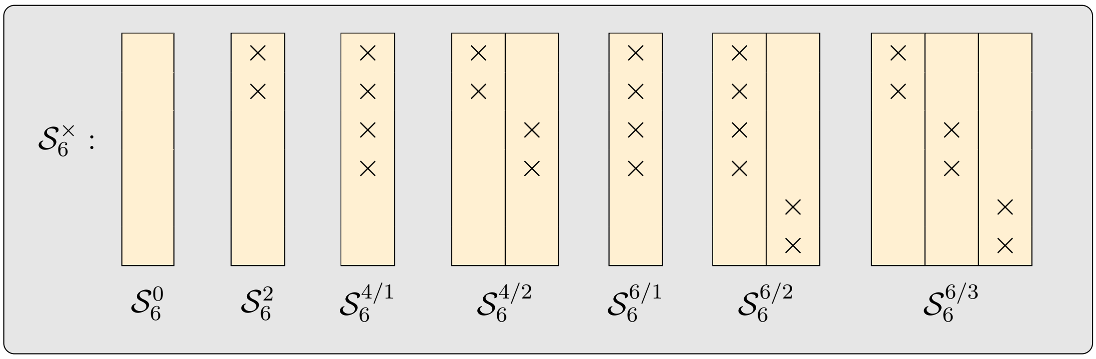


Figure 2.41: Structure of $\mathcal{S}_6^{\times}$ tables

2.8.3 Zero marks

We already know $S_6^0(t)$ since it equals $G_6(t)$ with $\mu_3 = 0$. That is,

$$S_6^0(t) = G_6(t)|_{\mu_3=0} = \frac{e^{t(\mu_6 - 15\mu_4 + 30)}}{(1+3t-\mu_4 t)^{15}} N_6^0 \left(\frac{t}{(1+3t-\mu_4 t)^3} \right). \tag{2.279}$$

Expanding the right hand side (see [5]), we get

$$s_6^0(n) = (n!)^2 \sum_{j=0}^{n} \sum_{i=0}^{j} \frac{(1+i)(2+i)(4+i)!}{48(n-j)!} \binom{14+j+2i}{j-i} (\mu_6 - 15\mu_4 + 30)^{n-j} (\mu_4 - 3)^{j-i}. \tag{2.280}$$

2.8.4 Two marks

Proposition 201. *Tables $S^2_{6,n}$ are formed by marking one pair of numbers in $S_{6,n}$ in a single column.*

Proof. Let $\tau \in S^2_{6,n}$, then the numbers which are covered by one pair of marks are the same numbers. If they were different, say a, b there would have been another a elsewhere in the table, making the table trivial in $S^2_{6,n}$ (since we would have odd number of a's uncovered). ■

Corollary 201.1. *For any distribution X_{ij} with $\mu_2 = 1, \mu_3 = 0$,*

$$S^2_6(t) = \frac{e^{t(\mu_6-15\mu_4+30)}}{(1+3t-\mu_4 t)^{16}}\left[N^2_6\left(\frac{t}{(1+3t-\mu_4 t)^3}\right) + 15t(\mu_4-3)N^0_6\left(\frac{t}{(1+3t-\mu_4 t)^3}\right)\right], \tag{2.281}$$

from which, via Taylor expansion,

$$s^2_6(n) = n!^2\sum_{j=0}^{n}\sum_{i=0}^{j}\frac{(j+2i)(1+i)(2+i)(4+i)!}{48(n-j)!}\binom{14+j+2i}{j-i}(\mu_6-15\mu_4+30)^{n-j}(\mu_4-3)^{j-i}. \tag{2.282}$$

Proof. Let $\tau' \in S_{6,n}$ have c six-columns d four-columns. Thus, there are $n-c-d$ two-columns. The weight of this table is given as $w(\tau') = \mu_6^c\mu_4^d$. Let us find the weights $w(\tau)$ of all marked tables $\tau \in S^2_{6,n}$ created from τ' by marking. There are the following possibilities where we can put those two marks:

- in 6-column of τ' in 15 ways, creating a table τ with weight $\mu_6^{c-1}\mu_4^{d+1}$,
- in 4-column of τ' in 6 ways by covering one pair of four identical numbers, creating a table τ with weight $\mu_6^c\mu_4^{d-1}$,
- in 4-column of τ' in 1 way by covering the remaining pair of two numbers, creating a table τ with weight $\mu_6^c\mu_4^d$
- in 2-column of τ' in 3 ways, creating a table τ with weight $\mu_6^c\mu_4^d$.

Thus, from τ', we get the following contribution to $s^2_6(n) = \sum_{\tau\in S^2_{6,n}} w(\tau)\operatorname{sgn}(\tau)$,

$$15c\mu_6^{c-1}\mu_4^{d+1} + 6d\mu_6^c\mu_4^{d-1} + d\mu_6^c\mu_4^d + 3(n-c-d)\mu_6^c\mu_4^d. \tag{2.283}$$

Grouping the terms, this is equal to

$$c\mu_6^{c-1}\mu_4^d(15\mu_4-3\mu_6) + d\mu_6^c\mu_4^{d-1}(6-2\mu_4) + 3n\mu_6^c\mu_4^d. \tag{2.284}$$

Summing up this contribution over all tables $\tau' \in S_{6,n}$, we get

$$s^2_6(n) = (15\mu_4-3\mu_6)\frac{\partial s^0_6(n)}{\partial\mu_6} + (6-2\mu_4)\frac{\partial s^0_6(n)}{\partial\mu_4} + 3ns^0_6(n) \tag{2.285}$$

or in terms of generating functions,

$$S^2_6(t) = (15\mu_4-3\mu_6)\frac{\partial S^0_6(t)}{\partial\mu_6} + (6-2\mu_4)\frac{\partial S^0_6(t)}{\partial\mu_4} + 3t\frac{\partial S^0_6(t)}{\partial t}. \tag{2.286}$$

Using Remark 148, we get the statement of the corollary. ■

Remark 202. Note that when $\mu_4 = 3$ and $\mu_6 = 15$, then $S^2_6(t) = N^2_6(t)$ as expected.

Alternative proof of Corollary 201.1. Another derivation of $S_6^2(t)$ can be done via the addition technique described earlier in section devoted to $F_4(t)$. In this way, we would get a very simple relation $S_6^2(t) = \frac{1}{t}\frac{\partial S_6^0(t)}{\partial \mu_4}$. To see how it is derived, let $\tau \in S_{6,n}^2$. We then construct $\tau' \in S_{6,n+1}^0$ in such a way we replace two $\times$'s in τ by the number "$n+1$" and add an extra column filled with "$n+1$"'s and the covered numbers in τ. The crucial observation is that these two now exposed numbers must be the same, so the added column is always a (nontrivial) 4-column (see Figure 2.42 below).

	3	8	1	4	2	7	5	6
	3	8	1	4	2	7	6	5
2	3	6	1	$\times$	4	7	8	5
	4	6	2	1	3	7	8	5
2	3	8	1	$\times$	4	7	6	5
	4	8	2	1	3	7	5	6

$\longleftrightarrow$

3	8	1	4	2	**9**	7	5	6
3	8	1	4	2	**9**	7	6	5
3	6	1	**9**	4	2	7	8	5
4	6	2	1	3	**9**	7	8	5
3	8	1	**9**	4	2	7	6	5
4	8	2	1	3	**9**	7	5	6

Figure 2.42: A correspondence between table $\tau \in S_{6,8}^2$ and table $\tau' \in S_{6,9}^0$

Vice versa, given a table $\tau' \in S_{4,n+1}^0$ with c 6-columns and d 4-columns (and thus with weight $\mu_6^c\mu_4^d$), there are d ways how we can select one of its 4-columns. We then erase this column and turn the remaining pair other pair found in τ' into two marks. That way, we get back our original τ (after appropriate shifting the names of all elements so the missing element is "$n+1$"). Since each table $\tau \in S_{6,n}^2$ is counted $(n+1)^2$ times, thus

$$s_6^2(n) = \sum_{\tau \in S_{6,n}^2} w(\tau)\operatorname{sgn}\tau = \frac{1}{(n+1)^2}\sum_{\tau' \in S_{6,n+1}^0} d\mu_6^c\mu_4^{d-1}\operatorname{sgn}\tau' = \frac{1}{(n+1)^2}\frac{\partial s_6^0(n+1)}{\partial \mu_4}. \tag{2.287}$$

Or in terms of generating functions,

$$S_6^2(t) = \frac{1}{t}\frac{\partial S_6^0(t)}{\partial \mu_4}. \tag{2.288}$$

Equation (2.282) is obtained from Equation (2.287) and by differentiating Equation (2.280) by μ_4.

■

Corollary 202.1. *For any distribution of X_{ij} with $\mu_2 = 1$,*

$$P_6(t) = \mu_4 t\left((1-\mu_4 t)\,S_6^2(t) - 15\mu_4 t S_6^0(t)\right).$$

Proof. Straightforwardly, as we already know from Proposition 82 that we have the following: $P_6(t) = \mu_4 t\,\frac{e^{t(\mu_6 - 15\mu_4 + 30)}}{(1+3t-t\mu_4)^{16}}\left[(1-\mu_4 t)\,N_6^2\left(\frac{t}{(1+3t-t\mu_4)^3}\right) - 45tN_6^0\left(\frac{t}{(1+3t-t\mu_4)^3}\right)\right]$. Alternatively, we could use Equations (2.286) and (2.288) and differentiate chain generating function $P_6(t)_z$ (see proof of Proposition 82). ■

2.8.5 Four marks

Proposition 203. *Similarly, tables $S_{6,n}^{4/1}$ are formed from $S_{6,n}$ by marking one of its columns with four marks.*

Corollary 203.1. *For any distribution X_{ij} with $\mu_2 = 1, \mu_3 = 0$,*

$$S_6^{4/1}(t) = (1 + t - \mu_4 t)\, S_6^2(t) - 15\,(\mu_4 - 1)\, t\, S_6^0(t)$$
$$= \frac{e^{t(\mu_6 - 15\mu_4 + 30)}}{(1+3t-\mu_4 t)^{16}} \left[(1+t-\mu_4 t)\, N_6^2 \left(\frac{t}{(1+3t-\mu_4 t)^3} \right) - 30t\, N_6^0 \left(\frac{t}{(1+3t-\mu_4 t)^3} \right) \right].$$

Proof. Again let $\tau' \in S_{6,n}$ have c six-columns d four-columns. Thus, there are $n - c - d$ two-columns. The weight of this table is given as $w(\tau') = \mu_6^c \mu_4^d$. Let us find the weights $w(\tau)$ of all marked tables $\tau \in S_{6,n}^{4/1}$ created from τ' by marking. There are the following possibilities where we can put those four marks:

- in 6-column of τ' in 15 ways, creating a table τ with weight $\mu_6^{c-1}\mu_4^d$,
- in 4-column of τ' in 1 way by covering its four identical numbers, creating a table τ with weight $\mu_6^c \mu_4^{d-1}$,
- in 4-column of τ' in 6 ways by covering one pair of its four identical numbers and the two different numbers, creating a table τ with weight $\mu_6^c \mu_4^{d-1}$,
- in 2-column of τ' in 3 ways, creating a table τ with weight $\mu_6^c \mu_4^d$.

Thus, from τ', we get the following contribution to $s_6^{4/1}(n) = \sum_{\tau \in S_{6,n}^{4/1}} w(\tau)\,\mathrm{sgn}(\tau)$,

$$15c\mu_6^{c-1}\mu_4^d + 7d\mu_6^c\mu_4^{d-1} + 3(n - c - d)\mu_6^c\mu_4^d. \tag{2.289}$$

Grouping the terms, this is equal to

$$c\mu_6^{c-1}\mu_4^d(15 - 3\mu_6) + d\mu_6^c\mu_4^{d-1}(7 - 3\mu_4) + 3n\mu_6^c\mu_4^d. \tag{2.290}$$

Summing up this contribution over all tables $\tau' \in S_{6,n}$, we get

$$s_6^{4/1}(n) = (15 - 3\mu_6)\frac{\partial s_6^0(n)}{\partial \mu_6} + (7 - 3\mu_4)\frac{\partial s_6^0(n)}{\partial \mu_4} + 3n s_6^0(n) \tag{2.291}$$

or in terms of generating functions,

$$S_6^{4/1}(t) = (15 - 3\mu_6)\frac{\partial S_6^0(t)}{\partial \mu_6} + (7 - 3\mu_4)\frac{\partial S_6^0(t)}{\partial \mu_4} + 3t\,\frac{\partial S_6^0(t)}{\partial t}. \tag{2.292}$$

Using Remark 148 and/or Equations (2.286) and (2.288), we get the statement of the corollary. ■

Remark 204. There is an alternative way how we can express $s_6^{4/2}(n)$ (will be useful later). We use the addition technique and the correspondence established in Figure 2.43.

2		4	3	×	7	8	1	5	6	
1		4	5	×	7	2	3	8	6	
1		7	3	×	8	6	2	5	4	⟷
2		7	5	×	8	6	3	1	4	
		3	5	6	7	2	1	8	4	
		3	5	6	7	8	2	1	4	

4	3	**9**	7	8	1	5	2	6
4	5	**9**	7	2	3	8	1	6
7	3	**9**	8	6	2	5	1	4
7	5	**9**	8	6	3	1	2	4
3	5	6	7	2	1	8	**9**	4
3	5	6	7	8	2	1	**9**	4

Figure 2.43: A correspondence between table $\tau \in S_{6,8}^{4/1}$ and table $\tau' \in S_{6,9}^0$

Let $\tau' \in S_{6,n+1}$ has c 6-columns and d 4-columns (and thus weight $w(\tau') = \mu_6^c \mu_4^d$). In order to get a table $\tau \in S_{6,n}^{4/1}$, we start by selecting a number i which appears in two different columns of τ'. These numbers form a set $I_2(t')$ (there are d such numbers). Note that in one column i appears in fours. The other column with two displaced i's can be either a 4-column or a 2-column. Next, we erase this other column and turn the remaining four i's into marks. That way, we get a table $\tau \in S_{6,n}^{4/1}$. The weight of τ is given as $w(\tau) = \mu_6^c \mu_4^{d-\nu_i(t')}$. However, since $\nu_i(t')$ equals to one or two only, it is convenient to write it as

$$w(\tau) = \mu_6^c \mu_4^{d-1} \left(2 - \frac{1}{\mu_4} + \nu_i(t') \left(\frac{1}{\mu_4} - 1\right)\right). \tag{2.293}$$

Thus, by adding all contributions,

$$\begin{aligned} s_6^{4/1}(n) &= \sum_{\tau \in S_{6,n}^{4/1}} w(\tau) \operatorname{sgn} \tau = \sum_{\tau' \in S_{6,n+1}^0} \operatorname{sgn} \tau' \tfrac{\mu_6^c \mu_4^{d-1}}{(n+1)^2} \sum_{i \in I_2(t')} \left(2 - \tfrac{1}{\mu_4} + \nu_i(t') \left(\tfrac{1}{\mu_4} - 1\right)\right) \\ &= \sum_{\tau' \in S_{6,n+1}^0} \tfrac{\mu_6^c \mu_4^{d-1}}{(n+1)^2} \operatorname{sgn} \tau' \left[d \left(2 - \frac{1}{\mu_4}\right) + \left(\tfrac{1}{\mu_4} - 1\right) \sum_{i \in I_2(t')} \nu_i(t') \right]. \end{aligned} \tag{2.294}$$

Proposition 205. *For any distribution X_{ij} with $\mu_2 = 1, \mu_3 = 0$,*

$$\begin{aligned} S_6^{4/2}(t) &= \left(\frac{1}{t} - 5 - (2+t)\mu_4 + \mu_4^2 t\right) S_6^2(t) - 15 S_6^0(t) \mu_4 (1 + t - \mu_4 t) \\ &= \frac{e^{t(\mu_6 - 15\mu_4 + 30)}}{t(1+3t-\mu_4 t)^{16}} \left[\left(1 - 5t - 2\mu_4 t - \mu_4 t^2 + \mu_4^2 t^2\right) N_6^2 \left(\frac{t}{(1+3t-\mu_4 t)^3}\right) \right. \\ &\qquad \left. - 45t(1 - 5t + \mu_4 t) N_6^0 \left(\frac{t}{(1+3t-\mu_4 t)^3}\right) \right]. \end{aligned}$$

Proof. We use the addition technique described in section on $F_4(t)$. First, we seek the correspondence between $S_{6,n}^{4/2}$ and $S_{6,n+1}^0$ (see Figure 2.44 below). The crucial observation is that no matter which table $\tau \in S_{6,n}^{4/2}$ we select, if we put the covered (under marks) numbers into a single column together with two $(n+1)$'s (the column in grey), then this column has nonzero weight.

$\begin{Bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{Bmatrix}$

4	3	×	7	8	1	5	6
4	5	×	7	2	3	8	6
7	3	6	8	×	2	5	4
7	5	6	8	×	3	1	4
3	5	6	7	2	1	8	4
3	5	6	7	8	2	1	4

$\longleftrightarrow$

4	3	**9**	7	8	1	5	2	6
4	5	**9**	7	2	3	8	1	6
7	3	6	8	**9**	2	5	1	4
7	5	6	8	**9**	3	1	2	4
3	5	6	7	2	1	8	**9**	4
3	5	6	7	8	2	1	**9**	4

Figure 2.44: A correspondence between table $\tau \in S_{6,8}^{4/2}$ and table $\tau' \in S_{6,9}^0$

Let $\tau' \in S_{6,n+1}^0$ have c 6-columns and d 4-columns. Hence, $w(\tau') = \mu_6^c \mu_4^d$. Counting the number of tables $\tau \in S_{6,n}^{4/2}$ is rather intricate. First, we select a number which appears in three different columns (three pairs). As the number of 6-columns is c and 4-columns is d, there is $n+1-c-d$ numbers satisfying that criterion. However, we don't know whether those pairs are in 4-columns or 2-columns. Say the number i was selected and denote $\nu_i(t')$ the number of pairs of i's which lie in 4-columns. In the example above, $\nu_i(t') = 1$. We mark every occurrence of i and then erase one of the columns with pairs of i's (in three ways as there are three such columns). The contribution of τ' to $(n+1)^2 \sum_{\tau \in S_{6,n}^{4/2}} w(\tau) \operatorname{sgn} \tau$ for a given i is then

$$\mu_6^c \mu_4^{d-1} \nu_i(t') + \mu_6^c \mu_4^d (3 - \nu_i(t')) = \mu_6^c \mu_4^d \left(3 + \nu_i(t') \left(\frac{1}{\mu_4} - 1\right)\right). \tag{2.295}$$

In total,

$$\begin{aligned} s_6^{4/2}(n) &= \sum_{\tau \in S_{6,n}^{4/2}} w(\tau) \operatorname{sgn} \tau = \sum_{\tau' \in S_{6,n+1}^0} \frac{\mu_6^c \mu_4^d}{(n+1)^2} \operatorname{sgn} \tau' \sum_{i \in I_3(t')} \left(3 + \nu_i(t') \left(\frac{1}{\mu_4} - 1\right)\right) \\ &= \sum_{\tau' \in S_{6,n+1}^0} \frac{\mu_6^c \mu_4^d}{(n+1)^2} \operatorname{sgn} \tau' \left[3(n+1-c-d) + \left(\frac{1}{\mu_4} - 1\right) \sum_{i \in I_3(t')} \nu_i(t')\right]. \end{aligned} \tag{2.296}$$

where $I_3(t')$ is the set of numbers of τ' which appear in three different columns. To sum the series exactly, we use Lemma 88. We get

$$s_6^{4/2}(n) = \sum_{\tau' \in S_{6,n+1}^0} \frac{\mu_6^c \mu_4^d \operatorname{sgn} \tau'}{(n+1)^2} \left[3(n+1-c-d) + p(\tau') \left(\frac{1}{\mu_4} - 1\right)\right]. \tag{2.297}$$

where $p(\tau')$ is the total number of 4-column chains in τ'. This can be written as

$$s_6^{4/2}(n) = \tfrac{3}{n+1} s_6^0(n+1) - \tfrac{3\mu_6}{(n+1)^2} \tfrac{\partial s_6^0(n+1)}{\partial \mu_6} - \tfrac{3\mu_4}{(n+1)^2} \tfrac{\partial s_6^0(n+1)}{\partial \mu_4} + \tfrac{\frac{1}{\mu_4} - 1}{(n+1)^2} p_6(n+1), \tag{2.298}$$

where $p_6(n) = \sum_{\tau' \in S_{6,n}} p(\tau') w(\tau') \operatorname{sgn} \tau'$. In terms of generating functions,

$$S_6^{4/2}(t) = 3 \frac{\partial S_6^0(t)}{\partial t} - \frac{3\mu_6}{t} \frac{\partial S_6^0(t)}{\partial \mu_6} - \frac{3\mu_4}{t} \frac{\partial S_6^0(t)}{\partial \mu_4} + \frac{\frac{1}{\mu_4} - 1}{t} P_6(t)$$

Since by Corollary 202.1 we know the value of $P_6(t)$, we are done (in the end we also employ Remark 148 and/or previous propositions). ∎

Alternative proof of Proposition 205. There is another derivation of $S_6^{4/2}(t)$ not involving knowing the value of $P_6(t)$. Let $\tau' \in S_{6,n+1}$ have c 6-columns and d 4-columns. Since

$$\sum_{i \in I_3(t')} \nu_i(t') + \sum_{i \in I_2(t')} \nu_i(t') = 2d, \tag{2.299}$$

by combining Equations (2.294) and (2.296), we get the following remarkable connection

$$\begin{aligned} s_6^{4/1}(n) + \tfrac{1}{\mu_4} s_6^{4/2}(n) &= \sum_{\tau' \in S_{6,n+1}^0} \tfrac{\mu_6^c \mu_4^{d-1}}{(n+1)^2} \operatorname{sgn} \tau' \left[d\left(2 - \tfrac{1}{\mu_4}\right) + 3(n+1-c-d) + \left(\tfrac{1}{\mu_4} - 1\right) 2d\right] \\ &= \frac{3}{(n+1)\mu_4} s_6^0(n+1) - \frac{3 - \frac{1}{\mu_4}}{(n+1)^2} \frac{\partial s_6^0(n+1)}{\partial \mu_4} - \frac{3\mu_6}{(n+1)^2 \mu_4} \frac{\partial s_6^0(n+1)}{\partial \mu_6} \end{aligned} \tag{2.300}$$

which in terms of generating functions gives

$$S_6^{4/1}(t) + \frac{1}{\mu_4} S_6^{4/2}(t) = \frac{3}{\mu_4}\frac{\partial S_6^0(t)}{\partial t} - \frac{3-\frac{1}{\mu_4}}{t}\frac{\partial S_6^0(t)}{\partial \mu_4} - \frac{3\mu_6}{\mu_4 t}\frac{\partial S_6^0(t)}{\partial \mu_6} \tag{2.301}$$

which gives, after simplifications of derivatives of $S_6^0(t)$ (see Equations (2.286) and (2.288)),

$$S_6^{4/1}(t) + \frac{1}{\mu_4} S_6^{4/2}(t) = \frac{(1-\mu_4 t - 5t)}{\mu_4 t} S_6^2(t) - 15 S_6^0(t). \tag{2.302}$$

Rearranging the terms and using the already known value for $S_6^{4/1}(t)$, we get

$$S_6^{4/2}(t) = \left(\tfrac{1}{t} - 5 - (2+t)\mu_4 + \mu_4^2 t\right) S_6^2(t) - 15 S_6^0(t)\mu_4\,(1+t-\mu_4 t) \tag{2.303}$$

as before. ■

Corollary 205.1. *Summing $S_6^{4/1}(t)$ and $S_6^{4/2}(t)$, we get*

$$\begin{aligned} S_6^4(t) &= 15\left(t - (1+2t)\mu_4 + t\mu_4^2\right) S_6^0(t) + \left(\tfrac{1}{t} - 4 + t - 2\mu_4 - 2t\mu_4 + t\mu_4^2\right) S_6^2(t) \\ &= \frac{e^{t(\mu_6 - 15\mu_4 + 30)}}{t\,(1+3t-\mu_4 t)^{16}}\Bigg[-15t\,(3-13t+3\mu_4 t)\, N_6^0\left(\frac{t}{(1+3t-\mu_4 t)^{3}}\right) \\ &\qquad + \left(1 - 4t + t^2 - 2\mu_4 t - 2\mu_4 t^2 + \mu_4^2 t^2\right) N_6^2\left(\frac{t}{(1+3t-\mu_4 t)^{3}}\right)\Bigg]. \end{aligned}$$

Remark 206. Note that the previous result gives $N_6^4(t)$ for $\mu_6 = 15$ and $\mu_4 = 3$ since

$$t\,N_6^4(t) = \left(4t^2 - 10t + 1\right) N_6^2(t) + 15t(4t-3) N_6^0(t).$$

2.8.6 Six marks

Proposition 207. *Similarly, by marking, tables $S_{6,n}^{6/1}$ are formed from $S_{6,n}$ by marking one of its columns with six marks.*

Corollary 207.1. *For any distribution X_{ij} with $\mu_2 = 1, \mu_3 = 0$,*

$$\begin{aligned} S_6^{6/1}(t) &= (1-5\mu_4)\,t\,S_6^0(t) + \frac{1}{3}\,(1-3t-\mu_4 t)\,S_6^2(t) \\ &= \tfrac{e^{t(\mu_6-15\mu_4+30)}}{3(1+3t-\mu_4 t)^{16}}\left[(1-3t-\mu_4 t)\,N_6^2\left(\tfrac{t}{(1+3t-\mu_4 t)^3}\right) - 6t\,(7-24t+8\mu_4 t)\,N_6^0\left(\tfrac{t}{(1+3t-\mu_4 t)^3}\right)\right]. \end{aligned}$$

Proof. Again let $\tau' \in S_{6,n}$ have c six-columns and d four-columns and thus $n-c-d$ two-columns. Its weight is then $w(\tau') = \mu_6^c \mu_4^d$. To create a table $\tau \in S_{6,n}^{6/1}$, we can put six marks

- in 6-column of τ' in 1 way, creating a table τ with weight $\mu_6^{c-1}\mu_4^d$,
- in 4-column of τ' in 1 way, creating a table τ with weight $\mu_6^c\mu_4^d$,
- in 2-column of τ' in 1 way, creating a table τ with weight $\mu_6^c\mu_4^d$,

Thus, from τ', we get the following contribution to $s_6^{6/1}(n) = \sum_{\tau \in S_{6,n}^{6/1}} w(\tau)\,\mathrm{sgn}(\tau)$,

$$c\mu_6^{c-1}\mu_4^d + d\mu_6^c\mu_4^{d-1} + (n-c-d)\mu_6^c\mu_4^d. \tag{2.304}$$

Grouping the terms, this is equal to

$$c\mu_6^{c-1}\mu_4^d(1-\mu_6)+d\mu_6^c\mu_4^{d-1}(1-\mu_4)+n\mu_6^c\mu_4^d. \tag{2.305}$$

Summing up this contribution over all tables $\tau' \in S_{6,n}$, we get

$$s_6^{6/1}(n)=(1-\mu_6)\frac{\partial s_6^0(n)}{\partial\mu_6}+(1-\mu_4)\frac{\partial s_6^0(n)}{\partial\mu_4}+ns_6^0(n) \tag{2.306}$$

or in terms of generating functions,

$$S_6^{6/1}(t)=(1-\mu_6)\frac{\partial S_6^0(t)}{\partial\mu_6}+(1-\mu_4)\frac{\partial S_6^0(t)}{\partial\mu_4}+t\,\frac{\partial S_6^0(t)}{\partial t}. \tag{2.307}$$

■

Corollary 207.2. *After some simplification, we have for any distribution X_{ij} with $\mu_2=1$,*

$$s_6^{6/1}(n)=(n!)^2\sum_{j=0}^{n}\sum_{i=0}^{j}\frac{(1+i)(2+i)(4+i)!}{48(n-j)!}\binom{14+j+2i}{j-i}\Big(2(7+2i+j)(j-n) + i\,(\mu_6-15\mu_4+30)\Big)(\mu_6-15\mu_4+30)^{n-j-1}(\mu_4-3)^{j-i}. \tag{2.308}$$

Proposition 208. *Tables $S_{6,n}^{6/2}$ are formed from $S_{6,n}^{6/1}$ by swapping two marks in $\times^6$ column with a pair of numbers in some other column. Via this swapping, each table from $S_{6,n}^{6/2}$ is counted once.*

Proof. Let $\tau \in S_{6,n}^{6/2}$. There are four options how the table can look like based on the uncovered numbers in $\times^4$ and $\times^2$ columns. Either

- Identical number a everywhere in both $\times^4$ and $\times^2$
- Number a in $\times^4$ column and numbers a, b in $\times^2$ column
- Number a in $\times^4$ column and four numbers b in $\times^2$ column
- Number a in $\times^4$ column and numbers b, e in $\times^2$ column

Swapping two marks in $\times^2$ column with numbers in $\times^4$ column, we get a corresponding table $\tau' \in S_{6,n}^{6/1}$ (see figures below).

×	a		×	a
×	a		×	a
×	a	←	×	a
×	a		×	a
a	×		×	a
a	×		×	a

Figure 2.45: First option for $S_{6,n}^{6/2}$

×	b		×	b
×	b		×	b
×	a	←	×	a
×	a		×	a
a	×		×	a
a	×		×	a

Figure 2.46: Second option for $S_{6,n}^{6/2}$

×	b		×	b
×	b		×	b
×	b	←	×	b
×	b		×	b
a	×		×	a
a	×		×	a

Figure 2.47: Third option for $S_{6,n}^{6/2}$

×	b		×	b
×	b		×	b
×	e	←	×	e
×	e		×	e
a	×		×	a
a	×		×	a

Figure 2.48: Fourth option for $S_{6,n}^{6/2}$

■

Corollary 208.1. *For any distribution X_{ij} with $\mu_2 = 1, \mu_3 = 0$,*

$$S_6^{6/2}(t) = \frac{e^{t(\mu_6-15\mu_4+30)}}{t\,(1+3t-\mu_4 t)^{\,17}}\Bigg[N_6^2\left(\frac{t}{(1+3t-\mu_4 t)^{\,3}}\right)\Big(1-3t-11t^2+36t^3-81t^4-4t\mu_4$$
$$-3t^2\mu_4-12t^3\mu_4+54t^4\mu_4+6t^2\mu_4^2+12t^3\mu_4^2-4t^3\mu_4^3-6t^4\mu_4^3+t^4\mu_4^4\Big)$$
$$-15tN_6^0\left(\frac{t}{(1+3t-\mu_4 t)^{\,3}}\right)\Big(3-9t-48t^2+144t^3-3\mu_4 t+16t^2\mu_4-96t^3\mu_4+16t^3\mu_4^2\Big)\Bigg]$$

Proof. Let $\tau' \in S_{6,n}^{6/1}$ have c six-columns and d four-columns, its weight is then $w(\tau') = \mu_6^c\mu_4^d$. However, now there are only $n-c-d-1$ two-columns as one column is covered with six marks. To create a table $\tau \in S_{6,n}^{6/2}$, we can swap two marks of $\times^6$-column with

- a 6-column of τ' in 15 ways, creating a table τ with weight $\mu_6^{c-1}\mu_4^{d+1}$,
- a 4-column of τ' in 6 ways, swapping marks with one pair of four identical numbers, creating a table τ with weight $\mu_6^c\mu_4^{d-1}$,
- a 4-column of τ' in 1 way, swapping marks with the remaining two numbers, creating a table τ with weight $\mu_6^c\mu_4^d$,
- a 2-column of τ' in 3 ways, creating a table τ with weight $\mu_6^c\mu_4^d$,

Thus, from τ', we get the following contribution to $s_6^{6/2}(n) = \sum_{\tau\in S_{6,n}^{6/2}} w(\tau)\operatorname{sgn}(\tau)$,

$$15c\mu_6^{c-1}\mu_4^{d+1} + 6d\mu_6^c\mu_4^{d-1} + d\mu_6^c\mu_4^d + 3(n-c-d-1)\mu_6^c\mu_4^d. \tag{2.309}$$

Grouping the terms, this is equal to

$$c\mu_6^{c-1}\mu_4^d(15\mu_4 - 3\mu_6) + d\mu_6^c\mu_4^{d-1}(6-2\mu_4) + 3n\mu_6^c\mu_4^d - 3\mu_6^c\mu_4^d. \tag{2.310}$$

Summing up this contribution over all tables $\tau' \in S_{6,n}^{6/1}$, we get

$$s_6^{6/2}(n) = (15\mu_4-3\mu_6)\frac{\partial s_6^{6/1}(n)}{\partial\mu_6}+(6-2\mu_4)\frac{\partial s_6^{6/1}(n)}{\partial\mu_4}+3ns_6^{6/1}(n)-3s_6^{6/1}(n) \tag{2.311}$$

or in terms of generating functions,

$$S_6^{6/2}(t) = (15\mu_4-3\mu_6)\frac{\partial S_6^{6/1}(t)}{\partial\mu_6} + (6-2\mu_4)\frac{\partial S_6^{6/1}(t)}{\partial\mu_4} + 3t\,\frac{\partial S_6^{6/1}(t)}{\partial t} - 3S_6^{6/1}(t). \tag{2.312}$$

Using Corollary 207.1 and Remark 148, we get the statement of our corollary. Note that in this proof we rely on the already derived $S_6^{6/1}(t)$ thus we need to compute the second derivative of $N_6^0(t)$. ■

Remark 209. If we instead used the addition technique, we would have found another relation for $S_6^{6/2}(t)$. First, we develop a correspondence between $S_{6,n}^{6/2}$ and $S_{6,n+1}^0$ (see Figure 2.49 below). The crucial observation is that if we put the covered number into a single column (in white), this column has nonzero weight.

2	4	3	×	7	8	1	5	6
1	4	5	×	7	2	3	8	6
1	7	3	×	8	6	2	5	4
2	7	5	×	8	6	3	1	4
3	×	5	6	7	2	1	8	4
3	×	5	6	7	8	2	1	4

$\longleftrightarrow$

4	3	**9**	7	8	1	5	2	6
4	5	**9**	7	2	3	8	1	6
7	3	**9**	8	6	2	5	1	4
7	5	**9**	8	6	3	1	2	4
9	5	6	7	2	1	8	3	4
9	5	6	7	8	2	1	3	4

Figure 2.49: A correspondence between table $\tau \in S^{6/2}_{6,8}$ and table $\tau' \in S^0_{6,9}$

Let $\tau' \in S^0_{6,n+1}$ have c 6-columns and d 4-columns. Hence, $w(\tau') = \mu_6^c\mu_4^d$. We now count the number of tables $\tau \in S^{6/2}_{6,n}$. First, we select a number i which appears in two different columns. There is d such numbers (they form a set $I_2(t')$). In the one of the columns, there are always four copies of i (making it a four column). The other column with two i's is either a 4-column or a 2-column. It is convenient to define $\nu_i(t')$ again as the number of four columns in which the selected number i appears (it is either one or two for $i \in I_2(t')$). We then select a column other that these two and erase it. Finally, we turn the selected number i to marks. That way, we get our table $\tau \in S^{6/2}_{6,n}$. To count the overall contribution of τ' to $w(\tau)\operatorname{sgn}\tau$, we could either select

- one 2-column (not the ones in which i's lie) in $n-1-c-d+\nu_i(t')$ ways and erase it, creating table τ with $w(\tau) = w(\tau')/\mu_4$ (remember all i's are turned into marks which have weight one)
- or one 4-column in $d-\nu_i(t')$ ways and erase it, creating table τ with $w(\tau) = w(\tau')/\mu_4^2$
- or one 6-column in c ways and erase it, creating table τ with $w(\tau) = w(\tau')/(\mu_6\mu_4)$

In total,

$$
\begin{aligned}
s_6^{6/2}(n) &= \sum_{\tau \in S^{6/2}_{6,n}} w(\tau)\operatorname{sgn}\tau = \sum_{\tau' \in S^0_{6,n+1}} \frac{\mu_6^c\mu_4^{d-1}\operatorname{sgn}\tau'}{(n+1)^2} \sum_{i \in I_2(t')} n-1-c-d+\nu_i(t')+\frac{d-\nu_i(t')}{\mu_4}+\frac{c}{\mu_6} \\
&= \sum_{\tau' \in S^0_{6,n+1}} \frac{\mu_6^c\mu_4^{d-1}\operatorname{sgn}\tau'}{(n+1)^2} \left[d\left(n-1-c-d+\frac{d}{\mu_4}+\frac{c}{\mu_6}\right) + \left(1-\frac{1}{\mu_4}\right) \sum_{i \in I_2(t')} \nu_i(t') \right].
\end{aligned}
\tag{2.313}
$$

This can be written as

$$
\begin{aligned}
s_6^{6/2}(n) = {} & \frac{n-\frac{1}{\mu_4}}{(n+1)^2}\frac{\partial s_6^0(n+1)}{\partial \mu_4} \\
& + \frac{1-\mu_4}{(n+1)^2}\frac{\partial^2 s_6^0(n+1)}{\partial \mu_4^2} - \frac{1-\frac{1}{\mu_4}}{(n+1)^2\mu_4} p_6(n+1).
\end{aligned}
\tag{2.314}
$$

This relation will be useful later.

Proposition 210. *For any distribution X_{ij} with $\mu_2 = 1, \mu_3 = 0$,*

$$S_6^{6/3}(t) = \frac{e^{t(\mu_6-15\mu_4+30)}}{3t^2\left(1+3t-t\mu_4\right)^{17}}\Big[(1-11t+4t^2+105t^3-315t^4+486t^5-5t\mu_4$$
$$+25t^2\mu_4+34t^3\mu_4+165t^4\mu_4-243t^5\mu_4+10t^2\mu_4^2-17t^3\mu_4^2-74t^4\mu_4^2$$
$$-54t^5\mu_4^2-10t^3\mu_4^3+3t^4\mu_4^3+36t^5\mu_4^3+5t^4\mu_4^4-t^5\mu_4^5)N_6^2\left(\frac{t}{(1+3t-\mu_4 t)^{3}}\right)$$
$$-45t(1-11t+t^2+159t^3-288t^4-2t\mu_4+10t^2\mu_4-56t^3\mu_4$$
$$+192t^4\mu_4+t^2\mu_4^2+t^3\mu_4^2-32t^4\mu_4^2)N_6^0\left(\frac{t}{(1+3t-\mu_4 t)^{3}}\right)\Big].$$

Proof. We use the addition technique described in section on $F_4(t)$. First, we seek the correspondence between $S_{6,n}^{6/3}$ and $S_{6,n+1}^{0}$ (see Figure 2.50 below). The crucial observation is that if we put the covered number into a single column (in grey), this column has nonzero weight.

$\{2, 1, 1, 2, 3, 3\}$

4	3	×	7	8	1	5	6
4	5	×	7	2	3	8	6
7	3	6	8	×	2	5	4
7	5	6	8	×	3	1	4
×	5	6	7	2	1	8	4
×	5	6	7	8	2	1	4

$\longleftrightarrow$

4	3	**9**	7	8	1	5	2	6
4	5	**9**	7	2	3	8	1	6
7	3	6	8	**9**	2	5	1	4
7	5	6	8	**9**	3	1	2	4
9	5	6	7	2	1	8	3	4
9	5	6	7	8	2	1	3	4

Figure 2.50: A correspondence between table $\tau \in S_{6,8}^{6/3}$ and table $\tau' \in S_{6,9}^{0}$

It depends on whether previously covered numbers (in the gray column above) form a 6-column, a 4-column or a 2-column. Let $\tau' \in S_{6,n+1}^0$ have c 6-columns and d 4-columns. Hence, $w(\tau') = \mu_6^c\mu_4^d$. We now count the number of tables $\tau \in S_{6,n}^{6/3}$. First, we select a number which appears in three columns (three pairs). As the number of 6-columns is c and 4-columns is d, there is $n+1-c-d$ numbers satisfying that criterion. However, we don't know whether those pairs themselves lie in 4-columns of 2-columns. Say the number i was selected and denote $\nu_i(t')$ the number of pairs of i's which lie in 4-columns. In the example above, $\nu_i(t') = 1$. We mark every occurrence of i and then select either

- one 2-column (not the ones in which i's lie) in $n+1-c-3+\nu_i(t)-d$ ways and erase it, creating table τ with $w(\tau) = w(\tau')$
- or one 4-column in $d-\nu_i(t')$ ways and erase it, creating table τ with $w(\tau) = w(\tau')/\mu_4$
- or one 6-column in c ways and erase it, creating table τ with $w(\tau) = w(\tau')/\mu_6$

In total,

$$s_6^{6/3}(n) = \sum_{\tau\in S_{6,n}^{6/3}} w(\tau)\operatorname{sgn}\tau = \sum_{\tau'\in S_{6,n+1}^0} \tfrac{\mu_6^c\mu_4^d \operatorname{sgn}\tau'}{(n+1)^2} \sum_{i\in I_3(t')} n-2-c-d+\nu_i(t') + \tfrac{d-\nu_i(t')}{\mu_4} + \tfrac{c}{\mu_6}$$
$$= \sum_{\tau'\in S_{6,n+1}^0} \tfrac{\mu_6^c\mu_4^{d-1}\operatorname{sgn}\tau'}{(n+1)^2}\left[(n+1-c-d)\left(n-2-c-d+\tfrac{d}{\mu_4}+\tfrac{c}{\mu_6}\right) + \left(1-\tfrac{1}{\mu_4}\right)\sum_{i\in I_3(t')}\nu_i(t')\right]. \tag{2.315}$$

where i is summed over all numbers which lie in three different columns (there are $n+1-c-d$ such numbers forming the set $I_3(t')$). By Lemma 88, $\sum_{i\in I_3(t')}\nu_i(t')=p(\tau')$, where $p(\tau')$ is the total number of chains of 4-columns in τ', thus

$$s_6^{6/3}(n)=\sum_{\tau'\in S^0_{6,n+1}}\frac{\mu_6^c\mu_4^d\operatorname{sgn}\tau'}{(n+1)^2}\left[(n+1-c-d)\left(n-2-c-d+\frac{d}{\mu_4}+\frac{c}{\mu_6}\right)+p(\tau')\left(1-\frac{1}{\mu_4}\right)\right] \tag{2.316}$$

Note that

$$(n+1-c-d)\left(n-2-c-d+\frac{d}{\mu_4}+\frac{c}{\mu_6}\right)=c(c-1)\left(1-\frac{1}{\mu_6}\right)+d(d-1)\left(1-\frac{1}{\mu_4}\right)$$
$$+cd\left(2-\frac{1}{\mu_4}-\frac{1}{\mu_6}\right)+c\left(2-2n+\frac{n}{\mu_6}\right)+d\left(2-2n+\frac{n}{\mu_4}\right)+(n+1)(n-2) \tag{2.317}$$

Thus, by Corollary 202.1,

$$s_6^{6/3}(n)=\sum_{\tau'\in S^0_{6,n+1}}\frac{\mu_6^c\mu_4^d\operatorname{sgn}\tau'}{(n+1)^2}\left[(n+1-c-d)\left(n-2-c-d+\frac{d}{\mu_4}+\frac{c}{\mu_6}\right)+p(\tau')\left(1-\frac{1}{\mu_4}\right)\right]$$
$$=\frac{\mu_6(\mu_6-1)}{(n+1)^2}\frac{\partial^2 s_6^0(n+1)}{\partial\mu_6^2}+\frac{\mu_4(\mu_4-1)}{(n+1)^2}\frac{\partial^2 s_6^0(n+1)}{\partial\mu_4^2}+\frac{2\mu_6\mu_4-\mu_6-\mu_4}{(n+1)^2}\frac{\partial^2 s_6^0(n+1)}{\partial\mu_6\partial\mu_4}$$
$$+\frac{(2-2n)\mu_6+n}{(n+1)^2}\frac{\partial s_6^0(n+1)}{\partial\mu_6}+\frac{(2-2n)\mu_4+n}{(n+1)^2}\frac{\partial s_6^0(n+1)}{\partial\mu_4}+\frac{n-2}{n+1}s_6^0(n+1)+\frac{1-\frac{1}{\mu_4}}{(n+1)^2}p_6(n+1).$$

Or in terms of generating functions,

$$S_6^{6/3}(t)=\frac{\mu_6(\mu_6-1)}{t}\frac{\partial^2 S_6^0(t)}{\partial\mu_6^2}+\frac{\mu_4(\mu_4-1)}{t}\frac{\partial^2 S_6^0(t)}{\partial\mu_4^2}+\frac{2\mu_6\mu_4-\mu_6-\mu_4}{t}\frac{\partial^2 S_6^0(t)}{\partial\mu_6\partial\mu_4}$$
$$+\frac{4\mu_6-1}{t}\frac{\partial S_6^0(t)}{\partial\mu_6}+(1-2\mu_6)\frac{\partial^2 S_6^0(t)}{\partial\mu_6\partial t}+\frac{4\mu_4-1}{t}\frac{\partial S_6^0(t)}{\partial\mu_4}+(1-2\mu_4)\frac{\partial^2 S_6^0(t)}{\partial\mu_4\partial t}$$
$$+t\frac{\partial^2 S_6^0(t)}{\partial t^2}-2\frac{\partial S_6^0(t)}{\partial t}+\frac{1-\frac{1}{\mu_4}}{t}P_6(t)$$

This finishes the calculation of $S_6^{6/3}(t)$. Simplification of derivatives is cumbersome, but straightforward. ■

Corollary 210.1. *Summing $S_6^{6/1}(t)$, $S_6^{6/2}(t)$ and $S_6^{6/3}(t)$, we get*

$$S_6^6(t)=\frac{e^{t(\mu_6-15\mu_4+30)}}{3t^2\left(1+3t-t\mu_4\right)^{17}}\Bigg[\Big(1-8t-4t^2+72t^3-216t^4+243t^5-5t\mu_4+13t^2\mu_4$$
$$+23t^3\mu_4+129t^4\mu_4-81t^5\mu_4+10t^2\mu_4^2+t^3\mu_4^2-37t^4\mu_4^2-54t^5\mu_4^2-10t^3\mu_4^3$$
$$-9t^4\mu_4^3+18t^5\mu_4^3+5t^4\mu_4^4+3t^5\mu_4^4-t^5\mu_4^5\Big)N_6^2\left(\frac{t}{(1+3t-t\mu_4)^3}\right)-3t\Big(15-120t$$
$$-106t^2+1659t^3-2304t^4-30t\mu_4+105t^2\mu_4-598t^3\mu_4+1536t^4\mu_4+15t^2\mu_4^2$$
$$+15t^3\mu_4^2-256t^4\mu_4^2\Big)N_6^0\left(\frac{t}{(1+3t-t\mu_4)^3}\right)\Bigg].$$

Remark 211. Note that the previous result indeed gives $N_6^6(t)$ for $\mu_6=15$ and $\mu_4=3$ since

$$3t^2N_6^6(t)=\left(1-23t+125t^2-120t^3\right)N_6^2(t)-3t\left(15-210t+344t^2\right)N_6^0(t).$$

2.8.7 Complete sixth moment and its generating function

Proposition 212. *For any distribution of X_{ij} with $\mu_2 = 1, \mu_3 = 0$, we get the statement of Theorem 196.*

Proof. Follows directly from Proposition 198, which states

$$F_6(t) = (1+m_1\mu_5 t)^6 S_6^0(t) + m_1^2(1+m_1\mu_5 t)^4 S_6^2(t) + m_1^4(1+m_1\mu_5 t)^2 S_6^4(t) + m_1^6 S_6^6(t).$$

wherein we insert $S_6^0(t)$, $S_6^2(t)$, $S_6^4(t)$ and $S_6^6(t)$ as expressed before. ■

3. Even Volumetric Moments

As we will see, the problem of finding $v_n^{(k)}(K_d)$ when k is even can be treated as a purely combinatorial problem. For even k and any $n \le d$, volumetric moments are trivial to obtain, especially for polytopes. First, note that Δ_n can be expressed as an absolute value of a determinant of the coordinates of the $n+1$ points forming the vertices of the convex hull $\mathbb{H}_n$ (or as a square root of Gram determinant when $n < d$). Rising this determinant to some (even) power k, we obtain some polynomial in coordinates. This is then integrated over the original polytope P_d. For completeness, we enlist in Table 3.1 the first three even moments $v_d^{(k)}(P_d)$ for the families of polytopes T_d, C_d and O_d and the unit ball $\mathbb{B}_d$ upto $d = 5$.

$v_d^{(k)}(T_d)$	$k=2$	$k=4$	$k=6$
$d=1$	$\frac{1}{6}$	$\frac{1}{15}$	$\frac{1}{28}$
$d=2$	$\frac{1}{72}$	$\frac{1}{900}$	$\frac{403}{2116800}$
$d=3$	$\frac{3}{4000}$	$\frac{871}{123480000}$	$\frac{2797}{11202105600}$
$d=4$	$\frac{1}{33750}$	$\frac{2083}{96808320000}$	$\frac{28517}{264649744800000}$
$d=5$	$\frac{5}{5445468}$	$\frac{24995}{682923373461504}$	$\frac{11490716929}{618668393733836328960000}$

$v_d^{(k)}(C_d)$	$k=2$	$k=4$	$k=6$
$d=2$	$\frac{1}{96}$	$\frac{1}{2400}$	$\frac{761}{27095040}$
$d=3$	$\frac{1}{2592}$	$\frac{701}{839808000}$	$\frac{29563}{7466363412480}$
$d=4$	$\frac{5}{497664}$	$\frac{887}{1146617856000}$	$\frac{6207797}{38533602917272780800}$
$d=5$	$\frac{1}{4976640}$	$\frac{2899}{7166361600000000}$	$\frac{3591192719}{1348676102104547328000000000}$

$v_d^{(k)}(O_d)$	$k=2$	$k=4$	$k=6$
$d=3$	$\frac{3}{8000}$	$\frac{4259}{5268480000}$	$\frac{7200523}{1835352981504000}$
$d=4$	$\frac{1}{108000}$	$\frac{3959}{5664669696000}$	$\frac{74002087}{462508951339008000000}$
$d=5$	$\frac{5}{29042496}$	$\frac{228685}{699313534424580096}$	$\frac{7261177207}{4059550791626730830069288140800000}$

$v_d^{(k)}(\mathbb{B}_d)$	$k=2$	$k=4$	$k=6$
$d=2$	$\frac{3}{32\pi^2}$	$\frac{1}{32\pi^4}$	$\frac{275}{16384\pi^6}$
$d=3$	$\frac{3}{1000\pi^2}$	$\frac{117}{2744000\pi^4}$	$\frac{17}{14817600\pi^6}$
$d=4$	$\frac{5}{7776\pi^4}$	$\frac{475}{191102976\pi^8}$	$\frac{161}{7644119040\pi^{12}}$
$d=5$	$\frac{45}{4302592\pi^4}$	$\frac{325}{401483464704\pi^8}$	$\frac{3875}{24632119418683392\pi^{12}}$

Table 3.1: Selected values of $v_d^{(k)}(P_d)$ with $P_d = T_d, C_d, O_d, \mathbb{B}_d$, even k and $d \le 5$.

3.1 d-Cube even volumetric moments

As a simple application of the results on moments of random determinants, we deduce a general formula for $v_d^{(k)}(C_d)$ when $k = 2, 4, 6$ and $d \geq 2$ arbitrary for the unit d-cube defined as $C_d = [0, 1]^d$.

$$v_d^{(2)}(C_d) = \frac{d+1}{12^d \, d!}. \tag{3.1}$$

Table 3.2 shows the second volumetric moments $v_d^{(2)}(C_d)$ for low values of d.

d	2	3	4	5	6	7	8
$v_d^{(2)}(C_d)$	$\frac{1}{96}$	$\frac{1}{2592}$	$\frac{5}{497664}$	$\frac{1}{4976640}$	$\frac{7}{2149908480}$	$\frac{1}{22574039040}$	$\frac{1}{1926317998080}$

Table 3.2: Second volumetric moment in d-cube

We are able to deduce the fourth moment

$$v_d^{(4)}(C_d) = \frac{d+1}{144^d (d!)^2} \sum_{j=0}^{d} \left(-\frac{6}{5}\right)^{d-j} \frac{(j+1)^2(j+2)}{2\,(d-j)!}, \tag{3.2}$$

Table 3.3 shows the fourth volumetric moments $v_d^{(4)}(C_d)$ for low values of d.

d	2	3	4	5	6
$v_d^{(4)}(C_d)$	$\frac{1}{2400}$	$\frac{701}{839808000}$	$\frac{887}{1146617856000}$	$\frac{2899}{7166361600000000}$	$\frac{24257989}{180551034077184000000000}$

Table 3.3: Fourth volumetric moment in d-cube

The case $k = 6$ with $P_d = C_d$ is somehow clearer than for $P_d = T_d$ as we will see later. In fact, we can find a relatively simple formula for $v_d^{(6)}(C_d)$ for any d,

$$\begin{aligned} v_d^{(6)}(C_d) = \frac{d+1}{252^d (d!)^4} \sum_{j=0}^{d+1} \sum_{i=0}^{j} \frac{(1+i)(2+i)(4+i)!}{168\,(d+1-j)!} \left(-\frac{7}{40}\right)^j \left(-\frac{5}{6}\right)^i \times \\ \binom{14+2i+j}{j-i} (24i + 7(j-1-d)(7+2i+j)). \end{aligned} \tag{3.3}$$

Table 3.4 shows the sixth volumetric moments $v_d^{(6)}(C_d)$ for low values of d.

d	2	3	4	5
$v_d^{(6)}(C_d)$	$\frac{761}{27095040}$	$\frac{29563}{7466363412480}$	$\frac{6207797}{38533602917272780800}$	$\frac{3591192719}{134867610210454732800000000 0}$

Table 3.4: Sixth volumetric moment in d-cube

3.1.1 Shifted determinant formula

Let $\mathbb{X} = (\mathbf{X}_0, \ldots, \mathbf{X}_d)$ be a collection of $(d+1)$ random points $\mathbf{X}_j = (X_{1j}, \ldots X_{dj})^\top \in \mathbb{R}^d$ with $X_{ij} \sim \mathsf{Unif}(0,1)$ i.i.d. and let $\mathbb{H}_d = \operatorname{conv}\mathbb{X}$ be their convex hull and $\Delta_d = \operatorname{vol}_d \mathbb{H}_d$ its volume, then

$$\Delta_d = \pm\frac{1}{d!}\det\left(\mathbf{X}_1 - \mathbf{X}_0 \mid \mathbf{X}_2 - \mathbf{X}_0 \mid \cdots \mid \mathbf{X}_d - \mathbf{X}_0\right) \tag{3.4}$$

and from which ($\operatorname{vol}_d C_d = 1$) we get $v_d^{(k)}(C_d) = \mathbb{E}\,\Delta_d^k$. It turns out we can express Δ_d in a different form. Let $\mathbf{X}'_j = (X_{1j}, \ldots, X_{dj}, 1)^\top \in \mathbb{R}^d$, $\mathbb{H}'_d = \operatorname{conv}(\mathbf{0}, \mathbf{X}'_0, \ldots, \mathbf{X}'_d) \subset \mathbb{R}^{d+1}$ and $\nabla_{d+1} = \operatorname{vol}_{d+1}\mathbb{H}'_d$ On one hand, by base-height splitting, $\nabla_{d+1} = \frac{1}{d+1}\Delta_d$. On the other, $\nabla_{d+1} = \frac{\pm 1}{(d+1)!}\det(\mathbf{X}'_0 \mid \cdots \mid \mathbf{X}'_d)$. Comparing, we get

$$\Delta_d = \pm\frac{1}{d!}\det\left(\mathbf{X}'_0 \mid \mathbf{X}'_1 \mid \mathbf{X}'_2 \mid \cdots \mid \mathbf{X}'_d\right). \tag{3.5}$$

By linearity of determinants, we can subtract the last row $m_1 = \mathbb{E}\,X_{ij}$ times from every other row. We then get

$$\Delta_d = \pm\frac{1}{d!}\det\left(\mathbf{Y}'_0 \mid \mathbf{Y}'_1 \mid \mathbf{Y}'_2 \mid \cdots \mid \mathbf{Y}'_d\right), \tag{3.6}$$

where $\mathbf{Y}'_j = (Y_{1j}, \ldots, Y_{dj}, 1)^\top$ and $Y_{ij} = X_{ij} - m_1$. Note that, when $X_{ij} \sim \mathsf{Unif}(0,1)$, we have $m_r = \mathbb{E}\,X_{ij}^r = 1/(r+1)$. Since $Y_{ij} \sim \mathrm{Unif}(-1/2, -1/2)$, we also have explicitly

$$\mu_r = \mathbb{E}\,Y_{ij}^r = \frac{1+(-1)^r}{2^{r+1}(r+1)}, \tag{3.7}$$

so $m_1 = 1/2, \mu_2 = 1/12, \mu_3 = 0, \mu_4 = 1/80, \mu_5 = 0, \mu_6 = 1/448$. Note that

$$t_k^{k/1}(d+1) = (d+1)\,\mathbb{E}\,\left(\det\left(\mathbf{Y}'_0 \mid \mathbf{Y}'_1 \mid \cdots \mid \mathbf{Y}'_d\right)\right)^k, \tag{3.8}$$

since the (even) k-th moment of the determinant on the right hand side corresponds to a marked permutation table with all k marks in the first column. The factor $(d+1)$ then comes from symmetry. Hence,

$$v_d^{(k)}(C_d) = \frac{1}{(d!)^k}\frac{1}{d+1}t_k^{k/1}(d+1) = \frac{d+1}{(d!)^{k-2}}[t^{d+1}]T_k^{k/1}(t). \tag{3.9}$$

When $k = 2$, we have by Equation (2.189) that $T_2^{2/1}(t) = T_2^2(t) = te^{\mu_2 t}$ and thus

$$v_d^{(2)}(C_d) = (d+1)[t^{d+1}]te^{\mu_2 t} = (d+1)[t^d]e^{\mu_2 t} = \frac{d+1}{d!}\mu_2^d. \tag{3.10}$$

When $k = 4$, we take advantage of the fact that $\mu_3 = 0$, so $T_4^{4/1}(t) = S_4^{4/1}(t)$. By Corollary 163.1 and by scaling,

$$S_4^{4/1}(t) = t\frac{1+2\mu_2^2 t}{(1-\mu_2^2 t)^4}e^{t(\mu_4 - 3\mu_2^2)} \tag{3.11}$$

and thus

$$v_d^{(4)}(C_d) = \frac{d+1}{(d!)^2}[t^d]\frac{1+2\mu_2^2 t}{(1-\mu_2^2 t)^4}e^{t(\mu_4-3\mu_2^2)} = \frac{d+1}{(d!)^2}\sum_{j=0}^{d}[t^{d-j}]e^{t(\mu_4-3\mu_2^2)} \\ \times [t^j]\frac{1+2\mu_2^2 t}{(1-\mu_2^2 t)^4} = \frac{d+1}{(d!)^2}\sum_{j=0}^{d}\frac{(\mu_4-3\mu_2^2)^{d-j}}{(d-j)!}\frac{1}{2}(j+1)^2(j+2)\mu_2^{2j}. \tag{3.12}$$

Finally, when $k=6$, we take advantage of the fact that $\mu_3=\mu_5=0$, so $T_6^{6/1}(t) = S_6^{6/1}(t)$. By Corollary 207.2 and by scaling,

$$v_d^{(6)}(C_d) = \frac{s_6^{6/1}(d+1)}{(d+1)(d!)^6} = \frac{d+1}{d!^4}\mu_2^{3d}\sum_{j=0}^{d+1}\sum_{i=0}^{j}\frac{(1+i)(2+i)(4+i)!}{48(d+1-j)!}\binom{14+j+2i}{j-i}\times \\ \left(2(7+2i+j)(j-d-1)+i\left(\frac{\mu_6}{\mu_2^3}-15\frac{\mu_4}{\mu_2^2}+30\right)\right)\left(\frac{\mu_6}{\mu_2^3}-15\frac{\mu_4}{\mu_2^2}+30\right)^{d-j}\left(\frac{\mu_4}{\mu_2^2}-3\right)^{j-i}. \tag{3.13}$$

3.2 d-Simplex even volumetric moments

The objective of this section is to deduce a general formula for $v_d^{(k)}(T_d)$ when $k=2,4$ and d arbitrary. The case $k=2$ was obtained by Reed [59]:

$$v_d^{(2)}(T_d) = \frac{d!}{(d+2)^d(d+1)^d}. \tag{3.14}$$

Table 3.5 shows the second volumetric moments $v_d^{(2)}(T_d)$ for low values of d.

d	0	1	2	3	4	5	6	7	8
$v_d^{(2)}(T_d)$	1	$\frac{1}{6}$	$\frac{1}{72}$	$\frac{3}{4000}$	$\frac{1}{33750}$	$\frac{5}{5445468}$	$\frac{45}{1927561216}$	$\frac{35}{69657034752}$	$\frac{7}{747338906250}$

Table 3.5: Second volumetric moment in d-simplex

In fact, Reed showed that the problem of determining $v_d^{(k)}(T_d)$ for even k is closely related to even moments of determinants of some random matrices (Reed's formula, Proposition 215). Using this connection, we are able to deduce

$$v_d^{(4)}(T_d) = \frac{(d+1)!^2}{((d+4)(d+3)(d+2)(d+1))^{d+1}} \\ \times \sum_{w=0}^{2}\sum_{s=0}^{4-2w}\sum_{c=0}^{d+1-s}\binom{4-2w}{s}\frac{(1+c)2^s 6^{d+1-c-s}d_w(c)}{(d+1-c-s)!(2-w)!w!}, \tag{3.15}$$

where

$$d_0(c) = (2+c), \qquad d_1(c) = c(2+c), \qquad d_2(c) = c^3. \tag{3.16}$$

Table 3.6 shows the fourth volumetric moments $v_d^{(4)}(T_d)$ for low values of d.

d	0	1	2	3	4	5	6
$v_d^{(4)}(T_d)$	1	$\frac{1}{15}$	$\frac{1}{900}$	$\frac{871}{123480000}$	$\frac{2083}{96808320000}$	$\frac{24995}{682923373461504}$	$\frac{54793}{1422757028044800000}$

Table 3.6: Fourth volumetric moment in d-simplex

Finally, for completeness, Table 3.7 shows the sixth volumetric moments $v_d^{(6)}(T_d)$ for low values of d. In this case, there also exists a general formula, the scope of which is however beyond this thesis[1].

d	0	1	2	3	4	5
$v_d^{(6)}(T_d)$	1	$\frac{1}{28}$	$\frac{403}{2116800}$	$\frac{2797}{11202105600}$	$\frac{28517}{264649744800000}$	$\frac{11490716929}{61866839373383632896000 0}$

Table 3.7: Sixth volumetric moment in d-simplex

Note that the values $v_3^{(4)}(T_3)$ and $v_3^{(6)}(T_3)$ were already known to Mannion [44].

3.2.1 Uniform and Dirichlet simplices

See section A.4 in the Appendix which covers the Dirichlet distribution first. In there, T_d^* is defined as $\operatorname{conv}\{\mathbf{e}_0, \mathbf{e}_1, \dots \mathbf{e}_d\}$, that is, T_d^* is a d-simplex embedded into $\mathbb{R}^{d+1}$.

Definition 213. We say a collection $\mathbb{Y} = (\mathbf{Y}_0, \dots, \mathbf{Y}_d)$ is a standard Dirichlet random sample if the points $\mathbf{Y}_j$ are independent and follow the same symmetric Dirichlet distribution with concentration parameter α. We call the convex hull $\mathbb{H}_d(T_d^*) = \operatorname{conv}(\mathbf{Y}_0, \dots, \mathbf{Y}_d)$ of those points as a Dirichlet random simplex with volume $\Delta_{d,\alpha} = \operatorname{vol}_d \mathbb{H}_d(T_d^*)$ and normalised volume $\underline{\Delta}_{d,\alpha} = \Delta_{d,\alpha} / \operatorname{vol}_d T_d^*$ with its usual moments $v_{d,\alpha}^{(k)}(T_d^*) = \mathbb{E}\, \underline{\Delta}_{d,\alpha}^k$.

Theorem 214. *Let $X_{ij} \sim \Gamma(\alpha)$ be i.i.d. random variables, $A = (X_{ij})_{n \times n}$ and $f_k(n) = \mathbb{E}\,(\det A)^k$ as usual. Let $\mathbf{Y}_0, \dots, \mathbf{Y}_d$ be a standard Dirichlet random sample with concentration parameter α. Then*

$$v_{d,\alpha}^{(k)}(T_d^*) = \left(\frac{\Gamma(\alpha(d+1))}{\Gamma(\alpha(d+1)+k)} \right)^{d+1} f_k(d+1). \tag{3.17}$$

Proof. Note that the distance form $\mathbf{0}$ to $\mathcal{A}(\mathbf{Y_0}, \dots, \mathbf{Y}_d)$ is

$$\left\| \left(\frac{1}{d+1}, \dots, \frac{1}{d+1} \right)^\top \right\| = \frac{1}{\sqrt{d+1}}. \tag{3.18}$$

Denote $B = (\mathbf{Y}_0 \mid \cdots \mid \mathbf{Y}_d)$, then, by base-height splitting,

$$\begin{aligned} |\det B| &= (d+1)! \operatorname{vol}_{d+1} \operatorname{conv}(\mathbf{0}, \mathbf{Y}_0, \dots, \mathbf{Y}_d) \\ &= (d+1)! \frac{1}{d+1} \frac{1}{\sqrt{d+1}} \operatorname{vol}_d \operatorname{conv}(\mathbf{Y}_0, \dots, \mathbf{Y}_d) = \underline{\Delta}_{d,\alpha}. \end{aligned} \tag{3.19}$$

[1]at the time of submission of this thesis, the formula has not yet been found, however, the current progress with Potechin and Lv suggests the formula will be available soon

Set $n = d+1$ and write $A = (\mathbf{X}_0 \,|\, \cdots \,|\, \mathbf{X}_d)$ with $\mathbf{X}_j = (X_{0j}, \ldots, X_{dj})^\top$. Denote $S_j = \sum_{i=0}^d X_{ij}$, that is, S_j equals the sum of coordinates of $\mathbf{X}_j$. Then, by Lemma 274, $S_j \sim \Gamma(\alpha(d+1))$ and

$$B = (\mathbf{Y}_0 \,|\, \cdots \,|\, \mathbf{Y}_d) \overset{d}{=} \left(\frac{\mathbf{X}_0}{S_0} \,\middle|\, \cdots \,\middle|\, \frac{\mathbf{X}_d}{S_d} \right), \tag{3.20}$$

from which, taking determinant and writing $S = \prod_{j=0}^d S_j$,

$$\det B \overset{d}{=} \det \left(\frac{\mathbf{X}_0}{S_0} \,\middle|\, \cdots \,\middle|\, \frac{\mathbf{X}_d}{S_d} \right) = \frac{\det(\mathbf{X}_0 \,|\, \cdots \,|\, \mathbf{X}_d)}{\prod_{j=0}^d S_j} = \frac{\det A}{S}. \tag{3.21}$$

Moreover, by Lemma 274, $\mathbf{X}_j / S_j$ and S_j are stochastically independent, so are B and S. Hence, for even k,

$$\begin{aligned} f_k(d+1) &= \mathbb{E}\,(\det A)^k = \mathbb{E}\,(S \det B)^k = \mathbb{E}\, S^k\, \mathbb{E}\,(\det B)^k \\ &= (\mathbb{E}\, S_0)^k\, \mathbb{E}\, \Delta^k_{d,\alpha} = \left(\frac{\Gamma(\alpha(d+1)+k)}{\Gamma(\alpha(d+1))} \right)^{d+1} v^{(k)}_{d,\alpha}(T^*_d). \end{aligned} \tag{3.22}$$

■

3.2.2 Reed's formula

There is a connection between moments of random matrices with a certain distribution of entries and moments of volume of a random simplices in a regular tetrahedron. As a consequence of Theorem 214, we obtain Reed's formula we have already seen in Introduction which establishes this connection:

Proposition 215 ([59] Reed 1974)**.** *Let $X_{ij} \sim \mathsf{Exp}(1)$ be i.i.d. random variables, $A = (X_{ij})_{n\times n}$ and $f_k(n) = \mathbb{E}\,(\det A)^k$ as usual. Let $\mathbf{Y}_0, \ldots, \mathbf{Y}_d$ be i.i.d. random points uniformly distributed in T_d. Then for k even,*

$$v^{(k)}_d(T_d) = \left(\frac{d!}{(d+k)!} \right)^{d+1} f_k(d+1). \tag{3.23}$$

Proof. From Theorem 214 with $\alpha = 1$ and upon noticing $v^{(k)}_{d,1}(T^*_d) = v^{(k)}_d(T_d)$. ■

> *Remark* 216. Note that the formula also holds for any real $k > -1$ if we replace determinant moments $f_k(d+1) = \mathbb{E}\,(\det A)^k$ with absolute determinant moments $\mathbb{E}\,|\det A|^k$. However, since we can expand the absolute values only for k being an even positive integer, the problem of finding these moments for general k is no longer a combinatorial problem.

Example 217. Let $X_{ij} \sim \Gamma(\alpha)$ i.i.d. and $A = (X_{ij})_{n\times n}$. Note that $m_r = \mathbb{E}\, X^r_{ij} = \Gamma(\alpha+k)/\Gamma(\alpha)$, so $m_1 = \alpha$ and $m_2 = \alpha(\alpha+1)$. Since we know that in general $f_2(n) = n!(m_2 - m_1^2)^{n-1}(m_2 + m_1(n-1))$, we get $f_2(n) = \alpha^n n!(\alpha n + 1)$ and thus

$$v^{(2)}_{d,\alpha}(T^*_d) = \left(\frac{\Gamma(\alpha(d+1))}{\Gamma(\alpha(d+1)+2)} \right)^{d+1} \alpha^{d+1}(d+1)!(\alpha(d+1)+1). \tag{3.24}$$

Especially, when $\alpha = 1$ (uniform simplices on T^*_d), we get (Reed [59])

$$v^{(2)}_d(T_d) = \frac{d!}{(d+2)^d(d+1)^d}. \tag{3.25}$$

Example 218. Using or previous results on random matrices, we can also obtain the value for the fourth moment for general d. Let $X_{ij} \sim \Gamma(\alpha)$ be i.i.d. random variables, $A = (X_{ij})_{n\times n}$, then $m_r = \Gamma(\alpha+r)/\Gamma(\alpha) = \alpha(\alpha+1)\cdots(\alpha+r-1)$. By Corollary 156.1 with $m_1 = \alpha$, $m_2 = \alpha(\alpha+1)$, $m_3 = \alpha(\alpha+1)(\alpha+2)$, $m_4 = \alpha(\alpha+1)(\alpha+2)(\alpha+3)$, from which we compute $\mu_2 = \alpha$, $\mu_3 = 2\alpha$, $\mu_4 = 3\alpha(2+\alpha)$, we get

$$f_4(n) = (n!)^2 \sum_{w=0}^{2} \sum_{s=0}^{4-2w} \sum_{c=0}^{n-s} \binom{4-2w}{s} \frac{(1+c)2^s 6^{n-c-s} \alpha^{n+s+w+c}}{(n-c-s)!(2-w)!w!} d_w(c), \tag{3.26}$$

where

$$d_0(c) = (2+c), \qquad d_1(c) = c(2+c), \qquad d_2(c) = c^3, \tag{3.27}$$

from which

$$v_{d,\alpha}^{(4)}(T_d^*) = \left(\frac{\Gamma(\alpha(d+1))}{\Gamma(\alpha(d+1)+4)} \right)^{d+1} f_4(d+1). \tag{3.28}$$

Especially for $\alpha = 1$, we get as promised

$$\begin{aligned} v_d^{(4)}(T_d) = & \frac{(d+1)!^2}{((d+4)(d+3)(d+2)(d+1))^{d+1}} \\ & \times \sum_{w=0}^{2} \sum_{s=0}^{4-2w} \sum_{c=0}^{d+1-s} \binom{4-2w}{s} \frac{(1+c)2^s 6^{d+1-c-s} d_w(c)}{(d+1-c-s)!(2-w)!w!}. \end{aligned} \tag{3.29}$$

3.3 d-Orthoplex's and even moments in general

The aim of this section is to deduce $v_d^{(k)}(O_d)$ for $k = 2, 4, 6$. We briefly discuss how we can obtain even volumetric moments for various polytopes efficiently in a computer. Note that $O_2 = C_2$ (although with different area), so we can restrict ourselves to the case $d \geq 3$. We got

$$v_d^{(2)}(O_d) = \frac{(d+1)!}{2^d (d+2)^d (d+1)^d}. \tag{3.30}$$

Table 3.8 shows the second volumetric moments $v_d^{(2)}(O_d)$ for low values of d. We

d	3	4	5	6	7	8
$v_d^{(2)}(O_d)$	$\frac{3}{8000}$	$\frac{1}{108000}$	$\frac{5}{29042496}$	$\frac{45}{17623416832}$	$\frac{35}{1114512556032}$	$\frac{7}{21257640000000}$

Table 3.8: Second volumetric moment in d-orthoplex

are able to deduce also the fourth moment

$$v_d^{(4)}(O_d) = \frac{d!(d+1)! \sum_{j=0}^{d} \frac{3^{d-j}(1+j)(2+j)}{2\,(d-j)!} \left(\frac{j(d+4)(d+3)}{(d+2)(d+1)} + 1 \right)}{2^{2d}(d+4)^d(d+3)^d(d+2)^d(d+1)^d}. \tag{3.31}$$

d	3	4	5	6
$v_d^{(4)}(O_d)$	$\frac{4259}{5268480000}$	$\frac{3959}{5664669696000}$	$\frac{228685}{699313534424580096}$	$\frac{1940773}{2072040101998755840000}$

Table 3.9: Fourth volumetric moment in d-orthoplex

Table 3.9 shows the fourth volumetric moments $v_d^{(4)}(O_d)$ for low values of d. Finally, for $k = 6$, we obtain the following formula for $v_d^{(6)}(O_d)$ for any d,

$$v_d^{(6)}(O_d) = \frac{d!^2}{48}\left(\frac{15}{4}\right)^d \left(\frac{d!}{(d+6)!}\right)^d \sum_{j=0}^{d+1}\sum_{i=0}^{j} \frac{(1+i)(2+i)(4+i)!}{3^i 10^j (3+d)(4+d)(2+d)} \times \frac{\binom{14+j+2i}{j-i}}{(d-j+1)!}\left\{\begin{matrix}(3+d)(4+d)((1+d)(2+d)+(5+d)(6+d)(2i+j))(1+d-j) \\ +30\,(5+d)^2\,(6+d)^2(1+d+2i-2j)\end{matrix}\right\}. \tag{3.32}$$

Table 3.10 shows the sixth volumetric moments $v_d^{(6)}(O_d)$ for low values of d.

d	3	4	5
$v_d^{(6)}(O_d)$	$\frac{7200523}{1835352981504000}$	$\frac{74002087}{46250895133900800000}$	$\frac{7261177207}{4059550791626730830069288140800000}$

Table 3.10: Sixth volumetric moment in d-orthoplex

3.3.1 General numerical technique

Let $\mathbb{x} = (\mathbf{x}_0, \ldots, \mathbf{x}_d)$ be a collection of points $\mathbf{x}_j \in P_d \subset \mathbb{R}^d, j = 0, \ldots, d$ with coordinates $\mathbf{x}_j = (x_{1j}, \ldots, x_{dj})^\top$ and let $\Delta_d = \operatorname{vol}_d \operatorname{conv}(\mathbb{x})$. Denote

$$e_d^{(k)} = \int_{(P_d)^{d+1}} \Delta_d^k \, \lambda_d^{d+1}(\mathrm{d}\mathbb{x}), \tag{3.33}$$

so then

$$v_d^{(k)}(P_d) = e_d^{(k)} / (\operatorname{vol}_d P_d)^{d+k+1}. \tag{3.34}$$

We have seen that we can express Δ_d using determinants as

$$\Delta_d = \pm\frac{1}{d!}\det(\mathbf{x}_1 - \mathbf{x}_0 \mid \cdots \mid \mathbf{x}_d - \mathbf{x}_0) = \pm\frac{1}{d!}\det(\mathbf{x}'_0 \mid \mathbf{x}'_1 \mid \cdots \mid \mathbf{x}'_d), \tag{3.35}$$

where $\mathbf{x}'_j = (x_{1j}, \ldots, x_{dj}, 1)^\top$. Denote $\mathbf{y} = (y_1, \ldots, y_d)^\top \in \mathbb{R}^d$, then

$$a_{i_1 i_2 i_3 \ldots i_d} = \int_{P_d} y_1^{i_1} y_2^{i_2} \cdots y_d^{i_d} \lambda_d(\mathrm{d}\mathbf{y}) \tag{3.36}$$

form a basis of $e_d^{(k)}$, in fact $e_d^{(k)}$ is a polynomial of homogeneity $d+1$ in $a_{i_1,\ldots,i_d}$ with $i_1 + \cdots + i_d \leq k$. The total sum of indices i_p in each product of a's must be equal to k for any position p. We can obtain these polynomials separately via combinatorics in a computer, for $d = 2$ with $k = 2, 3$ and $d = 3$ with $k = 2$,

$$e_2^{(2)} = \frac{3}{2}\left(2a_{10}a_{11}a_{01} - a_{20}a_{01}^2 - a_{02}a_{10}^2 - a_{00}a_{01}^2 + a_{00}a_{02}a_{20}\right), \tag{3.37}$$

$$e_2^{(4)} = \frac{3}{8}\left\{\begin{array}{c} 3a_{00}a_{22}^2 + 12a_{12}^2a_{20} + 12a_{11}^2a_{22} + 12a_{21}^2a_{02} + 12a_{03}a_{11}a_{30} \\ +3a_{40}a_{02}^2 - 4a_{04}a_{10}a_{30} + 6a_{20}a_{22}a_{02} + 4a_{01}a_{13}a_{30} + 4a_{03}a_{10}a_{31} \\ +3a_{04}a_{20}^2 + 12a_{01}a_{12}a_{31} - 4a_{00}a_{13}a_{31} - 4a_{01}a_{03}a_{40} + a_{00}a_{04}a_{40} \\ -12a_{12}a_{30}a_{02} - 12a_{11}a_{31}a_{02} - 12a_{11}a_{13}a_{20} - 12a_{11}a_{12}a_{21} \\ +12a_{10}a_{13}a_{21} - 12a_{03}a_{20}a_{21} - 12a_{10}a_{12}a_{22} - 12a_{01}a_{21}a_{22}, \end{array}\right\}, \tag{3.38}$$

$$e_3^{(2)} = \frac{2}{3}\left\{\begin{array}{c} a_{010}^2a_{101}^2 - a_{000}a_{011}^2a_{200} - a_{001}^2a_{020}a_{200} + a_{000}a_{002}a_{020}a_{200} \\ -2a_{010}a_{011}a_{100}a_{101} + 2a_{002}a_{010}a_{100}a_{110} - 2a_{001}a_{011}a_{100}a_{110} \\ +2a_{001}a_{010}a_{011}a_{200} - 2a_{001}a_{010}a_{101}a_{110} + 2a_{000}a_{011}a_{101}a_{110} \\ +a_{001}^2a_{110}^2 + a_{011}^2a_{100}^2 - a_{002}a_{020}a_{100}^2 + 2a_{001}a_{020}a_{100}a_{101} \\ -a_{002}a_{010}^2a_{200} - a_{000}a_{020}a_{101}^2 - a_{000}a_{002}a_{110}^2 \end{array}\right\} \tag{3.39}$$

and so on (see Code 1). A substantial simplification is achieved when we place the center of our coordinates in the centroid of P_d. In that case, all the values $a_{i_1 \ldots i_d}$ with exactly one index equal to one and with remaining indices equal to zero vanish. Then

$$e_2^{(2)} = \frac{3}{2}\left(a_{00}a_{02}a_{20} - a_{00}a_{11}^2\right), \tag{3.40}$$

$$e_2^{(4)} = \frac{3}{8}\left\{\begin{array}{c} 3a_{40}a_{02}^2 + 12a_{21}^2a_{02} + 6a_{20}a_{22}a_{02} - 12a_{12}a_{30}a_{02} - 12a_{11}a_{31}a_{02} \\ +3a_{00}a_{22}^2 + 12a_{12}^2a_{20} - 12a_{11}a_{13}a_{20} - 12a_{11}a_{12}a_{21} - 4a_{00}a_{13}a_{31} \\ +3a_{04}a_{20}^2 + 12a_{11}^2a_{22} + 12a_{03}a_{11}a_{30} - 12a_{03}a_{20}a_{21} + a_{00}a_{04}a_{40} \end{array}\right\}, \tag{3.41}$$

$$e_3^{(2)} = \frac{2}{3}\left\{\begin{array}{c} 2a_{000}a_{101}a_{110}a_{011} + a_{000}a_{002}a_{020}a_{200} \\ -a_{000}a_{002}a_{110}^2 - a_{000}a_{200}a_{011}^2 - a_{000}a_{020}a_{101}^2, \end{array}\right\} \tag{3.42}$$

$$e_4^{(2)} = \frac{5}{24}\left\{\begin{array}{c} a_{0000}a_{0002}a_{0020}a_{0200}a_{2000} - 2a_{0000}a_{0110}a_{1001}a_{1100}a_{0011} \\ +a_{0000}a_{0110}^2a_{1001}^2 + a_{0000}a_{0101}^2a_{1010}^2 + a_{0000}a_{1100}^2a_{0011}^2 \\ +2a_{0000}a_{0200}a_{1001}a_{1010}a_{0011} - a_{0000}a_{0200}a_{2000}a_{0011}^2 \\ -2a_{0000}a_{0101}a_{1010}a_{1100}a_{0011} - a_{0000}a_{0020}a_{0200}a_{1001}^2 \\ +2a_{0000}a_{0101}a_{0110}a_{2000}a_{0011} - a_{0000}a_{0002}a_{0200}a_{1010}^2 \\ -2a_{0000}a_{0101}a_{0110}a_{1001}a_{1010} - a_{0000}a_{0002}a_{0020}a_{1100}^2 \\ +2a_{0000}a_{0020}a_{0101}a_{1001}a_{1100} - a_{0000}a_{0020}a_{0101}^2a_{2000} \\ +2a_{0000}a_{0002}a_{0110}a_{1010}a_{1100} - a_{0000}a_{0002}a_{0110}^2a_{2000}, \end{array}\right\} \tag{3.43}$$

3.3.2 Octahedral symmetry

Finally, we assume that P_d has octahedral symmetry. That is, P_d is symmetrical with respect to any permutation of axes and any reflection $y_i \to -y_i$ for any i and $(y_1, \ldots, y_d)^\top \in P_d$. In that case, $a_{i_1 \ldots i_d}$ is invariant under any permutation of indices $i_1, \ldots, i_d$. Also, all $a_{i_1 \ldots i_d}$ with at least one odd i_s vanish. As a consequence, we get much more simplified formulae. When $k = 2, 4, 6$, we can write those polynomials explicitly in that case. Let us define $b_0 = a_{\ldots 000} = \mathrm{vol}_d\, P_d$, $b_2 = a_{\ldots 002}$, $b_4 = a_{\ldots 004}$, $b_6 = a_{\ldots 006}$, $b_{22} = a_{\ldots 022}$, $b_{42} = a_{\ldots 042}$, and so on. In general, the indices of b's are the nonzero indices of a's in decreasing order (with the only exception b_0 where there are no non-zero indices). That is for any $b_{i_1, i_2, \ldots, i_p}$, we have $i_1 \geq i_2 \geq \cdots \geq i_p > 0$ and $b_{i_1 \dots i_p} = a_{i_1,\dots,i_p,0,0,0,\dots,0}$ ($d-p$ zeros). Or explicitly,

$$b_{i_1 i_2 i_3 \dots i_p} = \int_{P_d} y_1^{i_1} y_2^{i_2} \cdots y_p^{i_p} \lambda_d(\mathrm{d}\mathbf{y}). \tag{3.44}$$

From this integral expression for $b_{i_1,\dots,i_p}$, it is clear why all indices must be even otherwise $b_{i_1,\dots,i_p}$ vanishes (follows from the substitution $y_s \to -y_s$ for a given s for which i_s would be odd and the octahedral symmetry of P_d). We may associate $e_d^{(k)}$ with a sum over marked permutation tables $E_{k,d}$, which we define as row permutations of $d+1$ elements $[0, 1, \dots, d]$ with k rows and in total k marks covering the element 0. This is contrary to the d-cube case as $x_{ij}, x_{i'j}$ may no longer be independent. We have

$$e_d^{(k)} = \frac{1}{(d!)^k} \sum_{\tau \in E_{k,d}} w(\tau) \operatorname{sgn} \tau, \tag{3.45}$$

where $w(\cdot)$ is given as a product of the corresponding $a_{i_1 \dots i_d}$ factors. When $k = 2$, there is one column with two marks in $E_{2,d}$ (weight b_0), the remaining columns are columns with weight b_2. The sum of weights over all those remaining columns yields the Fortet's second moment $s_2^0(d)$ with $\mu_2 = b_2$. Since there are $d+1$ positions for the marked column, we have

$$e_d^{(2)} = \frac{1}{d!^2} \sum_{\tau \in E_{2,d}} w(\tau) \operatorname{sgn} \tau = \frac{d+1}{d!^2} b_0 s_2^0(d) = \frac{d+1}{d!^2} b_0 d! b_2^d = \frac{d+1}{d!} b_0 b_2^d \tag{3.46}$$

and thus, by Equation (3.34),

$$v_d^{(2)}(P_d) = \frac{1}{b_0^2} \frac{d+1}{d!} \left(\frac{b_2}{b_0} \right)^d. \tag{3.47}$$

Tables $E_{4,d}$ (with octahedral symmetry assumed) have the following structure:

Type:	4-column	2-column	$\times^2$-column	$\times^4$-column
	a	a	$\times$	$\times$
	a	a	$\times$	$\times$
	a	b	a	$\times$
	a	b	a	$\times$
Weight w:	b_4	b_{22}	b_2	b_0

Note that the requirement of containing $k = 4$ marks means that a table $\tau \in E_{4,d}$ either contains one $\times^4$-column (covering four zeros) or two $\times^2$-columns. Let $\tau' \in S_{4,d}^0$ have c four-columns (and thus $d-c$ two-columns), we have $w(\tau) = b_4^c b_{22}^{d-c}$. The first case is obtained by appending a column filled with 4 marks (covering four 0's) to τ', we get $w(\tau) = b_0 b_4^c b_{22}^{d-c}$. The second case is obtained by additionally switching two marks with two elements from

- a four-column of τ' in 6 ways, yielding $w(\tau) = b_2^2 b_4^{c-1} b_{22}^{d-c}$
- a two-column in 2 ways, selecting one of its pairs, $w(\tau) = b_2^2 b_4^c b_{22}^{d-c-1}$.

In total, we get the following contribution of τ' to $\sum_{\tau \in E_{4,d}} w(\tau) \operatorname{sgn} \tau$. Note that by symmetry, we have to multiply the second case contribution by 1/2.

$$(d+1)\left[b_0 b_4^c b_{22}^{d-c} + \frac{1}{2}\left(6c b_2^2 b_4^{c-1} b_{22}^{d-c} + 2(d-c) b_2^2 b_4^c b_{22}^{d-c-1}\right)\right]. \tag{3.48}$$

Summing up the contribution from all tables $\tau' \in S_{4,d}^0$, we get

$$e_d^{(4)} = \frac{d+1}{d!^4}\left[b_0 s_4^0(d) + 3b_2^2 \frac{\partial s_4^0(d)}{\partial b_4} + b_2^2 \frac{\partial s_4^0(d)}{\partial b_{22}}\right], \tag{3.49}$$

where $s_4^0(d)$ is the Niquist, Rice and Riordan's fourth moment given by Corollary (68.1) with $\mu_4 = b_4$ and $\mu_2^2 = b_{22}$, so

$$s_4^0(d) = (d!)^2 \sum_{j=0}^{d} \frac{1}{j!}\binom{d-j+2}{2}(b_4 - 3b_{22})^j\, b_{22}^{d-j}. \tag{3.50}$$

Hence, after some simplifications,

$$e_d^{(4)} = \frac{d+1}{(d!)^2}\sum_{j=0}^{d} \frac{(1+j)(2+j) b_{22}^j}{2(d-j)!}(b_4 - 3b_{22})^{d-j}\left(\frac{j b_2^2}{b_{22}} + b_0\right), \tag{3.51}$$

from which, by Equation (3.34),

$$v_d^{(4)}(P_d) = \frac{1}{b_0^4}\frac{d+1}{(d!)^2}\left(\frac{b_{22}}{b_0}\right)^d \sum_{j=0}^{d} \frac{(1+j)(2+j)}{2\,(d-j)!}\left(\frac{b_4}{b_{22}} - 3\right)^{d-j}\left(\frac{j b_2^2}{b_{22} b_0} + 1\right). \tag{3.52}$$

Alternatively, we can extract $s_4^0(d)$ as $s_4^0(d) = (d!)^2 [t^d] S_4^0(t)$, where the generating function $S_4^0(t)$ is given by Proposition 68. Explicitly,

$$S_4^0(t) = \frac{e^{t(\mu_4 - 3\mu_2^2)}}{(1-\mu_2^2 t)^3} = \frac{e^{t(b_4 - 3b_{22})}}{(1-b_{22}t)^3}. \tag{3.53}$$

In particular,

$$e_2^{(4)} = \frac{3}{8}\left(6b_4 b_2^2 + 6b_{22} b_2^2 + b_0 b_4^2 + 3b_0 b_{22}^2\right), \tag{3.54}$$

$$e_3^{(4)} = \frac{1}{54}\left(b_0 b_4^3 + 9b_2^2 b_4^2 + 9b_0 b_{22}^2 b_4 + 18 b_2^2 b_{22} b_4 + 6b_0 b_{22}^3 + 45 b_2^2 b_{22}^2\right) \tag{3.55}$$

$$e_4^{(4)} = \frac{5}{13824}\left\{\begin{array}{c} b_0 b_4^4 + 18 b_0 b_{22}^2 b_4^2 + 24 b_0 b_{22}^3 b_4 + 45 b_0 b_{22}^4 \\ +12 b_2^2 b_4^3 + 36 b_2^2 b_{22} b_4^2 + 180 b_2^2 b_{22}^2 b_4 + 252 b_2^2 b_{22}^3 \end{array}\right\}. \tag{3.56}$$

Lastly, $E_{6,d}$ tables have the following structure:

6-column	4-column	2-column	$\times^2_4$-column	$\times^2_2$-column	$\times^4$-column	$\times^6$-column
a	a	a	$\times$	$\times$	$\times$	$\times$
a	a	a	$\times$	$\times$	$\times$	$\times$
a	a	b	a	a	$\times$	$\times$
a	a	b	a	a	$\times$	$\times$
a	b	c	a	b	a	$\times$
a	b	c	a	b	a	$\times$
b_6	b_{42}	b_{222}	b_4	b_{22}	b_2	b_0

Note that all marks cover the element 0 only. The requirement of $E_{6,d}$ containing $k=6$ marks means that a table $\tau \in E_{6,d}$ either contains

- case I: one $\times^6$-column (covering six zeros),
- case II: one $\times^2$ and one $\times^4$ column or
- case III: three $\times^2$-columns.

Let $\tau' \in S^0_{6,d}$ have s six-columns and f four-columns (and thus $d-s-f$ two-columns), we have $w(\tau) = b_6^s b_{42}^f b_{222}^{d-s-f}$. The first case is obtained by appending a column filled with 6 marks (covering six 0's) to τ', we get $w(\tau) = b_0 b_6^s b_{42}^f b_{222}^{d-s-f}$. The second case is obtained by additionally switching two marks with two elements from

- a six-column of τ' in 15 ways, yielding $w(\tau) = b_2 b_4 b_6^{s-1} b_{42}^f b_{222}^{d-s-f}$
- a four-column of τ' in 6 ways selecting two from its four identical elements, creating a $\times^2_2$-column and yielding $w(\tau) = b_2 b_{22} b_6^s b_{42}^{f-1} b_{222}^{d-s-f}$
- a four-column of τ' in 1 way selecting the remaining lonely pair of two elements, creating a $\times^4_2$-column and yielding $w(\tau) = b_2 b_4 b_6^s b_{42}^{f-1} b_{222}^{d-s-f}$
- a two-column in 3 ways, selecting one of its pairs, $w(\tau) = b_2 b_{22} b_6^s b_{42}^f b_{222}^{d-s-f-1}$.

We get the following contribution of τ' to $\sum_{\tau \in E_{6,d}} w(\tau) \operatorname{sgn} \tau$,

$$(d+1)\Big[b_0 b_6^s b_{42}^f b_{222}^{d-s-f} + 15 s b_2 b_4 b_6^{s-1} b_{42}^f b_{222}^{d-s-f} + 6 f b_2 b_{22} b_6^s b_{42}^{f-1} b_{222}^{d-s-f} \\ + f b_2 b_4 b_6^s b_{42}^{f-1} b_{222}^{d-s-f} + 3(d-s-f) b_2 b_{22} b_6^s b_{42}^f b_{222}^{d-s-f-1}\Big]. \tag{3.57}$$

Summing up the contribution from all tables $\tau' \in S^0_{6,d}$, we get that the contribution of cases I and II to $\sum_{\tau \in E_{6,d}} w(\tau) \operatorname{sgn} \tau$ is

$$(d+1)\left[b_0 s_6^0(d) + 15 b_2 b_4 \frac{\partial s_6^0(d)}{\partial b_6} + b_2(6b_{22}+b_4)\frac{\partial s_6^0(d)}{\partial b_{42}} + 3 b_2 b_{22} \frac{\partial s_6^0(d)}{\partial b_{222}}\right], \tag{3.58}$$

where $s_6^0(d)$ is the sixth moment for symmetrical distributions given as coefficients $s_6^0(d) = (d!)^2 [t^d] S_6^0(t)$ of the corresponding generating function

$$S_6^0(t) = \frac{e^{t\left(\mu_6 - 15\mu_4\mu_2 + 30\mu_2^3\right)}}{(1+3\mu_2^3 t - \mu_4\mu_2 t)^{15}} N_6\left(\frac{\mu_2^3 t}{(1+3\mu_2^3 t - \mu_4\mu_2 t)^3}\right).$$

with $\mu_6 = b_6$, $\mu_4\mu_2 = b_{42}$ and $\mu_2^3 = b_{222}$. Explicitly (our $f_6^{\text{sym}}(d)$ formula),

$$s_6^0(d) = d!^2 \sum_{j=0}^{d} \sum_{i=0}^{j} \frac{(1+i)(2+i)(4+i)!}{48(d-j)!} \binom{14+j+2i}{j-i} (b_6 - 15b_{42} + 30b_{222})^{d-j} (b_{42} - 3b_{222})^{j-i} b_{222}^i. \tag{3.59}$$

Lastly, the third case can be obtained by the position approach. We start with a larger table $\tau'' \in S^0_{6,d+1}$ and assume it has s six-columns and f four-columns (and thus $d+1-s-f$ two-columns), we have $w(\tau) = b_6^s b_{42}^f b_{222}^{d+1-s-f}$ for its weight. First, we select a number i which appears in three distinct columns (three pairs), these numbers form a set $I_3(t'')$. There is exactly $\#I_3(t'') = d+1-s-f$ of those numbers. However, we don't know whether these three distinct columns where the elements i reside are 4- or 2-columns. Denote $\nu_i(t'')$ the number of four columns from those three distinct columns. Then, we turn all six i's into marks. The weight of the resulting table $\tau \in E_{6,d}$, based on the number of four columns covered, is

$$b_6^s b_{42}^{f-\nu_i(t'')} b_{222}^{d+1-s-f-(3-\nu_i(t''))} b_4^{\nu_i(t'')} b_{22}^{3-\nu_i(t'')}. \tag{3.60}$$

Since $\nu_i(t'') \in \{0,1,2,3\}$, we can write this as a quintic polynomial in $\nu_i(t'')$. Unfortunately, we are currently only able to find explicit generating functions when the polynomial in $\nu_i(t'')$ is at most linear. Let us further assume $b_{222}b_4 = b_{22}b_{42}$ so the factor above equals

$$b_6^s b_{42}^{f-3} b_{222}^{d+1-s-f} b_4^3. \tag{3.61}$$

Summing over all $i \in I_3(t'')$ and by symmetry, we get the following contribution of τ'' to $(d+1)\sum_{\tau\in E_{6,d}} w(\tau)\operatorname{sgn}\tau$,

$$(d+1-s-f) b_6^s b_{42}^{f-3} b_{222}^{d+1-s-f} b_4^3. \tag{3.62}$$

Finally, summing over all $\tau'' \in S^0_{6,d+1}$, we find that the contribution of case III to $(d+1)\sum_{\tau\in E_{6,d}} w(\tau)\operatorname{sgn}\tau$ is

$$\frac{b_{222}b_4^3}{b_{42}^3}\frac{\partial s_6^0(d+1)}{\partial b_{222}}. \tag{3.63}$$

Note that the factor $(d+1)$ comes from symmetry, since we require a specific number (zero) to be covered. In total, putting together cases I, II and III,

$$\begin{aligned} e_d^{(6)} = \frac{d+1}{d!^6}\Big[& b_0 s_6^0(d) + 15 b_2 b_4 \frac{\partial s_6^0(d)}{\partial b_6} + b_2(6b_{22}+b_4)\frac{\partial s_6^0(d)}{\partial b_{42}} \\ &+ 3b_2 b_{22}\frac{\partial s_6^0(d)}{\partial b_{222}} + \frac{1}{(d+1)^2}\frac{b_{222}b_4^3}{b_{42}^3}\frac{\partial s_6^0(d+1)}{\partial b_{222}}\Big]. \end{aligned} \tag{3.64}$$

Keep in mind that even though s_6^0 appears as a function of d and $d+1$, we treat b's as a function of d (and not $d+1$) since they were derived combinatorially. Alternatively, we may write $s_6^0(d+1)$ derivative in terms of $s_6^0(d)$ derivatives. Plugging the explicit formula for $s_6^0(d)$, we arrive at the following formula

$$\begin{aligned} e_d^{(6)} &= \frac{1+d}{d!^4}\sum_{j=0}^{d+1}\sum_{i=0}^{j}\frac{(1+i)(2+i)(4+i)!}{48(d-j+1)!}\binom{14+j+2i}{j-i}\times \\ &\times b_{222}^{i-3}\left(b_6 - 15b_{42} + 30b_{222}\right)^{d-j}\left(b_{42}-3b_{222}\right)^{j-i-1}\times \\ &\times\left\{\begin{matrix}(d-j+1)b_{222}\left(b_{42}-3b_{222}\right)\left(b_2 b_{22} b_{222}(2i+j)+30b_{22}^3+b_0 b_{222}^2\right) \\ +b_{22}^3\left(b_6-15b_{42}+30b_{222}\right)\left(ib_{42}-3jb_{222}\right)\end{matrix}\right\} \end{aligned} \tag{3.65}$$

and from which by Equation (3.34) immediately

$$v_d^{(6)}(P_d) = \frac{1+d}{d!^4 b_0^{d+7}} \sum_{j=0}^{d+1} \sum_{i=0}^{j} \frac{(1+i)(2+i)(4+i)!}{48(d-j+1)!} \binom{14+j+2i}{j-i} \times$$
$$\times\, b_{222}^{i-3} \left(b_6 - 15b_{42} + 30b_{222}\right)^{d-j} \left(b_{42} - 3b_{222}\right)^{j-i-1} \times \tag{3.66}$$
$$\times \left\{ \begin{array}{c} (d-j+1)b_{222}\left(b_{42}-3b_{222}\right)\left(b_2 b_{22} b_{222}(2i+j)+30b_{22}^3+b_0 b_{222}^2\right) \\ +b_{22}^3\left(b_6 - 15b_{42} + 30b_{222}\right)\left(ib_{42} - 3jb_{222}\right) \end{array} \right\}$$

both valid as long as $b_{42}b_{22} = b_4 b_{222}$. Finally, let us list the values of $e_d^{(6)}$ for small d. With no assumption on b's (i.e. we relax $b_{42}b_{22} = b_4 b_{222}$), we have

$$e_2^{(6)} = \frac{3}{32}\Big(b_0 b_6^2 - 90b_{22}^3 + 90b_4^2 b_{22} + 180b_2 b_{42} b_{22} + 15b_0 b_{42}^2 + 30b_2 b_4 b_{42} + 30b_2 b_4 b_6\Big), \tag{3.67}$$

$$e_3^{(6)} = \frac{1}{1944} \left\{ \begin{array}{c} b_0 b_6^3 + 90b_{222}b_4^3 + 270b_6 b_{22} b_4^2 + 540b_{22}b_{42}b_4^2 + 45b_2 b_6^2 b_4 \\ +765b_2 b_{42}^2 b_4 + 3240b_{22}^2 b_{42} b_4 + 540b_2 b_{42} b_{222} b_4 + 90b_2 b_6 b_{42} b_4 \\ -810b_{22}^2 b_{222} b_4 + 3240b_2 b_{22} b_{42} b_{222} - 270b_6 b_{22}^3 + 30b_0 b_{42}^3 \\ +45b_0 b_6 b_{42}^2 + 1350b_2 b_{22} b_{42}^2 - 810b_2 b_{22} b_{222}^2 - 1620b_{22}^3 b_{42} \\ +540b_2 b_6 b_{22} b_{42} + 4320b_{22}^3 b_{222} + 270b_0 b_{42}^2 b_{222} - 90b_0 b_{222}^3 \end{array} \right\}. \tag{3.68}$$

3.3.3 d-Cube

As an example of a solid with octahedral symmetry, let us consider C_d^* a d-cube with vertices $[\pm1, \pm1, \dots, \pm1]$, so $C_d^* = [-1,1]^d$. Let $\mathbf{y} \in \mathbb{R}^d$. Via symmetry and by simple integration, $b_0 = \operatorname{vol}_d C_d^* = 2^d$ and

$$b_{i_1,\dots,i_p} = \int_{C_d^*} y_1^{i_1} \cdots y_p^{i_p} \lambda_d(\mathrm{d}\mathbf{y}) = \int_{[0,1]^d} y_1^{i_1} \cdots y_p^{i_p} \mathrm{d}y_1 \cdots \mathrm{d}y_d = 2^d \frac{1}{i_1+1} \cdots \frac{1}{i_p+1}, \tag{3.69}$$

from which we deduce the moments $v_d^{(k)}(C_d)$ as before (Equations (3.1), (3.2) and (3.3)).

3.3.4 d-Orthoplex

For d-orthoplex $O_d = \operatorname{conv}(\pm\mathbf{e}_1, \dots, \pm\mathbf{e}_d)$, it is an easy exercise (see Remark 277) to deduce the following general formula

$$b_{i_1,\dots,i_p} = \int_{O_d} y_1^{i_1} \cdots y_p^{i_p} \lambda_d(\mathrm{d}\mathbf{y}) = 2^d \int_{\mathbb{T}_d} y_1^{i_1} \cdots y_p^{i_p} \lambda_d(\mathrm{d}\mathbf{y}) = \frac{2^d\, i_1! \cdots i_p!}{(d+i_1+\cdots+i_p)!}. \tag{3.70}$$

Hence,

$$b_0 = \operatorname{vol}_d O_d = \frac{2^d}{d!}, \qquad b_2 = \frac{2^{d+1}}{(d+2)!} \tag{3.71}$$

and thus by Equation (3.47), summing over $E_{2,d}$ tables,

$$v_d^{(2)}(O_d) = \frac{1}{b_0^2} \frac{d+1}{d!} \left(\frac{b_2}{b_0}\right)^d = \frac{(d+1)!}{2^d (d+2)^d (d+1)^d}. \tag{3.72}$$

Similarly, we have

$$b_{22} = \frac{2^{d+2}}{(d+4)!}, \qquad b_4 = \frac{4!\, 2^d}{(d+4)!}. \tag{3.73}$$

Thus, by Equation (3.52), summing over $E_{4,d}$ tables,

$$v_d^{(4)}(O_d) = \frac{d!(d+1)!\sum_{j=0}^{d} \frac{3^{d-j}(1+j)(2+j)}{2\,(d-j)!}\left(\frac{j(d+4)(d+3)}{(d+2)(d+1)}+1\right)}{2^{2d}(d+4)^d(d+3)^d(d+2)^d(d+1)^d}. \tag{3.74}$$

Finally,

$$b_{222} = \frac{2^{d+3}}{(d+6)!}, \quad b_{42} = \frac{4!\,2^{d+1}}{(d+6)!}, \quad b_6 = \frac{6!\,2^d}{(d+6)!}. \tag{3.75}$$

Crucially, d-orthoplex also satisfies the condition on moments $b_{222}b_4 = b_{22}b_{42}$. Hence, we use Equation (3.66) to deduce

$$\begin{aligned} v_d^{(6)}(O_d) = {} & \frac{d!^2}{48}\left(\frac{15}{4}\right)^d\left(\frac{d!}{(d+6)!}\right)^d \sum_{j=0}^{d+1}\sum_{i=0}^{j} \frac{(1+i)(2+i)(4+i)!}{3^i 10^j (3+d)(4+d)(2+d)} \times \\ & \frac{\binom{14+j+2i}{j-i}}{(d-j+1)!}\left\{\begin{matrix}(3+d)(4+d)((1+d)(2+d)+(5+d)(6+d)(2i+j))(1+d-j)\\ +30\,(5+d)^2\,(6+d)^2(1+d+2i-2j)\end{matrix}\right\}. \end{aligned} \tag{3.76}$$

3.3.5 d-Ball

Another interesting example possessing (among others also) the octahedral symmetry is the unit ball $\mathbb{B}_d$. Let $\mathbf{y} \in \mathbb{R}^d$, then

$$b_{i_1,\dots,i_p} = \int_{\mathbb{B}_d} y_1^{i_1}\cdots y_p^{i_p}\,\lambda_d(\mathrm{d}\mathbf{y}). \tag{3.77}$$

A common trick how to solve types of integrals like this is using the Gaussian integral $\int_{-\infty}^{\infty}\exp(-x^2)\,\mathrm{d}x = \sqrt{\pi}$ and related integrals in higher dimensions. First, let us take advantage of the homogeneity of our integral by making the substitution $\mathbf{y} = r\mathbf{x}$, where $r \in (0,\infty)$ and $\mathbf{x} \in \mathbb{S}^{d-1}$. Splitting the measures into radial and spherical part, that is $\lambda_d(\mathrm{d}\mathbf{y}) = r^{d-1}\mathrm{d}r\,\sigma_d(\mathrm{d}\mathbf{x})$, we get

$$b_{i_1,\dots,i_p} = \int_{\mathbb{S}^d}\int_0^1 x_1^{i_1}\cdots x_p^{i_p}\,r^{d-1+i_1+\dots+i_p}\,\mathrm{d}r\,\sigma_d(\mathrm{d}\mathbf{x}) = \frac{\int_{\mathbb{S}^d} x_1^{i_1}\cdots x_p^{i_p}\sigma_d(\mathrm{d}\mathbf{x})}{d+i_1+\dots+i_p}. \tag{3.78}$$

Note that when $i_1 = \dots = i_p = 0$, we get $b_0 = \omega_d/d$ as expected since $b_0 = \mathrm{vol}_d\,\mathbb{B}_d = \kappa_d = \omega_d/d$. In order to utilize the Gaussian integral trick, let us consider the integral

$$I = \int_{\mathbb{R}^d} y_1^{i_1}\cdots y_p^{i_p} e^{-y_1^2-y_2^2-\dots-y_d^2}\lambda_d(\mathrm{d}\mathbf{y}). \tag{3.79}$$

This integral can be solved via two methods. First, we can split it into a product of one-dimensional integrals in each of the variables y_s. This gives

$$I = \pi^{d/2}\prod_{s=1}^{p}\frac{\Gamma(\frac{1+i_s}{2})}{\Gamma(\frac{1}{2})}. \tag{3.80}$$

On the other hand, by splitting the measures into radial and spherical part,

$$I = \int_{\mathbb{S}^d}\int_0^\infty x_1^{i_1}\cdots x_p^{i_p} r^{d-1+i_1+\dots+i_p}e^{-r^2}\mathrm{d}r\,\sigma_d(\mathrm{d}\mathbf{x}) = \frac{\Gamma(\frac{d+i_1+\dots+i_p}{2})}{2}\int_{\mathbb{S}^d} x_1^{i_1}\cdots x_p^{i_p}\sigma_d(\mathrm{d}\mathbf{x}). \tag{3.81}$$

Comparing, we get $\int_{\mathbb{S}^d} x_1^{i_1} \cdots x_p^{i_p}\, \sigma_d(\mathrm{d}\mathbf{x})$ explicitly and as a result,

$$b_{i_1,\dots,i_p} = \frac{2\pi^{d/2} \prod_{s=1}^p \Gamma(\frac{1+i_s}{2})/\Gamma(\frac{1}{2})}{(d+i_1+\cdots+i_p)\,\Gamma(\frac{d+i_1+\cdots+i_p}{2})} = \frac{\pi^{d/2} \prod_{s=1}^p \Gamma(\frac{1+i_s}{2})/\Gamma(\frac{1}{2})}{\Gamma\left(\frac{d+2+i_1+\cdots+i_p}{2}\right)}. \tag{3.82}$$

Hence,

$$b_0 = \mathrm{vol}_d\, \mathbb{B}_d = \kappa_d = \frac{\pi^{d/2}}{\Gamma(\frac{d+2}{2})}, \qquad b_2 = \frac{\kappa_d}{2+d} \tag{3.83}$$

and thus by Equation (3.47), summing over $E_{2,d}$ tables,

$$v_d^{(2)}(\mathbb{B}_d) = \frac{1}{b_0^2}\frac{d+1}{d!}\left(\frac{b_2}{b_0}\right)^d = \frac{1}{\kappa_d^2}\frac{d+1}{d!(d+2)^d}. \tag{3.84}$$

Similarly, we have

$$b_{22} = \frac{\kappa_d}{(4+d)(2+d)}, \qquad b_4 = \frac{3\kappa_d}{(4+d)(2+d)}. \tag{3.85}$$

Thus, by Equation (3.52) (only $j = d$ term survives), summing over $E_{4,d}$ tables,

$$v_d^{(4)}(\mathbb{B}_d) = \frac{1}{\kappa_d^4}\frac{(1+d)^2\,(d^2+5d+2)}{2(d!)^2\,((2+d)(4+d))^d}. \tag{3.86}$$

Finally,

$$b_{222} = \frac{\kappa_d}{(2+d)(4+d)(6+d)}, \quad b_{42} = \frac{3\kappa_d}{(2+d)(4+d)(6+d)}, \quad b_6 = \frac{15\kappa_d}{(2+d)(4+d)(6+d)}. \tag{3.87}$$

Crucially, d-ball also satisfies the condition on moments $b_{222}b_4 = b_{22}b_{42}$. Hence, we use Equation (3.66) to deduce

$$v_d^{(6)}(\mathbb{B}_d) = \frac{1}{\kappa_d^6}\frac{(1+d)^3(2+d)(3+d)\,(d^2+7d+2)\,(d^2+7d+4)}{48(d!)^3((2+d)(4+d)(6+d))^d}. \tag{3.88}$$

Note that the values $v_d^{(2)}(\mathbb{B}_d)$, $v_d^{(4)}(\mathbb{B}_d)$ and $v_d^{(6)}(\mathbb{B}_d)$ obtained via this method agree with the more general result of Miles expressing $v_d^{(k)}(\mathbb{B}_d)$ for any $k > -d$ (see the consequence of Theorem 220 in Chapter 4).

3.3.6 Polygon triangle even area moments

By similar treatment as before, we can find an explicit formula for $v_2^{(k)}(K_2)$ with even k in terms of planar moments

$$a_{r,s} = \int_{K_2} x^r y^s \,\mathrm{d}x\mathrm{d}y \tag{3.89}$$

for any planar shape K_2 (possibly not convex). By Equation (3.34), we have $v_2^{(k)}(K_2) = e_2^{(k)}/(\operatorname{vol}_2 K_2)^{3+k}$, where $e_2^{(k)}$ is given by Equation (3.45) as a sum over $E_{k,d}$ tables. Hence

$$v_2^{(k)}(K_2) = \frac{2^{-k}}{(\operatorname{vol}_2 K_2)^{3+k}} \sum_{\tau \in E_{k,2}} w(\tau) \operatorname{sgn} \tau. \tag{3.90}$$

Tables $E_{k,2}$ consist of three columns and k rows. Each row is a permutation of $\{0,1,2\}$. For each $\tau \in E_{k,2}$, the weight $w(\tau)$ is equal to a product of three a_{rs}'s (for each column) such that r is the number of 1's and s is the number of 2's in this column. Let $E_{k,2}^{(ijpq)} \subset E_{k,2}$ be the subset of tables with i ones and p twos in the first column and j ones and q twos in the second column and denote $E_{k,2}^{(ijpq)|l}$ as tables $E_{k,2}^{(ijpq)}$ whose total number of $(012), (201), (210)$ rows is l. Table 3.11 below shows the number of specific permutations appearing in any $\tau \in E_{k,2}^{(ijpq)|l}$.

row	(012)	(201)	(210)	(021)	(102)	(120)
sign	$+$	$+$	$-$	$-$	$-$	$+$
count	$l-p$	$l-j$	$j-l+p$	$k-i-l$	$k-l-q$	$i-k+l+q$

Table 3.11: Structure of $E_{k,2}^{(ijpq)|l}$ tables

For any $\tau \in E_{k,2}^{(ijpq)|l}$, we have $w(\tau) = a_{i,p}a_{j,q}a_{k-i-j,k-p-q}$ and $\operatorname{sgn}\tau = (-1)^{i+j+p+q+l}$. Therefore, we deduce that

$$v_2^{(k)}(K_2) = \frac{2^{-k}}{(\operatorname{vol}_2 K_2)^{3+k}} \sum_{p=0}^{k}\sum_{q=0}^{k-p}\sum_{j=0}^{k}\sum_{i=0}^{k-j} c_{ijpqk}\, a_{i,p}a_{j,q}a_{k-i-j,k-p-q}, \tag{3.91}$$

where

$$c_{ijpqk} = \sum_{\tau \in E_{k,2}^{(ijpq)}} \operatorname{sgn}\tau = \sum_{l=0}^{k} (-1)^{i+j+p+q+l} |E_{k,2}^{(ijpq)|l}|. \tag{3.92}$$

The number of $E_{k,2}^{(ijpq)|l}$ tables is equal to the number of placements of specific permutations into rows. By simple combinatorial argument, we deduce that

$$|E_{k,2}^{(ijpq)|l}| = \frac{k!}{(l-j)!(l-p)!(j+p-l)!(k-i-l)!(k-q-l)!(l+i+q-k)!} \tag{3.93}$$

and thus

$$c_{ijpqk} = \sum_{l=\max(j,p,k-q-i)}^{\min(k-i,j+p,k-q)} \frac{(-1)^{i+j+p+q+l}k!}{(l-j)!(l-p)!(j+p-l)!(k-i-l)!(k-q-l)!(l+i+q-k)!}. \tag{3.94}$$

Note that this result can be also derived more directly with the help of random variables. Let $\mathbf{V}_i = [X_i, Y_i] \sim \mathsf{Unif}(K_2)$, $i \in \{0,1,2\}$ be random points selected from K_2 independently. Note that $\mathbb{E}[X_i^r, Y_i^s] = a_{r,s}/\operatorname{vol}_2 K$, $i \in \{1,2,3\}$. For the area Δ_2 of the convex hull $\operatorname{conv}(\mathbf{V}_0, \mathbf{V}_1, \mathbf{V}_2)$, we have

$$\Delta_2 = \frac{1}{2}\left|\det\begin{pmatrix} X_0 & Y_0 & 1 \\ X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \end{pmatrix}\right| = \frac{1}{2}\left|X_0Y_1 + X_1Y_2 + X_2Y_0 - X_1Y_0 - X_2Y_1 - X_0Y_2\right|. \tag{3.95}$$

By *multinomial formula*, we get for even k,

$$\begin{aligned} v_2^{(k)}(K_2) &= \mathbb{E}\left[\left(\frac{\Delta_2}{\operatorname{vol}_2 K_2}\right)^k\right] = \frac{2^{-k}}{(\operatorname{vol}_2 K_2)^k}\,\mathbb{E}\left(X_0Y_1 + X_1Y_2 + X_2Y_0 - X_1Y_0 - X_2Y_1 - X_0Y_2\right)^k \\ &= \frac{2^{-k}}{(\operatorname{vol}_2 K_2)^k}\mathbb{E}\sum_{k_1+\cdots+k_6=k} \frac{(-1)^{k_4+k_5+k_6}k!}{k_1!k_2!k_3!k_4!k_5!}(X_0Y_1)^{k_1}(X_1Y_2)^{k_2}(X_2Y_0)^{k_3}(X_1Y_0)^{k_4}(X_2Y_1)^{k_5}(X_0Y_2)^{k_6} \\ &= \frac{2^{-k}}{(\operatorname{vol}_2 K_2)^k}\sum_{k_1+\cdots+k_6=k} \frac{(-1)^{k_4+k_5+k_6}k!}{k_1!k_2!k_3!k_4!k_5!}\mathbb{E}[X_0^{k_1+k_6}Y_0^{k_3+k_4}]\mathbb{E}[X_1^{k_2+k_4}Y_1^{k_1+k_5}]\mathbb{E}[X_2^{k_3+k_5}Y_2^{k_2+k_6}] \\ &= \frac{2^{-k}}{(\operatorname{vol}_2 K_2)^{k+3}}\sum_{k_1+\cdots+k_6=k} \frac{(-1)^{k_4+k_5+k_6}k!}{k_1!k_2!k_3!k_4!k_5!}a_{k_1+k_6,k_3+k_4}a_{k_2+k_4,k_1+k_5}a_{k_3+k_5,k_2+k_6}, \end{aligned} \tag{3.96}$$

which is precisely Equation (3.91).

Example 219. Let $\mathbb{U}_2^{\alpha\beta} = \operatorname{conv}([\alpha,0],[0,\beta],[1,0],[0,1])$ with $\alpha,\beta \in (0,1)$ be the *canonical truncated triangle* having $\operatorname{vol}_2(\mathbb{U}_2^{\alpha\beta}) = \frac{1}{2}(1-\alpha\beta)$. Hence, for even k,

$$v_2^{(k)}(\mathbb{U}_2^{\alpha\beta}) = \frac{8\,k!}{(1-\alpha\beta)^{3+k}}\sum_{p=0}^{k}\sum_{q=0}^{k-p}\sum_{j=0}^{k}\sum_{i=0}^{k-j} c_{ijpqk}\,a_{i,p}a_{j,q}a_{k-i-j,k-p-q}, \tag{3.97}$$

where, by inclusion/exclusion, by scaling and by Remark 277, we have

$$a_{rs} = \int_{\mathbb{U}_2^{\alpha\beta}} x^r y^s \,\mathrm{d}x\mathrm{d}y = (1-\alpha^{r+1}\beta^{s+1})\int_{\mathbb{T}_2} x^r y^s \,\mathrm{d}x\mathrm{d}y = r!s!\frac{1-\alpha^{r+1}\beta^{s+1}}{(2+r+s)!}, \tag{3.98}$$

from which we obtain when $k=2$ and $k=4$,

$$v_2^{(2)}(\mathbb{U}_2^{\alpha\beta}) = \frac{\left\{\begin{matrix} \alpha^4\beta^4 - 8\alpha^3\beta^3 + 8\alpha^3\beta^2 - 4\alpha^3\beta + 8\alpha^2\beta^3 \\ -10\alpha^2\beta^2 + 8\alpha^2\beta - 4\alpha\beta^3 + 8\alpha\beta^2 - 8\alpha\beta + 1 \end{matrix}\right\}}{72(1-\alpha\beta)^4}, \tag{3.99}$$

$$v_2^{(4)}(\mathbb{U}_2^{\alpha\beta}) = \frac{\left\{\begin{matrix} \alpha^6\beta^6 - 6\alpha^5\beta - 6\alpha\beta^5 + 18\alpha\beta^4 + 32\alpha\beta^2 - 19\alpha\beta + 1 \\ -31\alpha\beta^3 - 19\alpha^5\beta^5 + 32\alpha^5\beta^4 - 31\alpha^5\beta^3 + 18\alpha^5\beta^2 \\ -31\alpha^3\beta + 32\alpha^4\beta^5 - 47\alpha^4\beta^4 + 46\alpha^4\beta^3 - 34\alpha^4\beta^2 \\ +18\alpha^4\beta - 31\alpha^3\beta^5 + 46\alpha^3\beta^4 - 50\alpha^3\beta^3 + 46\alpha^3\beta^2 \\ +18\alpha^2\beta^5 - 34\alpha^2\beta^4 + 46\alpha^2\beta^3 - 47\alpha^2\beta^2 + 32\alpha^2\beta \end{matrix}\right\}}{900(1-\alpha\beta)^6}. \tag{3.100}$$

4. Odd Volumetric Moments

In this chapter, we are going to investigate how we can deduce polytopes P_d when k is odd and $d = 3$ and higher.

4.1 Summary of known and new results

4.1.1 Known results

Extending the work of Crofton, Hostinský [36, p. 22–26] considered and solved many problems concerning geometric probability. One of them is the ball tetrahedron picking, which was the first metric moment obtained in $d = 3$, it reads

$$v_3^{(1)}(\mathbb{B}_3) = \frac{9}{715}. \tag{4.1}$$

The result was generalised to higher dimensions by Kingman [40]. For the mean volume of a d-simplex picked from a d-ball, Kingman got

$$v_d^{(1)}(\mathbb{B}_d) = \frac{2^d \Gamma^2\left(\frac{(d+1)^2}{2}\right)\Gamma^{d+1}(d+1)}{(d+1)^{d-1}\,\Gamma((d+1)^2)\,\Gamma^{2(d+1)}\left(\frac{d+1}{2}\right)}. \tag{4.2}$$

The result above can be obtained as a special case of even more general formula by Miles [48, p. 363, Eq. (29)]

Theorem 220. *[Miles, 1971] Denote $V_d^{(i,j)}$ as the $r = i + j - 1$ dimensional content (vol_r) of an r-simplex formed by a convex hull of randomly selected i points from the interior and j points from the surface of $\mathbb{B}_d$ (ball with unit radius). If $2 \le r \le d + 1$, then for $k = 1, 2, 3, \ldots$*

$$\mathbb{E}\left[\left(V_d^{(i,j)}\right)^k\right] = \frac{1}{r!^k}\left(\frac{d}{d+k}\right)^i \frac{\Gamma\left(\frac{(r+1)(d+k)}{2} - j + 1\right)}{\Gamma\left(\frac{(r+1)d+rk}{2} - j + 1\right)}\left(\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+k}{2}\right)}\right)^r \prod_{l=1}^{r-1}\frac{\Gamma\left(\frac{d-r+k+l}{2}\right)}{\Gamma\left(\frac{d-r+l}{2}\right)}.$$

As a consequence of Miles' formula, we get for d-ball volumetric moments,

$$v_d^{(k)}(\mathbb{B}_d) = \left(\frac{\Gamma\left(\frac{d}{2}+1\right)}{\pi^{d/2} d!}\right)^k \left(\frac{d}{d+k}\right)^{d+1} \frac{\Gamma\left(\frac{(d+1)(d+k)}{2}+1\right)}{\Gamma\left(\frac{d(d+k+1)}{2}+1\right)}\left(\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+k}{2}\right)}\right)^d \prod_{l=1}^{d-1}\frac{\Gamma\left(\frac{k+l}{2}\right)}{\Gamma\left(\frac{l}{2}\right)}. \tag{4.3}$$

Table 4.1 shows odd volumetric moments obtained by this formula for small values of k (even volumetric moments for $k = 2, 4, 6$ are already shown in Table 3.1).

$v_d^{(k)}(\mathbb{B}_d)$	$k = 1$	$k = 3$	$k = 5$	$k = 7$
$d = 2$	$\frac{35}{48\pi^2}$	$\frac{1001}{6400\pi^4}$	$\frac{138567}{2007040\pi^6}$	$\frac{1062347}{24772608\pi^8}$
$d = 3$	$\frac{9}{715}$	$\frac{3}{29393\pi^2}$	$\frac{1}{475456\pi^4}$	$\frac{63}{909788000\pi^6}$
$d = 4$	$\frac{676039}{3888000\pi^4}$	$\frac{73465381}{212425113600\pi^8}$	$\frac{192875738341}{917466736125542400\pi^{12}}$	$\frac{32283434353859}{14035728178796738641920\pi^{16}}$
$d = 5$	$\frac{20000}{90751353}$	$\frac{3125}{390325604864\pi^4}$	$\frac{2025}{1929127875659776\pi^8}$	$\frac{2625}{94664358113583431680\pi^{12}}$

Table 4.1: Selected values of $v_d^{(k)}(\mathbb{B}_d)$ with odd k and $d \le 5$.

Less is known about polytopes. In two dimensions, however, Buchta and Reitzner [19] found a formula expressing $v_n^{(1)}(P_2)$ for any convex polygon P_2.

In three dimensions, there was a famous difficult problem proposed by Klee [41] and popularised by Blaschke, which concerns finding $v_3^{(1)}(T_3)$, the mean volume of a tetrahedron formed by four uniformly selected random points from the interior of a fixed unit volume tetrahedron. The first attempt was made by Reed. In [59], he uses the Crofton reduction technique [61] which enables him to express the exact value of $v_3^{(1)}(T_3) = V_{3333}$ as a linear combination of mean volumes of four irreducible configurations (3320), (2222), (3311), (3221), in which the points forming the random tetrahedron are chosen from sets of lower dimensions.

- (3320) : two points inside, one on a face and the fourth being a vertex,
- (2222) : points on faces only, one on each face,
- (3311) : two points inside and two on the opposite edges,
- (3221) : one point inside, two points on adjacent faces and the fourth being a vertex.

The specific form of the linear combination can be deduced as an easy exercise from the Crofton Reduction Technique developed in Chapter 1. First, we construct a reduction diagram corresponding to the aforementioned configurations (Figure 4.1 below). In this diagram, we also included the position of the scaling point **C** in cases reduction is possible. The arrows indicate which configurations reduce to which. Each arrow is labeled by a roman numeral corresponding to a given reduction equation in the system of reduction equations.

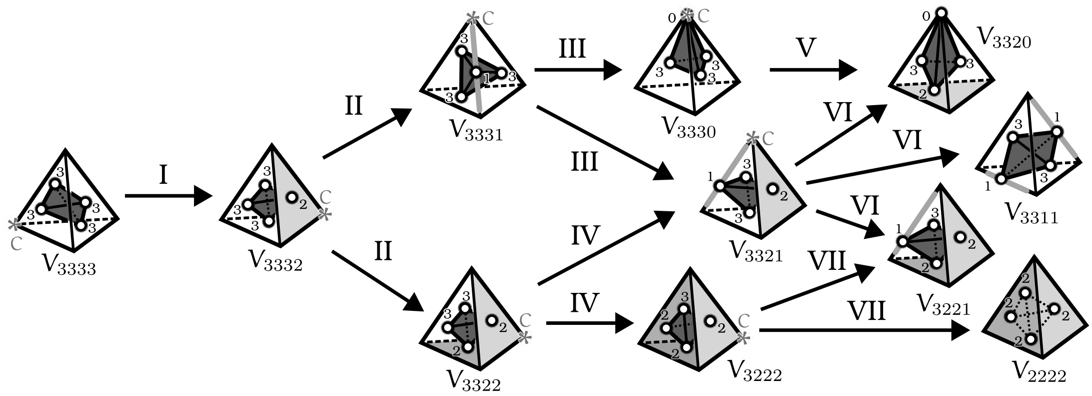


Figure 4.1: All different $(abcd)$ sub-configurations in T_3

The full system obtained by the Multivariate Crofton Reduction Technique is

$$\begin{aligned}
\mathbf{I}:&\ 3\,V_{3333} = 4\cdot 3(V_{3332} - V_{3333})\\
\mathbf{II}:&\ 3\,V_{3332} = 3\cdot 3(V_{3322} - V_{3332}) + 2(V_{3331} - V_{3332}),\\
\mathbf{III}:&\ 3\,V_{3331} = 3\cdot 3(V_{3321} - V_{3331}) + 1(V_{3330} - V_{3331})\\
\mathbf{IV}:&\ 3\,V_{3322} = 2\cdot 3(V_{3222} - V_{3322}) + 2\cdot 2(V_{3321} - V_{3322})\\
\mathbf{V}:&\ 3\,V_{3330} = 3\cdot 3(V_{3320} - V_{3330})\\
\mathbf{VI}:&\ 3\,V_{3321} = 2\cdot 3(V_{3221} - V_{3321}) + 2(V_{3311} - V_{3321}) + 1(V_{3320} - V_{3321})\\
\mathbf{VII}:&\ 3\,V_{3222} = 3(V_{2222} - V_{3222}) + 3\cdot 2(V_{3221} - V_{3222})
\end{aligned}$$

Solving the system for V_{3333}, we get,

$$V_{3333} = \frac{27V_{2222}}{455} + \frac{108V_{3221}}{455} + \frac{18V_{3311}}{455} + \frac{12V_{3320}}{455}. \tag{4.4}$$

Reed was, however, only able to express $V_{3320} = 3/64$ in a closed form (he also attempted to find V_{2222} but obtained an erroneous value). The remaining configurations were only solved by Mannion [44] using a clever handling of improper integrals. Their exact values are $V_{2222} = \frac{23}{486} + \frac{2\pi^2}{6237}$, $V_{3221} = \frac{11}{216} - \frac{\pi^2}{3465}$ and $V_{3320} = \frac{7}{144} - \frac{\pi^2}{2310}$. As a consequence, Mannion concluded that

$$v_3^{(1)}(T_3) = \frac{13}{720} - \frac{\pi^2}{15015} \approx 0.017398. \tag{4.5}$$

However, Buchta and Reitzner [18] obtained this value earlier using the Efron section formula [26], c.f. [46, p. 372], which relates the mean volume of a convex hull of random points picked from a given body with an integral over section planes. This integral over section planes can be then transformed, after some nontrivial algebraic manipulations, into set of some calculable double integrals. The same technique enabled Zinani [78] to deduce

$$v_3^{(1)}(C_3) = \frac{3977}{21600} - \frac{\pi^2}{2160} \approx 0.01384277. \tag{4.6}$$

The derivation of $v_3^{(1)}(C_3)$ itself is straightforward, but at the same time unworldly difficult, containing millions of intermediate integrals necessary to solve (to do so, Zinani used the package Mathematica 4.0). No other values of odd volumetric moments in three dimensions were known.

In higher dimensions, there were no results for polytopes. The Efron formula completely breaks down because of the existence of cyclic polytopes.

However, Efron's formula is not the only approach to volumetric moments. The original method by Reed and Mannion to obtain $v_3^{(1)}(T_3)$ was the Crofton's reduction technique. Another derivation of $v_3^{(1)}(T_3)$ and $v_3^{(1)}(C_3)$ which appeared recently and was not using Efron's formula (but equally difficult) was due to Philip [52, 53]. As we shall see later in this thesis, there is yet another way. Had it not been for Philip's work, the author of this thesis would not have been convinced that there might still be another method for obtaining volumetric moments.

4.1.2 New results

The objective of this section is to extend the number of polytopes for which the volumetric moments are expressed exactly and to present the method to find it efficiently. The key approach is the method of section integration. That is, instead of integrating over points, we integrate over a section in the spirit of Blaschke-Petkantschin formula (see Appendix B). In fact, there are two approaches. The first is based on the Efron section formula, which enables to deduce $v_n^{(1)}(P_d)$ for any integer $n \geq d$ in dimensions $d = 2$ and $d = 3$ (see Theorems 234 and 235). Efron's approach will be discussed later in Chapter 5. As there is no analog of the Efron section formula for higher moments and dimensions, we might use the second section integral approach applicable to volumetric moments $v_d^{(k)}(K_d)$ for any k (picking a d-simplex from a d-dimensional body K_d). The second approach is based on base-height splitting (Theorem 221) which is discussed in this chapter.

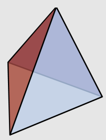
4.1.3 Three dimensions

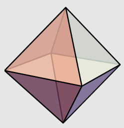
First, we found higher volumetric moments in the tetrahedron, cube, and octahedron. That is $v_3^{(k)}(T_3)$, $v_3^{(k)}(C_3)$ and $v_3^{(k)}(O_3)$. The results are summarised in Table 4.2 below.

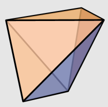
P_3	$v_3^{(1)}(P_3)$	$v_3^{(3)}(P_3)$	$v_3^{(5)}(P_3)$
T_3	$\frac{13}{720} - \frac{\pi^2}{15015}$	$\frac{733}{12600000} + \frac{79\pi^2}{2424922500}$	$\frac{5125739}{4356374400000} - \frac{547\pi^2}{8943995970000}$
C_3	$\frac{3977}{216000} - \frac{\pi^2}{2160}$	$\frac{8411819}{450084600000} - \frac{\pi^2}{3402000}$	$\frac{306749173351\pi^2}{124439390208000} - \frac{2225580641145943786613}{914796764569239552000000}$
O_3	$\frac{19297\pi^2}{3843840} - \frac{6619}{184320}$	$\frac{1628355709\pi^2}{19864965120000} - \frac{81932629}{103219200000}$	$\frac{6356364544399\pi^2}{1611922729697280000} - \frac{205491225433}{5287025049600000}$

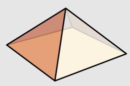
Table 4.2: Selected values of $v_3^{(k)}(T_3)$, $v_3^{(k)}(C_3)$ and $v_3^{(k)}(O_3)$ for odd k.

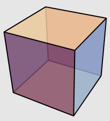
Next, we considered finding $v_3^{(1)}(P_3)$ for various other polyhedra P_3 shown in Table 4.3 (including the case of a tetrahedron and a cube).

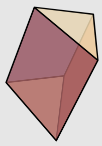

T_3, tetrahedron

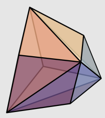

O_3, octahedron

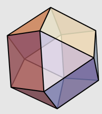

tetrahedron bipyramid

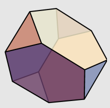

square pyramid

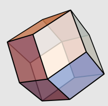

C_3, cube

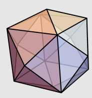

triangular prism


triakis tetrahedron*


cuboctahedron


truncated tetrahedron


rhombic dodecahedron


tetrakis hexahedron*

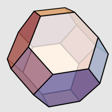

truncated octahedron*

Table 4.3: Polyhedra for which we considered $v_3^{(1)}(K_3)$

To be honest with the reader, the polyhedra indicated by * have not been computed yet (section integrals are availible only in some particular genealogies), but they will surely appear in an updated version of this thesis. Interestingly, in contrast to the well known tetrahedron and cube case, $v_3^{(1)}(P_3)$ often involves logarithms and special values of the so called *dilogarithm* function $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} x^2/n^2$, especially

$$\mathrm{Li}_2\left(\tfrac{1}{4}\right) \approx 0.267652639082732606919183828487811575819857066\,9\ldots \tag{4.7}$$

Table 4.4 below summarises all new results of exact mean tetrahedron volume in various 3-bodies K_3. For completeness, the previously known cases of a ball, tetrahedron and a cube have been added as well. Each K_3 is having volume one or alternatively, the right column displays $v_3^{(1)}(K_3)$.

K_3	$v_3^{(1)}(K_3)$
ball, [36] *0.012587413*	$\frac{9}{715}$
rhombic dodecahedron 0.012938482	$\frac{2421179003623}{17933819904000} + \frac{37061863\pi^2}{29889699840} - \frac{9406373047 \ln 2}{9340531200} - \frac{1757220593 \ln^2 2}{2490808320} + \frac{282589831 \ln 3}{283852800} - \frac{6078271\, \mathrm{Li}_2\left(\frac{1}{4}\right)}{8515584}$
cuboctahedron 0.013002516	$\frac{117410162173}{525525000000} + \frac{8752199\pi^2}{2402400000} - \frac{192940695481 \ln 2}{105105000000} - \frac{318759601 \ln^2 2}{250250000} + \frac{506316394917 \ln 3}{280280000000} - \frac{648098487\, \mathrm{Li}_2\left(\frac{1}{4}\right)}{500500000}$
octahedron 0.013637411	$\frac{19297\pi^2}{3843840} - \frac{6619}{184320}$
cube, [78] *0.013842776*	$\frac{3977}{216000} - \frac{\pi^2}{2160}$
truncated tetrahedron 0.014845102	$\frac{35604506258521}{162358039443600} - \frac{13447020779\pi^2}{96641690145} + \frac{9972537226592 \ln 2}{3382459155075} + \frac{3485442712 \ln^2 2}{1400604205} - \frac{8953623027 \ln 3}{7884520175} - \frac{53493528168 \ln 2 \ln 3}{32213896715} + \frac{53162662164\, \mathrm{Li}_2\left(\frac{1}{4}\right)}{32213896715}$
triangular bipyramid 0.015082427	$\frac{1712190037}{16812956160} + \frac{81471636487\pi^2}{907899632640} - \frac{185777703053 \ln 2}{50438868480} - \frac{909434448983 \ln^2 2}{121053284352} + \frac{3498264683 \ln 3}{2401850880} + \frac{20912895 \ln 2 \ln 3}{2050048} - \frac{1887867 \ln^2 3}{585728} - \frac{62045573287 \mathrm{Li}_2\left(\frac{1}{4}\right)}{57644421120}$
triangular prism 0.015357705	$\frac{1859}{116640} - \frac{\pi^2}{17010}$
square pyramid 0.015782681	$\frac{941\pi^2}{72072} - \frac{977}{8640}$
tetrahedron,[18] *0.017398239*	$\frac{13}{720} - \frac{\pi^2}{15015}$

Table 4.4: Mean tetrahedron volume $v_3^{(1)}(K_3)$ in various bodies K_3

4.1.4 Higher dimensions

Also, our another goal is to present a new technique and deduce the values of $v_d^{(k)}(P_d)$ for various odd k and $d = 3, 4, 5$ in the most elementary way (for even k, they are trivial). The results for T_d are shown in Table 4.5.

	$v_d^{(1)}(T_d)$
$d=4$	$\frac{97}{27000} - \frac{2173\pi^2}{52026975}$
$d=5$	$\frac{2207}{3265920} - \frac{244129\pi^2}{14522729760} + \frac{73522\pi^4}{541513323351}$
$d=6$	$\frac{26609}{217818720} - \frac{3396146609\pi^2}{621871356506400} + \frac{1318349152898\pi^4}{1218020640129839045 5}$

	$v_d^{(3)}(T_d)$
$d=4$	$\frac{1955399}{3403417500000} + \frac{63065881\pi^2}{39669996140775000}$
$d=5$	$\frac{362173019}{98363448852480000} + \frac{10217818563857\pi^2}{557436796045056999751680} + \frac{602363516243\pi^4}{5699340654659722793923 20}$

	$v_d^{(5)}(T_d)$
$d=4$	$\frac{12443146181}{980368514637120000 0} - \frac{1262701803371\pi^2}{3557043272871373325040000}$

Table 4.5: Selected values of $v_d^{(k)}(T_d)$ for odd k and $d = 4, 5, 6$.

In higher dimensions in general, other higher order *polylogarithm* functions will appear, that is $\mathrm{Li}_s(x) = \sum_{n=1}^{\infty} x^n/n^s$. As a consequence, in four dimensions for example, many exact formulae involve *Apéry's constant* (which coincides with $\mathrm{Li}_3(1)$):

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.202056903159594285399738161511, \ldots \tag{4.8}$$

An example is the volumetric moments of C_4, which are shown in Table 4.6.

	$v_4^{(k)}(C_4)$
$k=1$	$\frac{318746289625217532 37}{1058357013719040000000} - \frac{26003\pi^2}{1399680000} + \frac{610208\ln 2}{1913625} - \frac{536557\zeta(3)}{2592000}$
$k=3$	$\frac{19330626155629115959}{168272319220914585600 0000} - \frac{52276897\pi^2}{216801070940160000} + \frac{10004540239\ln 2}{77977156950000} - \frac{6155594561\zeta(3)}{73741860864000}$

Table 4.6: Values of $v_4^{(k)}(C_4)$ for $k = 1, 3$.

4.2 Canonical section integral

Theorem 221. *Let K_d be a d-dimensional convex body, $\mathbb{x}' = (\mathbf{x}_1, \ldots, \mathbf{x}_d)$ a collection of d points in K_d and $\boldsymbol{\sigma} = \mathcal{A}(\mathbb{x}') \in \mathbb{A}(d, d-1)$ be a hyperplane parametrised by $\boldsymbol{\eta} = (\eta_1 \ldots, \eta_d)^\top \in \mathbb{R}^d$ as $\mathbf{x} \in \boldsymbol{\sigma} \Leftrightarrow \boldsymbol{\eta}^\top \mathbf{x} = 1$, then*

$$v_d^{(k)}(K_d) = \frac{(d-1)!}{d^k} \int_{\mathbb{R}^d \setminus K_d^\circ} v_{d-1}^{(k+1)}(\boldsymbol{\sigma} \cap K_d)\, \zeta_d^{d+k+1}(\boldsymbol{\sigma}) \iota_d^{(k)}(\boldsymbol{\sigma}) \lambda_d(\mathrm{d}\boldsymbol{\eta}) \tag{4.9}$$

for any real $k > -1$, where

$$\zeta_d(\boldsymbol{\sigma}) = \frac{\mathrm{vol}_{d-1}(\boldsymbol{\sigma} \cap K_d)}{\|\boldsymbol{\eta}\| \,\mathrm{vol}_d K_d}, \qquad \iota_d^{(k)}(\boldsymbol{\sigma}) = \int_{K_d} |\boldsymbol{\eta}^\top \mathbf{x} - 1|^k \lambda_d(\mathrm{d}\mathbf{x}) \tag{4.10}$$

and $K_d^\circ = \{\mathbf{x} \in \mathbb{R}^d \,|\, \mathbf{x}^\top \mathbf{y} \leq 1, \mathbf{y} \in K_d\}$ is the polar body of K_d.

Proof. Let $\mathbb{X} = (\mathbf{X}_0, \ldots, \mathbf{X}_n)$ be a collection of random $n+1$ i.i.d. points taken uniformly from K_d, let $\mathbb{H}_n = \mathrm{conv}(\mathbb{X})$ be their convex hull and $\Delta_n = \mathrm{vol}_d \mathbb{H}_n$, then we have in general $(n \geq d)$

$$v_n^{(k)}(K_d) = \frac{\mathbb{E}\,[\Delta_n^k]}{(\mathrm{vol}_d K_d)^k}. \tag{4.11}$$

When $n = d$, $\mathbb{H}_d$ is almost surely a d-simplex. That means that any d-tuple of points $\mathbf{X}_i$ from $\mathbb{X}$ form a facet. Let $\mathbb{X}' = (\mathbf{X}_1, \ldots, \mathbf{X}_d)$, $\boldsymbol{\sigma} = \mathcal{A}(\mathbb{X}')$ as in the statement of the theorem and let $\mathrm{dist}_{\boldsymbol{\sigma}}(\mathbf{X}_0)$ be the distance from $\boldsymbol{\sigma}$ to the point $\mathbf{X}_0$, then by base-height splitting,

$$\Delta_d = \frac{1}{d} \mathrm{dist}_{\boldsymbol{\sigma}}(\mathbf{X}_0) \Delta_{d-1}, \tag{4.12}$$

where $\Delta_{d-1} = \mathrm{vol}_{d-1} \mathrm{conv}(\mathbb{X}')$. See Figure 4.2 below.

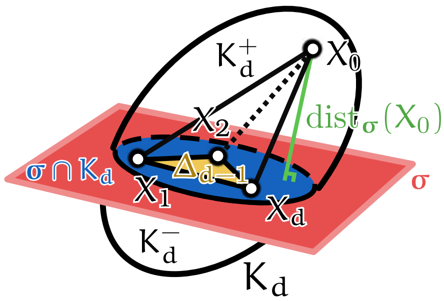


Figure 4.2: Base-height splitting

Fixing $\mathbb{X}'$, we get by conditioning,

$$v_d^{(k)}(K_d) = \frac{\mathbb{E}\,[\mathbb{E}\,[\mathrm{dist}_{\boldsymbol{\sigma}}^k(\mathbf{X}_0) \mid \mathbb{X}'\,] \Delta_{d-1}^k]}{d^k (\mathrm{vol}_d K_d)^k}, \tag{4.13}$$

where

$$\mathbb{E}\,[\mathrm{dist}_{\boldsymbol{\sigma}}^k(\mathbf{X}_0) \mid \mathbb{X}'\,] = \frac{1}{\mathrm{vol}_d K_d} \int_{K_d} \mathrm{dist}_{\boldsymbol{\sigma}}^k(\mathbf{x}_0) \lambda_d(\mathrm{d}\mathbf{x}_0) \tag{4.14}$$

is the k-th distance moment from (fixed) $\boldsymbol{\sigma}$. If $\boldsymbol{\sigma}$ is parametrised Cartesianely, that means by $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_d)^\top$ such that $\mathbf{x} \in \boldsymbol{\sigma} \Leftrightarrow \boldsymbol{\eta}^\top \mathbf{x}_0 = 1$, we may write

$$\operatorname{dist}_{\boldsymbol{\sigma}}(\mathbf{x}_0) = |\boldsymbol{\eta}^\top \mathbf{x}_0 - 1| / \|\boldsymbol{\eta}\| \tag{4.15}$$

and thus

$$\mathbb{E}\left[\operatorname{dist}^k_{\boldsymbol{\sigma}}(\mathbf{X}_0) \mid \mathbb{X}'\right] = \frac{1}{\|\boldsymbol{\eta}\|^k \operatorname{vol}_d K_d} \int_{K_d} |\boldsymbol{\eta}^\top \mathbf{x}_0 - 1|^k \lambda_d(\mathrm{d}\mathbf{x}_0). \tag{4.16}$$

Note that since $\mathbb{E}\left[\operatorname{dist}^k_{\boldsymbol{\sigma}}(\mathbf{X}_0) \mid \mathbb{X}'\right]$ is only a function of $\boldsymbol{\sigma}$, we may use Blaschke-Petkantschin formula in Cartesian parametrisation (Corollary 296.2), that is

$$\mathbb{E}\left[g(\boldsymbol{\sigma})\Delta^k_{d-1}\right] = (d-1)!(\operatorname{vol}_d K)^{k+1} \int_{\mathbb{R}^d \setminus K_d^\circ} v^{(k+1)}_{d-1}(\boldsymbol{\sigma}_{K_d}) \zeta_d^{d+k+1}(\boldsymbol{\sigma}) g(\boldsymbol{\sigma}) \|\boldsymbol{\eta}\|^k \lambda_d(\mathrm{d}\boldsymbol{\eta}),$$

where $\boldsymbol{\sigma}_{K_d} = \boldsymbol{\sigma} \cap K_d$. Selecting $g(\boldsymbol{\sigma}) = \mathbb{E}\left[\operatorname{dist}^k_{\boldsymbol{\sigma}}(\mathbf{X}_0) \mid \mathbb{X}'\right]$ and by definition of $\iota^{(k)}_d(\boldsymbol{\sigma})$, Equation (4.13) then becomes the desired assertion of the theorem. ■

4.2.1 Limit behaviour

Lemma 222. *For any $k > -1$, we can write in terms of geometric quantities*

$$\iota^{(k)}_d(\boldsymbol{\sigma}) = \|\boldsymbol{\eta}\|^k \int_{-\infty}^{\infty} \operatorname{vol}_{d-1}((\boldsymbol{\sigma} + t\hat{\boldsymbol{\eta}}) \cap K_d)\, |t|^k \,\mathrm{d}t. \tag{4.17}$$

Proof. By definition, we have for a given plane $\boldsymbol{\sigma} \in \mathbb{A}(d, d-1)$ parametrised by a corresponding $\boldsymbol{\eta} \in \mathbb{R}^d \setminus K_d^\circ$,

$$\iota^{(k)}_d(\boldsymbol{\sigma}) = \int_{K_d} |\boldsymbol{\eta}^\top \mathbf{x} - 1|^k \lambda_d(\mathrm{d}\mathbf{x}). \tag{4.18}$$

Let $\hat{\boldsymbol{\eta}} = \boldsymbol{\eta}/\|\boldsymbol{\eta}\|$ be the unit normal vector perpendicular to $\boldsymbol{\sigma}$. We can decompose any point $\mathbf{x} \in K_d$ as $\mathbf{x} = \mathbf{y} + t\hat{\boldsymbol{\eta}}$ for some $t \in \mathbb{R}$ and some $\mathbf{y} \in \boldsymbol{\sigma}$, which yields

$$|\boldsymbol{\eta}^\top \mathbf{x} - 1| = |\boldsymbol{\eta}^\top \mathbf{y} - 1 + t\boldsymbol{\eta}^\top \hat{\boldsymbol{\eta}}| = |t|\|\boldsymbol{\eta}\|. \tag{4.19}$$

By Fubini's theorem, we get, plugging into $\iota^{(k)}_d(\boldsymbol{\sigma})$,

$$\iota^{(k)}_d(\boldsymbol{\sigma}) = \|\boldsymbol{\eta}\|^k \int_{-\infty}^{\infty} \int_{\boldsymbol{\sigma} \cap K_d} |t|^k \mathbb{1}_{\mathbf{y} + t\hat{\boldsymbol{\eta}} \in K_d}\, \lambda_{d-1}(\mathrm{d}\mathbf{y})\,\mathrm{d}t. \tag{4.20}$$

The lemma follows by integrating $\mathbf{y}$ over $\boldsymbol{\sigma} \cap K_d$. ■

Remark 223. Let $r \in \mathbb{R} \setminus \{0\}$, we define $r\boldsymbol{\sigma}$ as another section plane whose Cartesian parametrization vector is $\boldsymbol{\eta}/r$ (the plane $\boldsymbol{\sigma}$ gets scaled by r). Then $\boldsymbol{\sigma} + t\hat{\boldsymbol{\eta}} = (1 + t\|\boldsymbol{\eta}\|)\boldsymbol{\sigma}$ and also

$$\zeta_d((1 + t\|\boldsymbol{\eta}\|)\boldsymbol{\sigma}) = |1 + t\|\boldsymbol{\eta}\|| \frac{\operatorname{vol}_{d-1}((\boldsymbol{\sigma} + t\hat{\boldsymbol{\eta}}) \cap K_d)}{\|\boldsymbol{\eta}\| \operatorname{vol}_d K_d}. \tag{4.21}$$

By substitution $r = 1 + t\|\boldsymbol{\eta}\|$ and by Lemma 222, we may also write $\iota^{(k)}_d(\boldsymbol{\sigma})$ in terms of $\zeta_d(\boldsymbol{\sigma})$ for any $k > -1$ as

$$\iota^{(k)}_d(\boldsymbol{\sigma}) = \operatorname{vol}_d K_d \int_{-\infty}^{\infty} \frac{\zeta_d(r\boldsymbol{\sigma})}{|r|} |r - 1|^k \,\mathrm{d}r. \tag{4.22}$$

Finally, let us obtain the limit behaviour of $v^{(k)}_d(K_d)$ when $k \to (-1)^+$.

Proposition 224.

$$\lim_{k\to(-1)^+}(1+k)\,v_d^{(k)}(K_d) = 2\,d!\,\mathrm{vol}_d K_d \int_{\mathbb{R}^d\setminus K_d^\circ} \zeta_d^{d+1}(\boldsymbol{\sigma})\lambda_d(\mathrm{d}\boldsymbol{\eta}). \tag{4.23}$$

Proof. The function $\iota_d^{(k)}(\boldsymbol{\sigma}) = \int_{K_d} |\boldsymbol{\eta}^\top \mathbf{x} - 1|^k \lambda_d(\mathrm{d}\mathbf{x})$ becomes singular as $k \to -1$ because of the points on $\boldsymbol{\sigma}$ which satisfy $\boldsymbol{\eta}^\top \mathbf{x} - 1 = 0$. For any fixed (small) $\varepsilon > 0$, we get by Lemma 222 and by continuity of $\mathrm{vol}_{d-1}((\boldsymbol{\sigma} + t\hat{\boldsymbol{\eta}}) \cap K_d)$,

$$\begin{aligned}\iota_d^{(k)}(\boldsymbol{\sigma}) &= \int_{-\varepsilon}^{\varepsilon} \mathrm{vol}_{d-1}((\boldsymbol{\sigma} + t\hat{\boldsymbol{\eta}}) \cap K_d)|t|^k\,\mathrm{d}t + O(1)\\ &= \frac{2\,\mathrm{vol}_{d-1}(\boldsymbol{\sigma}\cap K_d)}{\|\boldsymbol{\eta}\|(k+1)} + O(1) = \frac{2\zeta_d(\boldsymbol{\sigma})\,\mathrm{vol}_d K_d}{k+1} + O(1).\end{aligned} \tag{4.24}$$

as $k \to (-1)^+$. Since $\iota_d^{(k)}(\boldsymbol{\sigma})$ is the only singular term in $v_d^{(k)}(K_d)$ when k approaches -1 (see Theorem 221), the statement of the proposition follows. ■

Alternative proof. Alternatively, let $\boldsymbol{\gamma} = \{t\hat{\boldsymbol{\eta}} \mid t \in \mathbb{R}\}$ be a line passing through the origin in the direction of $\hat{\boldsymbol{\eta}}$ and $L_\pm(\hat{\boldsymbol{\eta}}, \mathbf{y}) = \mathrm{vol}_1(K_d^\pm \cap \boldsymbol{\gamma})$ be the lengths of line segments of $\boldsymbol{\gamma}$ in K_d below and above the section plane $\boldsymbol{\sigma}$, respectively. Then, integrating out t in Equation (4.20), we get for any $k > -1$,

$$\iota_d^{(k)}(\boldsymbol{\sigma}) = \frac{1}{1+k}\|\boldsymbol{\eta}\|^k \int_{\boldsymbol{\sigma}\cap K_d} L_+^{k+1}(\hat{\boldsymbol{\eta}}, \mathbf{y}) + L_-^{k+1}(\hat{\boldsymbol{\eta}}, \mathbf{y})\,\lambda_{d-1}(\mathrm{d}\mathbf{y}), \tag{4.25}$$

from which we get for the limit $\lim_{k\to(-1)^+} \iota_d^{(k)}(\boldsymbol{\sigma}) = 2\,\mathrm{vol}_{d-1}(\boldsymbol{\sigma}\cap K_d)/\|\boldsymbol{\eta}\|$. ■

Remark 225. In terms of invariant measures (see Lemma 293), we obtain

$$\lim_{k\to(-1)^+}(1+k)\,v_d^{(k)}(K_d) = \frac{\omega_d\,d!}{\mathrm{vol}_d^d K_d} \int_{\mathbb{A}(d,d-1)} \mathrm{vol}_{d-1}^{d+1}(\boldsymbol{\sigma}\cap K_d)\,\mu_{d-1}(\mathrm{d}\boldsymbol{\sigma}). \tag{4.26}$$

4.2.2 Symmetries and parametrization of configurations

By affine invariancy of volumetric moments and when $K_d = P_d$ is a polytope, we may take advantage of its symmetries (see Appendix C) to obtain

$$v_d^{(k)}(P_d) = \sum_{C\in\mathcal{C}(P_d)} w_{\mathrm{C}}\,v_d^{(k)}(P_d)_{\mathrm{C}}, \tag{4.27}$$

where the sum is carried over all representants C in the set of all equivalence classes $\mathcal{C}(P_d)$ of selections of vertices of P_d which could be separated by some section plane $\boldsymbol{\sigma}$ and which are equivalent under affine transformations (section-equivalent configurations). The weight w_{C} then represents the size of the orbit of C (see example of $\mathcal{C}(O_3)$ in Table C.6). Lastly,

$$v_d^{(k)}(P_d)_{\mathrm{C}} = \frac{(d-1)!}{d^k} \int_{(\mathbb{R}^d\setminus P_d^\circ)_{\mathrm{C}}} v_{d-1}^{(k+1)}(\boldsymbol{\sigma}\cap K_d)\,\zeta_d^{k+d+1}(\boldsymbol{\sigma})\iota_d^{(k)}(\boldsymbol{\sigma})\lambda_d(\mathrm{d}\boldsymbol{\eta}), \tag{4.28}$$

where $(\mathbb{R}^d \setminus P_d^\circ)_{\mathrm{C}}$ is the subset of $\mathbb{R}^d \setminus P_d^\circ$ of all $\boldsymbol{\eta}$-parametrisations of planes $\boldsymbol{\sigma}$ which only cut out vertices found in the given configuration C. It may seem that finding the precise integration domains $(\mathbb{R}^d \setminus P_d^\circ)_{\mathrm{C}}$ for various configurations is complicated. In fact, it is relatively easy. Recall that a configuration $C = P_d(S)$ is defined by the property of $\boldsymbol{\sigma}$ separating some given vertices from the set S out of the set of all vertices V of the polytope P_d. The domain $(\mathbb{R}^d \setminus P_d^\circ)_{\mathrm{C}}$ in $(\eta_1, \ldots, \eta_d)^\top$ is then the unique solution of the following inequalities

$$\boldsymbol{\eta}^\top \mathbf{v} < 1 \text{ for all } \mathbf{v} \in S, \qquad \boldsymbol{\eta}^\top \mathbf{v} > 1 \text{ for all } \mathbf{v} \in V \setminus S \tag{4.29}$$

or inequalities with $<, >$ flipped (we then take the union of those two options). Note that $\boldsymbol{\sigma}$ always separates P_d into disjoint union $P_d^+ \sqcup P_d^-$, where

$$P_d^+ = \{\mathbf{x} \in P_d \,|\, \boldsymbol{\eta}^\top \mathbf{x} < 1\}, \qquad P_d^- = \{\mathbf{x} \in P_d \,|\, \boldsymbol{\eta}^\top \mathbf{x} > 1\}. \tag{4.30}$$

We have $\mathrm{vol}_d\, P_d = \mathrm{vol}_d\, P_d^+ + \mathrm{vol}_d\, P_d^-$ trivially.

Remark 226. *Fundamental Lemma of Convex Geometry* tells us that a polytope is described equivalently either by linear inequalities or as a convex hull of its vertices (H- and V- representation equivalence). Hence, for example by linear programming techniques, we can deduce the vertices of P_d^+ from the inequalities for P_d^+ and vice versa. The same applies for the polytope $\boldsymbol{\sigma} \cap P_d$ whose number of vertices is n_{C} by definition.

4.2.3 Iota function splitting

Splitting P_d into $P_d^+ \sqcup P_d^-$ integration domains, we obtain that the computation of $\iota_d^{(k)}(\boldsymbol{\sigma})$ is also straightforward as

$$\iota_d^{(k)}(\boldsymbol{\sigma}) = \int_{P_d^+} (1 - \boldsymbol{\eta}^\top \mathbf{x})^k \lambda_d(\mathrm{d}\mathbf{x}) + \int_{P_d^-} (\boldsymbol{\eta}^\top \mathbf{x} - 1)^k \lambda_d(\mathrm{d}\mathbf{x}) \tag{4.31}$$

for any real $k > -1$. When k is an integer, let us denote

$$\iota_d^{(k)}(\boldsymbol{\sigma})_{\mathrm{N}} = \int_{P_d} (\boldsymbol{\eta}^\top \mathbf{x} - 1)^k \lambda_d(\mathrm{d}\mathbf{x}), \tag{4.32}$$

then, when k is even, we have $\iota_d^{(k)}(\boldsymbol{\sigma}) = \iota_d^{(k)}(\boldsymbol{\sigma})_{\mathrm{N}}$. For any general integer k, we get by inclusion/exclusion

$$\begin{aligned} \iota_d^{(k)}(\boldsymbol{\sigma}) &= \iota_d^{(k)}(\boldsymbol{\sigma})_{\mathrm{N}} - (1 - (-1)^k) \int_{P_d^+} (\boldsymbol{\eta}^\top \mathbf{x} - 1)^k \lambda_d(\mathrm{d}\mathbf{x}) \\ &= (-1)^k \iota_d^{(k)}(\boldsymbol{\sigma})_{\mathrm{N}} + (1 - (-1)^k) \int_{P_d^-} (\boldsymbol{\eta}^\top \mathbf{x} - 1)^k \lambda_d(\mathrm{d}\mathbf{x}). \end{aligned} \tag{4.33}$$

4.2.4 Geometric interpretation of iota

Let $\mathbf{M}$, $\mathbf{M}^+$ and $\mathbf{M}^-$ be the centerpoint (centre of mass) of P_d, P_d^+ and P_d^-, respectively. By mass balance, those centrepoints satisfy the vectorial equation

$$\mathbf{M}\, \mathrm{vol}_d\, P_d = \mathbf{M}^+\, \mathrm{vol}_d\, P_d^+ + \mathbf{M}^-\, \mathrm{vol}_d\, P_d^-. \tag{4.34}$$

On the other hand, by the definition of the centrepoints, $\int_{P_d} \mathbf{x}\lambda_d(\mathrm{d}\mathbf{x}) = \mathbf{M} \operatorname{vol}_d P_d$ (similarly for $\mathbf{M}^+$ and $\mathbf{M}^-$). Hence, we get for the iota function by Equations (4.31), (4.32) and (4.33) that $\iota_d^{(1)}(\boldsymbol{\sigma})_{\mathrm{N}} = (\boldsymbol{\eta}^\top \mathbf{M} - 1) \operatorname{vol}_d P_d$ and

$$\begin{aligned} \iota_d^{(1)}(\boldsymbol{\sigma}) &= (1 - \boldsymbol{\eta}^\top \mathbf{M}^+) \operatorname{vol}_d P_d^+ + (\boldsymbol{\eta}^\top \mathbf{M}^- - 1) \operatorname{vol}_d P_d^-, \\ &= (\boldsymbol{\eta}^\top \mathbf{M} - 1) \operatorname{vol}_d P_d + 2(1 - \boldsymbol{\eta}^\top \mathbf{M}^+) \operatorname{vol}_d P_d^+ \\ &= (1 - \boldsymbol{\eta}^\top \mathbf{M}) \operatorname{vol}_d P_d + 2(\boldsymbol{\eta}^\top \mathbf{M}^- - 1) \operatorname{vol}_d P_d^- \end{aligned} \tag{4.35}$$

For higher values of k, we are no longer able to express $\iota_d^{(k)}(\boldsymbol{\sigma})$ using centrepoints. However, we can always express it in terms of geometric quantities (see Lemma 222)

4.2.5 Zeta section function

Lastly, note that $\zeta_d(\boldsymbol{\sigma})$ is a rational function of $\boldsymbol{\eta}$. To see this, we know that $\operatorname{vol}_d P_d^+$ is a rational function in $(\eta_1, \dots, \eta_d)^\top$. From homogeneity (Remark 297),

$$\zeta_d(\boldsymbol{\sigma}) = -\frac{1}{\operatorname{vol}_d P_d} \sum_{j=1}^d \eta_j \frac{\partial \operatorname{vol}_d P_d^+}{\partial \eta_j} = \frac{1}{\operatorname{vol}_d P_d} \sum_{j=1}^d \eta_j \frac{\partial \operatorname{vol}_d P_d^-}{\partial \eta_j} \tag{4.36}$$

which is also rational since differentiation preserves rationality. Note that, denoting $\Gamma_d^+(\boldsymbol{\sigma}) = \operatorname{vol}_d P_d^- / \operatorname{vol}_d P_d$ and $\Gamma_d^-(\boldsymbol{\sigma}) = \operatorname{vol}_d P_d^- / \operatorname{vol}_d P_d$, we can write

$$\zeta_d(\boldsymbol{\sigma}) = -\sum_{j=1}^d \eta_j \frac{\partial \Gamma_d^+(\boldsymbol{\sigma})}{\partial \eta_j} = \sum_{j=1}^d \eta_j \frac{\partial \Gamma_d^-(\boldsymbol{\sigma})}{\partial \eta_j}. \tag{4.37}$$

Alternatively, $\zeta_d(r\boldsymbol{\sigma}) = r\frac{\partial}{\partial r}\Gamma_d^+(r\boldsymbol{\sigma}) = -r\frac{\partial}{\partial r}\Gamma_d^-(r\boldsymbol{\sigma})$.

4.2.6 Line distance moments

Consider a trivial example of $v_1^{(k)}(T_1)$, that is the k-th moment of a random line length. Parametrising $\boldsymbol{\eta} = (a)^\top$, $a > 1$, we get $\zeta_1(\boldsymbol{\sigma}) = 1/a$,

$$\iota_1^{(k)}(\boldsymbol{\sigma}) = \int_0^1 |ax - 1|^k \mathrm{d}x = \frac{(a-1)^{k+1} + 1}{a(1+k)} \tag{4.38}$$

and thus by Theorem 221 with $\mathbb{R}^1 \setminus T_1^\circ = (1, \infty)$ and $\lambda_1(\mathrm{d}\boldsymbol{\eta}) = \mathrm{d}a$,

$$v_1^{(k)}(T_1) = \int_1^\infty \frac{(a-1)^{k+1} + 1}{a^{k+3}(k+1)} \,\mathrm{d}a = \frac{2}{(1+k)(2+k)}. \tag{4.39}$$

4.3 Two dimensions

4.3.1 Triangle area moments

As a toy model, which already includes the Sylvester problem as its special case, is the derivation of volumetric moments $v_2^{(k)}(T_2)$ from the canonical section integral formula (Theorem 221). We obtain values shown in Table 4.7. Note that we already obtained those values in Chapter 1 via the Crofton Reduction Technique (Table 1.13).

k	0	1	2	3	4	5	6	7	8	9
$v_2^{(k)}(T_2)$	1	$\frac{1}{12}$	$\frac{1}{72}$	$\frac{31}{9000}$	$\frac{1}{900}$	$\frac{1063}{2469600}$	$\frac{403}{2116800}$	$\frac{211}{2268000}$	$\frac{13}{2646000}$	$\frac{2593}{93915360}$

Table 4.7: Volumetric moments $v_2^{(k)}(T_2)$ (triangle area moments)

First, from affine invariancy, $v_2^{(k)}(T_2)$ must be the same as $v_2^{(k)}(\mathbb{T}_2)$, where

$$\mathbb{T}_2 = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2) = \operatorname{conv}([0,0],[1,0],[0,1]) \tag{4.40}$$

is the canonical triangle. Trivialy, or by Proposition 276, we have $\operatorname{vol}_2 \mathbb{T}_2 = 1/2! = 1/2$. Let $\boldsymbol{\eta} = (a,b)^\top$ be the Cartesian parametrisation of the line $\boldsymbol{\sigma} \in \mathbb{A}(2,1)$ such that $\mathbf{x} \in \boldsymbol{\sigma} \Leftrightarrow \boldsymbol{\eta}^\top \mathbf{x} = 1$. We have $\|\boldsymbol{\eta}\| = \sqrt{a^2+b^2}$. Based on symmetries $\mathcal{G}(T_2)$, there is only one realisable configuration. Moreover, thanks to affine invariancy, we can consider the only configuration I in $\mathcal{C}(\mathbb{T}_2)$. Table 4.8 shows specifically which sets S of vertices are separated by a cutting plane $\boldsymbol{\sigma}$. The corresponding configurations in T_2 are shown in Figure 4.3.

C	I
S	$[0,0]$
w_C	3

Table 4.8: Configurations $\mathcal{C}(\mathbb{T}_2)$.

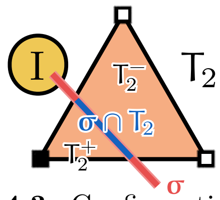


Figure 4.3: Configurations $\mathcal{C}(T_2)$

By Theorem 221 and for any $\mathrm{C} \in \mathcal{C}(\mathbb{T}_2)$,

$$v_2^{(k)}(\mathbb{T}_2)_\mathrm{C} = \frac{1}{2^k} \int_{(\mathbb{R}^2 \setminus \mathbb{T}_2^\circ)_\mathrm{C}} v_1^{(k+1)}(\boldsymbol{\sigma} \cap \mathbb{T}_2)\, \zeta_2^{k+3}(\boldsymbol{\sigma}) \iota_2^{(k)}(\boldsymbol{\sigma}) \lambda_2(\mathrm{d}\boldsymbol{\eta}), \tag{4.41}$$

where

$$\zeta_2(\boldsymbol{\sigma}) = \frac{\operatorname{vol}_1(\boldsymbol{\sigma} \cap \mathbb{T}_2)}{\|\boldsymbol{\eta}\| \operatorname{vol}_2 \mathbb{T}_2}, \qquad \iota_2^{(k)}(\boldsymbol{\sigma}) = \int_{\mathbb{T}_2} |\boldsymbol{\eta}^\top \mathbf{x} - 1|^k \lambda_2(\mathrm{d}\mathbf{x}). \tag{4.42}$$

To ensure $\boldsymbol{\sigma}$ separates only the point $[0,0]$ in Configuration I, we must force the plane intersection coordinates $\frac{1}{a}, \frac{1}{b}$ to lie in the interval $(0,1)$. Or, by Equation (4.29), we get $a > 1$ and $b > 1$ directly. Any way, that means $(\mathbb{R}^2 \setminus \mathbb{T}_2^\circ)_\mathrm{I} = (1,\infty)^2$ is our integration domain in a, b. See Figure 4.4.

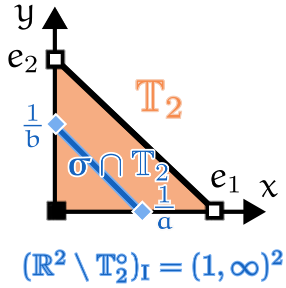


Figure 4.4: Configuration I in $\mathcal{C}(\mathbb{T}_2)$

Denote

$$\mathbb{T}_2^{ab} = \operatorname{conv}([0,0],[1/a,0],[0,1/b]). \tag{4.43}$$

The line $\boldsymbol{\sigma}$ splits $\mathbb{T}_2$ into disjoint union of two domains $\mathbb{T}_2^+ \sqcup \mathbb{T}_2^-$, where the one closer to the origin is precisely $\mathbb{T}_2^+ = \mathbb{T}_2^{ab}$. Therefore,

$$\iota_2^{(k)}(\boldsymbol{\sigma}) = \int_{\mathbb{T}_2^{ab}} (1-\boldsymbol{\eta}^\top \mathbf{x})^k \lambda_2(\mathrm{d}\mathbf{x}) + \int_{\mathbb{T}_2 \setminus \mathbb{T}_2^{ab}} (\boldsymbol{\eta}^\top \mathbf{x} - 1)^k \lambda_2(\mathrm{d}\mathbf{x}). \tag{4.44}$$

This integral is easy to compute. In fact, for any real $k > -1$, we get

$$\iota_2^{(k)}(\boldsymbol{\sigma}) = \frac{b(a-1)^{k+2} - a(b-1)^{k+2} + a - b}{ab(a-b)(1+k)(2+k)}. \tag{4.45}$$

Note that $\boldsymbol{\sigma} \cap \mathbb{T}_2 = \operatorname{conv}([1/a,0],[0,1/b])$ and thus

$$\operatorname{vol}_1(\boldsymbol{\sigma} \cap \mathbb{T}_2) = \frac{\sqrt{a^2+b^2}}{ab} = \frac{\|\boldsymbol{\eta}\|}{ab} \tag{4.46}$$

and hence

$$\zeta_2(\boldsymbol{\sigma}) = \frac{\operatorname{vol}_1(\boldsymbol{\sigma} \cap \mathbb{T}_2)}{\|\boldsymbol{\eta}\| \operatorname{vol}_2 \mathbb{T}_2} = \frac{2}{ab}. \tag{4.47}$$

Moreover, by affine invariancy of volumetric moments and using line distance moments (Equation (4.39)),

$$v_1^{(k+1)}(\boldsymbol{\sigma} \cap \mathbb{T}_2) = v_1^{(k+1)}(T_1) = \frac{2}{(2+k)(3+k)}. \tag{4.48}$$

Alternatively, we can obtain $\iota_2^{(k)}(\boldsymbol{\sigma})$ directly from $\zeta_2(\boldsymbol{\sigma})$. First, more generally and without the loss of generality assuming $a > b > 1$, we have for any $s \in (0,a)$,

$$\zeta_2(s\boldsymbol{\sigma}) = s\frac{\operatorname{vol}_1(s\boldsymbol{\sigma} \cap \mathbb{T}_2)}{\|\boldsymbol{\eta}\| \operatorname{vol}_2 \mathbb{T}_2} = \tfrac{2s^2}{ab} \mathbb{1}_{s<b} + \tfrac{2s(a-s)}{a(a-b)} \mathbb{1}_{b<s<a}, \tag{4.49}$$

from which. by Equation (4.22),

$$\iota_2^{(k)}(\boldsymbol{\sigma}) = \int_0^a \left(\tfrac{s}{ab} \mathbb{1}_{s<b} + \tfrac{a-s}{a(a-b)} \mathbb{1}_{b<s<a} \right) |s-1|^k \, \mathrm{d}s. \tag{4.50}$$

By Equation (C.118) and by affine invariancy,

$$v_2^{(k)}(T_2) = \sum_{C \in \mathcal{C}(T_2)} w_C \, v_2^{(k)}(T_2)_C = 3 v_2^{(k)}(\mathbb{T}_2)_{\mathrm{I}}, \tag{4.51}$$

from which, we get by Equation (4.41) for any real $k > -1$,

$$v_2^{(k)}(T_2) = 48 \int_1^\infty \int_1^\infty \frac{b(a-1)^{2+k} - a(b-1)^{2+k} + a - b}{a^{k+4} b^{k+4} (a-b)(1+k)(2+k)^2(3+k)} \, \mathrm{d}a \mathrm{d}b. \tag{4.52}$$

Let $a = 1/x$ and $b = 1/y$, then, after some simple manipulations,

$$v_2^{(k)}(T_2) = \frac{48}{(1+k)(2+k)^2(3+k)} \int_0^1 \int_0^1 \frac{(1-x)^{2+k}\left(x^{2+k} - y^{2+k}\right)}{x-y} \, \mathrm{d}x \, \mathrm{d}y \tag{4.53}$$

for any real $k > -1$. This integral can be computed explicitly when k is an integer. Dividing the numerator by $x - y$, we get

$$\begin{aligned} v_2^{(k)}(T_2) &= \frac{48}{(1+k)(2+k)^2(3+k)} \sum_{j=0}^{k+1} \int_0^1 \int_0^1 (1-x)^{2+k} x^{k-j+1} y^j \, \mathrm{d}x \, \mathrm{d}y \\ &= \frac{48}{(1+k)(2+k)^2(3+k)} \sum_{j=0}^{k+1} \frac{1}{j+1} \int_0^1 (1-x)^{2+k} x^j \, \mathrm{d}x, \end{aligned} \tag{4.54}$$

which is, of course, a Beta integral. Therefore, for any non-negative integer k,

$$v_2^{(k)}(T_2) = \frac{48}{(2+k)(3+k)} \sum_{j=0}^{k+1} \frac{k!\,(k+1-j)!}{(j+1)(2k-j+4)!}. \tag{4.55}$$

Alternatively, note that the integral

$$I_k = \int_0^1 \int_0^1 \frac{x^{2+k}(1-x)^{2+k} - y^{2+k}(1-y)^{2+k}}{x-y} \, \mathrm{d}x \mathrm{d}y \tag{4.56}$$

vanishes, since by substitution $x \to 1-x$ and $y \to 1-y$, we get $-I_k$. Hence, subtracting half of I_k from the integral in Equation (4.53) and by symmetry,

$$v_2^{(k)}(T_2) = \frac{24}{(1+k)(2+k)^2(3+k)} \int_0^1 \int_0^1 \frac{(x-xy)^{2+k} - (y-yx)^{2+k}}{x-y} \, \mathrm{d}x \, \mathrm{d}y. \tag{4.57}$$

Rewriting the numerator using the formula $A^{2+k} - B^{2+k} = \sum_{j=0}^{k+1} A^j B^{k+1-j}$,

$$v_2^{(k)}(T_2) = \frac{24}{(1+k)(2+k)^2(3+k)} \sum_{j=0}^{k+1} \left(\int_0^1 x^j (1-x)^{k+1-j} \, \mathrm{d}x \right)^2. \tag{4.58}$$

which is another Beta integral. Therefore, for any non-negative integer k,

$$v_2^{(k)}(T_2) = \frac{24}{(1+k)(2+k)^2(3+k)} \sum_{j=0}^{k+1} \frac{j!^2 (k+1-j)!^2}{(k+2)!^2}. \tag{4.59}$$

The result for $v_2^{(k)}(T_2)$ is not new, in fact, it has been derived several times, see Reed [59], Mathai [46, p. 391] or Alagar [2]. Finally, let us mention that

that the particular case of even moments is easy to obtain independently also by integrating even powers of the area over the unit triangle, see Figure 4.5 below.

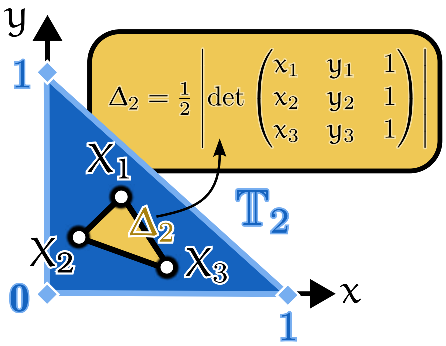


Figure 4.5: Random triangle area Δ_2 written as a determinant

In general, writing the expectation as an integral, we have for even k and $\mathbf{x}_i = (x_i, y_i)^\top, i = 1, 2, 3,$

$$v_2^{(k)}(\mathbb{T}_2) = 2^{k+3} \int_{\mathbb{T}_2^3} \Delta_2^k \, \mathrm{d}\mathbf{x}_0 \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2. \tag{4.60}$$

Density

The density can be recovered from moments using inverse Mellin transform (see appendix A.5). For the probability density $f(s)$ of the random variable $S = \underline{\Delta}_2 = \Delta_2 / \operatorname{vol}_2 T_2$, we have by Equation (4.57)

$$\mathcal{M}[f] = v_2^{(k-1)}(T_2) = \frac{24}{k(1+k)^2(2+k)} \int_0^1 \int_0^1 \frac{(x - xy)^{1+k} - (y - yx)^{1+k}}{x - y} \, \mathrm{d}x \, \mathrm{d}y, \tag{4.61}$$

so formally,

$$\begin{aligned} f(s) &= 24\, \mathcal{I}_0 \mathcal{I}_1^2 \mathcal{I}_2 \mathcal{M}^{-1}\left[\int_0^1 \int_0^1 \frac{(x - xy)^{1+k} - (y - yx)^{1+k}}{x - y} \, \mathrm{d}x \, \mathrm{d}y\right] \\ &= 24\, \mathcal{I}_0 \mathcal{I}_1^2 \mathcal{I}_2 \int_0^1 \int_0^1 \frac{x^2(1-y)^2 \delta(s - x(1-y)) - y^2(1-x)^2 \delta(s - y(1-x))}{x - y} \mathrm{d}x \mathrm{d}y. \end{aligned} \tag{4.62}$$

From Table A.5 (see Appendix A),

$$\mathcal{I}_0 \mathcal{I}_1^2 \mathcal{I}_2 \delta(s - \alpha) = \frac{\alpha^2 - s^2 - 2\alpha s \ln \frac{\alpha}{s}}{2\alpha^3} \mathbb{1}_{s<\alpha}. \tag{4.63}$$

via which we can deduce, with $\alpha = x(1 - y)$ and $\alpha = y(1 - x)$,

$$\begin{aligned} f(s) = 12 \int_0^1 \int_0^1 & \frac{(1 - y)^2 x^2 - s^2 - 2s(1 - y)x \ln \frac{(1-y)x}{s}}{x(1 - y)(x - y)} \mathbb{1}_{s<x(1-y)} \\ & - \frac{(1 - x)^2 y^2 - s^2 - 2s(1 - x)y \ln \frac{(1-x)y}{s}}{y(1 - x)(x - y)} \mathbb{1}_{s<y(1-x)} \, \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{4.64}$$

We can deduce that $f(s)$ is nonzero only when $s \in (0, 1)$. Evaluating this integral is cumbersome. After a lot of simplifications, we arrive at the same formula as derived in Chapter 1 on Crofton Reduction Technique (Equation (1.345)) namely

$$f(s) = \begin{cases} \left\{ \begin{array}{c} 12(1-s) - 6(1+24s+6s\ln s)\ln s \\ -12(1+26s)\sqrt{1-4s}\,\mathrm{argtanh}\,\sqrt{1-4s} \\ -144s(1+s)(\frac{\pi^2}{9} - \mathrm{argtanh}^2\sqrt{1-4s}) \end{array} \right\}, & 0 < s < 1/4, \\ \left\{ \begin{array}{c} 12(1-s) - 6(1+24s+6s\ln s)\ln s \\ -12(1+26s)\sqrt{4s-1}(\frac{\pi}{3} - \arctan\sqrt{4s-1}) \\ -144s(1+s)(\frac{\pi}{3} - \arctan\sqrt{4s-1})^2 \end{array} \right\}, & 1/4 \le s < 1. \end{cases} \tag{4.65}$$

4.3.2 Square area moments

As another example, we deduce the volumetric moments $v_2^{(k)}(C_2)$ from Theorem 221. We obtain values shown in Table 4.9.

k	1	2	3	4	5	6	7	8	9
$v_2^{(k)}(C_2)$	$\frac{11}{144}$	$\frac{1}{96}$	$\frac{137}{72000}$	$\frac{1}{2400}$	$\frac{363}{3512320}$	$\frac{761}{27095040}$	$\frac{7129}{870912000}$	$\frac{61}{24192000}$	$\frac{83711}{103038566400}$

Table 4.9: Volumetric moments $v_2^{(k)}(C_2)$ (square area moments)

We may parametrise C_2 with $\mathrm{vol}_2\, C_2 = 1$ as

$$C_2 = \mathrm{conv}([0,0],[1,0],[0,1],[1,1]), \tag{4.66}$$

Let $\boldsymbol{\eta} = (a,b)^\top$ be the Cartesian parametrisation of the line $\boldsymbol{\sigma} \in \mathbb{A}(2,1)$ such that $\mathbf{x} \in \boldsymbol{\sigma} \Leftrightarrow \boldsymbol{\eta}^\top \mathbf{x} = 1$. We have $\|\boldsymbol{\eta}\| = \sqrt{a^2+b^2}$. Based on symmetries $\mathcal{G}(C_2)$, there are two configurations. Table 4.10 shows specifically which sets S of vertices are separated by a cutting plane $\boldsymbol{\sigma}$ in which configurations in our local representation of C_2 above. Note that there is an ambiguity how to select those vertices as long it is the same configuration.

C	I	II
S	$[0,0]$	$[0,0]$ $[0,1]$
w_{C}	4	2

Table 4.10: Configurations $\mathcal{C}(C_2)$ in a local representation.

By Theorem 221 and for any $\mathrm{C} \in \mathcal{C}(C_2)$,

$$v_2^{(k)}(C_2)_{\mathrm{C}} = \frac{1}{2^k} \int_{(\mathbb{R}^2 \setminus C_2^\circ)_{\mathrm{C}}} v_1^{(k+1)}(\boldsymbol{\sigma} \cap C_2)\, \zeta_2^{k+3}(\boldsymbol{\sigma}) \iota_2^{(k)}(\boldsymbol{\sigma}) \lambda_2(\mathrm{d}\boldsymbol{\eta}), \tag{4.67}$$

where

$$\zeta_2(\boldsymbol{\sigma}) = \frac{\mathrm{vol}_1(\boldsymbol{\sigma} \cap C_2)}{\|\boldsymbol{\eta}\|\, \mathrm{vol}_2\, C_2}, \qquad \iota_2^{(k)}(\boldsymbol{\sigma}) = \int_{C_2} |\boldsymbol{\eta}^\top \mathbf{x} - 1|^k \lambda_2(\mathrm{d}\mathbf{x}). \tag{4.68}$$

Configuration I

By Equation (4.29), we get the following set of inequalities which ensure $\boldsymbol{\sigma}$ separates only the point $[0,0]$,

$$0 < 1, \qquad a > 1, \qquad b > 1, \qquad a+b > 1, \tag{4.69}$$

hence, our a, b integration domain is $(\mathbb{R}^2 \setminus C_2^\circ)_\mathrm{I} = (1,\infty)^2$. Denote

$$\mathbb{T}_2^{ab} = \operatorname{conv}([0,0],[1/a,0],[0,1/b]), \tag{4.70}$$

then the line $\boldsymbol{\sigma}$ splits C_2 into disjoint union of two domains $C_2^+ \sqcup C_2^-$, where the one closer to the origin is precisely $C_2^+ = \mathbb{T}_2^{ab}$. Therefore,

$$\iota_2^{(k)}(\boldsymbol{\sigma}) = \int_{\mathbb{T}_2^{ab}} (1-\boldsymbol{\eta}^\top \mathbf{x})^k \lambda_2(\mathrm{d}\mathbf{x}) + \int_{C_2\setminus\mathbb{T}_2^{ab}} (\boldsymbol{\eta}^\top \mathbf{x}-1)^k \lambda_2(\mathrm{d}\mathbf{x}). \tag{4.71}$$

This integral is easy to compute. In fact, for any real $k > -1$, we get

$$\iota_2^{(k)}(\boldsymbol{\sigma}) = \frac{(a+b-1)^{k+2}-(a-1)^{k+2}-(b-1)^{k+2}+1}{ab(k+1)(k+2)}. \tag{4.72}$$

By Equation (4.47) from the $P_2 = \mathbb{T}_2$ case,

$$\zeta_2(\boldsymbol{\sigma}) = \frac{\operatorname{vol}_1(\boldsymbol{\sigma}\cap C_2)}{\|\boldsymbol{\eta}\| \operatorname{vol}_2 C_2} = \frac{1}{ab} \tag{4.73}$$

and by affine invariancy, as $\boldsymbol{\sigma}\cap C_2$ is a line segment,

$$\upsilon_1^{(k+1)}(\boldsymbol{\sigma}\cap C_2) = \upsilon_1^{(k+1)}(T_1) = \frac{2}{(2+k)(3+k)}. \tag{4.74}$$

from which, we get by Equation (4.67) for any real $k > -1$,

$$\upsilon_2^{(k)}(C_2)_\mathrm{I} = 2^{1-k}\int_1^\infty\int_1^\infty \frac{(a+b-1)^{k+2}-(a-1)^{k+2}-(b-1)^{k+2}+1}{a^{k+4}b^{k+4}(1+k)(2+k)^2(3+k)}\,\mathrm{d}a\,\mathrm{d}b. \tag{4.75}$$

Integrating out b and substituting $a = 1/x$ and after some simplifications, we get

$$\upsilon_2^{(k)}(C_2)_\mathrm{I} = \frac{2^{1-k}}{(1+k)(2+k)^2(3+k)^2}\int_0^1 \frac{1-x^{2+k}}{1-x}\,\mathrm{d}x, \tag{4.76}$$

for any real $k > -1$. When k is an integer, we get

$$\upsilon_2^{(k)}(C_2)_\mathrm{I} = \frac{16 H_{k+2}}{(1+k)(2+k)^2(3+k)^2}, \tag{4.77}$$

where $H_k = \sum_{j=1}^k 1/j$ is the k-th *harmonic number*.

Configuration II

By Equation (4.29), we get the following set of inequalities which ensure $\boldsymbol{\sigma}$ separates points $[0,0]$ and $[0,1]$,

$$0<1, \qquad a>1, \qquad b<1, \qquad a+b>1, \tag{4.78}$$

however, by symmetry, we may additionally require $b>0$. In fact, both options $b>0$ and $b<0$ give the same factor since they correspond to two possibilities where $\boldsymbol{\sigma}$ hits $\mathcal{A}([0,0],[0,1])$. Therefore we only consider the following integration half-domain (indicated by $*$)

$$(\mathbb{R}^2 \setminus C_2^\circ)_{\mathrm{II}}^* = (1,\infty)\times(0,1) \tag{4.79}$$

and in the end multiply the result twice. The plane $\boldsymbol{\sigma}$ splits C_3 into disjoint union of two domains $C_3^+ \sqcup C_3^-$, where the one closer to the origin can be described as

$$C_2^+ = \operatorname{conv}\left([0,0], \left[\frac{1}{a},0\right], \left[\frac{1-b}{a},1\right], [0,1]\right), \tag{4.80}$$

from which, by elementary geometry $\operatorname{vol}_2 C_2^+ = (2-b)/(2a)$ and as a consequence of Equation (4.36),

$$\zeta_2(\boldsymbol{\sigma}) = -a\frac{\partial}{\partial a}\left(\frac{2-b}{2a}\right) - b\frac{\partial}{\partial b}\left(\frac{2-b}{2a}\right) = \frac{1}{a}. \tag{4.81}$$

Next, again, the following integrals

$$\iota_2^{(k)}(\boldsymbol{\sigma}) = \int_{C_2^+}(1-\boldsymbol{\eta}^\top\mathbf{x})^k\lambda_2(\mathrm{d}\mathbf{x}) + \int_{C_2\setminus C_2^+}(\boldsymbol{\eta}^\top\mathbf{x}-1)^k\lambda_2(\mathrm{d}\mathbf{x}). \tag{4.82}$$

are easy to compute for any real $k>-1$, we get

$$\iota_2^{(k)}(\boldsymbol{\sigma}) = \frac{(a+b-1)^{k+2}-(a-1)^{k+2}-(1-b)^{k+2}+1}{ab(k+1)(k+2)}. \tag{4.83}$$

and by affine invariancy, as $\boldsymbol{\sigma}\cap C_2$ is again a line segment,

$$v_1^{(k+1)}(\boldsymbol{\sigma}\cap C_2) = v_1^{(k+1)}(T_1) = \frac{2}{(2+k)(3+k)}. \tag{4.84}$$

from which, we get by Equation (4.67) for any real $k>-1$ (counted twice!),

$$v_2^{(k)}(C_2)_{\mathrm{II}} = \frac{4}{2^k}\int_0^\infty\!\!\int_1^\infty \frac{(a+b-1)^{k+2}-(a-1)^{k+2}-(1-b)^{k+2}+1}{a^{k+4}b(1+k)(2+k)^2(3+k)}\,\mathrm{d}a\,\mathrm{d}b. \tag{4.85}$$

Integrating out a and after some simplifications, we get

$$v_2^{(k)}(C_2)_{\mathrm{II}} = \frac{2^{3-k}}{(1+k)(2+k)^2(3+k)^2}\int_0^1 \frac{1-b^{2+k}}{1-b}\,\mathrm{d}b, \tag{4.86}$$

for any real $k>-1$. When k is an integer, we get

$$v_2^{(k)}(C_2)_{\mathrm{II}} = \frac{2^{3-k}H_{k+2}}{(1+k)(2+k)^2(3+k)^2}. \tag{4.87}$$

Contribution from all configurations

By Equation (C.118),

$$v_2^{(k)}(C_2) = \sum_{C\in\mathcal{C}(C_2)} w_{\mathrm{C}}\, v_2^{(k)}(C_2)_{\mathrm{C}} = 4v_2^{(k)}(C_2)_{\mathrm{I}} + 2v_2^{(k)}(C_2)_{\mathrm{II}}, \tag{4.88}$$

which gives for any real $k > -1$,

$$v_2^{(k)}(C_2) = \frac{24}{2^k(1+k)(2+k)^2(3+k)^2}\int_0^1 \frac{1-x^{k+2}}{1-x}\,\mathrm{d}x. \tag{4.89}$$

For k being an integer, we get

$$v_2^{(k)}(C_2) = \frac{24H_{k+2}}{2^k(1+k)(2+k)^2(3+k)^2} = \frac{24\sum_{j=1}^{k+2}\frac{1}{j}}{2^k(1+k)(2+k)^2(3+k)^2}. \tag{4.90}$$

This result is also not new, see Reed [59] or Henze [35]. We can also deduce this result independently from the Canonical section integral by using Crofton Reduction Technique (see Section 1.6.2 in Chapter 1).

Density

The density can be recovered using inverse Mellin transform (see appendix A.5). For the density $f(s)$ of the random variable $S = \underline{\Delta}_2 = \Delta_2/\operatorname{vol}_2 C_2$, we have by Equation (4.89)

$$\mathcal{M}[f] = v_2^{(k-1)}(C_2) = \frac{24}{2^{k-1}k(1+k)^2(2+k)^2}\int_0^1 \frac{1-x^{k+1}}{1-x}\,\mathrm{d}x, \tag{4.91}$$

so formally,

$$f(s) = 24\,\mathcal{I}_0\mathcal{I}_1^2\mathcal{I}_2^2\mathcal{M}^{-1}\left[\int_0^1 \frac{1-x^{k+1}}{2^{k-1}(1-x)}\mathrm{d}x\right] = 24\,\mathcal{I}_0\mathcal{I}_1^2\mathcal{I}_2^2\int_0^1 \frac{\delta(s-\frac{1}{2})-x^2\delta(s-\frac{x}{2})}{1-x}\mathrm{d}x. \tag{4.92}$$

From Table A.5 (see Appendix A),

$$\mathcal{I}_0\mathcal{I}_1^2\mathcal{I}_2^2\delta(s-\alpha) = \frac{(\alpha-s)(\alpha+5s)-2s(2\alpha+s)\ln\frac{\alpha}{s}}{4\alpha^3}\mathbb{1}_{s<\alpha}. \tag{4.93}$$

via which we can deduce, with $\alpha = 1/2$ and $\alpha = x/2$,

$$\begin{aligned} f(s) = 12\int_0^1 &\frac{1-20s^2+8s-8(s+1)s\ln\frac{1}{2s}}{1-x}\mathbb{1}_{s<\frac{1}{2}} \\ &- \frac{x^2-20s^2+8sx-8s(s+x)\ln\frac{x}{2s}}{x(1-x)}\mathbb{1}_{s<\frac{x}{2}}\,\mathrm{d}x. \end{aligned} \tag{4.94}$$

We can deduce that $f(s)$ is nonzero only when $s \in (0, 1/2)$. Calculating the integral (for example, using Mathematica),

$$\begin{aligned} f(s) = {} & 12\left(1-2s-25s^2\right) - 16\pi^2 s(1+s) + 12s^2(5-2\ln(2s))^2 \\ & - 12\left(1+8s-20s^2\right)\ln(1-2s) + 96s(1+s)\operatorname{Li}_2(2s), \end{aligned} \tag{4.95}$$

where $\operatorname{Li}_2(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^2}$ is the dilogarithm function. This result is not new, see Philip [55].

4.3.3 General 2-body area moments

Note that for any convex 2-body K_2, we have thanks to affine invariancy,

$$v_1^{(k+1)}(\boldsymbol{\sigma} \cap K_2) = v_1^{(k+1)}(T_1) = \frac{2}{(2+k)(3+k)}, \tag{4.96}$$

provided of course $\boldsymbol{\sigma} \cap K_2 \neq \emptyset$. Hence by Theorem 221

$$v_2^{(k)}(K_2) = \frac{2^{1-k}}{(2+k)(3+k)} \int_{\mathbb{R}^2 \setminus K_2^\circ} \zeta_2^{k+3}(\boldsymbol{\sigma}) \iota_2^{(k)}(\boldsymbol{\sigma}) \lambda_2(\mathrm{d}\boldsymbol{\eta}), \tag{4.97}$$

where

$$\zeta_2(\boldsymbol{\sigma}) = \frac{\operatorname{vol}_1(\boldsymbol{\sigma} \cap K_2)}{\|\boldsymbol{\eta}\| \operatorname{vol}_2 K_2}, \qquad \iota_2^{(k)}(\boldsymbol{\sigma}) = \int_{K_2} |\boldsymbol{\eta}^\top \mathbf{x} - 1|^k \lambda_2(\mathrm{d}\mathbf{x}). \tag{4.98}$$

Let us consider the special case when $k = 1$. By the geometrical interpretation of $\iota_2^{(1)}(\boldsymbol{\sigma})$ (Equation (4.35)),

$$\iota_2^{(1)}(\boldsymbol{\sigma}) = (1 - \boldsymbol{\eta}^\top \mathbf{M}^+) \operatorname{vol}_2 K_2^+ + (\boldsymbol{\eta}^\top \mathbf{M}^- - 1) \operatorname{vol}_2 K_2^- \tag{4.99}$$

from which

$$v_2^{(1)}(K_2) = \frac{\operatorname{vol}_2 K_2}{12} \int_{\mathbb{R}^2 \setminus K_2^\circ} \zeta_2^4(\boldsymbol{\sigma}) \Big((1 - \boldsymbol{\eta}^\top \mathbf{M}^+) \Gamma_2^+(\boldsymbol{\sigma}) + (\boldsymbol{\eta}^\top \mathbf{M}^- - 1) \Gamma_2^-(\boldsymbol{\sigma}) \Big) \lambda_2(\mathrm{d}\boldsymbol{\eta}), \tag{4.100}$$

where $\Gamma_2^+(\boldsymbol{\sigma}) = \operatorname{vol}_2 K_2^+ / \operatorname{vol}_2 K_2$ and $\Gamma_2^-(\boldsymbol{\sigma}) = \operatorname{vol}_2 K_2^- / \operatorname{vol}_2 K_2$.

For higher moments, first, by polar coordinates, let $\boldsymbol{\eta} = \hat{\boldsymbol{\eta}}/q$ with $q \in (0, \infty)$ and $\hat{\boldsymbol{\eta}} \in \mathbb{S}^1$, so $\lambda_2(\mathrm{d}\boldsymbol{\eta}) = \frac{1}{q} \mathrm{d}q \sigma_2(\mathrm{d}\hat{\boldsymbol{\eta}})$. In a slight abuse of notation, we identity $\boldsymbol{\sigma}$ with its closest point $\boldsymbol{\xi}$ from the origin. Hence $\boldsymbol{\sigma} = q\hat{\boldsymbol{\eta}}$ and Equation (4.97) becomes

$$v_2^{(k)}(K_2) = \frac{2^{1-k}}{(2+k)(3+k)} \int_{\mathbb{S}^1} \int_0^\infty \zeta_2^{k+3}(q\hat{\boldsymbol{\eta}}) \iota_2^{(k)}(q\hat{\boldsymbol{\eta}}) \tfrac{1}{q^3} \mathrm{d}q \, \sigma_2(\mathrm{d}\hat{\boldsymbol{\eta}}), \tag{4.101}$$

Note that we can express $\iota_2^{(k)}(q\hat{\boldsymbol{\eta}})$ using the following geometric integral (By Equation (4.22))

$$\iota_2^{(k)}(q\hat{\boldsymbol{\eta}}) = \operatorname{vol}_2 K_2 \int_{-\infty}^\infty \frac{\zeta_2(rq\hat{\boldsymbol{\eta}})}{|r|} |r-1|^k \ \mathrm{d}r. \tag{4.102}$$

Therefore, we get

$$v_2^{(k)}(K_2) = \frac{2^{1-k} \operatorname{vol}_2 K_2}{(2+k)(3+k)} \int_{\mathbb{S}^1} \int_0^\infty \int_{-\infty}^\infty \zeta_2^{k+3}(q\hat{\boldsymbol{\eta}}) \zeta_2(rq\hat{\boldsymbol{\eta}}) \frac{|s-1|^k}{|r|q^3} \ \mathrm{d}r \ \mathrm{d}q \, \sigma_2(\mathrm{d}\hat{\boldsymbol{\eta}}), \tag{4.103}$$

from which we can deduce the formula for density by inverse Mellin transform.

Density

The density can be recovered using inverse Mellin transform. For its basic properties and techniques, see appendix A.5. In our case of two-dimensional bodies, we have for the density $f(s)$ of the random variable $S = \underline{\Delta}_2 = \Delta_2/\operatorname{vol}_2 K_2$,

$$\mathcal{M}[f] = v_2^{(k-1)}(K_2) = \frac{2^{2-k}}{(1+k)(2+k)} \int_{\mathbb{R}^2\setminus K_2^\circ} \zeta_2^{k+2}(\boldsymbol{\sigma})\iota_2^{(k-1)}(\boldsymbol{\sigma})\lambda_2(\mathrm{d}\boldsymbol{\eta}), \tag{4.104}$$

or by using Equation (4.103). We can write the formal inversion as

$$f(s) = \operatorname{vol}_2 K_2\, \mathcal{I}_1\mathcal{I}_2\mathcal{M}^{-1}\left[\int_{\mathbb{S}^1}\int_0^\infty\int_{-\infty}^\infty \zeta_2^{k+2}(q\hat{\boldsymbol{\eta}})\zeta_2(rq\hat{\boldsymbol{\eta}})\frac{|r-1|^{k-1}}{2^{k-2}|r|q^3}\,\mathrm{d}r\,\mathrm{d}q\,\sigma_2(\mathrm{d}\hat{\boldsymbol{\eta}})\right]. \tag{4.105}$$

From Table A.5 (see Appendix A),

$$\mathcal{I}_1\mathcal{I}_2\mathcal{M}^{-1}[\alpha^{k-1}] = \mathcal{I}_1\mathcal{I}_2\delta(s-\alpha) = s\alpha^{-3}(\alpha - s)\mathbb{1}_{s<\alpha}, \tag{4.106}$$

we immediately get with $\alpha = \frac{1}{2}|r-1|\zeta_2(q\hat{\boldsymbol{\eta}})$,

$$f(s) = 8s\operatorname{vol}_2 K_2 \int_{\mathbb{S}^1}\int_0^\infty\int_{-\infty}^\infty \zeta_2(rq\hat{\boldsymbol{\eta}})\frac{|r-1|\zeta_2(q\hat{\boldsymbol{\eta}})-2s}{|r|q^3|r-1|^3}\mathbb{1}_{s<\frac{1}{2}|r-1|\zeta_2(q\hat{\boldsymbol{\eta}})}\,\mathrm{d}r\,\mathrm{d}q\,\sigma_2(\mathrm{d}\hat{\boldsymbol{\eta}}). \tag{4.107}$$

Let us make a substitution $r = 1 + t/q$, we get

$$f(s) = 8s\operatorname{vol}_2 K_2 \int_{\mathbb{S}^1}\int_0^\infty\int_{-\infty}^\infty \zeta_2((q+t)\hat{\boldsymbol{\eta}})\frac{|t|\zeta_2(q\hat{\boldsymbol{\eta}})-2sq}{|q+t||t|^3 q}\mathbb{1}_{2sq<|t|\zeta_2(q\hat{\boldsymbol{\eta}})}\,\mathrm{d}t\,\mathrm{d}q\,\sigma_2(\mathrm{d}\hat{\boldsymbol{\eta}}). \tag{4.108}$$

4.4 Three dimensions

4.4.1 Tetrahedron odd volumetric moments

Let us investigate how we can obtain the volumetric moments $v_3^{(k)}(T_3)$. First, since $v_3^{(k)}(T_3)$ is an affine invariant, then it must be the same as $v_3^{(k)}(\mathbb{T}_3)$, where

$$\mathbb{T}_3 = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \operatorname{conv}([0,0,0],[1,0,0],[0,1,0],[0,0,1]) \tag{4.109}$$

is the canonical tetrahedron. By Proposition 276, we have $\operatorname{vol}_3 \mathbb{T}_3 = 1/3! = 1/6$. Let $\boldsymbol{\eta} = (a,b,c)^\top$ be the Cartesian parametrisation of $\boldsymbol{\sigma} \in \mathbb{A}(3,2)$ such that $\mathbf{x} \in \boldsymbol{\sigma} \Leftrightarrow \boldsymbol{\eta}^\top\mathbf{x} = 1$. We have $\|\boldsymbol{\eta}\| = \sqrt{a^2+b^2+c^2}$. Based on symmetries $\mathcal{G}(T_3)$, there are two realisable configurations we need to consider (see its genealogy at Figure 4.7 or Figure D.2 in Appendix D). Moreover, thanks to affine invariancy, we can consider instead the two $\mathcal{C}(\mathbb{T}_3)$ configurations (see Table 4.11 below, Figure shows the correspoding configurations on the non-deformed T_3).

C	I	II
S	$[0,0,0]$	$[0,0,0]$ $[0,0,1]$
w_{C}	4	3
n_{C}	3	4

Table 4.11: Configurations $\mathcal{C}(\mathbb{T}_3)$.

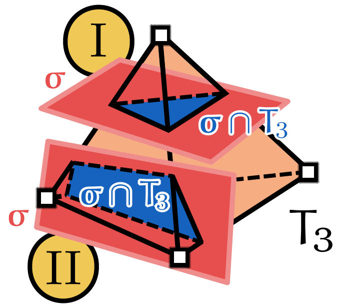


Figure 4.6: Configurations $\mathcal{C}(T_3)$

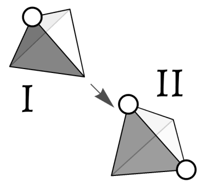


Figure 4.7: Tetrahedron genealogy

By Theorem 221 and for any $\mathrm{C} \in \mathcal{C}(\mathbb{T}_3)$,

$$v_3^{(k)}(\mathbb{T}_3)_{\mathrm{C}} = \frac{2}{3^k} \int_{(\mathbb{R}^3 \setminus \mathbb{T}_3^\circ)_{\mathrm{C}}} v_2^{(k+1)}(\boldsymbol{\sigma} \cap \mathbb{T}_3)\, \zeta_3^{k+4}(\boldsymbol{\sigma}) \iota_3^{(k)}(\boldsymbol{\sigma}) \lambda_3(\mathrm{d}\boldsymbol{\eta}), \tag{4.110}$$

where

$$\zeta_3(\boldsymbol{\sigma}) = \frac{\mathrm{vol}_2(\boldsymbol{\sigma} \cap \mathbb{T}_3)}{\|\boldsymbol{\eta}\| \,\mathrm{vol}_3 \mathbb{T}_3}, \qquad \iota_3^{(k)}(\boldsymbol{\sigma}) = \int_{\mathbb{T}_3} |\boldsymbol{\eta}^\top \mathbf{x} - 1|^k \lambda_3(\mathrm{d}\mathbf{x}). \tag{4.111}$$

In order to distinguish between configurations, we also write $\zeta_3(\boldsymbol{\sigma})_{\mathrm{C}}$ and $\iota_3^{(k)}(\boldsymbol{\sigma})_{\mathrm{C}}$ instead of just $\zeta_3(\boldsymbol{\sigma})$ and $\iota_3^{(k)}(\boldsymbol{\sigma})$. Here, C is only a subscript and does not imply any decomposition of those functions.

Configuration I

To ensure $\boldsymbol{\sigma}$ separates only the point $[0,0,0]$, plugging the remaining points into Equation (4.29), we get $a > 1$, $b > 1$ and $c > 1$. That means $(\mathbb{R}^3 \setminus \mathbb{T}_3^\circ)_{\mathrm{I}} = (1,\infty)^3$ is our integration domain in a, b, c. See Figure 4.8.

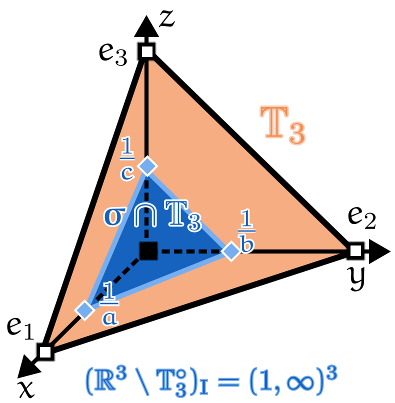


Figure 4.8: Configuration I in $\mathcal{C}(\mathbb{T}_3)$

Denote

$$\mathbb{T}_3^{abc} = \mathrm{conv}([0,0,0],[1/a,0,0],[0,1/b,0],[0,0,1/c]). \tag{4.112}$$

The plane $\boldsymbol{\sigma}$ splits $\mathbb{T}_3$ into disjoint union of two domains $\mathbb{T}_3^+ \sqcup \mathbb{T}_3^-$, where the one closer to the origin is precisely $\mathbb{T}_3^+ = \mathbb{T}_3^{abc}$. Therefore, by inclusion/exclusion,

$$\begin{aligned} \iota_3^{(k)}(\boldsymbol{\sigma})_{\mathrm{I}} &= \int_{\mathbb{T}_3^+} (1 - \boldsymbol{\eta}^\top \mathbf{x})^k \lambda_3(\mathrm{d}\mathbf{x}) + \int_{\mathbb{T}_3^-} (\boldsymbol{\eta}^\top \mathbf{x} - 1)^k \lambda_3(\mathrm{d}\mathbf{x}) \\ &= \int_{\mathbb{T}_3} (\boldsymbol{\eta}^\top \mathbf{x} - 1)^k \lambda_3(\mathrm{d}\mathbf{x}) - (1 - (-1)^k) \int_{\mathbb{T}_3^{abc}} (\boldsymbol{\eta}^\top \mathbf{x} - 1)^k \lambda_3(\mathrm{d}\mathbf{x}). \end{aligned} \tag{4.113}$$

for any k integer. These integrals are easy to compute. Mathematica Code 2 computes $\iota_3^{(k)}(\boldsymbol{\sigma})_{\mathrm{I}}$ for various values of k. Running the code for $k = 1, 2, 3$, we get

$$\iota_3^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{1}{24}\left(\frac{2}{abc} + a + b + c - 4\right), \tag{4.114}$$

$$\iota_3^{(2)}(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{1}{60}\left(a^2 + ab + bc + ac + b^2 + c^2 - 5a - 5b - 5c + 10\right), \tag{4.115}$$

$$\begin{aligned}\iota_3^{(3)}(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{1}{120}\bigg(&\frac{2}{abc} + 15a + 15b + 15c - 6a^2 - 6b^2 - 6c^2 - 6ab - 6ac \\ &-6bc + a^2b + ab^2 + a^2c + b^2c + ac^2 + bc^2 + a^3 + b^3 + c^3 + abc - 20\bigg).\end{aligned} \tag{4.116}$$

In fact, we can also deduce a general formula for $\iota_3^{(k)}(\boldsymbol{\sigma})$. Rescaling the second integral and applying Equation (A.28),

$$\begin{aligned}\iota_3^{(k)}(\boldsymbol{\sigma})_{\mathrm{I}} &= \int_{\mathbb{T}_3} (ax_1 + bx_2 + cx_3 - 1)^k - \frac{(-1)^k - 1}{abc}(1 - x_1 - x_2 - x_3)^k \lambda_3(\mathrm{d}\mathbf{x}) \\ &= \tfrac{1}{(k+1)(k+2)(k+3)}\left(\tfrac{1}{abc} + \tfrac{(a-1)^{3+k}}{a(a-b)(a-c)} + \tfrac{(b-1)^{3+k}}{b(b-a)(b-c)} + \tfrac{(c-1)^{3+k}}{c(c-a)(c-b)}\right).\end{aligned} \tag{4.117}$$

Alternatively, at least for the first moment, we can utilize our knowledge of the geometric interpretation of $\iota_3^{(1)}(\boldsymbol{\sigma})$ to derive it more easily. Let $\mathbf{M}$ and $\mathbf{M}^+$ be the centerpoints of $\mathbb{T}_3$ and $\mathbb{T}_3^+$, respectively. Clearly, since $\mathbf{M}$ and $\mathbf{M}^+$ are both centerpoints of tetrahedra,

$$\mathbf{M} = \tfrac{1}{4}(\mathbf{0} + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = [\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}], \quad \mathbf{M}^+ = \tfrac{1}{4}(\mathbf{0} + \tfrac{1}{a}\mathbf{e}_1 + \tfrac{1}{b}\mathbf{e}_2 + \tfrac{1}{c}\mathbf{e}_3) = [\tfrac{1}{4a}, \tfrac{1}{4b}, \tfrac{1}{4c}]. \tag{4.118}$$

Then, by Equation (4.35) and since $\mathrm{vol}_3\,\mathbb{T}_3 = \frac{1}{6}$ and $\mathrm{vol}_3\,\mathbb{T}_3^+ = \frac{1}{6abc}$,

$$\begin{aligned}\iota_3^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}} &= (\boldsymbol{\eta}^\top \mathbf{M} - 1)\,\mathrm{vol}_3\,\mathbb{T}_3 + 2(1 - \boldsymbol{\eta}^\top \mathbf{M}^+)\,\mathrm{vol}_3\,\mathbb{T}_3^+ \\ &= (\tfrac{a+b+c}{4} - 1)\tfrac{1}{6} + 2(1 - \tfrac{3}{4})\tfrac{1}{6abc} = \tfrac{1}{24}(a + b + c - 4 + \tfrac{2}{abc}).\end{aligned} \tag{4.119}$$

Denote T_2^{abc} as the triangle $\mathrm{conv}([1/a, 0, 0], [0, 1/b, 0], [0, 0, 1/c])$. Then the intersection of the plane $\boldsymbol{\sigma}$ with $\mathbb{T}_3$ is precisely T_2^{abc}. That is,

$$\boldsymbol{\sigma} \cap \mathbb{T}_3 = T_2^{abc}. \tag{4.120}$$

By Equation (4.15), the distance from T_2^{abc} to the origin is $\mathrm{dist}_{\boldsymbol{\sigma}}(\mathbf{0}) = 1/\|\boldsymbol{\eta}\|$. By base-height splitting,

$$\frac{\mathrm{vol}_3\,\mathbb{T}_3}{abc} = \mathrm{vol}_3\,\mathbb{T}_3^+ = \frac{1}{3}\,\mathrm{dist}_{\boldsymbol{\sigma}}(\mathbf{0})\,\mathrm{vol}_2\,T_2^{abc} = \frac{\mathrm{vol}_2(\boldsymbol{\sigma} \cap \mathbb{T}_3)}{3\|\boldsymbol{\eta}\|}, \tag{4.121}$$

from which we immediately get

$$\zeta_3(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{\mathrm{vol}_2(\boldsymbol{\sigma} \cap \mathbb{T}_3)}{\|\boldsymbol{\eta}\|\,\mathrm{vol}_3\,\mathbb{T}_3} = \frac{3}{abc}. \tag{4.122}$$

Finally, by scale affinity (we have $n_{\mathrm{I}} = 3$),

$$v_2^{(k+1)}(\boldsymbol{\sigma} \cap \mathbb{T}_3) = v_2^{(k+1)}(T_2^{abc}) = v_2^{(k+1)}(T_2), \tag{4.123}$$

which implies for $k = 1, 2, 3$ that (see Table 4.7 or Tables 3.5 and 3.6)

$$v_2^{(2)}(\boldsymbol{\sigma} \cap \mathbb{T}_3) = \frac{1}{72}, \qquad v_2^{(3)}(\boldsymbol{\sigma} \cap \mathbb{T}_3) = \frac{31}{9000}, \qquad v_2^{(4)}(\boldsymbol{\sigma} \cap \mathbb{T}_3) = \frac{1}{900}. \tag{4.124}$$

Putting everything into the integral in Equation (4.110), we get when $k = 1$,

$$v_3^{(1)}(\mathbb{T}_3)_{\mathrm{I}} = \frac{3}{32} \int_1^\infty \int_1^\infty \int_1^\infty \frac{2 + abc(a+b+c-4)}{a^6 b^6 c^6} \,\mathrm{d}a \,\mathrm{d}b \,\mathrm{d}c = \frac{3}{2000}. \tag{4.125}$$

For higher values of k, we get

$$\begin{aligned} v_3^{(2)}(\mathbb{T}_3)_{\mathrm{I}} &= \frac{279}{4000000}, & v_3^{(3)}(\mathbb{T}_3)_{\mathrm{I}} &= \frac{37193}{6174000000}, \\ v_3^{(4)}(\mathbb{T}_3)_{\mathrm{I}} &= \frac{681383}{847072800000}, & v_3^{(5)}(\mathbb{T}_3)_{\mathrm{I}} &= \frac{3674957}{25092716544000}. \end{aligned} \tag{4.126}$$

Configuration II

In this scenario, $\boldsymbol{\sigma}$ separates two points $[0,0,0]$ and $[0,0,1]$ from $\mathbb{T}_3$. By Equation (4.29), we get $a > 1$, $b > 1$ and $c < 1$. We can split the condition for c into to cases: either $0 < c < 1$ or $c < 0$. In fact, both options give the same factor since they are symmetrical as they correspond to two possibilities where $\boldsymbol{\sigma}$ might intersect $\mathcal{A}([0,0,0],[0,0,1])$. Therefore we only consider the integration half-domain (indicated by $*$)

$$(\mathbb{R}^3 \setminus \mathbb{T}_3^\circ)_{\mathrm{II}}^* = (1,\infty)^2 \times (0,1) \tag{4.127}$$

and in the end multiply the result twice.

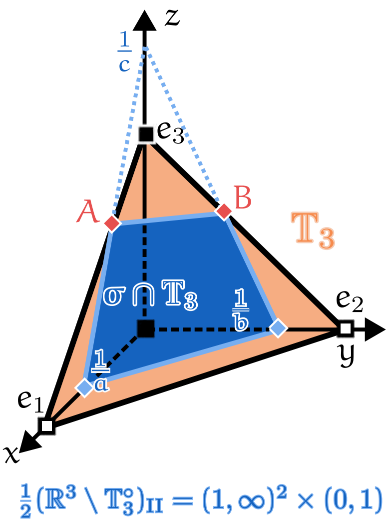


Figure 4.9: Configuration II in $\mathcal{C}(\mathbb{T}_3)$

From Figure 4.9 above, we can see the plane $\boldsymbol{\sigma}$ intersects $\mathbb{T}_3$ at points $\frac{1}{a}\mathbf{e}_1$, $\frac{1}{b}\mathbf{e}_2$ (already in Configuration I) and additionally at

$$\begin{aligned} \mathbf{A} &= \tfrac{1}{c}\mathbf{e}_3 + \alpha(\tfrac{1}{a}\mathbf{e}_1 - \tfrac{1}{c}\mathbf{e}_3) = [\tfrac{1-c}{a-c}, 0, \tfrac{a-1}{a-c}], \\ \mathbf{B} &= \tfrac{1}{c}\mathbf{e}_3 + \beta(\tfrac{1}{b}\mathbf{e}_2 - \tfrac{1}{c}\mathbf{e}_3) = [0, \tfrac{1-c}{b-c}, \tfrac{b-1}{b-c}], \end{aligned} \tag{4.128}$$

where we denote $\alpha = \frac{a(1-c)}{a-c}$ and $\beta = \frac{b(1-c)}{b-c}$. Thus, the plane $\boldsymbol{\sigma}$ splits $\mathbb{T}_3$ into disjoint union of two domains $\mathbb{T}_3^+ \sqcup \mathbb{T}_3^-$, where $\mathbb{T}_3^+$ being the one closer to the origin. Denote $\mathbb{T}_3^{abc} = \operatorname{conv}(\mathbf{0}, \frac{1}{a}\mathbf{e}_1, \frac{1}{b}\mathbf{e}_2, \frac{1}{c}\mathbf{e}_3)$ and $\mathbb{T}_3^* = \operatorname{conv}(\mathbf{e_3}, \mathbf{A}, \mathbf{B}, \frac{1}{c}\mathbf{e}_3)$, or explicitly

$$\mathbb{T}_3^{abc} = \operatorname{conv}\left([0,0,0], \left[\tfrac{1}{a},0,0\right], \left[0,\tfrac{1}{b},0\right], \left[0,0,\tfrac{1}{c}\right]\right), \tag{4.129}$$

$$\mathbb{T}_3^* = \operatorname{conv}\left([0,0,1], \left[\tfrac{1-c}{a-c},0,\tfrac{a-1}{a-c}\right], \left[0,\tfrac{1-c}{b-c},\tfrac{b-1}{b-c}\right], \left[0,0,\tfrac{1}{c}\right]\right). \tag{4.130}$$

Then we can write $\mathbb{T}_3^+ = \mathbb{T}_3^{abc} \setminus \mathbb{T}_3^* = \operatorname{conv}(\mathbf{0}, \mathbf{e}_3, \frac{1}{a}\mathbf{e}_1, \frac{1}{b}\mathbf{e}_2, \mathbf{A}, \mathbf{B})$, that is

$$\mathbb{T}_3^+ = \operatorname{conv}\left([0,0,0],[0,0,1],[\tfrac{1}{a},0,0],[0,\tfrac{1}{b},0],[\tfrac{1-c}{a-c},0,\tfrac{a-1}{a-c}],[0,\tfrac{1-c}{b-c},\tfrac{b-1}{b-c}]\right). \tag{4.131}$$

By inclusion/exclusion,

$$\begin{aligned} \iota_3^{(k)}(\boldsymbol{\sigma})_{\mathrm{II}} = & \int_{\mathbb{T}_3} (\boldsymbol{\eta}^\top \mathbf{x} - 1)^k \lambda_3(\mathrm{d}\mathbf{x}) - (1-(-1)^k) \int_{\mathbb{T}_3^{abc}} (\boldsymbol{\eta}^\top \mathbf{x} - 1)^k \lambda_3(\mathrm{d}\mathbf{x}) \\ & + (1-(-1)^k) \int_{\mathbb{T}_3^*} (\boldsymbol{\eta}^\top \mathbf{x} - 1)^k \lambda_3(\mathrm{d}\mathbf{x}). \end{aligned} \tag{4.132}$$

for any k integer. These integrals are again easy to compute. Mathematica Code 3 computes $\iota_3^{(k)}(\boldsymbol{\sigma})_{\mathrm{II}}$ for various values of k. Running the code for $k=1$ and $k=3$, we obtain

$$\iota_3^{(1)}(\boldsymbol{\sigma})_{\mathrm{II}} = \iota_3^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}} - \frac{(1-c)^4}{12c(a-c)(b-c)}, \tag{4.133}$$

$$\iota_3^{(3)}(\boldsymbol{\sigma})_{\mathrm{II}} = \iota_3^{(3)}(\boldsymbol{\sigma})_{\mathrm{I}} - \frac{(1-c)^6}{60c(a-c)(b-c)}. \tag{4.134}$$

where the functions $\iota_3^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}}$ and $\iota_3^{(3)}(\boldsymbol{\sigma})_{\mathrm{I}}$ are given by Equations (4.114) and (4.116). In general case for any k integer, we have

$$\iota_3^{(k)}(\boldsymbol{\sigma})_{\mathrm{II}} = \iota_3^{(k)}(\boldsymbol{\sigma})_{\mathrm{I}} - \frac{(1-(-1)^k)(1-c)^{3+k}}{(k+1)(k+2)(k+3)c(a-c)(b-c)}. \tag{4.135}$$

For k even, we have $\iota_3^{(k)}(\boldsymbol{\sigma})_{\mathrm{II}} = \iota_3^{(k)}(\boldsymbol{\sigma})_{\mathrm{I}} = \iota_3^{(k)}(\boldsymbol{\sigma})_{\mathrm{N}}$ since the part with $1-(-1)^k$ vanishes. However, since the even metric moments $v_3^{(k)}(T_3)$ are trivial to compute by integration alone, we will proceed by assuming k is odd. The calculation of $\iota_3^{(k)}(\boldsymbol{\sigma})_{\mathrm{II}}$ is again trivial when $k=1$ and can be done by hand from its geometric interpretation. Note that $\frac{1}{6abc} = \operatorname{vol}_3 \mathbb{T}_3^{abc} = \frac{1}{3!}|\det(\frac{1}{a}\mathbf{e}_1 - \frac{1}{c}\mathbf{e}_3 \mid \frac{1}{b}\mathbf{e}_2 - \frac{1}{c}\mathbf{e}_3 \mid \mathbf{0} - \frac{1}{c}\mathbf{e}_3)|$ and thus

$$\operatorname{vol}_3 \mathbb{T}_3^* = \tfrac{1}{3!}\left|\det\left(\alpha(\tfrac{1}{a}\mathbf{e}_1 - \tfrac{1}{c}\mathbf{e}_3) \,|\, \beta(\tfrac{1}{b}\mathbf{e}_2 - \tfrac{1}{c}\mathbf{e}_3) \,|\, (1-c)(\mathbf{0} - \tfrac{1}{c}\mathbf{e}_3)\right)\right| = \tfrac{\alpha\beta(1-c)}{6abc}. \tag{4.136}$$

Let $\mathbf{M}$, $\mathbf{M}^{abc}$, $\mathbf{M}^*$ and $\mathbf{M}^+$ be the centerpoints of $\mathbb{T}_3$, $\mathbb{T}_3^{abc}$, $\mathbb{T}_3^*$ and $\mathbb{T}_3^+$, respectively. Trivially, $\mathbf{M} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$ and $\mathbf{M}^{abc} = [\frac{1}{4a}, \frac{1}{4b}\frac{1}{4c}]$. Since also $\mathbf{M}^*$ is a centerpoint of a tetrahedron, namely $\mathbb{T}_3^* = \operatorname{conv}(\mathbf{e_3}, \mathbf{A}, \mathbf{B}, \mathbf{e}_3/c)$,

$$\mathbf{M}^* = \tfrac{1}{4}(\mathbf{e}_3 + \mathbf{A} + \mathbf{B} + \tfrac{1}{c}\mathbf{e}_3) = [\tfrac{\alpha}{4a}, \tfrac{\beta}{4b}, \tfrac{3+c-\alpha-\beta}{4c}]. \tag{4.137}$$

Since $\mathbb{T}_3^+ = \mathbb{T}_3^{abc} \setminus \mathbb{T}_3^*$, we have $\operatorname{vol}_3 \mathbb{T}_3^+ = \operatorname{vol}_3 \mathbb{T}_3^{abc} - \operatorname{vol}_3 \mathbb{T}_3^*$ and by mass balance,

$$\mathbf{M}^+ \operatorname{vol}_3 \mathbb{T}_3^+ = \mathbf{M}^{abc} \operatorname{vol}_3 \mathbb{T}_3^{abc} - \mathbf{M}^* \operatorname{vol}_3 \mathbb{T}_3^*. \tag{4.138}$$

Solving for $\mathbf{M}^+$ is left as an exercise for the reader, but it turns out one does not need its knowledge to obtain $\iota_3^{(1)}(\boldsymbol{\sigma})_{\text{II}}$. To see this, plugging the mass balance directly into Equation (4.35) and by our previous relation for $\iota_3^{(1)}(\boldsymbol{\sigma})_{\text{I}}$, we get

$$\begin{aligned}\iota_3^{(1)}(\boldsymbol{\sigma})_{\text{I}} &= (\boldsymbol{\eta}^\top\mathbf{M}-1)\operatorname{vol}_3\mathbb{T}_3 + 2(1-\boldsymbol{\eta}^\top\mathbf{M}^+)\operatorname{vol}_3\mathbb{T}_3^+ \\ &= \iota_3^{(1)}(\boldsymbol{\sigma})_{\text{I}} - 2(1-\boldsymbol{\eta}^\top\mathbf{M}^*)\operatorname{vol}_3\mathbb{T}_3^* = \iota_3^{(1)}(\boldsymbol{\sigma})_{\text{I}} - \tfrac{\alpha\beta(1-c)^2}{12abc},\end{aligned} \tag{4.139}$$

which matches Equation (4.133). Denote

$$T_2^{abc} = \operatorname{conv}\left(\left[\frac{1}{a},0,0\right],\left[0,\frac{1}{b},0\right],\left[0,0,\frac{1}{c}\right]\right), \tag{4.140}$$

$$T_2^* = \operatorname{conv}\left(\left[\frac{1-c}{a-c},0,\frac{a-1}{a-c}\right],\left[0,\frac{1-c}{b-c},\frac{b-1}{b-c}\right],\left[0,0,\frac{1}{c}\right]\right), \tag{4.141}$$

we have for the intersection of $\boldsymbol{\sigma}$ with $\mathbb{T}_3$,

$$\boldsymbol{\sigma}\cap\mathbb{T}_3 = T_2^{abc}\setminus T_2^* = \operatorname{conv}\Big([\tfrac{1}{a},0,0],[0,\tfrac{1}{b},0],[\tfrac{1-c}{a-c},0,\tfrac{a-1}{a-c}],[0,\tfrac{1-c}{b-c},\tfrac{b-1}{b-c}]\Big), \tag{4.142}$$

so $n_{\text{II}} = 4$. By scale affinity

$$v_2^{(k+1)}(\boldsymbol{\sigma}\cap\mathbb{T}_3) = v_2^{(k+1)}(T_2^{abc}\setminus T_2^*) = v_2^{(k+1)}(\mathbb{U}_2^{\alpha\beta}), \tag{4.143}$$

where $\mathbb{U}_2^{\alpha\beta} = \operatorname{conv}([\alpha,0],[0,\beta],[0,1],[1,0])$ is a *canonical truncated triangle* with

$$\alpha = \frac{a(1-c)}{a-c}, \qquad \beta = \frac{b(1-c)}{b-c}. \tag{4.144}$$

See Figure 4.10 below for an illustration of $\mathbb{U}_2^{\alpha\beta}$ and its volumetric moments.

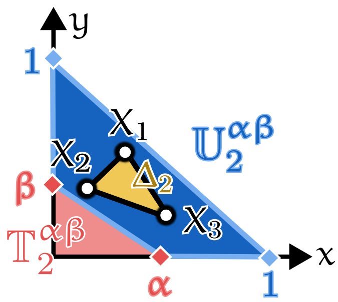


Figure 4.10: Mean section moments in the second $\mathcal{C}(\mathbb{T}_3)$ configuration

Since $\operatorname{vol}_2\mathbb{U}_2^{\alpha\beta} = \frac{1}{2}(1-\alpha\beta)$, we can write in general,

$$v_2^{(k+1)}(\mathbb{U}_2^{\alpha\beta}) = \left(\frac{2}{1-\alpha\beta}\right)^{k+4}\int_{(\mathbb{U}_2^{\alpha\beta})^3}\Delta_2^{k+1}\,\mathrm{d}\mathbf{x}_0\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2, \tag{4.145}$$

We would like to find $v_2^{(k+1)}(\mathbb{U}_2^{\alpha\beta})$ for odd k. This is, luckily, trivial, since we are now integrating even powers of

$$\Delta_2 = \frac{1}{2!}\left|\det(\mathbf{x}_1-\mathbf{x}_0\,|\,\mathbf{x}_2-\mathbf{x}_0)\right|. \tag{4.146}$$

The calculation can be carried out in Mathematica using Code 4, which exploits the symmetries and uses inclusion/exclusion. Running the code for $k = 1$ and $k = 3$, we obtain

$$\upsilon_2^{(2)}(\mathbb{U}_2^{\alpha\beta}) = \frac{\left\{\begin{matrix} \alpha^4\beta^4 - 8\alpha^3\beta^3 + 8\alpha^3\beta^2 - 4\alpha^3\beta + 8\alpha^2\beta^3 \\ -10\alpha^2\beta^2 + 8\alpha^2\beta - 4\alpha\beta^3 + 8\alpha\beta^2 - 8\alpha\beta + 1 \end{matrix}\right\}}{72(1-\alpha\beta)^4}, \tag{4.147}$$

$$\upsilon_2^{(4)}(\mathbb{U}_2^{\alpha\beta}) = \frac{\left\{\begin{matrix} \alpha^6\beta^6 - 6\alpha^5\beta - 6\alpha\beta^5 + 18\alpha\beta^4 + 32\alpha\beta^2 - 19\alpha\beta + 1 \\ -31\alpha\beta^3 - 19\alpha^5\beta^5 + 32\alpha^5\beta^4 - 31\alpha^5\beta^3 + 18\alpha^5\beta^2 \\ -31\alpha^3\beta + 32\alpha^4\beta^5 - 47\alpha^4\beta^4 + 46\alpha^4\beta^3 - 34\alpha^4\beta^2 \\ +18\alpha^4\beta - 31\alpha^3\beta^5 + 46\alpha^3\beta^4 - 50\alpha^3\beta^3 + 46\alpha^3\beta^2 \\ +18\alpha^2\beta^5 - 34\alpha^2\beta^4 + 46\alpha^2\beta^3 - 47\alpha^2\beta^2 + 32\alpha^2\beta \end{matrix}\right\}}{900(1-\alpha\beta)^6}. \tag{4.148}$$

Alternatively, since we have turned our problem to essentially finding the even moments by recursions, we can use Example 219 in Chapter 3 on even volumetric moments. Finally, by definition (alternatively by Equation (4.36))

$$\zeta_3(\boldsymbol{\sigma})_{\mathrm{II}} = \frac{\operatorname{vol}_2(\boldsymbol{\sigma} \cap \mathbb{T}_3)}{\|\boldsymbol{\eta}\| \operatorname{vol}_3 \mathbb{T}_3} = (1-\alpha\beta)\frac{\operatorname{vol}_2 T_2^{abc}}{\|\boldsymbol{\eta}\| \operatorname{vol}_3 \mathbb{T}_3} = (1-\alpha\beta)\zeta_3(\boldsymbol{\sigma})_{\mathrm{I}}. \tag{4.149}$$

Before we proceed to evaluate the final integral, we make the following change of variables $(a, b, c) \to (\alpha, \beta, c)$ via transformation Equations (4.144), which transform the integration half-domain into

$$(\mathbb{R}^3 \setminus \mathbb{T}_3^\circ)_{\mathrm{II}}^* |_{\alpha,\beta,c} = (1-c, 1)^2 \times (0, 1). \tag{4.150}$$

Note that, if c is treated as a parameter, the variables a, b depend on α, β separately. As a consequence,

$$\mathrm{d}a = \frac{c(1-c)\,\mathrm{d}\alpha}{(1-c-\alpha)^2}, \qquad \mathrm{d}b = \frac{c(1-c)\,\mathrm{d}\beta}{(1-c-\beta)^2} \tag{4.151}$$

and thus one has for the of transformation of measure

$$\lambda_3(\mathrm{d}\boldsymbol{\eta}) = \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c = \frac{c^2(1-c)^2\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}c}{(1-c-\alpha)^2(1-c-\beta)^2}. \tag{4.152}$$

Our functions in variables a, b, c are transformed into

$$\zeta_3(\boldsymbol{\sigma})_{\mathrm{II}} = \frac{3(1-c-\alpha)(1-c-\beta)(1-\alpha\beta)}{c^3\alpha\beta}, \tag{4.153}$$

$$\iota_3^{(1)}(\boldsymbol{\sigma})_{\mathrm{II}} = \frac{1}{24}\left(\tfrac{2(1-c-\alpha)(1-c-\beta)\left(1-(1-c)^2\alpha\beta\right)}{c^3\alpha\beta} + c\left(1 - \tfrac{\alpha}{1-c-\alpha} - \tfrac{\beta}{1-c-\beta}\right) - 4\right) \tag{4.154}$$

and so on for $\iota_3^{(1)}(\boldsymbol{\sigma})_{\mathrm{II}}$ with larger k. Putting everything into the integral in Equation (4.110) with prefactor 2, we get when $k = 1$,

$$\upsilon_3^{(1)}(\mathbb{T}_3)_{\mathrm{II}} = \frac{3}{16}\int_0^1\int_{1-c}^1\int_{1-c}^1 \frac{(1-c)^2(1-c-\alpha)^3(1-c-\beta)^3(1-\alpha\beta)}{c^{13}\alpha^5\beta^5} \times$$
$$\left(\frac{2(1-c-\alpha)(1-c-\beta)\left(1-(1-c)^2\alpha\beta\right)}{c^3\alpha\beta} + c\left(1 - \frac{\alpha}{1-c-\alpha} - \frac{\beta}{1-c-\beta}\right) - 4\right) \tag{4.155}$$
$$\times \Big(1 - 8\alpha\beta + 8\alpha^2\beta - 4\alpha^3\beta + 8\alpha\beta^2 - 10\alpha^2\beta^2 + 8\alpha^3\beta^2 - 4\alpha\beta^3 + 8\alpha^2\beta^3$$
$$- 8\alpha^3\beta^3 + \alpha^4\beta^4\Big)\,\mathrm{d}\alpha\,\mathrm{d}\alpha\,\mathrm{d}c.$$

Integrating out α, β can be done relatively easily, we end up with

$$v_3^{(1)}(\mathbb{T}_3)_{\mathrm{II}} = \int_0^1 \frac{c^2 p_0 + 4800c(1-c)^2 p_1 \ln(1-c) + 3600(1-c)^2 p_2 \ln^2(1-c)}{19200c^{16}} \mathrm{d}c, \quad (4.156)$$

where

$$\begin{aligned} p_0 &= 8265600 - 49593600c + 111530400c^2 - 103044000c^3 \\ &\quad - 10353200c^4 + 114147200c^5 - 115229200c^6 + 58917200c^7 \\ &\quad - 17280824c^8 + 2861248c^9 - 220122c^{10} - 702c^{11} + 213c^{12}, \end{aligned} \quad (4.157)$$

$$\begin{aligned} p_1 &= 3444 - 15498c + 22076c^2 - 4942c^3 - 18060c^4 + 21343c^5 \\ &\quad - 11086c^6 + 3147c^7 - 496c^8 + 36c^9, \end{aligned} \quad (4.158)$$

$$\begin{aligned} p_2 &= 2296 - 11480c + 19692c^2 - 9888c^3 - 11350c^4 + 20442c^5 \\ &\quad - 13971c^6 + 5296c^7 - 1191c^8 + 154c^9 - 9c^{10}. \end{aligned} \quad (4.159)$$

The last c integration can be carried out by Mathematica (alternatively, we can use derivatives of the Beta function). We get

$$v_3^{(1)}(\mathbb{T}_3)_{\mathrm{II}} = \frac{217}{54000} - \frac{\pi^2}{45045}. \quad (4.160)$$

For higher values of k, the integration possesses similar difficulty, we got

$$\begin{aligned} v_3^{(3)}(\mathbb{T}_3)_{\mathrm{II}} &= \frac{105199}{9261000000} + \frac{79\pi^2}{7274767500}, \\ v_3^{(5)}(\mathbb{T}_3)_{\mathrm{II}} &= \frac{1890871}{9601804800000} - \frac{547\pi^2}{26831987910000}. \end{aligned} \quad (4.161)$$

Contribution from all configurations

By Equation (C.118) and by affine invariancy,

$$v_3^{(k)}(T_3) = \sum_{C \in \mathcal{C}(T_3)} w_{\mathrm{C}}\, v_3^{(k)}(T_3)_{\mathrm{C}} = 4v_3^{(k)}(\mathbb{T}_3)_{\mathrm{I}} + 3v_3^{(k)}(\mathbb{T}_3)_{\mathrm{II}}, \quad (4.162)$$

from which, immediately, we get Buchta and Reitzner's [18], Mannion's [44] and Philip's [52] result for $v_3^{(1)}(T_3)$ and also some of its further generalisations

$$v_3^{(1)}(T_3) = \frac{13}{720} - \frac{\pi^2}{15015} \approx 0.01739823925, \quad (4.163)$$

$$v_3^{(3)}(T_3) = \frac{733}{12600000} + \frac{79\pi^2}{2424922500} \approx 0.0000584961, \quad (4.164)$$

$$v_3^{(5)}(T_3) = \frac{5125739}{4356374400000} - \frac{547\pi^2}{8943995970000} \approx 0.000001176003. \quad (4.165)$$

4.4.2 Octahedron odd volumetric moments

By affine invariancy, it does not matter how large is the volume of an octahedron as long as the octahedron stays regular. Hence, we may select the following representation of a regular octahedron

$$O_3 = \operatorname{conv}([\pm 1,0,0],[0,\pm 1,0],[0,0,\pm 1]), \tag{4.166}$$

which has $\operatorname{vol}_3 O_3 = 4/3$. According to its genealogy $\mathcal{C}(O_3)$, it has three configurations as shown in Figure 4.11 (or D.3 in Appendix D). Table 4.12 shows specifically which sets S of vertices are separated by a cutting plane $\boldsymbol{\sigma}$ in which configurations in our local representation of O_3 above. Note that there is an ambiguity how to select those vertices as long it is the same configuration.

C	I	II	III
S	$[0,0,1]$	$[1,0,0]$ $[0,1,0]$	$[1,0,0]$ $[0,1,0]$ $[0,0,1]$
w_{C}	6	12	4

Table 4.12: Configurations $\mathcal{C}(O_3)$ in a local representation.

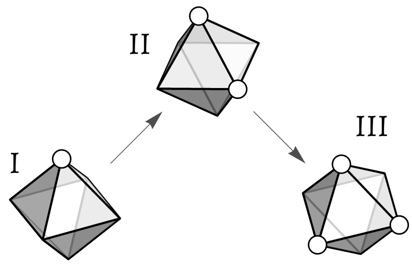


Figure 4.11: Octahedron genealogy

By Theorem 221 and for any $\mathrm{C} \in \mathcal{C}(O_3)$,

$$\upsilon_3^{(k)}(O_3)_{\mathrm{C}} = \frac{2}{3^k} \int_{(\mathbb{R}^3 \setminus O_3^\circ)_{\mathrm{C}}} \upsilon_2^{(k+1)}(\boldsymbol{\sigma} \cap O_3)\, \zeta_3^{k+4}(\boldsymbol{\sigma}) \iota_3^{(k)}(\boldsymbol{\sigma}) \lambda_3(\mathrm{d}\boldsymbol{\eta}), \tag{4.167}$$

where

$$\zeta_3(\boldsymbol{\sigma}) = \frac{\operatorname{vol}_2(\boldsymbol{\sigma} \cap O_3)}{\|\boldsymbol{\eta}\| \operatorname{vol}_3 O_3}, \qquad \iota_3^{(k)}(\boldsymbol{\sigma}) = \int_{O_3} |\boldsymbol{\eta}^\top \mathbf{x} - 1|^k \lambda_3(\mathrm{d}\mathbf{x}). \tag{4.168}$$

We can describe the relation $\mathbf{x} = (x,y,z)^\top \in O_3$ by the following set of eight linear inequalities (all of them keep $\mathbf{0} \in O_3$)

$$\begin{aligned} &x+y+z<1, \quad -x+y+z<1, \quad x+y-z<1, \quad -x+y-z<1, \\ &x-y+z<1, \quad -x-y+z<1, \quad x-y-z<1, \quad -x-y-z<1. \end{aligned} \tag{4.169}$$

Configuration I

First, we find $(\mathbb{R}^3 \setminus O_3^\circ)_{\mathrm{I}}$. By Equation (4.29), plugging the configurations points from S into $\boldsymbol{\eta}^\top \mathbf{x} > 1$ and from $V \setminus S$ into $\boldsymbol{\eta}^\top \mathbf{x} < 1$ (flipped inequalities give the empty set), we get that a, b, c must satisfy

$$c > 1, \quad -c < 1, \quad a < 1, \quad -a < 1, \quad b < 1, \quad -b < 1, \tag{4.170}$$

so $(\mathbb{R}^3 \setminus O_3^\circ)_{\mathrm{I}} = (-1,1)^2 \times (1,\infty)$. Next, $\boldsymbol{\sigma}$ splits O_3 into $O_3^+ \sqcup O_3^-$. We can parametrise those domains by simultaneously solving Equation (4.30) and (4.169). From those inequalities, we get by linear programming

$$O_3^+ = \operatorname{conv}\Big([-1,0,0],[0,-1,0],[0,0,-1],[0,0,1],[0,1,0], \\ [\tfrac{b-1}{b-a},\tfrac{a-1}{a-b},0],[\tfrac{b+1}{a+b},\tfrac{1-a}{a+b},0],[\tfrac{c-1}{c-a},0,\tfrac{a-1}{a-c}],[\tfrac{c+1}{a+c},0,\tfrac{1-a}{a+c}]\Big). \tag{4.171}$$

Note that a simultaneous system of inequalities can be reduced using the eponymous `Reduce` command in Mathematica (used also in the case above). As a direct consequence of this parametrisation, we get

$$\operatorname{vol}_3 O_3^+ = \frac{2\,(c^4+3c^3-3c^2-2a^2c^2-2b^2c^2+c+2a^2b^2)}{3(c-a)(c+a)(c-b)(c+b)} \tag{4.172}$$

from which, by Equation (4.36),

$$\zeta_3(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{3c(c-1)^2}{2(c-a)(c+a)(c-b)(c+b)}. \tag{4.173}$$

Also, thanks to our parametrisation, we get

$$\iota_3^{(k)}(\boldsymbol{\sigma})_{\mathrm{I}} = \int_{O_3^+}(1-\boldsymbol{\eta}^\top\mathbf{x})^k\lambda_3(\mathrm{d}\mathbf{x}) + \int_{O_3^-}(\boldsymbol{\eta}^\top\mathbf{x}-1)^k\lambda_3(\mathrm{d}\mathbf{x}) \tag{4.174}$$

for any real $k>-1$ almost for free, namely for $k=1$ and $k=3$,

$$\iota_3^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{c^5+6c^3+4a^2b^2-4c^2\,(1+a^2+b^2)+c}{3(c-a)(c+a)(c-b)(c+b)}, \tag{4.175}$$

$$\iota_3^{(3)}(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{\left\{\begin{matrix} c^7+15c^5+15c^3-6c^2+6a^4c^2-20b^2c^2-6b^4c^2 \\ -20a^2c^2-6a^2b^2c^2+c+20a^2b^2+6a^4b^2+6a^2b^4\end{matrix}\right\}}{15(c-a)(c+a)(c-b)(c+b)} \tag{4.176}$$

and also $n_{\mathrm{I}}=4$ since

$$\boldsymbol{\sigma}\cap O_3 = \operatorname{conv}\Big([\tfrac{b-1}{b-a},\tfrac{a-1}{a-b},0],[\tfrac{b+1}{a+b},\tfrac{1-a}{a+b},0],[\tfrac{c-1}{c-a},0,\tfrac{a-1}{a-c}],[\tfrac{c+1}{a+c},0,\tfrac{1-a}{a+c}]\Big). \tag{4.177}$$

We can use a computer to deduce the following even moments

$$v_2^{(2)}(\boldsymbol{\sigma}\cap O_3) = \frac{3c^4+c^2(a^2+b^2)-a^2b^2}{288c^4}, \tag{4.178}$$

$$v_2^{(4)}(\boldsymbol{\sigma}\cap O_3) = \frac{\left\{\begin{matrix} 12c^6+17a^2c^4+17b^2c^4+3a^4c^2 \\ -14a^2b^2c^2+3b^4c^2-3a^4b^2-3a^2b^4\end{matrix}\right\}}{28800c^6}. \tag{4.179}$$

Therefore, putting everything together,

$$\begin{aligned} v_3^{(1)}(O_3)_{\mathrm{I}} =& \frac{3}{512}\int_1^\infty\int_{-1}^1\int_{-1}^1 \frac{c\,(c-1)^{10}\,(c^2a^2+c^2b^2-a^2b^2+3c^4)}{(c-a)^6(c+a)^6(c-b)^6(c+b)^6} \\ &\times\Big(4a^2b^2-4a^2c^2-4b^2c^2-4c^2+c^5+6c^3+c\Big)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c, \end{aligned} \tag{4.180}$$

similarly for $v_3^{(3)}(O_3)_{\mathrm{I}}$. Integration in Mathematica then reveals

$$v_3^{(1)}(O_3)_{\mathrm{I}} = \frac{2569561}{230400} - \frac{11571\pi^2}{10240}, \tag{4.181}$$

$$v_3^{(3)}(O_3)_{\mathrm{I}} = \frac{3260724307264561}{433954160640000} - \frac{109143647\pi^2}{143360000}, \tag{4.182}$$

$$v_3^{(5)}(O_3)_{\mathrm{I}} = \frac{130691428618025026209 5927}{599658272376068505600000} - \frac{3676076446537\pi^2}{1664719257600}. \tag{4.183}$$

Configuration II

By Equation (4.29), plugging the configurations points from S into $\boldsymbol{\eta}^\top \mathbf{x} > 1$ and from $V \setminus S$ into $\boldsymbol{\eta}^\top \mathbf{x} < 1$ (flipped inequalities give empty set), we get that a, b, c must satisfy

$$c < 1, \quad -c < 1, \quad a > 1, \quad -a < 1, \quad b > 1, \quad -b < 1, \tag{4.184}$$

so $(\mathbb{R}^3 \setminus O_3^\circ)_{\mathrm{II}} = (1, \infty)^2 \times (-1, 1)$. Next, $\boldsymbol{\sigma}$ splits O_3 into $O_3^+ \sqcup O_3^-$. We can parametrise those domains by simultaneously solving Equation (4.30) and (4.169). Then, by linear programming, we get $n_{\mathrm{II}} = 6$ since we obtained

$$\begin{aligned} O_3^+ = \operatorname{conv}\Big(&[-1,0,0],[0,-1,0],[0,0,-1],[0,0,1],[0,\tfrac{c-1}{c-b},\tfrac{b-1}{b-c}],\\ &[0,\tfrac{c+1}{b+c},\tfrac{1-b}{b+c}],[\tfrac{1-b}{a+b},\tfrac{a+1}{a+b},0],[\tfrac{b+1}{a+b},\tfrac{1-a}{a+b},0],[\tfrac{c-1}{c-a},0,\tfrac{a-1}{a-c}],[\tfrac{c+1}{a+c},0,\tfrac{1-a}{a+c}]\Big), \end{aligned} \tag{4.185}$$

from which, using Mathematica,

$$\operatorname{vol}_3 O_3^+ = \frac{2\left\{\begin{array}{c} 3a^2b^2+a^3b^2+a^2b^3-c^2+3ac^2-3a^2c^2-a^3c^2+3bc^2 \\ -3abc^2-a^2bc^2-3b^2c^2-ab^2c^2-b^3c^2+2ac^4+2bc^4-ab \end{array}\right\}}{3(a+b)(a-c)(b-c)(a+c)(b+c)} \tag{4.186}$$

which further yields, by Equation (4.36),

$$\zeta_3(\boldsymbol{\sigma})_{\mathrm{II}} = \frac{3\,(2ac^2+2bc^2-ab+a^2b^2-a^2c^2-b^2c^2-abc^2-c^2)}{2(a+b)(a-c)(b-c)(a+c)(b+c)}. \tag{4.187}$$

Next, for $k = 1$, we obtain

$$\iota_3^{(1)}(\boldsymbol{\sigma})_{\mathrm{II}} = \frac{\left\{\begin{array}{c} a^4b^2-a^4c^2+a^3b^3-a^3bc^2+a^2b^4-a^2b^2c^2+6a^2b^2-6a^2c^2-ab \\ -ab^3c^2-6abc^2+4ac^4+4ac^2-b^4c^2-6b^2c^2+4bc^4+4bc^2-c^2 \end{array}\right\}}{3(a+b)(a-c)(b-c)(a+c)(b+c)}. \tag{4.188}$$

As $\iota_3^{(3)}(\boldsymbol{\sigma})$, $v_2^{(2)}(\boldsymbol{\sigma} \cap O_3)$ and $v_2^{(4)}(\boldsymbol{\sigma} \cap O_3)$ are rather long, we are not listing them here. Putting everything together and integrating over a, b, c, we get

$$v_3^{(1)}(O_3)_{\mathrm{II}} = \frac{72588071\pi^2}{92252160} - \frac{12023076361}{1548288000}, \tag{4.189}$$

$$v_3^{(3)}(O_3)_{\mathrm{II}} = \frac{388096633880599\pi^2}{95351832576000} - \frac{830108924076197}{206644838400000}, \tag{4.190}$$

$$v_3^{(5)}(O_3)_{\mathrm{II}} = \frac{247063831934862574481\pi^2}{221063688644198400000} - \frac{661447432765606661516951 9}{599658272376068505600000}. \tag{4.191}$$

Configuration III

By Equation (4.29), a, b, c must satisfy

$$c > 1, \quad -c < 1, \quad a > 1, \quad -a < 1, \quad b > 1, \quad -b < 1, \tag{4.192}$$

or with $<$ and $>$ flipped,

$$c < 1, \quad -c > 1, \quad a < 1, \quad -a > 1, \quad b < 1, \quad -b > 1, \tag{4.193}$$

so $(\mathbb{R}^3 \setminus O_3^\circ)_{\mathrm{II}} = ((-\infty,-1) \cup (1,\infty))^3$. By symmetry, we may integrate only over half-domain $(\mathbb{R}^3 \setminus O_3^\circ)_{\mathrm{II}}^* = (1,\infty)^3$. Next, $O_3^+ \sqcup O_3^-$, where, by simultaneously solving Equations (4.30) and (4.169) and by linear programming,

$$\begin{aligned} O_3^+ = \operatorname{conv}\Big(&[-1,0,0],[0,-1,0],[0,0,-1],[0,\tfrac{1-c}{b+c},\tfrac{b+1}{b+c}],[0,\tfrac{c+1}{b+c},\tfrac{1-b}{b+c}],\\ &[\tfrac{1-b}{a+b},\tfrac{a+1}{a+b},0],[\tfrac{b+1}{a+b},\tfrac{1-a}{a+b},0],[\tfrac{1-c}{a+c},0,\tfrac{a+1}{a+c}],[\tfrac{c+1}{a+c},0,\tfrac{1-a}{a+c}]\Big), \end{aligned} \tag{4.194}$$

which means $n_{\mathrm{III}} = 6$. Using Mathematica,

$$\operatorname{vol}_3 O_3^+ = \frac{2\left(3ab+a^2b+ab^2+3ac+a^2c+3bc+2abc+b^2c+ac^2+bc^2-1\right)}{3(a+b)(a+c)(b+c)} \tag{4.195}$$

from which, by Equation (4.36),

$$\zeta_3(\boldsymbol{\sigma})_{\mathrm{III}} = \frac{3(ab+ac+bc-1)}{2(a+b)(a+c)(b+c)}. \tag{4.196}$$

Next, for $k=1$ and $k=3$, we obtained

$$\iota_3^{(1)}(\boldsymbol{\sigma})_{\mathrm{III}} = \frac{\left\{\begin{matrix} 6ab-1+a^3b+a^2b^2+ab^3+6ac+a^3c+6bc \\ +2a^2bc+2ab^2c+b^3c+a^2c^2+2abc^2+b^2c^2+ac^3+bc^3 \end{matrix}\right\}}{3(a+b)(a+c)(b+c)} \tag{4.197}$$

and

$$v_2^{(2)}(\boldsymbol{\sigma}\cap O_3) = \frac{\left\{\begin{matrix} 3ab-1-6a^2+20a^3b-6b^2-18a^2b^2-6a^4b^2+20ab^3 \\ +18a^3b^3-6a^2b^4-21a^4b^4+3a^5b^5+3ac+20a^3c+3bc \\ -12a^2bc-12a^4bc-12ab^2c+18a^3b^2c+20b^3c+18a^2b^3c \\ -48a^4b^3c-12ab^4c-48a^3b^4c+15a^5b^4c+15a^4b^5c-6c^2 \\ -18a^2c^2-6a^4c^2-12abc^2+18a^3bc^2-18b^2c^2-6b^4c^2 \\ -54a^4b^2c^2+18ab^3c^2-84a^3b^3c^2+30a^5b^3c^2+108a^2b^2c^2 \\ -54a^2b^4c^2+54a^4b^4c^2+30a^3b^5c^2+20ac^3+18a^3c^3 \\ +20bc^3+18a^2bc^3-48a^4bc^3+18ab^2c^3-84a^3b^2c^3 \\ +30a^5b^2c^3+18b^3c^3-84a^2b^3c^3+78a^4b^3c^3-48ab^4c^3 \\ +78a^3b^4c^3+30a^2b^5c^3-6a^2c^4-21a^4c^4-12abc^4 \\ -48a^3bc^4+15a^5bc^4-21b^4c^4-54a^2b^2c^4+54a^4b^2c^4 \\ +3b^5c^5-6b^2c^4+78a^3b^3c^4+54a^2b^4c^4+15ab^5c^4+3a^5c^5 \\ -48ab^3c^4+15a^4bc^5+30a^3b^2c^5+30a^2b^3c^5+15ab^4c^5 \end{matrix}\right\}}{288(ab+ac+bc-1)^5}, \tag{4.198}$$

As $\iota_3^{(3)}(\boldsymbol{\sigma})$ and $v_2^{(4)}(\boldsymbol{\sigma}\cap O_3)$ are rather long, we are not listing them here. Putting everything together and integrating over a, b, c and multiplying by the factor of two (as $(1,\infty)^3$ is only a half-domain of integration),

$$v_3^{(1)}(O_3)_{\mathrm{III}} = \frac{376079789}{57344000} - \frac{2721\pi^2}{4096}, \tag{4.199}$$

$$v_3^{(3)}(O_3)_{\mathrm{III}} = \frac{752252545541087}{964342579200000} - \frac{90646167\pi^2}{1146880000}, \tag{4.200}$$

$$v_3^{(5)}(O_3)_{\mathrm{III}} = \frac{3995047725382306264583}{9994304539601141760000} - \frac{4195233727\pi^2}{103582531584}. \tag{4.201}$$

Contribution from all configurations

By Equation (C.118),

$$v_3^{(k)}(O_3) = \sum_{C\in\mathcal{C}(O_3)} w_{\mathrm{C}}\, v_3^{(k)}(O_3)_{\mathrm{C}} = 6v_3^{(k)}(O_3)_{\mathrm{I}} + 12v_3^{(k)}(O_3)_{\mathrm{II}} + 4v_3^{(k)}(O_3)_{\mathrm{III}}, \quad (4.202)$$

from which, immediately

$$v_3^{(1)}(O_3) = \frac{19297\pi^2}{3843840} - \frac{6619}{184320} \approx 0.013637411, \quad (4.203)$$

$$v_3^{(3)}(O_3) = \frac{1628355709\pi^2}{19864965120000} - \frac{81932629}{103219200000} \approx 0.0000152505, \quad (4.204)$$

$$v_3^{(5)}(O_3) = \frac{6356364544399\pi^2}{1611922729697280000} - \frac{205491225433}{5287025049600000} \approx 5.215748\cdot 10^{-8}. \quad (4.205)$$

4.4.3 Cube odd volumetric moments

We use the following standard representation of the unit cube ($\mathrm{vol}_3\, C_3 = 1$),

$$C_3 = \mathrm{conv}([0,0,0],[1,0,0],[0,1,0],[0,0,1],[0,1,1],[1,0,1],[1,1,0],[1,1,1]). \quad (4.206)$$

According to its genealogy $\mathcal{C}(C_3)$, it has five configurations as shown in Figure 4.12 below (or D.4 in Appendix D). Table 4.13 shows specifically which sets S of vertices in which configurations are separated by a cutting plane $\boldsymbol{\sigma}$ in our standard representation of C_3 above.

C	I	II	III	IV	V
S	[0, 0, 0]	[0, 0, 0] [0, 0, 1]	[0, 0, 0] [1, 0, 0] [0, 1, 0]	[0, 0, 0] [1, 0, 0] [0, 1, 0] [0, 0, 1]	[0, 0, 0] [1, 0, 0] [0, 1, 0] [1, 1, 0]
w_{C}	8	12	24	4	3

Table 4.13: Configurations $\mathcal{C}(C_3)$ in the standard representation of C_3.

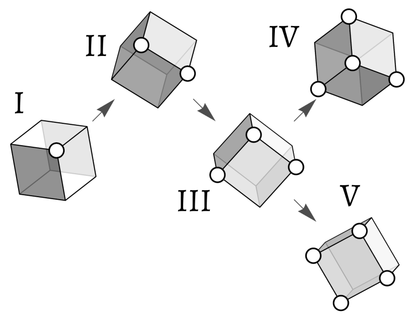


Figure 4.12: Cube genealogy

By Theorem 221 and for any $\mathrm{C} \in \mathcal{C}(C_3)$,

$$v_3^{(k)}(C_3)_{\mathrm{C}} = \frac{2}{3^k}\int_{(\mathbb{R}^3\setminus C_3^\circ)_{\mathrm{C}}} v_2^{(k+1)}(\boldsymbol{\sigma}\cap C_3)\,\zeta_3^{k+4}(\boldsymbol{\sigma})\iota_3^{(k)}(\boldsymbol{\sigma})\lambda_3(\mathrm{d}\boldsymbol{\eta}), \quad (4.207)$$

where

$$\zeta_3(\boldsymbol{\sigma}) = \frac{\mathrm{vol}_2(\boldsymbol{\sigma}\cap C_3)}{\|\boldsymbol{\eta}\|\,\mathrm{vol}_3\, C_3}, \qquad \iota_3^{(k)}(\boldsymbol{\sigma}) = \int_{C_3} |\boldsymbol{\eta}^\top\mathbf{x} - 1|^k \lambda_3(\mathrm{d}\mathbf{x}). \quad (4.208)$$

We can describe the relation $\mathbf{x} = (x,y,z)^\top \in C_3$ by the following set of three linear inequalities

$$0 < x < 1, \qquad 0 < y < 1, \qquad 0 < z < 1. \quad (4.209)$$

Configurations I – V

For Configuration I, by Equation (4.29), a, b, c must satisfy

$$a > 1, \quad b > 1, \quad a+b > 1, \quad a+c > 1, \quad b+c > 1, \quad a+b+c > 1, \tag{4.210}$$

so $(\mathbb{R}^3 \setminus C_3^\circ)_{\mathrm{I}} = (1, \infty)^3$. Similarly for other configurations. Since the analysis is similar as in the case of P_3 being a regular octahedron O_3, we only list the results from all configurations, see Table 4.14.

C	$v_3^{(1)}(C_3)_{\mathrm{C}}$	$v_3^{(3)}(C_3)_{\mathrm{C}}$	$v_3^{(5)}(C_3)_{\mathrm{C}}$
I	$\frac{391}{82944000}$	$\frac{8717}{1800338400000}$	$\frac{932274811}{50575353828920524800}$
II	$\frac{34309}{186624000}$	$\frac{648789871}{3089380694400000}$	$\frac{36816619074923}{5122861881587741 4912000}$
III	$\frac{3191\pi^2}{207360} - \frac{792503149}{5225472000}$	$\frac{182029\pi^2}{195955200} - \frac{113292736592927}{12357522777600000}$	$\frac{213033619\pi^2}{634894848000} - \frac{47144185844633987}{14235866239795200000}$
IV	$\frac{198785357}{217728000} - \frac{71\pi^2}{768}$	$\frac{22659798780677}{411917425920000} - \frac{910157\pi^2}{163296000}$	$\frac{26487208076498306317}{1921073205595403059200} - \frac{27814438817\pi^2}{19910302433280}$
V	$\frac{7}{5184}$	$\frac{29}{21870000}$	$\frac{22473091}{6271745266483200}$

Table 4.14: Sections integrals in various configurations $\mathcal{C}(C_3)$.

Contribution from all configurations

Summing up the contributions from all configurations with appropriate weights,

$$\begin{aligned} v_3^{(k)}(C_3) = \sum_{C \in \mathcal{C}(C_3)} w_{\mathrm{C}}\, v_3^{(k)}(C_3)_{\mathrm{C}} &= 8 v_3^{(k)}(C_3)_{\mathrm{I}} + 12 v_3^{(k)}(C_3)_{\mathrm{II}} \\ &+ 24 v_3^{(k)}(C_3)_{\mathrm{III}} + 4 v_3^{(k)}(C_3)_{\mathrm{IV}} + 3 v_3^{(k)}(C_3)_{\mathrm{V}}, \end{aligned} \tag{4.211}$$

from which immediately

$$v_3^{(1)}(C_3) = \frac{3977}{216000} - \frac{\pi^2}{2160} \approx 0.01384277574, \tag{4.212}$$

$$v_3^{(3)}(C_3) = \frac{8411819}{450084600000} - \frac{\pi^2}{3402000} \approx 0.0000157883, \tag{4.213}$$

$$v_3^{(5)}(C_3) = \frac{306749173351\pi^2}{124439390208000} - \frac{222558064114594378661 3}{9147967645692395520000 0} \approx 3.673225 \cdot 10^{-7}. \tag{4.214}$$

We find it striking that even though an octahedron has fewer number of configurations than a cube, the value $v_3^{(1)}(C_3)$ has been obtained by Zinani [78] by carrying out the contributions from all configurations while the octahedron case $v_3^{(1)}(O_3)$ remained unknown. Keep in mind that the configurations are the same in our canonical approach as well as in the original method using the Efron section formula.

4.5 Four dimensions

4.5.1 Pentachoron odd volumetric moments

By a *pentachoron*, we mean a 4-simplex. The regular pentachoron is then T_4. The analysis is somewhat analogous to the three-dimensional case. Now, we obtain the volumetric moments $v_4^{(k)}(T_4)$ for odd k. First, since $v_4^{(k)}(T_4)$ is an affine invariant, it must be the same as $v_4^{(k)}(\mathbb{T}_4)$, where

$$\mathbb{T}_4 = \operatorname{conv}([0,0,0,0],[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]) \tag{4.215}$$

is the canonical pentachoron. By Proposition 276, we have $\operatorname{vol}_4 \mathbb{T}_4 = 1/4! = 1/24$. Let $\boldsymbol{\eta} = (a,b,c,d)^\top$ be the Cartesian parametrisation of $\boldsymbol{\sigma} \in \mathbb{A}(4,3)$ such that $\mathbf{x} \in \boldsymbol{\sigma} \Leftrightarrow \boldsymbol{\eta}^\top \mathbf{x} = 1$. We have $\|\boldsymbol{\eta}\| = \sqrt{a^2+b^2+c^2+d^2}$. Based on symmetries $\mathcal{G}(T_4)$, there are two realisable configurations we need to consider. Thanks to affine invariancy, we can again consider instead the two $\mathcal{C}(\mathbb{T}_4)$ configurations (see Table 4.15 below).

$\mathbb{T}_4$	I	II
S	$[0,0,0,0]$	$[0,0,0,0]$ $[0,0,0,1]$
w_{C}	5	10

Table 4.15: Configurations $\mathcal{C}(\mathbb{T}_4)$.

By Theorem 221 and for any $\mathrm{C} \in \mathcal{C}(\mathbb{T}_4)$,

$$v_4^{(k)}(\mathbb{T}_4)_{\mathrm{C}} = \frac{6}{4^k} \int_{(\mathbb{R}^4 \setminus \mathbb{T}_4^\circ)_{\mathrm{C}}} v_3^{(k+1)}(\boldsymbol{\sigma} \cap \mathbb{T}_4)\, \zeta_4^{k+5}(\boldsymbol{\sigma}) \iota_4^{(k)}(\boldsymbol{\sigma}) \lambda_4(\mathrm{d}\boldsymbol{\eta}), \tag{4.216}$$

where

$$\zeta_4(\boldsymbol{\sigma}) = \frac{\operatorname{vol}_3(\boldsymbol{\sigma} \cap \mathbb{T}_4)}{\|\boldsymbol{\eta}\| \operatorname{vol}_4 \mathbb{T}_4}, \qquad \iota_4^{(k)}(\boldsymbol{\sigma}) = \int_{\mathbb{T}_4} |\boldsymbol{\eta}^\top \mathbf{x} - 1|^k \lambda_4(\mathrm{d}\mathbf{x}). \tag{4.217}$$

Again, in order to distinguish between configurations, we write $\zeta_4(\boldsymbol{\sigma})_{\mathrm{C}}$ and $\iota_4^{(k)}(\boldsymbol{\sigma})_{\mathrm{C}}$ instead of just $\zeta_4(\boldsymbol{\sigma})$ and $\iota_4^{(k)}(\boldsymbol{\sigma})$.

Configuration I

To ensure $\boldsymbol{\sigma}$ separates only the point $[0,0,0,0]$, we get from Equation (4.29), that $a > 1$, $b > 1$, $c > 1$ and $d > 1$. That means $(\mathbb{R}^4 \setminus \mathbb{T}_4^\circ)_{\mathrm{I}} = (1,\infty)^4$ is our integration domain in a,b,c,d. Denote

$$\mathbb{T}_4^{abcd} = \operatorname{conv}([0,0,0,0],[\tfrac{1}{a},0,0,0],[0,\tfrac{1}{b},0,0],[0,0,\tfrac{1}{c},0],[0,0,0,\tfrac{1}{d}]). \tag{4.218}$$

The hyperplane $\boldsymbol{\sigma}$ splits $\mathbb{T}_4$ into disjoint union of two domains $\mathbb{T}_4^+ \sqcup \mathbb{T}_4^-$, where the one closer to the origin is precisely $\mathbb{T}_4^+ = \mathbb{T}_4^{abcd}$. Therefore

$$\begin{aligned} \iota_4^{(k)}(\boldsymbol{\sigma})_{\mathrm{I}} &= \int_{\mathbb{T}_4^+} (1 - \boldsymbol{\eta}^\top \mathbf{x})^k \lambda_4(\mathrm{d}\mathbf{x}) + \int_{\mathbb{T}_4^-} (\boldsymbol{\eta}^\top \mathbf{x} - 1)^k \lambda_4(\mathrm{d}\mathbf{x}) \\ &= \int_{\mathbb{T}_4} (\boldsymbol{\eta}^\top \mathbf{x} - 1)^k \lambda_4(\mathrm{d}\mathbf{x}) - (1-(-1)^k) \int_{\mathbb{T}_4^{abcd}} (\boldsymbol{\eta}^\top \mathbf{x} - 1)^k \lambda_4(\mathrm{d}\mathbf{x}). \end{aligned} \tag{4.219}$$

for any k integer. These integrals are easy to compute. Mathematica Code 5 computes $\iota_4^{(k)}(\boldsymbol{\sigma})_{\mathrm{I}}$ for various values of k. Running the code for $k = 1$ and $k = 3$, we obtain

$$\iota_4^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{1}{120}\left(\frac{2}{abcd} + a + b + c + d - 5\right), \tag{4.220}$$

$$\begin{aligned}\iota_4^{(3)}(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{1}{840}\bigg(&\frac{2}{abcd} + a^3 + a^2b + a^2c + a^2d - 7a^2 + ab^2 + abc + abd \\ &- 7ab + ac^2 + acd - 7ac + ad^2 - 7ad + 21a + b^3 + b^2c + b^2d \\ &- 7b^2 + bc^2 + bcd - 7bc + bd^2 - 7bd + 21b + c^3 + c^2d - 7c^2 \\ &+ cd^2 - 7cd + 21c + d^3 - 7d^2 + 21d - 35\bigg).\end{aligned} \tag{4.221}$$

Alternatively, at least for $\iota_4^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}}$, we can use its geometric interpretation to derive it by hand. Let $\mathbf{M}$ and $\mathbf{M}^+$ be the centerpoints of $\mathbb{T}_4$ and $\mathbb{T}_4^+$, respectively. Clearly, since $\mathbf{M}$ and $\mathbf{M}^+$ are both centerpoints of pentachora (4-simplices),

$$\begin{aligned}\mathbf{M} &= \tfrac{1}{5}(\mathbf{0} + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) = [\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}], \\ \mathbf{M}^+ &= \tfrac{1}{5}(\mathbf{0} + \tfrac{1}{a}\mathbf{e}_1 + \tfrac{1}{b}\mathbf{e}_2 + \tfrac{1}{c}\mathbf{e}_3 + \tfrac{1}{d}\mathbf{e}_4) = [\tfrac{1}{5a}, \tfrac{1}{5b}, \tfrac{1}{5c}, \tfrac{1}{5d}].\end{aligned} \tag{4.222}$$

Then, by Equation (4.35) and since $\operatorname{vol}_4 \mathbb{T}_4 = \frac{1}{24}$ and $\operatorname{vol}_4 \mathbb{T}_4^+ = \frac{1}{24abcd}$,

$$\begin{aligned}&\iota_4^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}} = (\boldsymbol{\eta}^\top \mathbf{M} - 1)\operatorname{vol}_4 \mathbb{T}_4 + 2(1 - \boldsymbol{\eta}^\top \mathbf{M}^+)\operatorname{vol}_4 \mathbb{T}_4^+ \\ &= (\tfrac{a+b+c+d}{5} - 1)\tfrac{1}{24} + 2(1 - \tfrac{4}{5})\tfrac{1}{24abcd} = \tfrac{1}{120}(a + b + c + d - 5 + \tfrac{2}{abcd}).\end{aligned} \tag{4.223}$$

Denote $T_3^{abcd} = \operatorname{conv}([1/a, 0, 0, 0], [0, 1/b, 0, 0], [0, 0, 1/c, 0], [0, 0, 0, 1/d])$, then the intersection of the hyperplane $\boldsymbol{\sigma}$ with $\mathbb{T}_4$ is precisely tetrahedron T_3^{abcd}. That is,

$$\boldsymbol{\sigma} \cap \mathbb{T}_4 = T_3^{abcd}. \tag{4.224}$$

By Equation (4.15), the distance from T_3^{abcd} to the origin is $\operatorname{dist}_{\boldsymbol{\sigma}}(\mathbf{0}) = 1/\|\boldsymbol{\eta}\|$. By base-height splitting,

$$\frac{\operatorname{vol}_4 \mathbb{T}_4}{abc} = \operatorname{vol}_4 \mathbb{T}_4^+ = \frac{1}{4}\operatorname{dist}_{\boldsymbol{\sigma}}(\mathbf{0})\operatorname{vol}_3 T_3^{abcd} = \frac{\operatorname{vol}_3(\boldsymbol{\sigma} \cap \mathbb{T}_4)}{4\|\boldsymbol{\eta}\|}, \tag{4.225}$$

from which we immediately get

$$\zeta_4(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{\operatorname{vol}_3(\boldsymbol{\sigma} \cap \mathbb{T}_4)}{\|\boldsymbol{\eta}\| \operatorname{vol}_4 \mathbb{T}_4} = \frac{4}{abcd}. \tag{4.226}$$

Finally, by scale affinity (we have $n_{\mathrm{I}} = 4$),

$$v_3^{(k+1)}(\boldsymbol{\sigma} \cap \mathbb{T}_4) = v_3^{(k+1)}(T_3^{abcd}) = v_3^{(k+1)}(T_3), \tag{4.227}$$

which implies for $k = 1, 2, 3$ that (see Table 3.1 or Tables 3.5 and 3.6 and Equation (4.164)),

$$\begin{aligned}&v_3^{(2)}(\boldsymbol{\sigma} \cap \mathbb{T}_4) = \frac{3}{4000}, \qquad v_3^{(3)}(\boldsymbol{\sigma} \cap \mathbb{T}_4) = \frac{733}{12600000} + \frac{79\pi^2}{2424922500}, \\ &v_3^{(4)}(\boldsymbol{\sigma} \cap \mathbb{T}_4) = \frac{871}{123480000}.\end{aligned} \tag{4.228}$$

Putting everything into the integral in Equation (4.216), we get when $k = 1$,

$$v_4^{(1)}(\mathbb{T}_4)_{\mathrm{I}} = \frac{24}{625}\int_1^\infty\int_1^\infty\int_1^\infty\int_1^\infty \frac{2+abcd(a+b+c+d-5)}{a^7b^7c^7d^7}\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\,\mathrm{d}d = \frac{1}{16875}. \tag{4.229}$$

For $k = 3$ and $k = 5$, we get

$$v_4^{(3)}(\mathbb{T}_4)_{\mathrm{I}} = \tfrac{26061191}{1600967592000000}, \qquad v_4^{(5)}(\mathbb{T}_4)_{\mathrm{I}} = \tfrac{27909940019}{504189521813376000000}. \tag{4.230}$$

Configuration II

In this scenario, $\boldsymbol{\sigma}$ separates two points $[0,0,0,0]$ and $[0,0,0,1]$ from $\mathbb{T}_4$. By Equation (4.29), we get $a > 1$, $b > 1$, $c > 1$ and $d < 1$. We can split the condition for d into to cases: either $0 < d < 1$ or $d < 0$. In fact, both options give the same factor since they are symmetrical as they correspond to two possibilities where $\boldsymbol{\sigma}$ hits $\mathcal{A}([0,0,0,0],[0,0,0,1])$. Therefore we only consider the integration half-domain

$$(\mathbb{R}^4 \setminus \mathbb{T}_4^\circ)^*_{\mathrm{II}} = (1,\infty)^3 \times (0,1) \tag{4.231}$$

and in the end multiply the result twice. The hyperplane $\boldsymbol{\sigma}$ intersects $\mathbb{T}_4$ at points $\frac{1}{a}\mathbf{e}_1$, $\frac{1}{b}\mathbf{e}_2$, $\frac{1}{c}\mathbf{e}_3$ (already in Configuration I) and additionally at

$$\begin{aligned}\mathbf{A} &= \tfrac{1}{d}\mathbf{e}_4 + \alpha(\tfrac{1}{a}\mathbf{e}_1 - \tfrac{1}{d}\mathbf{e}_4) = [\tfrac{1-d}{a-d}, 0, 0, \tfrac{a-1}{a-d}],\\ \mathbf{B} &= \tfrac{1}{d}\mathbf{e}_4 + \beta(\tfrac{1}{b}\mathbf{e}_2 - \tfrac{1}{d}\mathbf{e}_4) = [0, \tfrac{1-d}{b-d}, 0, \tfrac{b-1}{b-d}],\\ \mathbf{C} &= \tfrac{1}{d}\mathbf{e}_4 + \gamma(\tfrac{1}{c}\mathbf{e}_3 - \tfrac{1}{d}\mathbf{e}_4) = [0, 0, \tfrac{1-d}{c-d}, \tfrac{c-1}{c-d}],\end{aligned} \tag{4.232}$$

where we denote $\alpha = \frac{a(1-d)}{a-d}$, $\beta = \frac{b(1-d)}{b-d}$ and $\gamma = \frac{c(1-d)}{c-d}$. Thus, the hyperplane $\boldsymbol{\sigma}$ splits $\mathbb{T}_4$ into disjoint union of two domains $\mathbb{T}_4^+ \sqcup \mathbb{T}_4^-$, where $\mathbb{T}_4^+$ being the one closer to the origin. Let $\mathbb{T}_4^{abcd} = \operatorname{conv}(\mathbf{0}, \frac{1}{a}\mathbf{e}_1, \frac{1}{b}\mathbf{e}_2, \frac{1}{c}\mathbf{e}_3, \frac{1}{d}\mathbf{e}_4)$ and $\mathbb{T}_4^* = \operatorname{conv}(\mathbf{e_4}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \frac{1}{d}\mathbf{e}_4)$, or explicitly

$$\mathbb{T}_4^{abcd} = \operatorname{conv}\left([0,0,0,0], \left[\tfrac{1}{a},0,0,0\right], \left[0,\tfrac{1}{b},0,0\right], \left[0,0,\tfrac{1}{c},0\right], \left[0,0,0,\tfrac{1}{d}\right]\right), \tag{4.233}$$

$$\mathbb{T}_4^* = \operatorname{conv}\left([0,0,0,1], \left[\tfrac{1-d}{a-d},0,0,\tfrac{a-1}{a-d}\right], \left[0,\tfrac{1-d}{b-d},0,\tfrac{b-1}{b-d}\right], \left[0,0,\tfrac{1-d}{c-d},\tfrac{c-1}{c-d}\right], \left[0,0,0,\tfrac{1}{d}\right]\right), \tag{4.234}$$

Then we can write $\mathbb{T}_4^+ = \mathbb{T}_4^{abcd} \setminus \mathbb{T}_4^* = \operatorname{conv}(\mathbf{0}, \mathbf{e}_4, \frac{1}{a}\mathbf{e}_1, \frac{1}{b}\mathbf{e}_2, \frac{1}{c}\mathbf{e}_3, \mathbf{A}, \mathbf{B}, \mathbf{C})$ and thus, by inclusion/exclusion

$$\begin{aligned}\iota_4^{(k)}(\boldsymbol{\sigma})_{\mathrm{II}} = &\int_{\mathbb{T}_4} (\boldsymbol{\eta}^\top\mathbf{x} - 1)^k \lambda_4(\mathrm{d}\mathbf{x}) - (1-(-1)^k)\int_{\mathbb{T}_4^{abcd}} (\boldsymbol{\eta}^\top\mathbf{x} - 1)^k \lambda_4(\mathrm{d}\mathbf{x})\\ &+ (1-(-1)^k)\int_{\mathbb{T}_4^*} (\boldsymbol{\eta}^\top\mathbf{x} - 1)^k \lambda_4(\mathrm{d}\mathbf{x}).\end{aligned} \tag{4.235}$$

for any k integer. These integrals are again easy to compute. Mathematica Code 6 computes $\iota_4^{(k)}(\boldsymbol{\sigma})_{\mathrm{II}}$ for various values of k. Running the code for $k = 1$ and $k = 3$, we obtain

$$\iota_4^{(1)}(\boldsymbol{\sigma})_{\mathrm{II}} = \iota_4^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}} - \frac{(1-d)^5}{60d(a-d)(b-d)(c-d)}, \tag{4.236}$$

$$\iota_4^{(3)}(\boldsymbol{\sigma})_{\mathrm{II}} = \iota_4^{(3)}(\boldsymbol{\sigma})_{\mathrm{I}} - \frac{(1-d)^7}{420d(a-d)(b-d)(c-d)}. \tag{4.237}$$

where the functions $\iota_4^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}}$ and $\iota_4^{(3)}(\boldsymbol{\sigma})_{\mathrm{I}}$ are given by Equations (4.220) and (4.221) from the configuration I. The calculation of $\iota_4^{(k)}(\boldsymbol{\sigma})_{\mathrm{II}}$ is again trivial when $k = 1$ and can be done by hand from its geometric interpretation. Scaling the volume of $\mathbb{T}_4^{abcd}$, we get

$$\mathrm{vol}_4\,\mathbb{T}_4^* = \tfrac{\alpha\beta\gamma(1-d)}{24abcd}. \tag{4.238}$$

Let $\mathbf{M}$, $\mathbf{M}^{abcd}$, $\mathbf{M}^*$ and $\mathbf{M}^+$ be the centerpoints of $\mathbb{T}_4$, $\mathbb{T}_4^{abcd}$, $\mathbb{T}_4^*$ and $\mathbb{T}_4^+$, respectively. Trivially, $\mathbf{M} = [\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}]$ and $\mathbf{M}^{abcd} = [\frac{1}{5a},\frac{1}{5b},\frac{1}{5c},\frac{1}{5d}]$. Since also $\mathbf{M}^*$ is a centerpoint of a 4-simplex, namely $\mathbb{T}_4^* = \mathrm{conv}(\mathbf{e_4},\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{e}_4/d)$,

$$\mathbf{M}^* = \tfrac{1}{5}(\mathbf{e}_4 + \mathbf{A} + \mathbf{B} + \mathbf{C} + \tfrac{1}{d}\mathbf{e}_4) = [\tfrac{\alpha}{5a},\tfrac{\beta}{5b},\tfrac{\gamma}{5c},\tfrac{4+d-\alpha-\beta-\gamma}{5d}]. \tag{4.239}$$

Since $\mathbb{T}_4^+ = \mathbb{T}_4^{abcd}\setminus\mathbb{T}_4^*$, we have $\mathrm{vol}_4\,\mathbb{T}_4^+ = \mathrm{vol}_4\,\mathbb{T}_4^{abcd} - \mathrm{vol}_4\,\mathbb{T}_4^*$ and by mass balance,

$$\mathbf{M}^+\,\mathrm{vol}_4\,\mathbb{T}_4^+ = \mathbf{M}^{abcd}\,\mathrm{vol}_4\,\mathbb{T}_4^{abcd} - \mathbf{M}^*\,\mathrm{vol}_4\,\mathbb{T}_4^*. \tag{4.240}$$

Solving for $\mathbf{M}^+$ is left as an exercise for the reader, but it turns out one does not need its knowledge to obtain $\iota_4^{(1)}(\boldsymbol{\sigma})_{\mathrm{II}}$. To see this, plugging the mass balance directly into Equation (4.35) and by our previous relation for $\iota_4^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}}$, we get

$$\begin{aligned}\iota_4^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}} &= (\boldsymbol{\eta}^\top\mathbf{M} - 1)\,\mathrm{vol}_4\,\mathbb{T}_4 + 2(1-\boldsymbol{\eta}^\top\mathbf{M}^+)\,\mathrm{vol}_4\,\mathbb{T}_4^+ \\ &= \iota_4^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}} - 2(1-\boldsymbol{\eta}^\top\mathbf{M}^*)\,\mathrm{vol}_4\,\mathbb{T}_4^* = \iota_4^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}} - \tfrac{\alpha\beta\gamma(1-d)^2}{60abcd},\end{aligned} \tag{4.241}$$

which matches Equation (4.236). By denoting

$$T_3^{abcd} = \mathrm{conv}\left(\left[\tfrac{1}{a},0,0,0\right],\left[0,\tfrac{1}{b},0,0\right],\left[0,0,0,\tfrac{1}{c}\right],\left[0,0,0,\tfrac{1}{d}\right]\right), \tag{4.242}$$

$$T_3^* = \mathrm{conv}\left(\left[\tfrac{1-d}{a-d},0,0,\tfrac{a-1}{a-d}\right],\left[0,\tfrac{1-d}{b-d},0,\tfrac{b-1}{b-d}\right],\left[0,0,\tfrac{1-d}{c-d},\tfrac{c-1}{c-d}\right],\left[0,0,0,\tfrac{1}{d}\right]\right), \tag{4.243}$$

we have for the intersection of $\boldsymbol{\sigma}$ with $\mathbb{T}_4$,

$$\begin{aligned}\boldsymbol{\sigma}\cap\mathbb{T}_4 = T_3^{abcd}\setminus T_3^* = \mathrm{conv}\Big(&\left[\tfrac{1}{a},0,0,0\right],\left[0,\tfrac{1}{b},0,0\right],\left[0,0,0,\tfrac{1}{c}\right],\\ &\left[\tfrac{1-d}{a-d},0,0,\tfrac{a-1}{a-d}\right],\left[0,\tfrac{1-d}{b-d},0,\tfrac{b-1}{b-d}\right],\left[0,0,\tfrac{1-d}{c-d},\tfrac{c-1}{c-d}\right]\Big),\end{aligned} \tag{4.244}$$

so $n_{\mathrm{II}} = 6$. By scale affinity

$$v_3^{(k+1)}(\boldsymbol{\sigma}\cap\mathbb{T}_4) = v_3^{(k+1)}(T_3^{abcd}\setminus T_3^*) = v_3^{(k+1)}(\mathbb{U}_3^{\alpha\beta\gamma}), \tag{4.245}$$

where

$$\mathbb{U}_3^{\alpha\beta\gamma} = \mathrm{conv}([\alpha,0,0],[0,\beta,0],[0,0,\gamma],[1,0,0],[0,1,0],[0,0,1])$$

is a canonical truncated tetradedron with the already introduced

$$\alpha = \frac{a(1-d)}{a-d}, \qquad \beta = \frac{b(1-d)}{b-d}, \qquad \gamma = \frac{c(1-d)}{c-d}. \tag{4.246}$$

See Figure 4.13 below for an illustration of $\mathbb{U}_3^{\alpha\beta\gamma}$ and its volumetric moments.

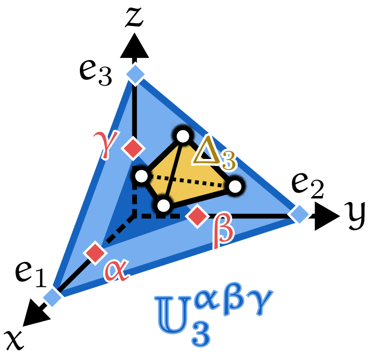


Figure 4.13: Mean section moments in the second $\mathcal{C}(\mathbb{T}_4)$ configuration

Since $\operatorname{vol}_3 \mathbb{U}_3^{\alpha\beta\gamma} = \frac{1}{3!}(1-\alpha\beta\gamma)$, we can write in general,

$$v_3^{(k+1)}(\mathbb{U}_3^{\alpha\beta\gamma}) = \left(\frac{6}{1-\alpha\beta\gamma}\right)^{k+5} \int_{(\mathbb{U}_3^{\alpha\beta\gamma})^4} \Delta_3^{k+1} \, \mathrm{d}\mathbf{x}_0 \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2 \mathrm{d}\mathbf{x}_3, \tag{4.247}$$

We would like to find $v_3^{(k+1)}(\mathbb{U}_3^{\alpha\beta\gamma})$ for odd k. This is, luckily, trivial, since we are now integrating even powers of

$$\Delta_3 = \frac{1}{3!} \left|\det(\mathbf{x}_1 - \mathbf{x}_0 \,|\, \mathbf{x}_2 - \mathbf{x}_0 \,|\, \mathbf{x}_3 - \mathbf{x}_0)\right|. \tag{4.248}$$

The calculation can be carried out in Mathematica using Code 7, which exploits the symmetries and uses inclusion/exclusion. Running the code for $k = 1$, we get

$$v_3^{(2)}(\mathbb{U}_3^{\alpha\beta\gamma}) = \frac{3\left\{\begin{array}{c} +10\alpha^2\beta\gamma-16\alpha^5\beta^5\gamma^5+10\alpha^5\beta^5\gamma^4+10\alpha^5\beta^4\gamma^5 \\ -4\alpha^5\beta^5\gamma^3-2\alpha^5\beta^4\gamma^4-4\alpha^5\beta^3\gamma^5+10\alpha^4\beta^5\gamma^5-2\alpha^4\beta^5\gamma^4 \\ -2\alpha^4\beta^4\gamma^5+9\alpha^4\beta^4\gamma^2+2\alpha^4\beta^3\gamma^3-10\alpha^4\beta^3\gamma^2+9\alpha^4\beta^2\gamma^4 \\ -10\alpha^4\beta^2\gamma^3+9\alpha^4\beta^2\gamma^2-4\alpha^3\beta^5\gamma^5+2\alpha^3\beta^4\gamma^3-10\alpha^3\beta^4\gamma^2 \\ +2\alpha^3\beta^3\gamma^4+2\alpha^3\beta^3\gamma^2-10\alpha^3\beta^2\gamma^4+2\alpha^3\beta^2\gamma^3-4\alpha^3\beta\gamma \\ +\alpha^6\beta^6\gamma^6-10\alpha^2\beta^4\gamma^3+9\alpha^2\beta^4\gamma^2-10\alpha^2\beta^3\gamma^4+2\alpha^2\beta^3\gamma^3 \\ +9\alpha^2\beta^4\gamma^4+9\alpha^2\beta^2\gamma^4-2\alpha^2\beta^2\gamma-2\alpha^2\beta\gamma^2-4\alpha\beta^3\gamma \\ -2\alpha\beta^2\gamma^2+10\alpha\beta^2\gamma-4\alpha\beta\gamma^3+10\alpha\beta\gamma^2-16\alpha\beta\gamma+1 \end{array}\right\}}{4000(1-\alpha\beta\gamma)^6}, \tag{4.249}$$

Finally, by definition (alternatively by Equation (4.36))

$$\zeta_4(\boldsymbol{\sigma})_{\mathrm{II}} = \frac{\operatorname{vol}_3(\boldsymbol{\sigma} \cap \mathbb{T}_4)}{\|\boldsymbol{\eta}\| \operatorname{vol}_4 \mathbb{T}_4} = (1-\alpha\beta\gamma)\frac{\operatorname{vol}_3 T_3^{abc}}{\|\boldsymbol{\eta}\| \operatorname{vol}_4 \mathbb{T}_4} = (1-\alpha\beta\gamma)\zeta_4(\boldsymbol{\sigma})_{\mathrm{I}}. \tag{4.250}$$

Before we proceed to evaluate the final integral, we make the following change of variables $(a,b,c,d) \to (\alpha,\beta,\gamma,d)$ via transformation Equations (4.246), which transform the integration half-domain into

$$(\mathbb{R}^4 \setminus \mathbb{T}_4^\circ)^*_{\mathrm{II}} \,|_{\alpha,\beta\gamma,d} = (1-d,1)^3 \times (0,1). \tag{4.251}$$

Note that, if d is treated as a parameter, the variables a,b,c depend on α,β,γ separately. As a consequence,

$$\mathrm{d}a = \frac{d(1-d)\;\mathrm{d}\alpha}{(1-d-\alpha)^2}, \qquad \mathrm{d}b = \frac{d(1-d)\;\mathrm{d}\beta}{(1-d-\beta)^2}, \qquad \mathrm{d}c = \frac{d(1-d)\;\mathrm{d}\gamma}{(1-d-\alpha)^2} \tag{4.252}$$

and thus one has for the of transformation of measure

$$\lambda_4(\mathrm{d}\boldsymbol{\eta}) = \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\,\mathrm{d}d = \frac{d^3(1-d)^3\ \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\mathrm{d}d}{(1-d-\alpha)^2(1-d-\beta)^2(1-d-\gamma)^2}. \tag{4.253}$$

Putting everything into the integral in Equation (4.216), we get when $k = 1$ and after integrating out α, β, γ,

$$\begin{aligned} v_4^{(1)}(\mathbb{T}_4)_{\mathrm{II}} &= \frac{1}{1406250}\int_0^1 \Big(d^3 p_0 + 180d^2(1-d)^3 p_1 \ln(1-d) \\ &+ 10800d(1-d)^3 p_2 \ln^2(1-d) + 216000(1-d)^3 p_3 \ln^3(1-d)\Big)\frac{\mathrm{d}d}{d^{25}}, \end{aligned} \tag{4.254}$$

where

$$\begin{aligned} p_0 &= 32480784000 - 324807840000d + 1556229024000d^2 \\ &- 4749037776000d^3 + 10279357367400d^4 - 16555175611200d^5 \\ &+ 20253161331700d^6 - 18987688381900d^7 + 13740024940130d^8 \\ &- 7798431753680d^9 + 3604300565845d^{10} - 1440768739775d^{11} \\ &+ 518639866862d^{12} - 161581999478d^{13} + 39317696413d^{14} \\ &- 6685392751d^{15} + 700753210d^{16} - 34837616d^{17} \\ &+ 6112d^{18} - 3272d^{19} + 784d^{20}, \end{aligned} \tag{4.255}$$

$$\begin{aligned} p_1 &= 541346400 - 4060098000d + 14794437000d^2 - 34585687500d^3 \\ &+ 56747312360d^4 - 67139592080d^5 + 57686267770d^6 - 36408101115d^7 \\ &+ 17574730626d^8 - 7114914681d^9 + 2659305113d^{10} - 888330365d^{11} \\ &+ 229856455d^{12} - 40385468d^{13} + 4279933d^{14} - 213224d^{15}, \end{aligned} \tag{4.256}$$

$$\begin{aligned} p_2 &= 9022440 - 72179520d + 279656230d^2 - 694452010d^3 + 1216036193d^4 \\ &- 1552509188d^5 + 1460599749d^6 - 1021377960d^7 + 544097150d^8 \\ &- 234903968d^9 + 90292498d^{10} - 32050399d^{11} + 9632345d^{12} \\ &- 2161105d^{13} + 327799d^{14} - 30254d^{15} + 1312d^{16}, \end{aligned} \tag{4.257}$$

$$\begin{aligned} p_3 &= 150374 - 1278179d + 5249902d^2 - 13810685d^3 + 25712115d^4 \\ &- 35209551d^5 + 35968805d^6 - 27633760d^7 + 16221440d^8 - 7575685d^9 \\ &+ 3035423d^{10} - 1117957d^{11} + 369741d^{12} - 99030d^{13} + 19440d^{14} \\ &- 2588d^{15} + 211d^{16} - 8d^{17}. \end{aligned} \tag{4.258}$$

The last d integration can be carried out by Mathematica (or tediously using Beta function derivatives). We get

$$v_4^{(1)}(\mathbb{T}_4)_{\mathrm{II}} = \frac{89}{270000} - \frac{2173\pi^2}{520269750}. \tag{4.259}$$

For higher values of k, the integration possesses similar difficulty, we got

$$\begin{aligned} v_4^{(3)}(\mathbb{T}_4)_{\mathrm{II}} &= \tfrac{3947568673}{80048379600000000} + \tfrac{63065881\pi^2}{396699961407750000}, \\ v_4^{(5)}(\mathbb{T}_4)_{\mathrm{II}} &= \tfrac{700536944899}{70586533053872640000000} - \tfrac{1262701803371\pi^2}{35570432728713733250400000}. \end{aligned} \tag{4.260}$$

Contribution from all configurations

By Equation (C.118) and by affine invariancy,

$$v_4^{(k)}(T_4) = \sum_{C \in \mathcal{C}(T_4)} w_{\mathrm{C}}\, v_4^{(k)}(T_4)_{\mathrm{C}} = 5v_4^{(k)}(\mathbb{T}_4)_{\mathrm{I}} + 10v_4^{(k)}(\mathbb{T}_4)_{\mathrm{II}}, \tag{4.261}$$

from which immediately

$$v_4^{(1)}(T_4) = \frac{97}{27000} - \frac{2173\pi^2}{52026975} \approx 0.0031803708487, \tag{4.262}$$

$$v_4^{(3)}(T_4) = \frac{1955399}{3403417500000} + \frac{63065881\pi^2}{39669996140775000} \approx 5.9023 \cdot 10^{-7}, \tag{4.263}$$

$$v_4^{(5)}(T_4) = \frac{12443146181}{9803685146371200000} - \frac{1262701803371\pi^2}{3557043272871373325040000} \approx 1.26573 \cdot 10^{-9}. \tag{4.264}$$

Monte-Carlo simulation shows that the value $v_4^{(1)}(T_4)$ fits withing the 95% confidence interval $(0.00318034, 0.00318043)$ obtained from 4×10^{10} trials of randomly generated 4-simplices in T_4 (We wrote and run Fortran program `simplex.f90` for that purpose, see Attachements).

Moreover, by Buchta's relation (Equation (5.36)), we get the value of mean 4-volume of a convex hull of 6 points in the unit pentachoron as

$$v_5^{(1)}(T_4) = 3v_4^{(1)}(T_4) = \frac{97}{9000} - \frac{2173\pi^2}{17342325} \approx 0.00954111. \tag{4.265}$$

4.5.2 Hexadecachoron first volumetric moment

The *hexadecachoron*, 16-cell or the 4-cross-polytope are alternative names of 4-orthoplex O_4, a polychoron with standard representation with $\mathrm{vol}_4\, O_4 = 2/3$,

$$\begin{aligned} O_4 = \operatorname{conv}(&[1,0,0,0],[0,1,0,0][0,0,1,0],[0,0,0,1],\\ &[-1,0,0,0],[0,-1,0,0],[0,0,-1,0],[0,0,0,-1]). \end{aligned} \tag{4.266}$$

The symmetry group $\mathcal{G}(O_4)$ is isomorphic to Coxeter group $\mathcal{B}_4$ of order $|\mathcal{B}_4| = 384$. We can describe the symmetry group using its four generators (one reflection, two rotations and one double rotation) of permutations acting on vertices indexed as in Equation (4.266). In cycle notation (excluding fixed points), we have

$$\mathcal{G}(O_4) = \langle (48), (2367), (1256), (1256), (3478) \rangle < \mathcal{S}_8, \tag{4.267}$$

where $\langle \cdot \rangle$ denotes the *algebraic closure* and $<$ the relation of being a subgroup. From this group, we can generate 14 configurations, out of which only 4 are realisable and section equivalent. These consist the genealogy $\mathcal{C}(C_4)$. Table 4.16 shows specifically which sets S of vertices in which configurations are separated by a cutting plane $\boldsymbol{\sigma}$ in our standard representation of O_4 in Equation (4.266). By similar treatment as in the case of O_3, we can easily find inequalities which describe O_4^+ and thus $\boldsymbol{\sigma} \cap O_4^+$. We only list the section integrals obtained from all configurations, see Table 4.17.

C	I	II	III	IV
S	$[0,0,0,1]$	$[0,0,1,0]$ $[0,0,0,1]$	$[0,1,0,0]$ $[0,0,1,0]$ $[0,0,0,1]$	$[1,0,0,0]$ $[0,1,0,0]$ $[0,0,1,0]$ $[0,0,0,1]$
w_{C}	8	24	32	16
n_{C}	6	10	12	0

Table 4.16: Configurations $\mathcal{C}(O_4)$ in the standard representation of O_4.

C	$v_4^{(1)}(O_4)_{\mathrm{C}}$
I	$\frac{2400441939\zeta(3)}{320000} - \frac{7176576945806282575133 9}{8136689713152000000} + \frac{173050612310219\pi^2}{3547315200000} - \frac{127327345788535137\ln 2}{130068224000000}$
II	$\frac{1157792018850958716538 9181}{2072472081039360000000} - \frac{13611484420925379\zeta(3)}{2928808960000} + \frac{69987566888072781151\pi^2}{1461358518681600000} - \frac{71866300533\pi^2\ln 2}{1040060000}$
III	(not yet derived)
IV	(not yet derived)

Table 4.17: Sections integrals in various configurations $\mathcal{C}(O_4)$.

By Equation (C.118), considering the contributions from all configurations,

$$v_4^{(k)}(O_4) = \sum_{C\in\mathcal{C}(O_4)} w_{\mathrm{C}}\, v_4^{(k)}(O_4)_{\mathrm{C}} = 8v_4^{(k)}(O_4)_{\mathrm{I}} + 24v_4^{(k)}(O_4)_{\mathrm{II}} + 32v_4^{(k)}(O_4)_{\mathrm{III}} + 16v_4^{(k)}(O_4)_{\mathrm{IV}} \tag{4.268}$$

from which immediately for $k=1$,

$$\begin{aligned} v_4^{(1)}(O_4) &= XXX \\ &\approx XXX, \end{aligned} \tag{4.269}$$

Remark 227. As of now, we have not found the expressions for $v_4^{(1)}(O_4)_{\mathrm{C}}$ for configurations $C \in \{\mathrm{III}, \mathrm{IV}\}$, we have succeeded in writing them as explicit double integrals, but the shear scope of them have not enabled us to calculate using our own computers. However, we think this might be doable and will be part of our future papers. We have also attempted to find higher odd moments, however, the section integrals became too complicated. The third and the fifth moment are in principle derivable but it would be extraordinarily time consuming. We found at least in the first configuration

$$\begin{aligned} v_4^{(3)}(O_4)_{\mathrm{I}} = &\frac{8928188080691679\zeta(3)}{7867596800000} - \frac{137576799361704969614180657626378751495 11}{1009767941418745678003804569600000000 0} \\ &+ \frac{420783881199433246283869\pi^2}{1357358340088791040000000} - \frac{1382007704595015894993581933 29\ln 2}{20380735476433197465600000000} \end{aligned} \tag{4.270}$$

4.5.3 Tesseract odd volumetric moments

By *tesseract*, we mean C_4 (4-cube). The standard representation of the unit tesseract with $\mathrm{vol}_4\, C_4 = 1$ is

$$C_4 = \mathrm{conv}([0,0,0,0],[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1],[1,1,0,0], [1,0,1,0],[1,0,0,1],[0,1,1,0],[0,1,0,1],[0,0,1,1],[1,1,1,0], [1,1,0,1],[1,0,1,1],[0,1,1,1],[1,1,1,1]). \tag{4.271}$$

The symmetry group $\mathcal{G}(C_4)$ is isomorphic to Coxeter group $\mathcal{B}_4$ of order $|\mathcal{B}_4| = 384$. We can describe the symmetry group using its four generators (one reflection, two rotations and one double rotation) of permutations acting on vertices indexed as in Equation (4.271). In cycle notation, we have

$$\begin{aligned}\mathcal{G}(C_4) = \langle &(1,5),(2,8),(3,10),(4,11),(6,13),(7,14),(9,15),(12,16),\\ &(1,3,9,4),(2,6,12,7),(5,10,15,11),(8,13,16,14),\\ &(1,2,6,3),(4,7,12,9),(5,8,13,10)(11,14,16,15),\\ &(1,7,16,10),(2,12,15,5),(3,4,14,13),(6,9,11,8)\rangle < \mathcal{S}_{16}.\end{aligned} \tag{4.272}$$

From this group, we can generate 402 configurations, out of which 14 are realisable and section equivalent. These consist the genealogy $\mathcal{C}(C_4)$. Table 4.18 shows specifically which sets S of vertices in which configurations are separated by a cutting plane $\boldsymbol{\sigma}$ in our standard representation of C_4 in Equation (4.271).

C	I	II	III	IV	V	VI	VII
S	[0, 0, 0, 0]	[0, 0, 0, 0] [1, 0, 0, 0]	[0, 0, 0, 0] [1, 0, 0, 0] [0, 1, 0, 0]	[0, 0, 0, 0] [1, 0, 0, 0] [0, 1, 0, 0] [1, 1, 0, 0]	[0, 0, 0, 0] [1, 0, 0, 0] [0, 1, 0, 0] [0, 0, 1, 0]	[0, 0, 0, 0] [1, 0, 0, 0] [0, 1, 0, 0] [0, 0, 1, 0] [0, 0, 0, 1]	[0, 0, 0, 0] [1, 0, 0, 0] [0, 1, 0, 0] [0, 0, 1, 0] [1, 1, 0, 0]
w_C	16	32	96	24	64	16	192
n_C	4	6	8	8	10	12	10

C	VIII	IX	X	XI	XII	XIII	XIV
S	[0, 0, 0, 0] [1, 0, 0, 0] [0, 1, 0, 0] [0, 0, 1, 0] [1, 0, 1, 0] [1, 1, 0, 0]	[0, 0, 0, 0] [1, 0, 0, 0] [0, 1, 0, 0] [0, 0, 1, 0] [0, 0, 0, 1] [1, 1, 0, 0]	[0, 0, 0, 0] [1, 0, 0, 0] [0, 1, 0, 0] [0, 0, 1, 0] [0, 1, 1, 0] [1, 0, 1, 0] [1, 1, 0, 0]	[0, 0, 0, 0] [1, 0, 0, 0] [0, 1, 0, 0] [0, 0, 1, 0] [0, 0, 0, 1] [1, 0, 1, 0] [1, 1, 0, 0]	[0, 0, 0, 0] [1, 0, 0, 0] [0, 1, 0, 0] [0, 0, 1, 0] [0, 1, 1, 0] [1, 0, 1, 0] [1, 1, 0, 0] [1, 1, 1, 0]	[0, 0, 0, 0] [1, 0, 0, 0] [0, 1, 0, 0] [0, 0, 1, 0] [0, 0, 0, 1] [1, 1, 0, 0] [1, 0, 1, 0] [1, 0, 0, 1]	[0, 0, 0, 0] [1, 0, 0, 0] [0, 1, 0, 0] [0, 0, 1, 0] [0, 0, 0, 1] [0, 1, 1, 0] [1, 0, 1, 0] [1, 1, 0, 0]
w_C	96	96	64	192	4	32	64
n_C	10	12	10	12	8	12	12

Table 4.18: Configurations $\mathcal{C}(C_4)$ in the standard representation of C_4.

By similar treatment as in the case of O_4, we can easily find inequalities which describe C_4^+ and thus $\boldsymbol{\sigma} \cap C_4^+$. We only list the section integrals obtained from all configurations, see Table 4.19. Also, for brevity, we only enlist the first volumetric moment, although we found also $v_4^{(3)}(C_4)_{\mathrm{C}}$ for all configurations. For example $v_4^{(3)}(C_4)_{\mathrm{I}} = \frac{573495143}{78323115855555297075200000}$. It turns out the last configuration XIV is tricky to integrate. In the end, one has to use the identity involving *trilogarithms* found (rediscovered) by Shobhit Bhatnagar [11], the identity states that

$$\mathrm{Li}_3\left(-\frac{1}{3}\right) - 2\,\mathrm{Li}_3\left(\frac{1}{3}\right) = -\frac{\ln^3 3}{6} + \frac{\pi^2}{6}\ln 3 - \frac{13\zeta(3)}{6}. \tag{4.273}$$

.

C	$v_4^{(1)}(C_4)_{\mathrm{C}}$
I	$\frac{65598041}{338674244390092800000}$
II	$\frac{102608713871}{3292649334374400000}$
III	$\frac{256081766015430731}{3457281801093120000000} - \frac{6302191\pi^2}{83980800000}$
IV	$\frac{7383631}{1862358220800}$
V	$\frac{74369\zeta(3)}{92160000} - \frac{154271921776554505 93}{230485453406208000000} + \frac{31318807\pi^2}{149299200000} + \frac{482072643302197\ln 2}{91462481510400000}$
VI	$\frac{200717066493911 4317\ln 2}{38109367296000000} - \frac{1663466629\zeta(3)}{622080000} - \frac{2109541607172182933 47879}{6338349968670720000000} - \frac{133847\pi^2}{124416000}$
VII	$\frac{388451\zeta(3)}{29859840} + \frac{596684331816745397}{2993317576704000000 0} + \frac{4354897\pi^2}{1343692800000} - \frac{23489337302150729\ln 2}{457312407552000000}$
VIII	$\frac{188122446351063331}{1097549778124800000 0} - \frac{1170683\pi^2}{671846400} + \frac{221036483033\ln 2}{2494431313920000}$
IX	$\frac{373791108546507725849549}{38030099812024320000000} - \frac{618197167\zeta(3)}{1866240000} + \frac{74238971\pi^2}{671846400000} - \frac{1333435310218723619\ln 2}{97995515904000000}$
X	$\frac{2274497329\zeta(3)}{69120000} - \frac{21609245552433862937}{4390199112499200000} - \frac{1523317655658026279\ln 2}{30487493836800000}$
XI	$\frac{24570427\zeta(3)}{55296000} - \frac{1574405955292326930 16981}{7606019962404864000000 0} + \frac{47205929\pi^2}{24883200000} + \frac{3002774140883958709\ln 2}{137193722265600000 0}$
XII	$\frac{17}{311040}$
XIII	$\frac{746581063847040871}{660244788403200000 0} - \frac{641346209\pi^2}{55987200000}$
XIV	$\frac{106059672721680228148 03}{115242726703104000000 0} - \frac{41203109797\zeta(3)}{622080000} - \frac{12193153\pi^2}{27993600000} + \frac{4645960252158518597\ln 2}{45731240755200000}$

Table 4.19: Sections integrals in various configurations $\mathcal{C}(C_4)$.

By Equation (C.118), considering the contributions from all configurations,

$$\begin{aligned} v_4^{(k)}(C_4) = \sum_{C\in\mathcal{C}(C_4)} w_{\mathrm{C}}\, v_4^{(k)}(C_4)_{\mathrm{C}} =& 16v_4^{(k)}(C_4)_{\mathrm{I}} + 32v_4^{(k)}(C_4)_{\mathrm{II}} + 96v_4^{(k)}(C_4)_{\mathrm{III}} \\ &+ 24v_4^{(k)}(C_4)_{\mathrm{IV}} + 64v_4^{(k)}(C_4)_{\mathrm{V}} + 16v_4^{(k)}(C_4)_{\mathrm{VI}} + 192v_4^{(k)}(C_4)_{\mathrm{VII}} \\ &+ 96v_4^{(k)}(C_4)_{\mathrm{VIII}} + 96v_4^{(k)}(C_4)_{\mathrm{IX}} + 64v_4^{(k)}(C_4)_{\mathrm{X}} + 192v_4^{(k)}(C_4)_{\mathrm{XI}} \\ &+ 4v_4^{(k)}(C_4)_{\mathrm{XII}} + 32v_4^{(k)}(C_4)_{\mathrm{XIII}} + 64v_4^{(k)}(C_4)_{\mathrm{XIV}}, \end{aligned} \tag{4.274}$$

from which immediately

$$v_4^{(1)}(C_4) = \frac{31874628962521753237}{105835701371904000000} - \frac{26003\pi^2}{1399680000} + \frac{610208\ln 2}{1913625} - \frac{536557\zeta(3)}{2592000} \approx 0.0021295294356445791857, \tag{4.275}$$

$$v_4^{(3)}(C_4) = \frac{19330626155629115959}{168272319220914585600000} - \frac{52276897\pi^2}{216801070940160000} + \frac{10004540239\ln 2}{77977156950000} - \frac{6155594561\zeta(3)}{73741860864000} \approx 7.5157\cdot 10^{-8}, \tag{4.276}$$

4.6 Higher dimensions

4.6.1 Hexateron odd volumetric moments

By the *hexateron*, we mean T_5 (5-simplex). By affine invariancy, we may consider

$$\mathbb{T}_5 = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5) \tag{4.277}$$

with configurations and $\mathcal{C}(T_5)$ weights given by Table 4.20.

C	I	II	III
S	$[0,0,0,0,0]$	$[0,0,0,0,0]$ $[0,0,0,0,1]$	$[0,0,0,0,0]$ $[0,0,0,1,0]$ $[0,0,0,0,1]$
w_{C}	6	15	10

Table 4.20: Configurations $\mathcal{C}(\mathbb{T}_5)$ in a local representation with $\mathcal{C}(T_5)$ weights.

By Theorem 221 and for any $\mathrm{C} \in \mathcal{C}(\mathbb{T}_5)$,

$$v_5^{(k)}(\mathbb{T}_5)_{\mathrm{C}} = \frac{24}{5^k} \int_{(\mathbb{R}^5 \setminus \mathbb{T}_5^\circ)_{\mathrm{C}}} v_4^{(k+1)}(\boldsymbol{\sigma} \cap \mathbb{T}_5)\, \zeta_5^{k+6}(\boldsymbol{\sigma}) \iota_5^{(k)}(\boldsymbol{\sigma}) \lambda_5(\mathrm{d}\boldsymbol{\eta}), \tag{4.278}$$

where

$$\zeta_5(\boldsymbol{\sigma}) = \frac{\operatorname{vol}_4(\boldsymbol{\sigma} \cap \mathbb{T}_5)}{\|\boldsymbol{\eta}\| \operatorname{vol}_5 \mathbb{T}_5}, \qquad \iota_5^{(k)}(\boldsymbol{\sigma}) = \int_{\mathbb{T}_5} |\boldsymbol{\eta}^\top \mathbf{x} - 1|^k \lambda_5(\mathrm{d}\mathbf{x}). \tag{4.279}$$

Configurations I and II are analogous to the first two configurations of $\mathbb{T}_3$ and $\mathbb{T}_4$, we have $n_{\mathrm{I}} = 5$ and $n_{\mathrm{II}} = 2n_{\mathrm{I}} - 2 = 8$ (truncated 4-simplex). The last configuration III, for which we have $n_{\mathrm{III}} = 9$, has no analogue in lower dimensions. However, by similar procedure as before, we obtained contributions from all configurations, see Table 4.21.

C	$v_5^{(1)}(\mathbb{T}_5)_{\mathrm{C}}$
I	$\frac{5}{2722734}$
II	$\frac{12732911}{653456160000}-\frac{1394234873\pi^2}{3353951824423200}+\frac{1622\pi^4}{2707566616755}$
III	$\frac{146034151}{3920736960000}-\frac{3546684881\pi^2}{3353951824423200}+\frac{4904\pi^4}{386795230965}$

C	$v_5^{(3)}(\mathbb{T}_5)_{\mathrm{C}}$
I	$\frac{9097367105}{359796813461446459392}$
II	$\frac{25351944803581}{245954852952160665600000}+\frac{204046383487590493\pi^2}{98081004264127779106308096000}+\frac{13583435573\pi^4}{17098021963979168381769600}$
III	$\frac{173514729599507}{874506143829904588800000}-\frac{12027338819078269\pi^2}{9341048025155026581553152000}+\frac{1191143596913\pi^4}{11398681309319445587846400}$

Table 4.21: Sections integrals in various configurations $\mathcal{C}(\mathbb{T}_5)$.

As a consequence, summing up the contributions from all configurations and by affine invariancy,

$$v_5^{(k)}(T_5) = \sum_{C \in \mathcal{C}(T_5)} w_{\mathrm{C}}\, v_5^{(k)}(T_5)_{\mathrm{C}} = 6v_5^{(k)}(\mathbb{T}_5)_{\mathrm{I}} + 15v_5^{(k)}(\mathbb{T}_5)_{\mathrm{II}} + 10v_5^{(k)}(\mathbb{T}_5)_{\mathrm{III}}, \quad (4.280)$$

from which immediately

$$v_5^{(1)}(T_5) = \frac{2207}{3265920} - \frac{244129\pi^2}{14522729760} + \frac{73522\pi^4}{541513323351} \approx 0.00052308272, \quad (4.281)$$

$$v_5^{(3)}(T_5) = \frac{362173019}{98363448852480000} + \frac{10217818563857\pi^2}{557436796045056999751680} + \frac{602363516243\pi^4}{569934065465972279392320} \approx 3.96585 \cdot 10^{-9}. \quad (4.282)$$

Remark 228. Higher volumetric moments are difficult to compute. For the fifth moment, we would need $v_5^{(5)}(\mathbb{T}_5)_{\mathrm{III}}$. However, even $v_5^{(3)}(\mathbb{T}_5)_{\mathrm{III}}$ was already extremely difficult to compute (the file we worked with exceeded 1GB of storage memory). The intricacy of the third configuration stems partly from its asymmetry and from lacking the decoupling substitution ($a \to \alpha, b \to \beta, c \to \gamma, d \to \delta$), which we found in the second configuration of T_4 (and which generalises as well into higher dimensions) and which enables us to integrate out $\alpha, \beta, \gamma, \delta$ immediately. We have not attempted to obtain the fifth moment, such calculation is surely within our grasp but the shear monstrosity of $v_4^{(6)}(\boldsymbol{\sigma} \cap \mathbb{T}_5)$ in Configuration III discourages us to finish the computation.

4.6.2 Heptapeton first volumetric moment

By the *heptapeton*, we mean T_6 (6-simplex). By affine invariancy, we may consider

$$\mathbb{T}_6 = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6) \quad (4.283)$$

with configurations and $\mathcal{C}(T_6)$ weights given by Table 4.22.
By Theorem 221 and for any $\mathrm{C} \in \mathcal{C}(\mathbb{T}_6)$,

$$v_6^{(k)}(\mathbb{T}_6)_{\mathrm{C}} = \frac{120}{6^k} \int_{(\mathbb{R}^6 \setminus \mathbb{T}_6^\circ)_{\mathrm{C}}} v_5^{(k+1)}(\boldsymbol{\sigma} \cap \mathbb{T}_6)\, \zeta_6^{k+7}(\boldsymbol{\sigma}) \iota_6^{(k)}(\boldsymbol{\sigma}) \lambda_6(\mathrm{d}\boldsymbol{\eta}), \quad (4.284)$$

C	I	II	III
S	$[0,0,0,0,0,0]$	$[0,0,0,0,0,0]$ $[0,0,0,0,0,1]$	$[0,0,0,0,0,0]$ $[0,0,0,0,0,1]$ $[0,0,0,0,1,0]$
w_{C}	7	21	35

Table 4.22: Configurations $\mathcal{C}(\mathbb{T}_6)$ in a local representation with $\mathcal{C}(T_6)$ weights.

where

$$\zeta_6(\boldsymbol{\sigma}) = \frac{\mathrm{vol}_5(\boldsymbol{\sigma} \cap \mathbb{T}_6)}{\|\boldsymbol{\eta}\| \,\mathrm{vol}_6 \,\mathbb{T}_6}, \qquad \iota_6^{(k)}(\boldsymbol{\sigma}) = \int_{\mathbb{T}_6} |\boldsymbol{\eta}^\top \mathbf{x} - 1|^k \lambda_6(\mathrm{d}\mathbf{x}). \tag{4.285}$$

Configurations I and II are analogous to the first two configurations of $\mathbb{T}_3$, $\mathbb{T}_4$ and $\mathbb{T}_5$, we have $n_{\mathrm{I}} = 6$ and $n_{\mathrm{II}} = 2n_{\mathrm{I}} - 2 = 10$ (truncated 5-simplex). The last configuration III is analogous to third configuration of $\mathbb{T}_5$. We have $n_{\mathrm{III}} = 12$. Thanks to this similarity, since we already know how to handle this configuration in the $\mathbb{T}_5$ case, we obtained contributions of all $\mathbb{T}_6$ configurations, see Table 4.23.

C	$v_6^{(1)}(\mathbb{T}_6)_{\mathrm{C}}$
I	$\frac{45}{963780608}$
II	$\frac{3826171}{4182119424000} - \frac{12560362004329\pi^2}{44356226537150079552 0} + \frac{6607326855286\pi^4}{85176183364279644451815}$
III	$\frac{71529389}{24395696640000} - \frac{4625576448278719\pi^2}{33267169902862559664000} + \frac{432402941059748\pi^4}{14196030560713274075302 5}$

Table 4.23: Sections integrals in various configurations $\mathcal{C}(\mathbb{T}_6)$.

As a consequence, summing up the contributions from all configurations and by affine invariancy,

$$v_6^{(k)}(T_6) = \sum_{C \in \mathcal{C}(T_6)} w_{\mathrm{C}}\, v_6^{(k)}(T_6)_{\mathrm{C}} = 7v_6^{(k)}(\mathbb{T}_6)_{\mathrm{I}} + 21 v_6^{(k)}(\mathbb{T}_6)_{\mathrm{II}} + 35 v_6^{(k)}(\mathbb{T}_6)_{\mathrm{III}}, \tag{4.286}$$

from which immediately

$$\begin{aligned} v_6^{(1)}(T_6) &= \frac{26609}{217818720} - \frac{3396146609\pi^2}{621871356506400} + \frac{1318349152898\pi^4}{12180206401298390455} \\ &\approx 0.00007880487647920397. \end{aligned} \tag{4.287}$$

We have not attempted to derive the higher moments. We leave this for our readers and humbly add that this task will be extraordinarily difficult.

4.7 Unsolved problems

An obvious question is to deduce the volumetric moments $v_d^{(k)}(T_d)$ for $d \geq 6$. When $d = 7$, there are four section equivalent configurations $C \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}$ in $\mathcal{C}(T_7)$. Evaluating the section integral $v_7^{(1)}(T_7)_{\mathrm{IV}}$ for the fourth configuration is beyond the capabilities of our computer. At least, since $\boldsymbol{\sigma} \cap T_d$ is always a

T_{d-1} simplex in the first configuration of T_d, that is $n_{\mathrm{I}} = d$ with $w_{\mathrm{I}} = d + 1$. By Theorem 221,

$$v_d^{(k)}(T_d)_{\mathrm{I}} = v_{d-1}^{(k+1)}(T_{d-1}) \frac{(d-1)!}{d^k} \int_{\mathbb{R}^d \setminus K_d^\circ} \zeta_d^{k+d+1}(\boldsymbol{\sigma}) \iota_d^{(k)}(\boldsymbol{\sigma}) \lambda_d(\mathrm{d}\boldsymbol{\eta}) \tag{4.288}$$

since $v_{d-1}^{(k+1)}(T_{d-1})$ are constants. More specifically, for $k = 1$ by using Reed's formula, we found the following surprising relation

$$v_d^{(1)}(T_d)_{\mathrm{I}} = 2v_d^{(2)}(T_d) = \frac{2(d!)}{(d+1)^d(d+2)^d}. \tag{4.289}$$

Based on the result we have seen so far for d-simplices, we conjecture

$$v_{r+1}^{(k)}(T_{r+1}) = \sum_{s=0}^{\lfloor r/2 \rfloor} p_{rs}^{(k)} \pi^{2s} \tag{4.290}$$

for some rationals $p_{rs}^{(k)}$ and $r = 0, 1, 2, 3, \ldots$ Since $\mathcal{G}(T_d)$ is isomorphic to the symmetry group on $d+1$ elements (any permutation of vertices is a valid symmetry), we have for the weights $o_{\mathrm{C}} = \binom{d+1}{|\mathrm{S}|}$, where $|\mathrm{S}|$ is the number of vertices separated by the section plane $\boldsymbol{\sigma}$ in configuration C.

At time of submission of this thesis (May 30, 2025), we had some partial results for $v_4^{(1)}(O_4)$, where O_4 is the *hexadecachoron*, the 4-dimensional analogy of an octahedron, known also as 16-cell, 4-orthoplex or a 4-cross-polytope. However, we were not able to solve all configurations.

5. First-order Metric Moments

Let us clarify the name of this chapter. By *order*, we mean the power k in metric moments $v_n^{(k)}(P_d)$. Hence, we are interested in $v_n^{(k)}(P_d)$ for general n but with $k = 1$ only. Moreover, we also restrict the dimension to be $d = 2$ or $d = 3$. Hence, there is a natural overlap with Chapter 4 on Odd volumetric moments $v_n^{(d)}(P_d)$. These two particular cases and the methods shown here are known and well explored in literature.

5.1 Efron's formulae

Let $K_3 \subset \mathbb{R}^3$ be a convex 3-body, from which we pick a random selection $\mathbb{X} = (\mathbf{X}_0, \mathbf{X}_1 \dots, \mathbf{X}_n)$ of $(n+1)$ random points uniformly and independently, $n \geq 3$. The convex hull $\mathbb{H}_n = \operatorname{conv}(\mathbb{X})$ of these points has volume $\operatorname{vol}_3(\mathbb{H}_n)$. When $n < 3$, we get $\mathbb{E}[\operatorname{vol}_3(\mathbb{H}_n)] = 0$ trivially. When $n = 3$, $\mathbb{H}_3$ is almost surely a tetrahedron. It turns out we can express the mean tetrahedron volume $\mathbb{E}[\operatorname{vol}_3(\mathbb{H}_3)]$ by an integral over all possible cutting planes. By affine invariancy, we have $\mathbb{E}[\operatorname{vol}_3(\mathbb{H}_3)] = v_3^{(1)}(K_3)\operatorname{vol}_3(K_3)$ and

$$v_3^{(1)}(K_3) = \frac{3}{5} - \mathbb{E}\left[\Gamma_3^+(\mathbb{X}')^2 + \Gamma_3^-(\mathbb{X}')^2\right], \tag{5.1}$$

where $\Gamma_3^+(\mathbb{X}') = \operatorname{vol}_3 K_3^+ / \operatorname{vol}_3 K_3$ and $\Gamma_3^-(\mathbb{X}') = \operatorname{vol}_3 K_3^- / \operatorname{vol}_3 K_3$ are the volume fractions of the two parts $K_3^+ \sqcup K_3^-$ into which K_3 is divided by a cutting plane $\boldsymbol{\sigma}$ passing through the collection $\mathbb{X}' = (\mathbf{X}_1', \mathbf{X}_2', \mathbf{X}_3') \in K_3^3$. That is, K_3 is split by $\boldsymbol{\sigma} = \mathcal{A}(\mathbb{X}')$ into disjoint union $K_3^+ \sqcup K_3^-$ with $\operatorname{vol}_3 K_3^+ + \operatorname{vol}_3 K_3^- = \operatorname{vol}_3 K_3$. Note that this result can be written out as an integral

$$v_3^{(1)}(K_3) = \frac{3}{5} - \frac{1}{(\operatorname{vol}_3 K_3)^3} \int_{K_3^3} \Gamma_3^+(\mathbb{x}')^2 + \Gamma_3^-(\mathbb{x}')^2 \, \lambda_3^3(\mathrm{d}\mathbb{x}'), \tag{5.2}$$

where $\mathbb{x}' = (\mathbf{x}_1', \mathbf{x}_2', \mathbf{x}_3')$ is the collection of points $\mathbf{x}_j' = (x_{1j}', x_{2j}', x_{3j}')^\top$, $j \in \{1,2,3\}$ and $\lambda_3^3(\mathrm{d}\mathbb{x}') = \lambda_3(\mathrm{d}\mathbf{x}_1')\lambda_3(\mathrm{d}\mathbf{x}_2')\lambda(\mathrm{d}\mathbf{x}_3') = \prod_{i,j=0}^3 \mathrm{d}x_{ij}'$ is the usual Lebesgue measure on $(\mathbb{R}^3)^3$. This formula is a special case of the more general **Efron section formula** [26] as stated in Theorem 235. Similar result holds in dimension two (Theorem 234). In order to prove those theorems, let us recall some definitions and show two intermediate results, the Efron vertex and facet identities.

5.1.1 Polytopes and their f-vector

First, we recall the following facts (cf. [71]) for any convex d-polytope $P_d \subset \mathbb{R}^d$ (convex d-dimensional polytope).

Definition 229 (f-vector)**.** We denote by $f_k(P_d)$ the total number of k-faces of P_d, where $f_0(P_d)$ stands for the number of its **vertices**, $f_1(P_d)$ the number of **edges**, $f_2(P_d)$ the number of **faces** and so on. The last value $f_{d-1}(P_d)$ denotes the number of **facets** of P_d. Together, the values can be combined into a single vector $(f_0(P_d), f_1(P_d), \dots, f_{d-1}(P_d))$ called the f-vector of P_d. Lastly, we denote by $\mathrm{vol}_d(P_d)$ the volume (d-volume) of P_d as usual.

The f-vector values $f_k(P_d)$ are not independent and are connected via linear relations. In $d = 2$, we have trivially for any P_2 that

$$f_0(P_2) = f_1(P_2). \tag{5.3}$$

In $d = 3$, we have for any convex 3-polytope P_3 the **Euler polyhedral formula**

$$f_0(P_3) - f_1(P_3) + f_2(P_3) = 2. \tag{5.4}$$

In higher dimensions, there exists an analogue of the Euler's polyhedral formula called the **Schläfli** or the **Euler-Poincaré formula** [64]

$$\sum_{k=0}^{d-1}(-1)^k f_k(P_d) = 2(-1)^d \tag{5.5}$$

valid for any convex d-polytope P_d. However, convex hulls $\mathbb{H}_n = \mathrm{conv}(\mathbb{X})$ of a random point collection $\mathbb{X} = (\mathbf{X}_0, \ldots, \mathbf{X}_n)$ form a more special class of d-polytopes as their facets are just $(d-1)$-simplices almost surely. Polytopes whose facets are $(d-1)$-simplices are called *simplical polytopes*. For any simplical d-polytope S_d, by counting the total number of facets in two different ways,

$$2\, f_{d-2}(S_d) = d\, f_{d-1}(S_d). \tag{5.6}$$

More generaly, the **Dehn–Sommerville equations** [66] form a complete set of linear relations between the numbers of k-faces of S_d. If we define $f_{-1}(S_d) = f_d(S_d) = 1$, they take the form

$$\sum_{j=k}^{d-1}(-1)^j \binom{j+1}{k+1} f_j(S_d) = (-1)^{d-1} f_k(S_d)$$

valid for $k = -1, 0, 1, \ldots, d-2$. The Schläfli formula is a special case when $k = -1$. Dehn-Sommerville equations imply that the knowledge of all $f_i(S_d)$ for $0 \leq i < \lfloor d/2 \rfloor$ uniquely determines all $f_i(S_d)$ with $i \geq \lfloor d/2 \rfloor$ and vice versa.

5.1.2 Vertex identity

Proposition 230 (Extended Efron vertex identity). *Let $K_d \subset \mathbb{R}^d$ be a convex d-body and let the points $\mathbf{X}_j$, $j = 0, ..., n$, $n \geq d$ be uniformly selected from K_d. Denote $\mathbb{H}_n$ their convex hull, $\mathrm{vol}_d\, \mathbb{H}_n$ its volume and $f_0(\mathbb{H}_n)$ its number of vertices as usual. Then $\mathbb{E}\,(\mathrm{vol}_d\, \mathbb{H}_{n-k})^k = v^{(k)}_{n-k}(K_d)(\mathrm{vol}_d\, K_d)^k$ with*

$$v^{(k)}_{n-k}(K_d) = \mathbb{E} \prod_{i=0}^{k-1}\left(1 - \frac{f_0(\mathbb{H}_n)}{n-i+1}\right). \tag{5.7}$$

Proof. Without the loss of generality, we assume that $\mathrm{vol}_d K_d = 1$. Let us select k indices from $\{0, 1, 2, 3, ..., n\}$, that is $J \subset \{0, 1, 2, 3, ..., n\}$ with the number of elements $|J| = k$ is our set of indices. There are two ways how to express the probability P that points $\mathbf{X}_j$ with selected indices $j \in J$ do not form vertices of $\mathbb{H}_n$.

First, we can condition on the realisation of the remaining $n+1-k$ points. That means, in K_d, we fix the position of those remaining $n+1-k$ points $\mathbf{X}_j$ with indices j not in J. Then, the probability that those k given points with J indices do not form vertices of $\mathbb{H}_n$ is simply the probability that all those k points fall into convex hull of the remaining $n+1-k$ points. Since they are independent, that is

$$(\mathrm{vol}_d \mathbb{H}_{n-k})^k \tag{5.8}$$

By the law of total probability (or expectation), in order to get P, we must average this over all conditions we have fixed, that is, over all realisations of $n+1-k$ points and thus we get

$$P = \mathbb{E} \, (\mathrm{vol}_d \mathbb{H}_{n-k})^k \tag{5.9}$$

Second, by symmetry, since the points $\mathbf{X}_j$ are indistinguishable, the probability P that points with **given** k indices do not form vertices of $\mathbb{H}_n$ must be the same as the probability Q that points with **random** k indices do not form vertices (uniformly selected from the set $\{0, 1, \dots, n\}$). We can compute this probability Q in two steps: First, we condition with respect to a given realisation of **all** $\mathbf{X}_j$ points according to the uniform distribution in K_d. Let the convex hull of this particular realisation of points have $f_0(\mathbb{H}_n)$ vertices (not random now). We then select k points at random from this realisation, that is, we randomly select k indices from $\{0, 1, 2, ..., n\}$. Number of points not being vertices now follow the *hypergeometric distribution*. That is, probability Q of randomly selected k points (among those $\mathbf{X}_j$'s already realised) not being vertices is equal to the probability of first point in not a vertex times the probability of the second point not being a vertex (given 1st point not being a vertex already) and so on, i.e.

$$\frac{n+1-f_0(\mathbb{H}_n)}{n+1} \cdot \frac{n-f_0(\mathbb{H}_n)}{n} \cdot \frac{n-1-f_0(\mathbb{H}_n)}{n-1} \cdots \frac{n+2-k-f_0(\mathbb{H}_n)}{n+2-k}. \tag{5.10}$$

By the law of total probability, we must average over all realisations and thus

$$Q = \mathbb{E}\left[\prod_{i=0}^{k-1} \frac{n-i+1-f_0(\mathbb{H}_n)}{n-i+1}\right]. \tag{5.11}$$

Since $P = Q$, we get the statement of the proposition. ∎

> *Remark* 231. The special case of $k = 1$ gives
>
> $$v_{n-1}^{(1)}(K_d) = 1 - \frac{\mathbb{E}\, f_0(\mathbb{H}_n)}{n+1}, \tag{5.12}$$
>
> or $\mathbb{E}\, f_0(\mathbb{H}_n) = (n+1)\left(1 - v_{n-1}^{(1)}(K_d)\right)$, which is the original Efron vertex identity [26], the extended case shown here was first proven by Buchta [15], who recently also provided a geometrical explanation for its dual version [16].

5.1.3 Facet identity

Proposition 232 (Efron facet identity)**.** *Let $K_d \subset \mathbb{R}^d$ be a convex body and let $\mathbb{X} = (\mathbf{X}_0, \ldots, \mathbf{X}_n)$ be the collection of points $\mathbf{X}_j$, $j = 0, ..., n$, $n \geq d$ uniformly selected from K_d. Denote $\mathbb{H}_n$ their convex hull and let $f_{d-1}(\mathbb{H}_n)$ denote its number of facets as usual. Then*

$$\mathbb{E}\, f_{d-1}(\mathbb{H}_n) = \binom{n+1}{d} \mathbb{E}\left[\Gamma_d^+(\mathbb{X}')^{n-d+1} + \Gamma_d^-(\mathbb{X}')^{n-d+1}\right], \tag{5.13}$$

where $\Gamma_d^+(\mathbb{X}') = \operatorname{vol}_d K_d^+ / \operatorname{vol}_d K_d$ and $\Gamma_d^-(\mathbb{X}') = \operatorname{vol}_d K_d^- / \operatorname{vol}_d K_d$ are the d-volume fractions of the two parts $K_d^+ \sqcup K_d^-$ into which K_d is divided by a hyperplane $\boldsymbol{\sigma} = \mathcal{A}(\mathbb{X}')$ going through the collection $\mathbb{X}' = (\mathbf{X}_1', \mathbf{X}_2', \ldots, \mathbf{X}_d')$ of random points $\mathbf{X}_j'$, $j \in \{1, 2, \ldots, d\}$ drawn from K_d uniformly and independently.

Remark 233. We may write out the expectation into an integral to get the following form of the proposition:

$$\mathbb{E}\, f_{d-1}(\mathbb{H}_n) = \binom{n+1}{d} \frac{1}{(\operatorname{vol}_d K_d)^d} \int_{K_d^d} \Gamma_d^+(\mathbb{x}')^{n-d+1} + \Gamma_d^-(\mathbb{x}')^{n-d+1} \lambda_d^d(\mathrm{d}\mathbb{x}'), \tag{5.14}$$

where $\mathbb{x}' = (\mathbf{x}_1', \mathbf{x}_2', \ldots, \mathbf{x}_d')$ is the collection of points $\mathbf{x}_j' = (x_{1j}', \ldots, x_{dj}')^\top$, $j \in \{1, 2, \ldots, d\}$ and $\lambda_d^d(\mathrm{d}\mathbb{x}') = \lambda_d(\mathrm{d}\mathbf{x}_1')\lambda_d(\mathrm{d}\mathbf{x}_2') \cdots \lambda_d(\mathrm{d}\mathbf{x}_d') = \prod_{i,j=0}^d \mathrm{d}x_{ij}'$ is the usual Lebesgue measure on $(\mathbb{R}^d)^d$.

Proof. Select a sub-collection $\mathbb{X}' \subset \mathbb{X}$ of d points with a given set of fixed indices and let $\boldsymbol{\sigma} = \mathcal{A}(\mathbb{X}')$. Cutting plane $\boldsymbol{\sigma}$ divides body K_d into two parts $K_d^+ \sqcup K_d^-$ with d-volume fractions $\Gamma_d^+(\mathbb{X}')$ and $\Gamma_d^-(\mathbb{X}')$. Fixing the position of the points in collection $\mathbb{X}'$ in K_d, we see that $\mathbb{H}' = \operatorname{conv}(\mathbb{X}')$ is a facet of $\mathbb{H}_n$ if and only if all the remaining $n + 1 - d$ points lie either on one side of $\boldsymbol{\sigma}$ or on the other. Hence

$$\mathbb{P}\left[\mathbb{H}' \text{ is a facet of } \mathbb{H}_n \,|\, \mathbb{X}' \text{ fixed}\right] = \Gamma_d^+(\mathbb{X}')^{n-d+1} + \Gamma_d^-(\mathbb{X}')^{n-d+1}. \tag{5.15}$$

By the law of total probability, averaging over all collections $\mathbb{X}'$, we get,

$$\mathbb{P}\left[\mathbb{H}' \text{ is a facet of } \mathbb{H}_n\right] = \mathbb{E}\left[\Gamma_d^+(\mathbb{X}')^{n-d+1} + \Gamma_d^-(\mathbb{X}')^{n-d+1}\right], \tag{5.16}$$

Let $I_{n,d}$ be the set of subset of $\{0, 1, \ldots, n\}$ with exactly d elements and for a given $\tau \in I_{n,d}$ denote $\mathbb{H}_\tau = \operatorname{conv}(\mathbf{X}_j)_{j \in \tau}$, then

$$f_{d-1}(\mathbb{H}_n) = \sum_{\tau \in I_{n,d}} \mathbb{1}\{\mathbb{H}_\tau \text{ is a facet of } \mathbb{H}_n\}. \tag{5.17}$$

Taking expectation, by linearity and by symmetry,

$$\mathbb{E}\, f_{d-1}(\mathbb{H}_n) = \binom{n+1}{d} \mathbb{P}\left[\mathbb{H}' \text{ is a facet of } \mathbb{H}_n\right]. \tag{5.18}$$

Together with Equation (5.16), we get the statement of the proposition. ■

5.1.4 Section formulae

We are now well equipped to show the Efron section formulae.

Two-dimensional

Theorem 234 (Efron 1965). *Let $K_2 \subset \mathbb{R}^2$ be a convex 2-body, from which we pick a collection $\mathbb{X} = (\mathbf{X}_0, \dots, \mathbf{X}_n)$ of $n+1$ random points uniformly independently. Let $\mathrm{vol}_2(\mathbb{H}_n)$ denote the area of the convex hull $\mathbb{H}_n = \mathrm{conv}(\mathbb{X})$. Then for all $n = 2, 3, \dots$, we have $\mathbb{E}[\mathrm{vol}_2(\mathbb{H}_n)] = v_n^{(1)}(K_2)\,\mathrm{vol}_2(K_2)$ with*

$$v_n^{(1)}(K_2) = 1 - \frac{n+1}{2}\mathbb{E}\left[\Gamma_2^+(\mathbb{X}')^n + \Gamma_2^-(\mathbb{X}')^n\right], \tag{5.19}$$

where $\Gamma_2^+(\mathbb{X}') = \mathrm{vol}_2 K_2^+ / \mathrm{vol}_2 K_2$ and $\Gamma_2^-(\mathbb{X}') = \mathrm{vol}_2 K_2^- / \mathrm{vol}_2 K_2$ are the area fraction of the two parts $K_2^+ \sqcup K_2^-$ into which K_2 is divided by a line $\boldsymbol{\sigma} = \mathcal{A}(\mathbb{X}')$ going through the collection $\mathbb{X}' = (\mathbf{X}_1', \mathbf{X}_2')$ of random points $\mathbf{X}_j'$, $j \in \{1, 2\}$ drawn from K_2 uniformly and independently.

Proof. Since $\mathbb{H}_n$ is a polygon almost surely, we may write $f_0(\mathbb{H}_n) = f_1(\mathbb{H}_n)$. Taking expectation of both sides and by the Efron vertex and facet identities,

$$(n+1)\left(1 - v_{n-1}^{(1)}(K_2)\right) = \binom{n+1}{2}\mathbb{E}\left[\Gamma_2^+(\mathbb{X}')^{n-1} + \Gamma_2^-(\mathbb{X}')^{n-1}\right]. \tag{5.20}$$

Rearranging and replacing n by $n+1$, we get the two-dimensional Efron section formula. ∎

Three-dimensional

Theorem 235 (Efron 1965). *Let $K_3 \subset \mathbb{R}^3$ be a convex 3-body, from which we pick a collection $\mathbb{X} = (\mathbf{X}_0, \dots, \mathbf{X}_n)$ of $n+1$ random points uniformly independently. Let $\mathrm{vol}_3(\mathbb{H}_n)$ denote the volume of the convex hull $\mathbb{H}_n = \mathrm{conv}(\mathbb{X})$. Then for all $n = 3, 4, \dots$, we have $\mathbb{E}[\mathrm{vol}_3(\mathbb{H}_n)] = v_n^{(1)}(K_3)\,\mathrm{vol}_3(K_3)$ with*

$$v_n^{(1)}(K_3) = \frac{n}{n+2} - \frac{n(n+1)}{12}\mathbb{E}\left[\Gamma_3^+(\mathbb{X}')^{n-1} + \Gamma_3^-(\mathbb{X}')^{n-1}\right], \tag{5.21}$$

where $\Gamma_3^+(\mathbb{X}') = \mathrm{vol}_3 K_3^+ / \mathrm{vol}_3 K_3$ and $\Gamma_3^-(\mathbb{X}') = \mathrm{vol}_3 K_3^- / \mathrm{vol}_3 K_3$ are the volume fractions of the two parts $K_3^+ \sqcup K_3^-$ into which K_3 is divided by a plane $\boldsymbol{\sigma} = \mathcal{A}(\mathbb{X}')$ going through the collection $\mathbb{X}' = (\mathbf{X}_1', \mathbf{X}_2', \mathbf{X}_3')$ of random points $\mathbf{X}_j'$, $j \in \{1, 2, 3\}$ drawn from K_3 uniformly and independently.

Proof. In $d = 3$, almost surely, $\mathbb{H}_n = \mathrm{conv}(\mathbb{X})$ is a simplical polyhedron whose faces are triangles. That means, by Equation (5.6),

$$2\,f_1(\mathbb{H}_n) = 3\,f_2(\mathbb{H}_n). \tag{5.22}$$

Moreover, by Euler's polyhedral formula (Equation (5.4)),

$$f_0(\mathbb{H}_n) - f_1(\mathbb{H}_n) + f_2(\mathbb{H}_n) = 2. \tag{5.23}$$

Combining these these two equations together, we get a linear relation

$$2f_0(\mathbb{H}_n) - f_2(\mathbb{H}_n) = 4. \tag{5.24}$$

Taking expectation of both sides and by using the Efron vertex and facet identities,

$$2(n+1)\left(1 - v^{(1)}_{n-1}(K_3)\right) - \binom{n+1}{3}\mathbb{E}\left[\Gamma_3^+(\mathbb{X}')^{n-2} + \Gamma_3^-(\mathbb{X}')^{n-2}\right] = 4. \tag{5.25}$$

Rearranging and replacing n by $n+1$, we get the three-dimensional Efron section formula. ■

5.1.5 Cartesian reparametrisation

Let $K_d \subset \mathbb{R}^d$ be a convex body and let $\mathbb{X}' = (\mathbf{X}'_1, \dots, \mathbf{X}'_d)$ be a collection of points $\mathbf{X}'_j$ drawn uniformly and independently from K_d. It is convenient to introduce the *gamma section functional*

$$\gamma_n(K_d) = \mathbb{E}\left[\Gamma_d^+(\mathbb{X}')^{n-d+2} + \Gamma_d^-(\mathbb{X}')^{n-d+2}\right], \tag{5.26}$$

where $\Gamma_d^+(\mathbb{X}') = \operatorname{vol}_d K_d^+ / \operatorname{vol}_d K_d$ and $\Gamma_d^-(\mathbb{X}') = \operatorname{vol}_d K_d^- / \operatorname{vol}_d K_d$ are the d-volume fractions of the two parts $K_d^+ \sqcup K_d^-$ into which K_d is divided by a hyperplane $\boldsymbol{\sigma} = \mathcal{A}(\mathbb{X}')$. Written as an integral, this is equivalent to

$$\gamma_n(K_d) = \frac{1}{(\operatorname{vol}_d K_d)^d} \int_{K_d^d} \Gamma_d^+(\mathbb{x}')^{n-d+2} + \Gamma_d^-(\mathbb{x}')^{n-d+2} \lambda_d^d(\mathrm{d}\mathbb{x}'), \tag{5.27}$$

where $\mathbb{x}' = (\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_d)$ is the collection of points $\mathbf{x}'_j = (x'_{1j}, \dots, x'_{dj})^\top$, $j \in \{1, 2, \dots, d\}$ and $\lambda_d^d(\mathrm{d}\mathbb{x}') = \lambda_d(\mathrm{d}\mathbf{x}'_1)\lambda_d(\mathrm{d}\mathbf{x}'_2)\cdots\lambda_d(\mathrm{d}\mathbf{x}'_d) = \prod_{i,j=0}^d \mathrm{d}x'_{ij}$ is the usual Lebesgue measure on $(\mathbb{R}^d)^d$.

Note that $\gamma_n(K_d)$ is an affine functional. If K_d is some sufficiently symmetric polytope P_d, we can further use genealogic decomposition (see Appendix C)

$$\gamma_n(P_d) = \sum_{C \in \mathcal{C}(P_d)} w_{\mathrm{C}}\, \gamma_n(P_d)_{\mathrm{C}}. \tag{5.28}$$

Efron section formulae (Theorems 234 and 235) then can be written in the following compact form

$$v_n^{(1)}(K_2) = 1 - \frac{n+1}{2}\gamma_n(K_2), \qquad v_n^{(1)}(K_3) = \frac{n}{n+2} - \frac{n(n+1)}{12}\gamma_n(K_3). \tag{5.29}$$

By Blaschke-Petkantschin formula (in the form of Corollary 296.2) with $k = 0$,

$$\gamma_n(K_d) = (d-1)!\operatorname{vol}_d K_d \int_{\mathbb{R}^d \setminus K_d^\circ} v_{d-1}^{(1)}(\boldsymbol{\sigma} \cap K_d)\zeta_d^{d+1}(\boldsymbol{\sigma})(\Gamma_d^+(\boldsymbol{\sigma})^{n-d+2} + \Gamma_d^-(\boldsymbol{\sigma})^{n-d+2})\lambda_d(\mathrm{d}\boldsymbol{\eta}),$$

where $\boldsymbol{\eta}$ is the Cartesian representation of $\boldsymbol{\sigma}$ defined by the relation $\boldsymbol{\eta}^\top \mathbf{x} = 1$. In this representation, we have $K_d^+ = \{\mathbf{x} \in K_d \,|\, \boldsymbol{\eta}^\top \mathbf{x} < 1\}$ and (by Remark 297)

$$\zeta_d(\boldsymbol{\sigma}) = \frac{\operatorname{vol}_{d-1}(\boldsymbol{\sigma} \cap K_d)}{\|\boldsymbol{\eta}\| \operatorname{vol}_d K_d} = -\frac{1}{\operatorname{vol}_d K_d} \sum_{j=1}^d \eta_j \frac{\partial \operatorname{vol}_d K_d^+}{\partial \eta_j} = -\sum_{j=1}^d \eta_j \frac{\partial \Gamma_d^+(K_d)}{\partial \eta_j}. \tag{5.30}$$

Dimension two

When $d = 2$, the formula is extraordinarily simple. Since $\boldsymbol{\sigma} \cap K_2$ is always a line segment, we get $v_1^{(1)}(\boldsymbol{\sigma} \cap K_2) = 1/3$ and so

$$\gamma_n(K_2) = \tfrac{1}{3} \operatorname{vol}_2 K_2 \int_{\mathbb{R}^2 \setminus K_2^\circ} \zeta_2^3(\boldsymbol{\sigma})(\Gamma_2^+(\boldsymbol{\sigma})^n + \Gamma_2^-(\boldsymbol{\sigma})^n)\lambda_2(\mathrm{d}\boldsymbol{\eta}) \tag{5.31}$$

Dimension three

When $d = 3$, we have

$$\gamma_n(K_3) = 2 \operatorname{vol}_3 K_3 \int_{\mathbb{R}^3 \setminus K_3^\circ} v_2^{(1)}(\boldsymbol{\sigma} \cap K_3)\zeta_3^4(\boldsymbol{\sigma})(\Gamma_3^+(\boldsymbol{\sigma})^{n-1} + \Gamma_3^-(\boldsymbol{\sigma})^{n-1})\lambda_3(\mathrm{d}\boldsymbol{\eta}), \tag{5.32}$$

This integral can be always solved when the integrand is a rational function. This happens when $K_3 = P_3$ a convex polygon. Then, $\boldsymbol{\sigma} \cap P_3$ is some convex polytope P_2. Since $v_2^{(1)}(P_2)$ is known for any convex polytope (due to Buchta and Reitzner [19]), in fact it is a rational function, we can plug this value into the integral and then integrate everything out. We can use this formula to deduce the first volume moment relatively easily regardless of the number of points in the convex hull. This is the method that we originally used to derive $v_3^{(1)}(P_3)$ for polyhedra in Table 4.3.

5.1.6 Generalisations of Efron's formula

Affentranger's recurrence relations

It turns out that the first volume moments $\mathbb{E}\left[\operatorname{vol}_d(\mathbb{H}_n)\right] = v_n^{(1)}(K_d) \operatorname{vol}_d K_d$ in a convex d-body K_d for $n \geq 3$ are related via the formula ($d = 3$ and $m \geq 1$)

$$v_{2m+d-1}^{(1)}(K_d) = \sum_{k=1}^{m} (4^k - 1) \frac{B_{2k}}{k} \binom{2m+d}{2k-1} v_{2m-2k+d}^{(1)}(K_d), \tag{5.33}$$

where B_{2k} are the Bernoulli numbers ($B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$ and so on). Special cases up to $m = 4$ are listed below

$$\begin{aligned}
v_4^{(1)}(K_3) &= \tfrac{5}{2} v_3^{(1)}(K_3),\\
v_6^{(1)}(K_3) &= \tfrac{7}{2} v_5^{(1)}(K_3) - \tfrac{35}{4} v_3^{(1)}(K_3),\\
v_8^{(1)}(K_3) &= \tfrac{9}{2} v_7^{(1)}(K_3) - 21 v_5^{(1)}(K_3) + 63 v_3^{(1)}(K_3),\\
v_{10}^{(1)}(K_3) &= \tfrac{11}{2} v_9^{(1)}(K_3) - \tfrac{165}{4} v_7^{(1)}(K_3) + 231 v_5^{(1)}(K_3) - \tfrac{2805}{4} v_3^{(1)}(K_3).
\end{aligned} \tag{5.34}$$

The identity is due to Affentranger [1] and Badertscher [3] and it is proven simply by comparing the coefficients of $\Gamma_d^+(\mathbb{x})$ by expanding $\Gamma_d^-(\mathbb{x})^{n-d+2}$ as $(1 - \Gamma_d^+(\mathbb{x}))^{n-d+2}$ in the Efron section formula. Note that the same formula holds also in two dimensions (put $d = 2$), from which we get up to $m = 4$

$$
\begin{aligned}
v_3^{(1)}(K_2) &= 2v_2^{(1)}(K_2),\\
v_5^{(1)}(K_2) &= 3v_4^{(1)}(K_2) - 5v_2^{(1)}(K_2),\\
v_7^{(1)}(K_2) &= 4v_6^{(1)}(K_2) - 14v_4^{(1)}(K_2) + 28v_2^{(1)}(K_2),\\
v_9^{(1)}(K_2) &= 5v_8^{(1)}(K_2) - 30v_6^{(1)}(K_2) + 126v_4^{(1)}(K_2) - 255v_2^{(1)}(K_2).
\end{aligned} \tag{5.35}
$$

Buchta's identity

However, Affentranger's recurrence relation does not generalise to higher dimensions. That means, there is no recurrence formula relating $\mathbb{E}[\text{vol}_d(\mathbb{H}_n)] = v_n^{(1)}(K_d)\,\text{vol}_d\,K_d$ to each other when $d \geq 4$. The only exception where Affentranger's formula holds is the case $m = 1$ for which indeed for any d,

$$
v_{d+1}^{(1)}(K_d) = \frac{d+2}{2}\,v_d^{(1)}(K_d) \tag{5.36}
$$

as proven by Buchta in [14, p. 96] by a simple projection argument. Neither the Efron section formula can be generalised to higher dimensions. This is because the values $f_0(\mathbb{H}_n)$ and $f_{d-1}(\mathbb{H}_n)$ are no longer connected by a simple linear relation. For example, when $d = 4$ and $f_0(\mathbb{H}_n) = 6$, then either $f_0(\mathbb{H}_n) = 8$ or $f_0(\mathbb{H}_n) = 9$. The second option corresponds to $\mathbb{H}_n$ being a cyclic polytope. It is thus believed there is no analogue of Efron's formula in higher dimensions [1], although partial results connecting expected values of various polyhedral elements have been found (see Cowan [22]).

Vertex-Facet polynomial

In $d = 4$, there are three ways how $\mathbb{H}_5$ can look like. Either $\mathbb{H}_5$ is

- a 4-simplex, $f_0(\mathbb{H}_5) = 5$, $f_0(\mathbb{H}_5) = 5$
- convex union of two 4-simplices sharing one facet, $f_0(\mathbb{H}_5) = 6$, $f_3(\mathbb{H}_3) = 8$
- or a cyclic polytope with $f_0(\mathbb{H}_5) = 6$, $f_0(\mathbb{H}_5) = 9$.

These three options can be combined into a single quadratic relation

$$
12 f_0(\mathbb{H}_5) = 17 f_3(\mathbb{H}_5) - f_3^2(\mathbb{H}_5). \tag{5.37}
$$

More generally, we have the following observation:

Proposition 236. *There is a polynomial $p_d : \mathbb{R} \to \mathbb{R}$ of order $\lfloor d/2 \rfloor$ such that*

$$
f_0(\mathbb{H}_{d+2}) = p_d(f_{d-1}(\mathbb{H}_{d+2})). \tag{5.38}
$$

Proof. The proof is based on the classification of simplical polytopes with low number of vertices (see [71, Chapter 15.]). A polytope P_d with $d+2$ vertices is simplical if and only if it can be written as a direct sum of two lower-dimensional simplices. That is, $P_d = T_k \oplus T_{d-k}$. There are $\lfloor \frac{d}{2} \rfloor$ such polytopes since by symmetry, $k = 1, \dots, \lfloor \frac{d}{2} \rfloor$. Based on the property of direct sums, $f_{d-1}(P_d) = f_{k-1}(T_k) f_{d-k-1}(T_{d-k}) = (k+1)(d-k+1)$. Together with $\mathbb{H}_{d+1}$ being d-simplex

($k = 0$), there are $1 + \lfloor \frac{d}{2} \rfloor$ possible simplical polytopes for $\mathbb{H}_{d+1}$ and thus there are $1 + \lfloor \frac{d}{2} \rfloor$ pairs of number of vertices and facets $\mathbb{H}_{d+1}$ can have. Therefore, in general, we can construct a polynomial $p_d : \mathbb{R} \to \mathbb{R}$ of order $\lfloor d/2 \rfloor$ as claimed by the proposition. ■

Example 237. For $d = 2$ upto $d = 11$, all possibilities are enlisted in Table 5.1.

(f_0, f_{d-1})	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$d = 2$	$(3, 3)$	$(4, 4)$				
$d = 3$	$(4, 4)$	$(5, 6)$				
$d = 4$	$(5, 5)$	$(6, 8)$	$(6, 9)$			
$d = 5$	$(6, 6)$	$(7, 10)$	$(7, 12)$			
$d = 6$	$(7, 7)$	$(8, 12)$	$(8, 15)$	$(8, 16)$		
$d = 7$	$(8, 8)$	$(9, 14)$	$(9, 18)$	$(9, 20)$		
$d = 8$	$(9, 9)$	$(10, 16)$	$(10, 21)$	$(10, 24)$	$(10, 25)$	
$d = 9$	$(10, 10)$	$(11, 18)$	$(11, 24)$	$(11, 28)$	$(11, 30)$	
$d = 10$	$(11, 11)$	$(12, 20)$	$(12, 27)$	$(12, 32)$	$(12, 35)$	$(12, 36)$
$d = 11$	$(12, 12)$	$(13, 22)$	$(13, 30)$	$(13, 36)$	$(13, 40)$	$(13, 42)$

Table 5.1: Number of vertices and facets of $P_d = \mathbb{H}_{d+1} = T_k \oplus T_{d-k}$.

From those values, we construct the following polynomials

$$p_2(x) = x, \tag{5.39}$$
$$p_3(x) = \frac{x}{2} + 2, \tag{5.40}$$
$$p_4(x) = \frac{17}{12}x - \frac{x^2}{12}, \tag{5.41}$$
$$p_5(x) = 2 + \frac{11}{12}x - \frac{x^2}{12}, \tag{5.42}$$
$$p_6(x) = \frac{17x}{10} - \frac{43x^2}{360} + \frac{x^3}{360}, \tag{5.43}$$
$$p_7(x) = 2 + \frac{223x}{180} - \frac{13x^2}{180} + \frac{x^3}{720}, \tag{5.44}$$
$$p_8(x) = \frac{537x}{280} - \frac{2749x^2}{20160} + \frac{43x^3}{10080} - \frac{x^4}{20160}, \tag{5.45}$$
$$p_9(x) = 2 + \frac{419x}{280} - \frac{103x^2}{1120} + \frac{5x^3}{2016} - \frac{x^4}{40320}, \tag{5.46}$$
$$p_{10}(x) = \frac{5281x}{2520} - \frac{32743x^2}{226800} + \frac{2971x^3}{604800} - \frac{x^4}{12096} + \frac{x^5}{1814400}, \tag{5.47}$$
$$p_{11}(x) = 2 + \frac{4307x}{2520} - \frac{5267x^2}{50400} + \frac{2857x^3}{907200} - \frac{17x^4}{362880} + \frac{x^5}{3628800}. \tag{5.48}$$

Remark 238. The leading coefficient of $p_d(x)$ is $\frac{(-1)^{\lfloor \frac{d}{2} \rfloor + 1}(d-1-\lfloor \frac{d}{2} \rfloor)!}{(d-1)!\lfloor \frac{d}{2} \rfloor!}$.

5.2 Two dimensions

5.2.1 Triangle first-order metric moments

Let us re-derive the solution of the Silvester problem, that is to find the (volu)metric moment $v_2^{(1)}(T_2)$, where T_2 is a triangle. By symmetry, $\gamma_n(T_2) = 3\gamma_n(T_2)_{\mathrm{I}}$, where I with $w_{\mathrm{I}} = 3$ is the only configuration in $\mathcal{C}(T_2)$ separating one of its vertices. By affinity, we may use $\mathbb{T}_2 = \mathrm{conv}([0,0],[1,0],[0,1])$ instead of T_2. In order $\boldsymbol{\sigma} : \boldsymbol{\eta}^\top \mathbf{x} = ax+by = 1$ to separate $S = \{[0,0]\}$, we must have $(\mathbb{R}^2 \setminus T_2^\circ)_{\mathrm{I}} = (1,\infty)^2$ to be the domain of integration of $\boldsymbol{\eta} = (a,b)^\top$ in configuration I. The area fraction (closer to the origin) is given by $\Gamma_2^+(\boldsymbol{\sigma})_{\mathrm{I}} = 1/(ab)$, so by Remark 297

$$\zeta_2(\mathbb{T})_{\mathrm{I}} = -a\frac{\partial \Gamma_2^+(\boldsymbol{\sigma})_{\mathrm{I}}}{\partial a} - b\frac{\partial \Gamma_2^+(\boldsymbol{\sigma})_{\mathrm{I}}}{\partial b} = \frac{2}{ab}. \tag{5.49}$$

Hence,

$$\gamma_n(\mathbb{T}_2)_{\mathrm{I}} = \frac{1}{6}\int_1^\infty \int_1^\infty \left(\frac{2}{ab}\right)^3 \left(\left(\frac{1}{ab}\right)^n + \left(1-\frac{1}{ab}\right)^n\right)\,\mathrm{d}a\,\mathrm{d}b = \frac{4H_{n+1}}{3(n+1)(n+2)}, \tag{5.50}$$

where $H_k = \sum_{j=1}^k 1/j$ is the k-th *harmonic number*, from which immediately

$$v_n^{(1)}(T_2) = v_n^{(1)}(\mathbb{T}_2) = 1 - \frac{n+1}{2}\gamma_n(\mathbb{T}_2) = 1 - \frac{2H_{n+1}}{n+2}. \tag{5.51}$$

Those values are tabulated in Table 5.2.

n	2	3	4	5	6	7	8	9	10
$v_n^{(1)}(T_2)$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{43}{180}$	$\frac{3}{10}$	$\frac{197}{560}$	$\frac{499}{1260}$	$\frac{5471}{12600}$	$\frac{589}{1260}$	$\frac{82609}{166320}$

Table 5.2: Convex hull area expectations $v_n^{(1)}(T_2)$

Silvester's problem is the special case when $n = 2$, that is $v_2^{(1)}(T_2) = 1 - \frac{1}{2}H_3 = \frac{1}{12}$.

5.2.2 Quadrilateral first area moment

We present a more elaborate example. In what follows, we find the first area moment in a *quadrilateral*. The first area moment in a quadrilateral was essential for Buchta and Reitzner (see their original 1992 paper [18]) to derive $v_3^{(1)}(\mathbb{T}_3)$ since the intersection of a section plane $\boldsymbol{\sigma}$ with $\mathbb{T}_3$ is either a triangle (treated in the previous section) or a quadrilateral – this is then plugged into three-dimensional Efron's section formula (Equation (5.32)). Although Buchta and Reitzner were able to get the first area moment in a quadrilateral from the general formula for $v_n^{(1)}(P_2)$ for P_2 being any polygon (which they described in [19]), they mentioned that the special case of P_2 being a quadrilateral is already contained in a textbook on geometric probability by Deltheil [23].

Any quadrilateral (apart from parallelograms) is affinely congruent to the *canonical truncated triangle* defined as

$$\mathbb{U}_2^{\alpha\beta} = \operatorname{conv}([\alpha,0],[0,\beta],[0,1],[1,0]) \tag{5.52}$$

with parameters $\alpha,\beta \in (0,1)$. Note that a parallelogram can still be obtained from $\mathbb{U}_2^{\alpha\beta}$ by the continuous limit $\alpha=\beta\to 1^-$. We have $\operatorname{vol}_2 \mathbb{U}_2^{\alpha\beta} = \frac{1}{2}(1-\alpha\beta)$. This time, our goal will be to derive only the first area moment $v_2^{(1)}(\mathbb{U}_2^{\alpha\beta})$. Let $\boldsymbol{\eta} = (a,b)^\top$ be the Cartesian parametrisation of the line $\boldsymbol{\sigma} \in \mathbb{A}(2,1)$ such that $\mathbf{x} \in \boldsymbol{\sigma} \Leftrightarrow \boldsymbol{\eta}^\top \mathbf{x} = 1$. We have $\|\boldsymbol{\eta}\| = \sqrt{a^2+b^2}$. Table 5.3 shows all possible configurations $\mathcal{C}(\mathbb{U}_2^{\alpha\beta})$ and their respective point selections S of vertices separated by a cutting plane $\boldsymbol{\sigma}$ which define those configurations.

C	I	II	III	IV	V	VI
S	$[\alpha,0]$	$[\beta,0]$	$[0,1]$	$[1,0]$	$[\alpha,0]$ $[1,0]$	$[1,0]$ $[0,1]$
w_{C}	1	1	1	1	1	1

Table 5.3: Configurations $\mathcal{C}(\mathbb{U}_2^{\alpha\beta})$.

Each configuration is unique, thus $w_{\mathrm{C}}=1$ for any C. Although there are in general no rigid symmetries of $\mathbb{U}_2^{\alpha\beta}$, we are still able to jump between configurations by using affine transformations. For any configuration C, we have for the gamma section functional (Equation (5.31)),

$$\gamma_2(\mathbb{U}_2^{\alpha\beta})_{\mathrm{C}} = \frac{1-\alpha\beta}{6}\int_{(\mathbb{R}^2\setminus(\mathbb{U}_2^{\alpha\beta})^\circ)_{\mathrm{C}}} \zeta_2^3(\boldsymbol{\sigma})(\Gamma_2^+(\boldsymbol{\sigma})^2+\Gamma_2^-(\boldsymbol{\sigma})^2)\lambda_2(\mathrm{d}\boldsymbol{\eta}), \tag{5.53}$$

where $\Gamma_2^+(\boldsymbol{\sigma})^2 = \operatorname{vol}_2(\mathbb{U}_2^{\alpha\beta})^+/\operatorname{vol}_2\mathbb{U}_2^{\alpha\beta}$ and $\Gamma_2^-(\boldsymbol{\sigma})^2 = \operatorname{vol}_2(\mathbb{U}_2^{\alpha\beta})^-/\operatorname{vol}_2\mathbb{U}_2^{\alpha\beta}$ are the area fractions of domains $(\mathbb{U}_2^{\alpha\beta})^+\sqcup(\mathbb{U}_2^{\alpha\beta})^- = \mathbb{U}_2^{\alpha\beta}$ onto which $\mathbb{U}_2^{\alpha\beta}$ is divided by line $\boldsymbol{\sigma}$. Furthermore, by definition of $\zeta_2(\boldsymbol{\sigma})$ and by Remark 297,

$$\zeta_2(\boldsymbol{\sigma})_{\mathrm{C}} = \frac{\operatorname{vol}_1(\boldsymbol{\sigma}\cap\mathbb{U}_2^{\alpha\beta})}{\|\boldsymbol{\eta}\|\operatorname{vol}_2\mathbb{U}_2^{\alpha\beta}} = -a\frac{\partial\Gamma_2^+(\boldsymbol{\sigma})_{\mathrm{C}}}{\partial a} - b\frac{\partial\Gamma_2^+(\boldsymbol{\sigma})_{\mathrm{C}}}{\partial b}. \tag{5.54}$$

Here, C is only a subscript to distinguish between configurations and does not imply any decomposition of $\zeta_2(\boldsymbol{\sigma})$ nor $\Gamma_2^\pm(\boldsymbol{\sigma})$.

Configuration I

Let us cut of the vertex $[\alpha,0]$. By Equation (4.29), we get the following set of inequalities which ensure $\boldsymbol{\sigma}$ separates only the point $[\alpha,0]$,

$$a\alpha<1, \qquad b\beta>1, \qquad a>1, \qquad b>1, \tag{5.55}$$

hence, our a,b integration domain is $(\mathbb{R}^2\setminus(\mathbb{U}_2^{\alpha\beta})^\circ)_{\mathrm{I}} = (1,1/\alpha)\times(1/\beta,\infty)$. The line $\boldsymbol{\sigma}$ splits $\mathbb{U}_2^{\alpha\beta}$ into disjoint union of two domains $(\mathbb{U}_2^{\alpha\beta})^+\sqcup(\mathbb{U}_2^{\alpha\beta})^-$, where

$$(\mathbb{U}_2^{\alpha\beta})^+ = \operatorname{conv}\left([\alpha,0],\left[\frac{1}{a},0\right],\left[\frac{\alpha(b\beta-1)}{b\beta-a\alpha},\frac{\beta(1-a\alpha)}{b\beta-a\alpha}\right]\right), \tag{5.56}$$

which has area equal to

$$\mathrm{vol}_2(\mathbb{U}_2^{\alpha\beta})^+ = \frac{\beta(1-a\alpha)^2}{2a(b\beta - a\alpha)} \tag{5.57}$$

from which we get for the area fraction (closer to the origin)

$$\Gamma_2^+(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{\beta(1-a\alpha)^2}{a(1-\alpha\beta)(b\beta-a\alpha)}. \tag{5.58}$$

Hence, by Equation (5.54),

$$\zeta_2(\mathbb{U}_2^{\alpha\beta})_{\mathrm{I}} = \frac{2\beta(1-a\alpha)}{a(1-\alpha\beta)(b\beta-a\alpha)}. \tag{5.59}$$

Finally, for the gamma section functional,

$$\begin{aligned}\gamma_2(\mathbb{U}_2^{\alpha\beta})_{\mathrm{I}} = \int_{1/\beta}^{\infty}\int_{1}^{1/\alpha} &\frac{4(1-a\alpha)^3\beta^3}{3a^5(a\alpha-b\beta)^5(1-\alpha\beta)^4}\times \\ &\left[(1-a\alpha)^4\beta^2 + \left(a^2\alpha+\beta-ab\beta(1-\alpha\beta)-2a\alpha\beta\right)^2\right]\,\mathrm{d}a\,\mathrm{d}b.\end{aligned} \tag{5.60}$$

Integrating out a and b, we get

$$\gamma_2(\mathbb{U}_2^{\alpha\beta})_{\mathrm{I}} = \frac{(1-\alpha)^2\beta^2\left(18-16\beta-20\alpha\beta+9\beta^2-2\alpha\beta^2+11\alpha^2\beta^2\right)}{54(1-\alpha\beta)^4}. \tag{5.61}$$

Configuration II

Note that Configuration II is obtained from Configuration I by reflection, that is by replacing α with β and vice versa in Equation (5.61). We get

$$\gamma_2(\mathbb{U}_2^{\alpha\beta})_{\mathrm{II}} = \frac{\alpha^2(1-\beta)^2\left(18-16\alpha+9\alpha^2-20\alpha\beta-2\alpha^2\beta+11\alpha^2\beta^2\right)}{54(1-\alpha\beta)^4}. \tag{5.62}$$

Configuration III

This configuration can be deduced from configuration II. Let $\mathbf{x}, \mathbf{v} \in \mathbb{R}^2$ and $M \in \mathbb{R}^{2\times 2}$ be a non-singular matrix. Consider an affine transformation $\mathbf{x} \mapsto M\mathbf{x} + \mathbf{v}$ with

$$M = \frac{1}{1-\alpha}\begin{pmatrix} -1 & -1 \\ 1 & \alpha/\beta \end{pmatrix}, \qquad \mathbf{v} = \frac{1}{1-\alpha}\begin{pmatrix} 1 \\ -\alpha \end{pmatrix}. \tag{5.63}$$

Applying the transformation on $\mathbb{U}_2^{\alpha\beta}$, we get

$$\mathbb{U}_2^{\alpha\beta} \mapsto \mathrm{conv}\left(\left[\tfrac{1-\beta}{1-\alpha}, 0\right], \left[0, \tfrac{\alpha(1-\beta)}{\beta(1-\alpha)}\right], [0,1], [1,0]\right), \tag{5.64}$$

which is another canonical truncated triangle $\mathbb{U}_2^{\gamma\delta}$ with

$$\gamma = \frac{1-\beta}{1-\alpha}, \qquad \delta = \frac{\alpha(1-\beta)}{\beta(1-\alpha)}. \tag{5.65}$$

Since the vertex $[0,1]$ to be cut away by $\boldsymbol{\sigma}$ maps to $\left[0, \frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right]^\top$ under our affine transformation, we realise that configuration III is equivalent to configuration II with γ, δ instead of α, β. Replacing $\alpha \to \frac{1-\beta}{1-\alpha}$ and $\beta \to \frac{\alpha(1-\beta)}{\beta(1-\alpha)}$ in Equation (5.62),

$$\gamma_2(\mathbb{U}_2^{\alpha\beta})_{\mathrm{III}} = \frac{(1-\beta)^2\,(11 - 2\beta - 20\alpha\beta + 9\beta^2 - 16\alpha\beta^2 + 18\alpha^2\beta^2)}{54(1-\alpha\beta)^4}. \tag{5.66}$$

Although we should assume that $\alpha < \beta$ (otherwise γ and δ are negative), it turns out the formula above is in fact valid for any α, β in $[0,1)$.

Configuration IV

By symmetry, configuration IV is obtained from III by replacing α with β and vice versa. Equation (5.66) therefore yields

$$\gamma_2(\mathbb{U}_2^{\alpha\beta})_{\mathrm{IV}} = \frac{(1-\alpha)^2\,(11 - 2\alpha + 9\alpha^2 - 20\alpha\beta - 16\alpha^2\beta + 18\alpha^2\beta^2)}{54(1-\alpha\beta)^4}. \tag{5.67}$$

Configuration V

By Equation (4.29), we get the following set of inequalities which ensure $\boldsymbol{\sigma}$ separates points $[1,0]$ and $[0,1]$,

$$a\alpha < 1, \qquad b\beta < 1, \qquad a > 1, \qquad b > 1, \tag{5.68}$$

from which we obtain the integration domain in (a,b) as

$$(\mathbb{R}^2 \setminus (\mathbb{U}_2^{\alpha\beta})^\circ)_{\mathrm{V}} = (1, 1/\alpha) \times (1, 1/\beta). \tag{5.69}$$

The plane $\boldsymbol{\sigma}$ splits $\mathbb{U}_2^{\alpha\beta}$ into disjoint union of two domains $(\mathbb{U}_2^{\alpha\beta})^+ \sqcup (\mathbb{U}_2^{\alpha\beta})^-$, where

$$(\mathbb{U}_2^{\alpha\beta})^- = \operatorname{conv}\left(\left[\tfrac{1}{a}, 0\right], \left[0, \tfrac{1}{b}\right], [0,1], [1,0]\right), \tag{5.70}$$

which is again a canonical truncated triangle with area $\mathrm{vol}_2(\mathbb{U}_2^{\alpha\beta})^- = \frac{1}{2}(1 - \frac{1}{ab})$ and the corresponding area fraction

$$\Gamma_2^-(\boldsymbol{\sigma})_{\mathrm{V}} = \frac{\mathrm{vol}_2(\mathbb{U}_2^{\alpha\beta})^-}{\mathrm{vol}_2\, \mathbb{U}_2^{\alpha\beta}} = \frac{1 - \frac{1}{ab}}{1-\alpha\beta}, \tag{5.71}$$

from which, by Equation (5.54),

$$\zeta_2(\boldsymbol{\sigma})_{\mathrm{V}} = a\frac{\partial \Gamma_2^-(\boldsymbol{\sigma})_{\mathrm{V}}}{\partial a} + b\frac{\partial \Gamma_2^-(\boldsymbol{\sigma})_{\mathrm{V}}}{\partial b} = \frac{2}{ab(1-\alpha\beta)}. \tag{5.72}$$

Finally, for the gamma section functional, we get by Equation (5.53)

$$\gamma_2(\mathbb{U}_2^{\alpha\beta})_{\mathrm{V}} = \int_1^{1/\beta}\int_1^{1/\alpha} \frac{4\,(2 - 2ab + a^2b^2 - 2ab\alpha\beta + a^2b^2\alpha^2\beta^2)}{3a^5b^5(1-\alpha\beta)^4}\,\mathrm{d}a\,\mathrm{d}b. \tag{5.73}$$

Integrating out a and b, we get

$$\gamma_2(\mathbb{U}_2^{\alpha\beta})_{\mathrm{V}} = \frac{\begin{Bmatrix} 11 - 18\alpha^2 - 18\beta^2 + 16\alpha^3 + 16\beta^3 - 9\alpha^4 - 9\beta^4 - 16\alpha\beta + 36\alpha^2\beta^2 \\ -16\alpha^3\beta^3 + 11\alpha^4\beta^4 + 16\alpha^4\beta + 16\alpha\beta^4 - 18\alpha^2\beta^4 - 18\alpha^4\beta^2 \end{Bmatrix}}{54(1-\alpha\beta)^4}. \tag{5.74}$$

Note that $\gamma_2(\mathbb{U}_2^{\alpha\beta})_{\mathrm{V}} = \gamma_2(\mathbb{U}_2^{\beta\alpha})_{\mathrm{V}}$ as expected by symmetry.

Configuration VI

Let $\mathbf{x} \mapsto M\mathbf{x}+\mathbf{v}$ be the affine transformation from configuration III, that is with $M, \mathbf{v}$ given by Equation (5.63). Again we have $\mathbb{U}_2^{\alpha\beta} \mapsto \mathbb{U}_2^{\gamma\delta}$ with

$$\gamma = \frac{1-\beta}{1-\alpha}, \qquad \delta = \frac{\alpha(1-\beta)}{\beta(1-\alpha)}. \tag{5.75}$$

Therefore, configuration VI is equivalent to configuration V with γ, δ instead of α, β. Replacing $\alpha \to \frac{1-\beta}{1-\alpha}$ and $\beta \to \frac{\alpha(1-\beta)}{\beta(1-\alpha)}$ in Equation (5.74),

$$\gamma_2(\mathbb{U}_2^{\alpha\beta})_{\text{VI}} = \frac{\begin{Bmatrix} 24\alpha+24\beta-24\alpha^2-24\beta^2+20\alpha^3+20\beta^3-9\alpha^4-9\beta^4-72\alpha\beta \\ +106\alpha^2\beta^2-72\alpha^3\beta^3+4\alpha^2\beta+4\alpha\beta^2-20\alpha^3\beta-20\alpha\beta^3+20\alpha^4\beta \\ +20\alpha\beta^4+4\alpha^3\beta^2+4\alpha^2\beta^3-24\alpha^4\beta^2-24\alpha^2\beta^4+24\alpha^4\beta^3+24\alpha^3\beta^4 \end{Bmatrix}}{54(1-\alpha\beta)^4}. \tag{5.76}$$

Note that again $\gamma_2(\mathbb{U}_2^{\alpha\beta})_{\text{VI}} = \gamma_2(\mathbb{U}_2^{\beta\alpha})_{\text{VI}}$ as expected by symmetry.

Contribution from all configurations

By Equation (C.118), we get after some simplifications,

$$\gamma_2(\mathbb{U}_2^{\alpha\beta}) = \sum_{C\in\mathcal{C}(\mathbb{U}_2^{\alpha\beta})} w_{\text{C}}\, \gamma_2(\mathbb{U}_2^{\alpha\beta})_{\text{C}} = \frac{11}{18} + \frac{2\alpha\beta(1-\alpha)^2(1-\beta)^2}{27(1-\alpha\beta)^4}, \tag{5.77}$$

which yields by Equation (5.29) with $n = 2$ and for any $\alpha, \beta \in [0, 1)$,

$$v_2^{(1)}(\mathbb{U}_2^{\alpha\beta}) = 1 - \frac{3}{2}\gamma_2(\mathbb{U}_2^{\alpha\beta}) = \frac{1}{12} - \frac{\alpha\beta(1-\alpha)^2(1-\beta)^2}{9(1-\alpha\beta)^4}. \tag{5.78}$$

Remark 239. Note that as $\alpha = \beta = 0$, we get Silvester's $v_2^{(1)}(T_2) = 1/12$ and when $\alpha = \beta \to 1^-$, we get $v_2^{(1)}(C_2) = 11/144$ as expected.

5.2.3 Half-disk first-order metric moments

What is the value of $v_2^{(1)}(\mathbb{D}_2)$ where $\mathbb{D}_2$ is the half-disk? Henze proposed me this problem while I was at a conference in Bad Herrenbald. More concretely, let

$$\mathbb{D}_2 = \{(x, y)^\top \in \mathbb{R}^2 \mid x^2 + y^2 \le 1 \wedge y > 0\} \tag{5.79}$$

and $\boldsymbol{\eta} = (a, b)^\top$ be the Cartesian parametrization of cutting plane $\boldsymbol{\sigma}$, that is $\mathbf{x} = (x, y)^\top \in \boldsymbol{\sigma} \Leftrightarrow \boldsymbol{\eta}^\top\mathbf{x} = ax + by = 1$. Although the top part of the boundary of $\mathbb{D}_2$ is smooth, we may still recognize two distinct configurations. See Figure 5.1 below.

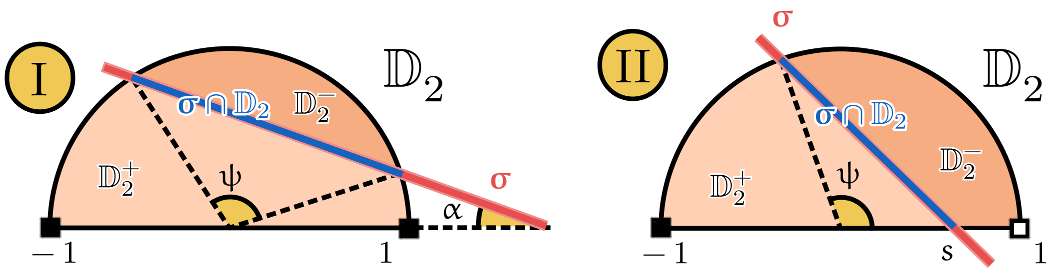


Figure 5.1: Half-disk configurations $\mathcal{C}(\mathbb{D}_2)$

Configuration I

Consider the following change of variables $(a, b) \to (\psi, \alpha)$, where $\alpha \in [0, \pi)$ is the angle of (left) incidence of $\boldsymbol{\sigma}$ and ψ is the angle of a circular sector on $\mathbb{D}_2$ intersected by $\boldsymbol{\sigma}$ (see Figure 5.1 on the left). By symmetry (symmetry factor of 2), we assume that $\alpha \in [0, \frac{\pi}{2})$ and thus $0 < \psi < \pi - 2\alpha$. Point $\boldsymbol{\xi}$ (the closest point of $\boldsymbol{\sigma}$ to the origin) can be expressed as

$$\boldsymbol{\xi} = (\cos\tfrac{\psi}{2}\sin\alpha, \cos\tfrac{\psi}{2}\cos\alpha)^\top, \tag{5.80}$$

which follows from the slope angle being minus $\tan\alpha$ and from $\|\boldsymbol{\xi}\| = \cos\frac{\psi}{2}$ or equivalently $1/\|\boldsymbol{\eta}\| = \cos\frac{\psi}{2}$. Since $\boldsymbol{\eta} = \boldsymbol{\xi}/\|\boldsymbol{\xi}\|^2$ and $\boldsymbol{\eta} = (a, b)^\top$, we get the following transformation rules

$$a = \frac{\sin\alpha}{\cos\frac{\psi}{2}}, \qquad b = \frac{\cos\alpha}{\cos\frac{\psi}{2}} \tag{5.81}$$

with the Lebesgue measure transforming as (by calculating the Jacobian)

$$\lambda_2(\mathrm{d}\boldsymbol{\eta}) = \mathrm{d}a\ \mathrm{d}b = \frac{\sin\frac{\psi}{2}}{2\cos^3\frac{\psi}{2}}\ \mathrm{d}\alpha\mathrm{d}\psi. \tag{5.82}$$

By simple geometry, we get for the length of intersection

$$\mathrm{vol}_1(\boldsymbol{\sigma} \cap \mathbb{D}_2) = 2\sqrt{1 - \|\boldsymbol{\xi}\|^2} = 2\sin\tfrac{\psi}{2} \tag{5.83}$$

and thus

$$\zeta_2(\boldsymbol{\sigma}) = \frac{\mathrm{vol}_1(\boldsymbol{\sigma} \cap \mathbb{D}_2)}{\|\boldsymbol{\eta}\|\,\mathrm{vol}_2\,\mathbb{D}_2} = \frac{4}{\pi}\sin\tfrac{\psi}{2}\cos\tfrac{\psi}{2} = \frac{2\sin\psi}{\pi}. \tag{5.84}$$

For the area of the circular segment $\mathbb{D}_2^-$ (above $\boldsymbol{\sigma}$), we have

$$\mathrm{vol}_2\,\mathbb{D}_2^- = \tfrac{\psi}{2} - \tfrac{1}{2}\sin\psi \tag{5.85}$$

and thus, since $\mathrm{vol}_2\,\mathbb{D}_2 = \pi/2$,

$$\Gamma_3^-(\boldsymbol{\sigma}) = \frac{\mathrm{vol}_2\,\mathbb{D}_2^-}{\mathrm{vol}_2\,\mathbb{D}_2} = \frac{1}{\pi}(\psi - \sin\psi). \tag{5.86}$$

By Equation (5.31) (including the symmetry factor 2),

$$\gamma_n(\mathbb{D}_2)_{\mathrm{I}} = \frac{32}{3\pi^{2+n}}\int_0^\pi\int_0^{\frac{\pi}{2}-\frac{\psi}{2}} \sin^4\tfrac{\psi}{2}((\psi - \sin\psi)^n + (\pi - \psi + \sin\psi)^n)\ \mathrm{d}\sigma\mathrm{d}\psi. \tag{5.87}$$

Integrating out α and writing $\sin^4\frac{\psi}{2} = (1 - \cos\psi)^2/4$, we get

$$\gamma_n(\mathbb{D}_2)_{\mathrm{I}} = \frac{4}{3\pi^{2+n}}\int_0^\pi (\pi - \psi)(1 - \cos\psi)^2((\psi - \sin\psi)^n + (\pi - \psi + \sin\psi)^n)\mathrm{d}\psi. \tag{5.88}$$

This integral is elementary for a given n. Especially, when $n = 2$, we get

$$\gamma_2(\mathbb{D}_2)_{\mathrm{I}} = \frac{2}{3} + \frac{2816}{81\pi^4} - \frac{131}{18\pi^2}. \tag{5.89}$$

Configuration II

Consider another change of variables $(a, b) \to (\psi, s)$, where ψ is again the angle of the circular sector and $s = 1/a$ is the x-intercept of $\boldsymbol{\sigma}$ (see Figure 5.1 on the right). Our integration domain in those variables is $0 < \psi < \pi$ and $-1 < s < 1$. By symmetry, we can restrict $s \in (0, \pi)$ (symmetry factor 2). By simple geometry,

$$\mathrm{vol}_2\, \mathbb{D}_2^- = \tfrac{\psi}{2} - \tfrac{s}{2} \sin\psi, \tag{5.90}$$

from which

$$\Gamma_3^-(\boldsymbol{\sigma}) = \frac{\mathrm{vol}_2\, \mathbb{D}_2^-}{\mathrm{vol}_2\, \mathbb{D}_2} = \frac{1}{\pi}(\psi - s\sin\psi). \tag{5.91}$$

By the rule of cosines,

$$\mathrm{vol}_1(\boldsymbol{\sigma} \cap \mathbb{D}_2) = \sqrt{1 + s^2 - 2s\cos\psi}. \tag{5.92}$$

Equating the area of triangle $[0, 0], [0, s], [\cos\psi, \sin\psi]$ in two ways, we get for the distance of the closest point $\boldsymbol{\xi}$ on $\boldsymbol{\sigma}$ from the origin,

$$\|\boldsymbol{\xi}\| = \frac{s\sin\psi}{\sqrt{1 + s^2 - 2s\cos\psi}} \tag{5.93}$$

and thus

$$\zeta_2(\boldsymbol{\sigma}) = \|\boldsymbol{\xi}\| \frac{\mathrm{vol}_1(\boldsymbol{\sigma} \cap \mathbb{D}_2)}{\mathrm{vol}_2\, \mathbb{D}_2} = \frac{2s\sin\psi}{\pi}. \tag{5.94}$$

Since $\|\boldsymbol{\xi}\| = 1/\|\boldsymbol{\eta}\| = 1/\sqrt{a^2 + b^2}$ and $a = 1/s$, we get, solving for b,

$$b = \frac{s - \cos\psi}{s\sin\psi}, \tag{5.95}$$

from which (by calculating the Jacobian)

$$\lambda_2(\mathrm{d}\boldsymbol{\eta}) = \mathrm{d}a\ \mathrm{d}b = \frac{1 - s\cos\psi}{s^3 \sin^2\psi}\ \mathrm{d}s\mathrm{d}\psi. \tag{5.96}$$

By Equation (5.31), with the symmetry factor 2,

$$\gamma_n(\mathbb{D}_2)_{\mathrm{II}} = \frac{8}{3\pi^{2+n}} \int_0^\pi \int_0^1 \sin\psi(1 - s\cos\psi)((\psi - s\sin\psi)^n + (\pi - \psi + s\sin\psi)^n)\mathrm{d}s\mathrm{d}\psi. \tag{5.97}$$

This integral is again elementary for a given n. Especially, when $n = 2$, we get

$$\gamma_2(\mathbb{D}_2)_{\mathrm{II}} = \frac{16}{3\pi^2} - \frac{1664}{81\pi^4}. \tag{5.98}$$

Contribution from all configurations

Adding up the contributions from the two configurations,

$$v_n^{(1)}(\mathbb{D}_2) = 1 - \frac{n+1}{2}\gamma_n(\mathbb{D}_2) = 1 - \frac{n+1}{2}\left(\gamma_n(\mathbb{D}_2)_{\mathrm{I}} + \gamma_n(\mathbb{D}_2)_{\mathrm{II}}\right). \tag{5.99}$$

When $n = 2$, we get the answer for Henze's question to be

$$v_2^{(1)}(\mathbb{D}_2) = \frac{35}{12\pi^2} - \frac{64}{3\pi^4} \approx 0.0765125. \tag{5.100}$$

The first-order metric moments with higher n are tabulated in Table 5.4 below.

n	2	3	4	5
$v_n^{(1)}(\mathbb{D}_2)$	$\frac{35}{12\pi^2} - \frac{64}{3\pi^4}$	$\frac{35}{6\pi^2} - \frac{128}{3\pi^4}$	$\frac{175}{18\pi^2} - \frac{69103}{432\pi^4} + \frac{204800}{243\pi^6}$	$\frac{204800}{81\pi^6} - \frac{53743}{144\pi^4} + \frac{175}{12\pi^2}$

n	6	7
$v_n^{(1)}(\mathbb{D}_2)$	$\frac{245}{12\pi^2} - \frac{322441}{432\pi^4} + \frac{954438853}{86400\pi^6} - \frac{24090300416}{455625\pi^8}$	$\frac{245}{9\pi^2} - \frac{290185}{216\pi^4} + \frac{6296189677}{194400\pi^6} - \frac{96361201664}{455625\pi^8}$

Table 5.4: Convex hull area expectations $v_n^{(1)}(\mathbb{D}_2)$

Note that only the values of $v_n^{(1)}(\mathbb{D}_2)$ with even n are independent. Values with odd n can be deduces from other even values by Affentranger's relations (Equation (5.35)).

5.3 Three dimensions

5.3.1 Tetrahedron first-order metric moments

Let us rederive the result of Buchta and Reitzner, namely $v_3^{(1)}(T_3)$ (see their original paper [18] from 1992) and $v_n^{(1)}(T_3)$ for general n (see their follow-up paper [20] from 2001). The auxiliary value of $\gamma_n(T_3)$ is split into configurations $\mathcal{C}(T_3) = \{\mathrm{I}, \mathrm{II}\}$ (shown in Figure 4.6). By affine invariancy, we can instead calculate $\gamma_n(\mathbb{T}_3)_\mathrm{C}$, where

$$\mathbb{T}_3 = \operatorname{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \operatorname{conv}([0,0,0],[1,0,0],[0,1,0],[0,0,1]) \tag{5.101}$$

is the canonical tetrahedron with $\operatorname{vol}_3 \mathbb{T}_3 = 1/6$. Its configurations are determined by the points which are separated by the cutting plane $\boldsymbol{\sigma} \in \mathbb{A}(3,2)$ (see Table 4.11). Since the analysis is the same as in the Canonical integral approach, we skip the unnecessary details. By Equation (5.32) and for any $\mathrm{C} \in \mathcal{C}(\mathbb{T}_3)$,

$$\gamma_n(\mathbb{T}_3)_\mathrm{C} = \tfrac{1}{3} \int_{\mathbb{R}^3 \setminus \mathbb{T}_3^\circ} v_2^{(1)}(\boldsymbol{\sigma} \cap \mathbb{T}_3) \zeta_3^4(\boldsymbol{\sigma}) (\Gamma_3^+(\boldsymbol{\sigma})^{n-1} + \Gamma_3^-(\boldsymbol{\sigma})^{n-1}) \lambda_3(\mathrm{d}\boldsymbol{\eta}), \tag{5.102}$$

where $\boldsymbol{\eta} = (a,b,c)^\top$ is the Cartesian parametrisation of $\boldsymbol{\sigma}$ such that $\mathbf{x} \in \boldsymbol{\sigma} \Leftrightarrow \boldsymbol{\eta}^\top \mathbf{x} = 1$ and

$$\zeta_3(\boldsymbol{\sigma}) = \frac{\operatorname{vol}_2(\boldsymbol{\sigma} \cap \mathbb{T}_3)}{\|\boldsymbol{\eta}\| \operatorname{vol}_3 \mathbb{T}_3}, \qquad \Gamma_3^+(\boldsymbol{\sigma}) = \frac{\operatorname{vol}_3 \mathbb{T}_3^+}{\operatorname{vol}_3 \mathbb{T}_3}, \qquad \Gamma_3^-(\boldsymbol{\sigma}) = \frac{\operatorname{vol}_3 \mathbb{T}_3^-}{\operatorname{vol}_3 \mathbb{T}_3} \tag{5.103}$$

with $\mathbb{T}_3^+ = \{\mathbf{x} \in \mathbb{T}_3 \mid \boldsymbol{\eta}^\top \mathbf{x} < 1\}$ and $\mathbb{T}_3^- = \{\mathbf{x} \in \mathbb{T}_3 \mid \boldsymbol{\eta}^\top \mathbf{x} > 1\}$. In order to distinguish between configurations, we also write $\zeta_3(\boldsymbol{\sigma})_\mathrm{C}$, $\Gamma_3^+(\boldsymbol{\sigma})_\mathrm{C}$ and $\Gamma_3^-(\boldsymbol{\sigma})_\mathrm{C}$ instead of just $\zeta_3(\boldsymbol{\sigma})$, $\Gamma_3^+(\boldsymbol{\sigma})$ and $\Gamma_3^-(\boldsymbol{\sigma})$. Here, C is only a subscript and does not imply any decomposition of those functions.

Configuration I

We already calculated functions related with the canonical tetrahedron in configuration I. However, let us derive some of them again in a different way. First,

$$\mathbb{T}_3^+ = \operatorname{conv}([0,0,0],[1/a,0,0],[0,1/b,0],[0,0,1/c]) \tag{5.104}$$

from which its trivial to see that

$$\Gamma_3^+(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{\operatorname{vol}_3 \mathbb{T}_3^+}{\operatorname{vol}_3 \mathbb{T}_3} = \frac{1}{abc}. \tag{5.105}$$

By handy Remark 297, we get

$$\zeta_3(\boldsymbol{\sigma})_{\mathrm{I}} = -\sum_{j=1}^{3} \eta_j \frac{\partial \Gamma_3^+(\boldsymbol{\sigma})}{\partial \eta_j} = -\left(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}\right)\Gamma_3^+(\boldsymbol{\sigma}) = \frac{3}{abc}. \tag{5.106}$$

Next, for any $\boldsymbol{\sigma}$ by scale affinity and by the already obtained solution of the Silvester problem (see Table 5.2),

$$v_2^{(1)}(\boldsymbol{\sigma} \cap \mathbb{T}_3) = v_2^{(1)}(T_2) = \frac{1}{12}. \tag{5.107}$$

Finally, our integration domain in a, b, c is $(\mathbb{R}^3 \setminus \mathbb{T}_3^\circ)_{\mathrm{I}} = (1,\infty)^3$ and therefore, putting everything into the integral in Equation (5.102), we get

$$\gamma_n(\mathbb{T}_3)_{\mathrm{I}} = \frac{9}{4}\int_1^\infty\int_1^\infty\int_1^\infty \tfrac{1}{a^4b^4c^4}\left(\left(\tfrac{1}{abc}\right)^{n-1} + \left(1-\tfrac{1}{abc}\right)^{n-1}\right)\,\mathrm{d}a\;\mathrm{d}b\;\mathrm{d}c. \tag{5.108}$$

This triple integral can be solved exactly for any n. Consider the following substitution $(a,b,c) \to (x,y,z)$ where

$$a = \tfrac{y}{x}, \qquad b = \tfrac{z}{y}, \qquad c = \tfrac{1}{z}. \tag{5.109}$$

This substitution enables us easily to integrate out y and z to get a single integral

$$\gamma_n(\mathbb{T}_3)_{\mathrm{I}} = \frac{9}{8}\int_0^1 x^2\left((1-x)^{n-1} + x^{n-1}\right)\ln^2 x\;\mathrm{d}x. \tag{5.110}$$

After some manipulations, we arrive at the following formula

$$\gamma_n(\mathbb{T}_3)_{\mathrm{I}} = \frac{9}{4}\left(\frac{(H_{n+2})^2 - 3H_{n+2} + H'_{n+2} + 1}{n(n+1)(n+2)} + \frac{1}{(n+2)^3}\right), \tag{5.111}$$

where $H_k = \sum_{j=1}^k 1/j$ is the k-th *harmonic number* and $H'_k = \sum_{j=1}^k 1/j^2$ is the k-th *diharmonic number*. Table 5.5 below shows $\gamma_n(\mathbb{T}_3)_{\mathrm{I}}$ for low values of n.

n	3	4	5	6	7	8	9	10
$\gamma_n(\mathbb{T}_3)_{\mathrm{I}}$	$\frac{2353}{48000}$	$\frac{3059}{96000}$	$\frac{182431}{8232000}$	$\frac{106583}{6585600}$	$\frac{8723171}{711244800}$	$\frac{9721567}{1016064000}$	$\frac{1291624303}{169047648000}$	$\frac{1402000513}{225396864000}$

Table 5.5: Auxiliary integral $\gamma_n(\mathbb{T}_3)_{\mathrm{I}}$

Configuration II

In this scenario, $\boldsymbol{\sigma}$ separates two points $[0,0,0]$ and $[0,0,1]$ (see Figure 4.9). We can split the integration domain $(\mathbb{R}^3 \setminus \mathbb{T}_3^\circ)_{\mathrm{II}}$ into two halfs, each of which contributes the same amount to $\gamma_n(\mathbb{T}_3)_{\mathrm{II}}$. One of the two domains is the following

$$(\mathbb{R}^3 \setminus \mathbb{T}_3^\circ)_{\mathrm{II}}^* = (1,\infty)^2 \times (0,1). \tag{5.112}$$

The plane $\boldsymbol{\sigma}$ splits $\mathbb{T}_3$ into disjoint union of two domains $\mathbb{T}_3^+ \sqcup \mathbb{T}_3^-$, where

$$\mathbb{T}_3^+ = \operatorname{conv}\Big([0,0,0],[0,0,1],[\tfrac{1}{a},0,0],[0,\tfrac{1}{b},0],[\tfrac{1-c}{a-c},0,\tfrac{a-1}{a-c}],[0,\tfrac{1-c}{b-c},\tfrac{b-1}{b-c}]\Big), \tag{5.113}$$

from which we obtain

$$\Gamma_3^+(\boldsymbol{\sigma})_{\mathrm{II}} = \frac{\operatorname{vol}_3 \mathbb{T}_3^+}{\operatorname{vol}_3 \mathbb{T}_3} = \frac{c-a-b+3ab-3abc+abc^2}{ab(a-c)(b-c)}. \tag{5.114}$$

and thus by Remark 297, we get

$$\zeta_3(\boldsymbol{\sigma})_{\mathrm{II}} = -\left(a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}\right)\Gamma_3^+(\boldsymbol{\sigma}) = \frac{3(c-a-b+2ab-abc)}{ab(a-c)(b-c)}. \tag{5.115}$$

We make the following change of variables $(a,b,c) \to (\alpha,\beta,c)$ via transformation

$$\alpha = \frac{a(1-c)}{a-c}, \qquad \beta = \frac{b(1-c)}{b-c}, \tag{5.116}$$

by which

$$\Gamma_3^+(\boldsymbol{\sigma})_{\mathrm{II}} = \frac{(1-c-\alpha)(1-c-\beta)(1-(1-c)\alpha\beta)}{c^3\alpha\beta}, \tag{5.117}$$

$$\zeta_3(\boldsymbol{\sigma})_{\mathrm{II}} = \frac{3(1-c-\alpha)(1-c-\beta)(1-\alpha\beta)}{c^3\alpha\beta}. \tag{5.118}$$

For the intersection of $\boldsymbol{\sigma}$ with $\mathbb{T}_3$, we have

$$\boldsymbol{\sigma} \cap \mathbb{T}_3 = \operatorname{conv}\Big([\tfrac{1}{a},0,0],[0,\tfrac{1}{b},0],[\tfrac{1-c}{a-c},0,\tfrac{a-1}{a-c}],[0,\tfrac{1-c}{b-c},\tfrac{b-1}{b-c}]\Big), \tag{5.119}$$

By scale affinity and recalling Equation (5.78),

$$v_2^{(1)}(\boldsymbol{\sigma} \cap \mathbb{T}_3) = v_2^{(1)}(\mathbb{U}_2^{\alpha\beta}) = \frac{1}{12} - \frac{\alpha\beta(1-\alpha)^2(1-\beta)^2}{9(1-\alpha\beta)^4} \tag{5.120}$$

where $\mathbb{U}_2^{\alpha\beta} = \operatorname{conv}([\alpha,0],[0,\beta],[0,1],[1,0])$ is the canonical truncated triangle (see Figure 4.10). Our change of variables transform the integration half-domain into

$$(\mathbb{R}^3 \setminus \mathbb{T}_3^\circ)_{\mathrm{II}}^*|_{\alpha,\beta,c} = (1-c,1)^2 \times (0,1). \tag{5.121}$$

Note that, if c is treated as a parameter, the variables a, b depend on α, β separately. As a consequence,

$$\mathrm{d}a = \frac{c(1-c)\ \mathrm{d}\alpha}{(1-c-\alpha)^2}, \qquad \mathrm{d}b = \frac{c(1-c)\ \mathrm{d}\beta}{(1-c-\beta)^2} \tag{5.122}$$

and thus one has for the of transformation of measure

$$\lambda_3(\mathrm{d}\boldsymbol{\eta}) = \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c = \frac{c^2(1-c)^2\ \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}c}{(1-c-\alpha)^2(1-c-\beta)^2}. \tag{5.123}$$

Putting everything into the integral in Equation (5.102) with prefactor 2, we get

$$\begin{aligned}
\gamma_n(\mathbb{T}_3)_{\mathrm{II}} = {} & \frac{3}{2}\int_0^1\int_{1-c}^1\int_{1-c}^1 \frac{(1-c)^2(1-c-\alpha)^2(1-c-\beta)^2}{c^{10}\alpha^4\beta^4}\times \\
& \Big(\big(\tfrac{(1-c-\alpha)(1-c-\beta)(1-(1-c)\alpha\beta)}{c^3\alpha\beta}\big)^{n-1} + \big(1-\tfrac{(1-c-\alpha)(1-c-\beta)(1-(1-c)\alpha\beta)}{c^3\alpha\beta}\big)^{n-1}\Big) \\
& \times \Big(3(1-\alpha\beta)^4 - 4\alpha\beta(1-\alpha)^2(1-\beta)^2\Big)\,\mathrm{d}\alpha\,\mathrm{d}\alpha\,\mathrm{d}c.
\end{aligned} \tag{5.124}$$

This integral is the same (apart from substitution) as $I_n(p_2, p_3)$ in Buchta and Reitzner [20, p. 23]. For a given n, integrating out α, β can be done relatively easily, when $n = 3$, we end up with

$$\gamma_3^{(1)}(\mathbb{T}_3)_{\mathrm{II}} = \int_0^1 \frac{c^2p_0 + 1200c(1-c)^2p_0\ln(1-c) + 3600(1-c)^2p_2\ln^2(1-c)}{1200c^{16}}\mathrm{d}c, \tag{5.125}$$

where

$$\begin{aligned}
p_0 = {} & 2318400 - 13910400c + 31299600c^2 - 28986000c^3 \\
& - 5018800c^4 + 40976800c^5 - 46746600c^6 + 28094200c^7 \\
& - 9678136c^8 + 1803672c^9 - 151833c^{10} - 903c^{11} + 357c^{12},
\end{aligned} \tag{5.126}$$

$$\begin{aligned}
p_1 = {} & 3864 - 17388c + 24796c^2 - 5642c^3 - 24680c^4 + 35233c^5 \\
& - 22818c^6 + 7765c^7 - 1358c^8 + 114c^9,
\end{aligned} \tag{5.127}$$

$$\begin{aligned}
p_2 = {} & 644 - 3220c + 5528c^2 - 2792c^3 - 4035c^4 + 8353c^5 \\
& - 6960c^6 + 3181c^7 - 814c^8 + 115c^9 - 8c^{10}.
\end{aligned} \tag{5.128}$$

The last c integration can be carried out by Mathematica (alternatively, we can use derivatives of the Beta function). We get

$$\gamma_3(\mathbb{T}_3)_{\mathrm{II}} = \frac{13891}{108000} + \frac{\pi^2}{45045}. \tag{5.129}$$

For higher values of n, the integration possesses similar difficulty, Table 5.6 below shows $\gamma_n(\mathbb{T}_3)_{\mathrm{II}}$ for low values of n.

n	4	5	6	7
$\gamma_n(\mathbb{T}_3)_{\mathrm{II}}$	$\frac{5891}{72000}+\frac{\pi^2}{30030}$	$\frac{343339}{6174000}+\frac{178\pi^2}{4849845}$	$\frac{588221}{14817600}+\frac{211\pi^2}{5819814}$	$\frac{3139907}{106686720}+\frac{22829\pi^2}{669278610}$

n	8	9	10
$\gamma_n(\mathbb{T}_3)_{\mathrm{II}}$	$\frac{17135963}{762048000}+\frac{461\pi^2}{14709420}$	$\frac{446479763}{25357147200}+\frac{6116122\pi^2}{214886239425}$	$\frac{475739497}{33809529600}+\frac{12890876\pi^2}{501401225325}$

Table 5.6: Auxiliary integral $\gamma_n(\mathbb{T}_3)_{\mathrm{II}}$

Contribution from all configurations

By affine invariancy and by decomposition into configurations,

$$\gamma_n(T_3) = \sum_{C \in \mathcal{C}(T_3)} w_{\mathrm{C}}\, \gamma_n(T_3)_{\mathrm{C}} = 4\gamma_n(\mathbb{T}_3)_{\mathrm{I}} + 3\gamma_n(\mathbb{T}_3)_{\mathrm{II}}, \tag{5.130}$$

Recall Efron's formula (Equation (5.29))

$$v_n^{(1)}(T_3) = \frac{n}{n+2} - \frac{n(n+1)}{12}\gamma_n(T_3). \tag{5.131}$$

When $n = 3$, we get $\gamma_3(T_3) = \frac{419}{720} + \frac{\pi^2}{15015}$, from which we immediately recover Buchta and Reitzner's [18]

$$v_3^{(1)}(T_3) = \frac{3}{5} - \gamma_3(T_3) = \frac{13}{720} - \frac{\pi^2}{15015} \approx 0.01739823925. \tag{5.132}$$

Moreover, for general $n \geq 3$, we get the same result as derived by Buchta and Reitzner in [20, Theorem 2]. Table 5.7 below shows $v_n^{(1)}(T_3)$ for low values of n.

n	3	4	5	6
$v_n^{(1)}(T_3)$	$\frac{13}{720} - \frac{\pi^2}{15015}$	$\frac{13}{288} - \frac{\pi^2}{6006}$	$\frac{127}{1680} - \frac{89\pi^2}{323323}$	$\frac{307}{2880} - \frac{211\pi^2}{554268}$
n	7	8	9	10
$v_n^{(1)}(T_3)$	$\frac{41369}{302400} - \frac{22829\pi^2}{47805615}$	$\frac{11129}{67200} - \frac{461\pi^2}{817190}$	$\frac{641303}{3326400} - \frac{3058061\pi^2}{4775249765}$	$\frac{37723}{172800} - \frac{6445438\pi^2}{9116385915}$

Table 5.7: Convex hull volume expectations $v_n^{(1)}(T_3)$

Note that only the values of $v_n^{(1)}(T_3)$ with odd n are independent. Values with even n can be deduces from other odd values by Affentranger's recurrence relations (Equation (5.34)).

5.4 Unsolved problems

We have seen that the metric moments $v_n^{(k)}(P_d)$ having $n = d$ can be computed for all odd k via our canonical section integral method whereas for $n > 1$ and $d = 3, k = 1$ we could use Efron's formula. However, we have shown that Efron's formula cannot be generalised in higher dimensions nor for higher moments because of the inability to obtain linear relations between the number of vertices and the number of facets. A natural question arises: How then can we compute $v_n^{(k)}(P_d)$ for odd $k > 1$ and $n > d$? Or when $d \geq 4$?

6. Radial Random Simplices

In this chapter, we deduce formulae for volumetric moments of random simplices whose vertices are drawn independently from radially symmetric distributions. Such simplices are called *radial simplices* (Miles [48] uses the term *isotropic*). There are, in fact, multiple approaches as we can express the volume of random simplices. For example, by using its facets, determinants of random matrices, integrals over section planes, Wishart (and other special) distributions, Grassmann/fermionic variables, etc.

Originally, the main purpose of this chapter is to deduce Theorem 181 without the knowledge of the properties of the (shifted) Wishart distribution. As a result, we establish even more tight connection between random determinant moments and their random simplices volumetric moments counterparts.

6.1 Definitions

We will be mainly interested in the following construction:

Definition 240. We call a collection $\mathbb{X}$ of random points $\mathbf{X}_j, j \in \mathbb{N}_0$ a (random) *sample* if the points are independent and identically distributed according to some distribution D_d in $\mathbb{R}^d$. Let $\mathbf{0} \in \mathbb{R}^d$ denotes the origin. We define random variables

$$\nabla_p(\mathsf{D}_d) \overset{\mathsf{def}}{=} \operatorname{vol}_p \operatorname{conv}(\mathbf{0}, \mathbf{X}_1, \ldots, \mathbf{X}_p), \tag{6.1}$$

$$\Delta_p(\mathsf{D}_d) \overset{\mathsf{def}}{=} \operatorname{vol}_p \operatorname{conv}(\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_p) \tag{6.2}$$

and their corresponding moments

$$\eta_p^{(k)}(\mathsf{D}_d) = \mathbb{E}\,\nabla_p^k(\mathsf{D}_d), \qquad \upsilon_p^{(k)}(\mathsf{D}_d) = \mathbb{E}\,\Delta_p^k(\mathsf{D}_d). \tag{6.3}$$

Remark 241. Again, we can see a tight connection to moments of random matrices. For the former, let $U = (\mathbf{X}_1 \mid \mathbf{X}_2 \mid \cdots \mid \mathbf{X}_p)$ be a (random) matrix whose columns are coordinates of the points $\mathbf{X}_j$. Using Gram matrix, we can write

$$\nabla_p(\mathsf{D}_d) = \frac{1}{p!}\sqrt{\det U^\top U}. \tag{6.4}$$

Remark 242. Note that if $\mathsf{D}_d = \mathsf{Unif}(K_d)$, that is a uniform distribution in some convex d-body K_d, then $\upsilon_p^{(k)}(\mathsf{Unif}(K_d))$ coincides with the volumetric moments $\upsilon_p^{(k)}(K_d)$ defined earlier in the Introduction of this thesis. The definition is hence consistent.

Definition 243 (Conditional radial simplices)**.** Let $\mathbf{X}_j, j = 1, \ldots d$ be i.i.d. random points following some generic distribution D_d as before and let $\mathbf{x}_0$ be some point in $\mathbb{R}^d$. We define

$$\Delta_p(\mathsf{D}_d \mid \mathbf{x}_0) \overset{\mathsf{def}}{=} \operatorname{vol}_p \operatorname{conv}(\mathbf{x}_0, \mathbf{X}_1, \ldots, \mathbf{X}_p) \tag{6.5}$$

and its corresponding moments

$$v_p^{(k)}(\mathsf{D}_d \mid \mathbf{x}_0) = \mathbb{E}\,\Delta_p^k(\mathsf{D}_d \mid \mathbf{x}_0). \tag{6.6}$$

Remark 244. By definition, the following relations hold

$$\eta_p^{(k)}(\mathsf{D}_d) = v_p^{(k)}(\mathsf{D}_d \mid \mathbf{0}), \tag{6.7}$$

$$v_p^{(k)}(\mathsf{D}_d) = \mathbb{E}\, v_p^{(k)}(\mathsf{D}_d \mid \mathbf{X}_0) = \int_{\mathbb{R}^d} v_p^{(k)}(\mathsf{D}_d \mid \mathbf{x}_0)\, \mathsf{D}_d(\mathrm{d}\mathbf{x}_0). \tag{6.8}$$

6.1.1 Radially symmetric functionals

Definition 245. Let $\mathbf{X}_j, j = 0, \dots, p$ be i.i.d. random points in $\mathbb{R}^d$ each distributed according to some common distribution D_d with a probability measure $\mathsf{D}_d(\mathrm{d}\mathbf{x}) = \rho_d(\mathbf{x})\lambda_d(\mathrm{d}\mathbf{x})$, where $\rho_d(\mathbf{x})$ represents its probability density. We say $\mathbf{X}_j \sim \mathsf{D}_d$ is radially symmetric if $\rho_d(\mathbf{x}) = \tilde{\rho}_d(\|\mathbf{x}\|)$ for some function $\tilde{\rho}_d : \mathbb{R} \to \mathbb{R}$. Since $\mathbf{x}$ and $\|\mathbf{x}\|$ are a vector and a scalar, we may denote the second function simply as $\rho_d(\|\mathbf{x}\|)$ without any additional diacritics. Similarly, we write for the joint probability density of the sample $\mathbb{x} = (\mathbf{X}_0, \dots, \mathbf{X}_p)$ using its joint probability measure $\mathsf{D}_d(\mathrm{d}\mathbb{x}) = \rho_d(\mathbb{x})\mathrm{d}\mathbb{x}$

$$\rho_d(\mathbb{x}) = \prod_{j=0}^{p} \rho_d(\mathbf{x}) = \prod_{j=0}^{p} \rho_d(\|\mathbf{x}\|), \tag{6.9}$$

even though the symbol ρ_d technically represents three different functions.

Let us select $f(\mathbb{x}) = \Delta_p^k\, g(\boldsymbol{\sigma})\rho_d(\mathbb{x})$ in the Blaschke-Petkantschin formula (Theorem 295) with $p = q$ and let us denote $\mathbf{y}$ as the closest point from the origin to $\boldsymbol{\sigma} = \mathcal{A}(\mathbf{x}_0, \dots, \mathbf{x}_p) \in \mathbb{G}(d, p)$. We will assume that the function $g(\boldsymbol{\sigma})$ is symmetric under actions of $\mathcal{SO}(d)$ group (group of rotations in $\mathbb{R}^d$) on the sample $\mathbb{x}$. As a result, g must be a function of the distance of $\boldsymbol{\sigma}$ to the origin only and we may thus write $g(\boldsymbol{\sigma}) = g(h)$, where $h = \|\mathbf{y}\|$. Under those assumptions of radial symmetry and thanks to decomposition $\mu_q(\mathrm{d}\boldsymbol{\sigma}) = \lambda_{d-p}(\mathrm{d}\mathbf{y})\nu_q(\mathrm{d}\boldsymbol{\gamma})$, where $\boldsymbol{\gamma} \in \mathbb{G}(d, p)$, $\mathbf{y} \in \boldsymbol{\gamma}_\perp$ and $\boldsymbol{\sigma} = \boldsymbol{\gamma} + \mathbf{y}$, Blaschke-Petkantschin formula (the special case of Lemma 295 with $p = q$ and $\beta_{dp} = \beta_{dpp}$) turns into

$$\mathbb{E}\left[\Delta_p^k(\mathsf{D}_d)\, g(H)\right] = \beta_{dp}\omega_{d-p}\int_0^\infty h^{d-p-1} g(h) \int_{\boldsymbol{\gamma}_0^{p+1}} \Delta_p^{d-p+k}\, \psi_p(\mathbb{u})\lambda_p^{p+1}(\mathrm{d}\mathbb{u})\mathrm{d}h, \tag{6.10}$$

where $\boldsymbol{\gamma}_0$ is any p-plane selected from $\mathbb{G}(d, p)$, $\mathbb{u} = (\mathbf{u}_0, \dots, \mathbf{u}_p)$ is a selection of points from $\boldsymbol{\gamma}_0$ and H is a random variable associated with h if the selection of points is random, that is H equals the distance to the affine p-plane $\mathcal{A}(\mathbf{X}_0, \dots, \mathbf{X}_p)$. Also, we decomposed $\mathbf{x}_j = \mathbf{y} + \mathbf{u}_j$ and defined

$$\psi_p(\mathbb{u}) = \prod_{j=0}^{p} \psi_p(\mathbf{u}_j) \tag{6.11}$$

with $\psi_p(\mathbf{u}) = \rho_d(\sqrt{h^2 + \|\mathbf{u}\|^2})$. There is an important special case to this formula:

Definition 246. Consider a family of radially symmetric distributions D_p on p-planes $\boldsymbol{\gamma} \in \mathbb{G}(d, p), p = 0, \dots, d$ with density $\rho_p(\mathbf{u})$. We say the family is *shape-preserving*, if for any $\mathbf{u} \in \boldsymbol{\gamma}$ and $\mathbf{y} \in \boldsymbol{\gamma}_\perp$, there exist functions $\varepsilon_{dp}(h)$ and $\xi(h)$, where $h = \|\mathbf{y}\|$, such that for any $p \leq d$,

$$\rho_d(\mathbf{y} + \mathbf{u}) = \varepsilon_{dp}(h)\rho_p(\mathbf{u}/\xi(h)). \tag{6.12}$$

Note that since both ρ_d and ρ_p are radial, we may write this, denoting $s = \|\mathbf{u}\|$,

$$\rho_d\left(\sqrt{h^2 + s^2}\right) = \varepsilon_{dp}(h)\rho_p(s/\xi(h)). \tag{6.13}$$

Proposition 247. *Let D_p be a family of shape-preserving radially symmetric distributions with density $\rho_p(\mathbf{u})$, where $\mathbf{u}$ lie on a p-plane $\boldsymbol{\gamma} \in \mathbb{G}(d,p)$, then*

$$\mathbb{E}\left[\Delta_p^k(\mathsf{D}_d)\, g(H)\right] = \beta_{dp}\, \omega_{d-p}\, \upsilon_p^{(d-p+k)}(\mathsf{D}_p)\, \mathcal{J}_{dp}^{(k)}[g(h)], \tag{6.14}$$

where $\Delta_p(\mathbb{u}) = \operatorname{vol}_p \operatorname{conv}(\mathbb{u})$ and $\mathcal{J}_{dp}^{(k)}[\cdot]$ is the functional defined as

$$\mathcal{J}_{dp}^{(k)}[g(h)] = \int_0^\infty h^{d-p-1} g(h)\xi(h)^{(d+k+1)p} \varepsilon_{dp}^{p+1}(h)\ \mathrm{d}h. \tag{6.15}$$

Proof. Denote $s_j = \|\mathbf{u}_j\|$. Assuming the shape-preserving property,

$$\psi_p(\mathbb{u})\lambda_p^{p+1}(\mathrm{d}\mathbb{u}) = \prod_{j=0}^{p} \psi_p(\mathbf{u}_j)\lambda_p(\mathrm{d}\mathbf{u}_j) = \varepsilon_{dp}^{p+1}(h) \prod_{j=0}^{p} \rho_p\left(\frac{\mathbf{u}_j}{\xi(h)}\right)\lambda_p(\mathrm{d}\mathbf{u}_j). \tag{6.16}$$

Let us make the change of variables $\mathbf{u}_j = \xi(h)\mathbf{u}_j'$. We transform the Lebesgue measures as $\lambda_p(\mathrm{d}\mathbf{u}_j) = \xi(h)^p\lambda_p(\mathrm{d}\mathbf{u}_j')$. Also, by scaling, $\Delta_p(\mathbb{u}) = \xi(h)^p\Delta_p(\mathbb{u}')$, where $\mathbb{u}' = (\mathbf{u}_0, \dots, \mathbf{u}_p)$. Overall,

$$\Delta_p^{d-p+k}(\mathbb{u})\, \psi_p(\mathbb{u})\lambda_p^{p+1}(\mathrm{d}\mathbb{u}) = \xi(h)^{(d+k+1)p}\varepsilon_{dp}^{p+1}(h)\Delta_p^{d-p+k}(\mathbb{u}')\, \rho_p(\mathbb{u}')\lambda_p^{p+1}(\mathrm{d}\mathbb{u}'). \tag{6.17}$$

Since the transformation does not affect $\boldsymbol{\gamma}$, we get, integrating over $\mathbb{u}' \in \boldsymbol{\gamma}^{p+1}$, the desired result. ∎

Corollary 247.1. *Let D_p be a family of shape-preserving distributions, then*

$$\upsilon_p^{(k)}(\mathsf{D}_d) = \frac{1}{p!^k} \frac{\mathcal{J}_{dp}^{(k)}[1]}{\mathcal{J}_{(d+k)p}^{(0)}[1]} \prod_{j=0}^{p} \frac{\omega_{d-j}}{\omega_{d+k-j}}, \tag{6.18}$$

$$\eta_{p+1}^{(k)}(\mathsf{D}_d) = \frac{1}{(p+1)!^k} \frac{\mathcal{J}_{dp}^{(k)}[h^k]}{\mathcal{J}_{(d+k)p}^{(0)}[1]} \prod_{j=0}^{p} \frac{\omega_{d-j}}{\omega_{d+k-j}}. \tag{6.19}$$

Remark 248. Alternatively, we can express these formulae in terms of γ's as

$$\upsilon_p^{(k)}(\mathsf{D}_d) = \frac{1}{p!^k \sqrt{2\pi}^{k(p+1)}} \frac{\mathcal{J}_{dp}^{(k)}[1]}{\mathcal{J}_{(d+k)p}^{(0)}[1]} \prod_{j=0}^{p} \frac{\gamma_{d+k-j}}{\gamma_{d-j}}, \tag{6.20}$$

$$\eta_{p+1}^{(k)}(\mathsf{D}_d) = \frac{1}{\sqrt{2\pi}^{k(p+1)}(p+1)!^k} \frac{\mathcal{J}_{dp}^{(k)}[h^k]}{\mathcal{J}_{(d+k)p}^{(0)}[1]} \prod_{j=0}^{p} \frac{\gamma_{d+k-j}}{\gamma_{d-j}}. \tag{6.21}$$

Proof. Set $g(h) = 1$ in Proposition 247, then

$$v_p^{(k)}(\mathsf{D}_p) = \beta_{dp}\,\omega_{d-p}\,v_p^{(d-p+k)}(\mathsf{D}_p)\,\mathcal{J}_{dp}^{(k)}[1], \tag{6.22}$$

plugging $k = 0$, we get $1 = \beta_{dp}\,\omega_{d-p}\,v_p^{(d-p)}(\mathsf{D}_p)\,\mathcal{J}_{dp}^{(0)}[1]$. Since d can be arbitrary, we replace d by $d+k$ and get $1 = \beta_{(d+k)p}\,\omega_{d+k-p}\,v_p^{(d-p+k)}(\mathsf{D}_p)\,\mathcal{J}_{(d+k)p}^{(0)}[1]$, from which we obtain,

$$v_p^{(k)}(\mathsf{D}_d) = \frac{\beta_{dp}\,\omega_{d-p}\,\mathcal{J}_{dp}^{(k)}[1]}{\beta_{(d+k)p}\,\omega_{d+k-p}\,\mathcal{J}_{(d+k)p}^{(0)}[1]}, \tag{6.23}$$

which further simplifies, since by Remark 296, $\beta_{dp}\,\omega_{d-p} = (p!)^{d-p}\prod_{j=0}^{p}\frac{\omega_{d-j}}{\omega_{p-j}}$. Next, by base-height splitting, we have when $p < d$,

$$\nabla_{p+1}(\mathsf{D}_d) = \frac{H}{p+1}\Delta_p(\mathsf{D}_d). \tag{6.24}$$

Thus, setting $g(h) = h^k/(p+1)^k$ in Proposition 247,

$$\eta_{p+1}^{(k)}(\mathsf{D}_p) = \frac{1}{(p+1)^k}\beta_{dp}\,\omega_{d-p}\,v_p^{(d-p+k)}(\mathsf{D}_p)\,\mathcal{J}_{dp}^{(k)}[h^k], \tag{6.25}$$

and therefore

$$\eta_{p+1}^{(k)}(\mathsf{D}_p) = \frac{1}{(p+1)^k}\frac{\beta_{dp}\,\omega_{d-p}\,\mathcal{J}_{dp}^{(k)}[h^k]}{\beta_{(d+k)p}\,\omega_{d+k-p}\,\mathcal{J}_{(d+k)p}^{(0)}[1]}. \tag{6.26}$$

■

Proposition 249. *Let D_p, $p = 0,\ldots,d$ be a family of shape-preserving radially symmetric distributions with density $\rho_p(\mathbf{u})$, where $\mathbf{u}$ lie on a p-plane $\boldsymbol{\gamma} \in \mathbb{G}(d,p)$, then for any function $g(h)$,*

$$\mathbb{E}\left[\Delta_p^k(\mathsf{D}_d)\,g(H)\right] = v_p^{(k)}(\mathsf{D}_d)\,\frac{\mathcal{J}_{dp}^{(k)}[g(h)]}{\mathcal{J}_{dp}^{(k)}[1]} = (p+1)^k\eta_{p+1}^{(k)}(\mathsf{D}_p)\frac{\mathcal{J}_{dp}^{(k)}[g(h)]}{\mathcal{J}_{dp}^{(k)}[h^k]}. \tag{6.27}$$

Shape-preserving distributions can either be Normal, Beta, Beta′ or Spherical shell distribution, see Ruben & Miles [62].

6.1.2 General conditional radial simplices

Finally, to conclude this section, we derive the following result

Theorem 250. *Let D_p, $p = 0,\ldots,d$ be a family of radially symmetric shape-preserving distributions, then for any $p < d$ and $k = 2m$, where m is an integer,*

$$v_{p+1}^{(2m)}(\mathsf{D}_d \mid \mathbf{x}_0) = \eta_{p+1}^{(2m)}(\mathsf{D}_d)\sum_{s=0}^{m}\frac{\gamma_d\gamma_{d-p+2m}\,\mathcal{J}_{dp}^{(2m)}[h^{2m-2s}]}{\gamma_{d+2s}\gamma_{d-p+2m-2s}\,\mathcal{J}_{dp}^{(2m)}[h^{2m}]}\binom{m}{s}\|\mathbf{x}_0\|^{2s}. \tag{6.28}$$

Proof. Let $\mathbb{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_{p+1})$ be a random sample of i.i.d. points $\mathbf{X}_j \sim \mathsf{D}_d$, $j = 1, \ldots, p+1$. By definition, $v_{p+1}^{(k)}(\mathsf{D}_d \mid \mathbf{x}_0) = \mathbb{E}\, \Delta_{p+1}^k(\mathsf{D}_d \mid \mathbf{x}_0)$, where $\Delta_{p+1}(\mathsf{D}_d \mid \mathbf{x}_0) = \operatorname{vol}_{p+1} \operatorname{conv}(\mathbf{x}_0, \mathbf{X}_1, \ldots, \mathbf{X}_{p+1})$. Denote $\mathbf{Y}$ the closest point from the origin to $\boldsymbol{\sigma} = \mathcal{A}(\mathbb{X}) \in \mathbb{A}(d,p)$. The points $\mathbf{U}_j = \mathbf{X}_j - \mathbf{Y}$, $j = 1, \ldots, p+1$ define a linear subspace $\boldsymbol{\gamma} \in \mathbb{G}(d,p)$ on which they lie. We have $\mathbf{Y} \in \boldsymbol{\gamma}_\perp$. By base-height splitting, we may write

$$\Delta_{p+1}(\mathsf{D}_d \mid \mathbf{x}_0) = \frac{1}{p+1} \operatorname{dist}_{\boldsymbol{\sigma}}(\mathbf{x}_0) \Delta_p(\mathsf{D}_d), \tag{6.29}$$

where $\Delta_p(\mathsf{D}_d) = \operatorname{vol}_p \operatorname{conv}(\mathbb{X})$ and $\operatorname{dist}_{\boldsymbol{\sigma}}(\mathbf{x}_0)$ is the distance from $\mathbf{x}_0$ to $\boldsymbol{\sigma}$ (see Figure 6.1).

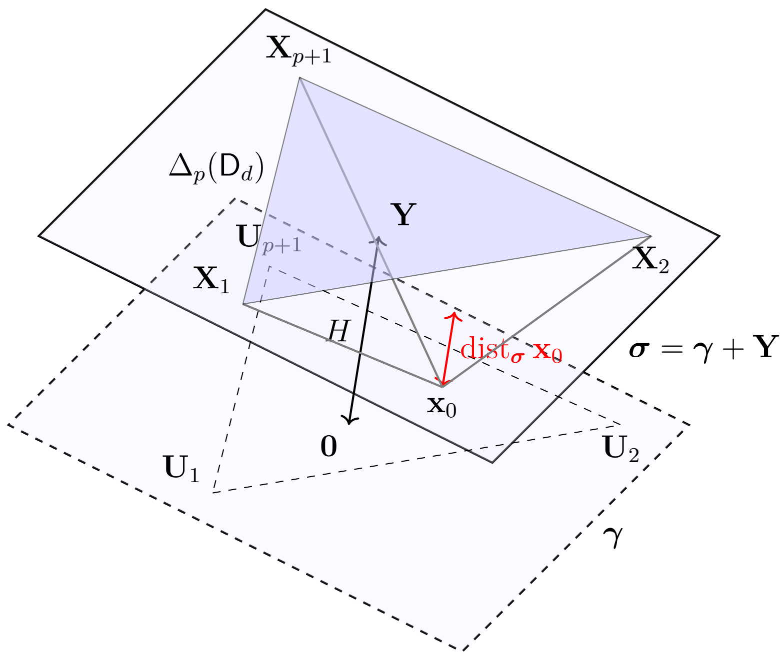


Figure 6.1: Base-height splitting of $\Delta_{p+1}(\mathsf{D}_d \mid \mathbf{x}_0)$

The distance is unchanged if we project it onto $\boldsymbol{\gamma}_\perp$,

$$\operatorname{dist}_{\boldsymbol{\sigma}}(\mathbf{x}_0) = \| \operatorname{proj}_{\boldsymbol{\gamma}_\perp} \mathbf{x}_0 - \mathbf{Y} \|. \tag{6.30}$$

This is a scalar function of $\mathbf{Y}$, but not $H = \|\mathbf{Y}\|$ only. However, we may use the following trick: By symmetry and the law of conditional expectation, we have

$$v_{p+1}^{(k)}(\mathsf{D}_d \mid \mathbf{x}_0) = \mathbb{E}\, v_{p+1}^{(k)}(\mathsf{D}_d \mid \|\mathbf{x}_0\| \mathbf{S}_0) = \frac{1}{(p+1)^k} \mathbb{E}\left[\Delta_p^k(\mathsf{D}_d) g(H)\right], \tag{6.31}$$

where $\mathbf{S}_0 \sim \mathsf{Unif}(\mathbb{S}^{d-1})$ (uniform distribution on the unit sphere $\mathbb{S}^{d-1}$) and

$$g(H) = \mathbb{E}\left[\| \|\mathbf{x}_0\| \operatorname{proj}_{\boldsymbol{\gamma}_\perp} \mathbf{S}_0 - \mathbf{Y}\|^k \mid \mathbb{X}\right] \tag{6.32}$$

is now a function of $H = \|\mathbf{Y}\|$ only. As a consequence, by Proposition 249,

$$v_{p+1}^{(k)}(\mathsf{D}_d \mid \mathbf{x}_0) = \eta_{p+1}^{(k)}(\mathsf{D}_d) \frac{\mathcal{J}_{dp}^{(k)}[g(h)]}{\mathcal{J}_{dp}^{(k)}[h^k]}. \tag{6.33}$$

Let us express the function $g(h)$. Since $\dim \boldsymbol{\gamma}_\perp = d - p$, we get by Lemma 269,

$$\mathbf{B}_0 = \operatorname{proj}_{\gamma_\perp} \mathbf{S}_0 \sim \mathsf{Beta}_{d-p}(2-p) \tag{6.34}$$

on $\boldsymbol{\gamma}_\perp$. Hence, for any fixed $\mathbf{y} \in \boldsymbol{\gamma}_\perp$ with $h = \|\mathbf{y}\|$,

$$g(h) = \mathbb{E}\left[\|\|\mathbf{x}_0\|\mathbf{B}_0 - \mathbf{y}\|^k\right]. \tag{6.35}$$

Denote $\boldsymbol{\delta} = \operatorname{span}(\mathbf{y}) \in \mathbb{G}(d-p,1)$ the line passing through $\mathbf{y}$ and the origin. By the Law of Cosines,

$$g(h) = \mathbb{E}\left[(\|\mathbf{x}_0\|^2\|\mathbf{B}_0\|^2 - 2\|\mathbf{x}_0\|h \operatorname{proj}_{\boldsymbol{\delta}} \mathbf{B}_0 + h^2)^{k/2}\right]. \tag{6.36}$$

We can split random variable $\mathbf{B}_0$ into the product $R_0\mathbf{S}_0'$ of two independent random variables R_0 and $\mathbf{S}_0' \sim \mathsf{S}_{d-p}$. Since

$$T = \operatorname{proj}_{\boldsymbol{\delta}} \mathbf{S}_0' \sim \mathsf{Beta}_1(3-d+p) \tag{6.37}$$

we get, by first taking expectation with respect to T,

$$g(h) = c_{1(3-d+p)}\mathbb{E}\int_{-1}^{1} \frac{(\|\mathbf{x}_0\|^2R_0^2 - 2\|\mathbf{x}_0\|R_0ht + h^2)^{k/2}}{(1-t^2)^{(3-d+p)/2}}\,\mathrm{d}t. \tag{6.38}$$

By substitution $t = \cos\theta$ and expressing the normalisation factor $c_{1(3-d+p)}$,

$$g(h) = \frac{\gamma_{d-p}}{\gamma_{d-p-1}}\mathbb{E}\,\frac{1}{\sqrt{2\pi}}\int_0^{\pi} (\|\mathbf{x}_0\|^2R_0^2 - 2\|\mathbf{x}_0\|R_0h + h^2)^{k/2}\sin^{d-p-2}\theta\;\mathrm{d}\theta. \tag{6.39}$$

This result is valid for any real k for which the expression makes sense. However, when k is an even integer as in the statement of the theorem, we can proceed further. From Gradshteyn and Ryzhik [33, (3.665)] for m integer and $p < d$,

$$\frac{1}{\sqrt{2\pi}}\int_0^{\pi}\Big(r^2 - 2rh\cos\theta + h^2\Big)^m \sin^{d-p-2}\theta\;\mathrm{d}\theta = \sum_{s=0}^{m}\frac{\gamma_{d-p-1}\gamma_{d-p+2m}}{\gamma_{d-p+2m-2s}\gamma_{d-p+2s}}\binom{m}{s}r^{2s}h^{2m-2s}. \tag{6.40}$$

Plugging $r = \|\mathbf{x}_0\|R_0$,

$$g(h) = \sum_{s=0}^{m}\frac{\gamma_{d-p}\gamma_{d-p+2m}}{\gamma_{d-p+2m-2s}\gamma_{d-p+2s}}\binom{m}{s}\|\mathbf{x}_0\|^{2s}h^{2m-2s}\mathbb{E}\,R_0^{2s}. \tag{6.41}$$

Recall that $R_0 = \|\mathbf{B}_0\|$, where $\mathbf{B}_0 \sim \mathsf{Beta}_{d-p}(2-p)$. So, by Lemma 267 (Equation (A.13)), we have $\mathbb{E}\,R_0^{2s} = \gamma_{d-p+2s}\gamma_d/(\gamma_{d-p}\gamma_{d+2s})$ and thus

$$g(h) = \sum_{s=0}^{m}\frac{\gamma_d\gamma_{d-p+2m}}{\gamma_{d+2s}\gamma_{d-p+2m-2s}}\binom{m}{s}\|\mathbf{x}_0\|^{2s}h^{2m-2s}. \tag{6.42}$$

We conclude the proof by plugging this result into Equation (6.33). ■

6.2 Gaussian simplices

In this section, we study random Gaussian simplices, that is, simplices whose vertices are points selected according to multivariate normal distribution N_d (see table of common distributions in Appendix). Using the standard method of base-height splitting (see [24]), we were able to obtain a new formula for the volume moments of Gaussian simplices with one vertex fixed. However, it turned out the formula has been known because of its connections with random determinant moments of some special matrix. More specfically, we provide an alternative derivation of Theorem 181 by reformulating it in terms of Gaussian simplices.

Definition 251. We call a collection $\mathbb{Z} = (\mathbf{Z}_1, \dots, \mathbf{Z}_p)$ of random points $\mathbf{Z}_i$ selected from $\mathbb{R}^d$ a *standard normal sample* if the points are independent and identically distributed according to the standard multivariate normal distribution N_d, that is with density function $\rho_d(\mathbf{z}) = e^{-\|\mathbf{z}\|^2/2}/(2\pi)^{d/2}$ for each $i \in \{1, \dots, p\}$ and with joint density function

$$\rho_d(\mathbf{z}_1, \dots, \mathbf{z}_p) = \prod_{i=1}^{p} \rho_d(\mathbf{z}_i). \tag{6.43}$$

We write $\mathsf{N}_d(\mathrm{d}\mathbf{z}) = e^{-\|\mathbf{z}\|^2/2}/(2\pi)^{d/2}\lambda_d(\mathrm{d}\mathbf{z}) = r^{d-1}e^{-r^2/2}/(2\pi)^{d/2}\sigma_d(\mathrm{d}\mathbf{u})\mathrm{d}r$ with $\mathbf{u} \in \mathbb{S}^{n-1}$ for the probability measure associated with the standard multivariate normal distribution and its decomposition into radial and spherical part.

Definition 252 (double factorial)**.** Let m be an integer, we define the *double factorial* standardly as

$$\begin{aligned} (2m)!! &= (2m) \times (2m-2) \times \cdots \times 4 \times 2 \quad \text{and} \\ (2m-1)!! &= (2m-1) \times (2m-3) \times \cdots \times 3 \times 1. \end{aligned} \tag{6.44}$$

Note that we can express the above in terms of the Gamma function as

$$(2m)!! = 2^m\,\Gamma(m+1) \qquad \text{and} \qquad (2m-1)!! = \frac{2^m}{\sqrt{\pi}}\,\Gamma\Big(m+\frac{1}{2}\Big). \tag{6.45}$$

Remark 253. If a and b are either both integers or both half-integers, one has

$$\frac{\Gamma(a)}{\Gamma(b)} = 2^{b-a}\frac{(2a-2)!!}{(2b-2)!!}. \tag{6.46}$$

6.2.1 Radial volumetric moments

A lot is known about Gaussian random simplices. The following formula is due to Miles [48, p. 377, (70)], which states

Proposition 254 (Miles, 1971)**.** *Let $\mathbf{Z}_0, \mathbf{Z}_1.\mathbf{Z}_2, \dots, \mathbf{Z}_p$ be a standard normal sample in $\mathbb{R}^d$. Assuming $p \le d$, we get for moments of the $p-$volume $\Delta_p(\mathsf{N}_d) = \mathrm{vol}_p \operatorname{conv}(\mathbf{Z}_0, \dots, \mathbf{Z}_p)$ of a simplex formed by a convex hull of those points*

$$\eta_p^{(k)}(\mathsf{N}_d) = \frac{v_p^{(k)}(\mathsf{N}_d)}{(p+1)^{k/2}} = \frac{1}{p!^k}\prod_{j=0}^{p-1}\frac{\gamma_{d+k-j}}{\gamma_{d-j}} = \frac{2^{pk/2}}{p!^k}\prod_{j=0}^{p-1}\frac{\Gamma\left(\frac{1}{2}(d+k-j)\right)}{\Gamma\left(\frac{1}{2}(d-j)\right)} \tag{6.47}$$

for any real $k > p-d-1$.

Proof. The crucial observation is that the normal distribution is shape-preserving. Let $\gamma \in \mathbb{G}(d,p)$, $\mathbf{u} \in \boldsymbol{\gamma}$, $s = \|\mathbf{u}\|$, $\mathbf{y} \in \boldsymbol{\gamma}_\perp$, $h = \|\mathbf{y}\|$ and $s = \|\mathbf{u}\|$. Let $r = \|\mathbf{x}\|$, $\mathbf{x} \in \mathbb{R}^d$, the density of the standard normal multivariate distribution N_d is $\rho_d(\mathbf{x}) =$

$\exp(-r^2/2)/\sqrt{2\pi}^d$, which we can identify as, plugging $r = \sqrt{h^2+s^2}$,

$$\rho_d(\sqrt{h^2+s^2}) = \frac{1}{\sqrt{2\pi}^d} e^{-\frac{1}{2}(h^2+s^2)} = \frac{e^{-\frac{1}{2}h^2}}{\sqrt{2\pi}^{d-p}} \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\pi}^p} = \frac{e^{-\frac{1}{2}h^2}}{\sqrt{2\pi}^{d-p}} \rho_p(\mathbf{u}), \tag{6.48}$$

where $\rho_p(\mathbf{u})$ is the density of the distribution N_p. Hence, we got a family of shape-preserving distributions with $\xi(h) = 1$ and $\varepsilon_{dp}(h) = e^{-\frac{1}{2}h^2}/\sqrt{2\pi}^{d-p}$. Now, let us compute $\mathcal{J}_{dp}^{(k)}[h^q]$. We have

$$\begin{aligned}\mathcal{J}_{dp}^{(k)}[h^q] &= \int_0^\infty h^{d+q-p-1}\xi(h)^{(d+k+1)p}\varepsilon_{dp}^{p+1}(h)\mathrm{d}h = \int_0^\infty \frac{h^{d-p+q-1}e^{-(p+1)h^2/2}}{\sqrt{2\pi}^{(d-p)(p+1)}}\mathrm{d}h \\ &= \frac{\gamma_{d-p+q}}{(2\pi)^{\frac{(d-p)(p+1)}{2}}(p+1)^{\frac{d-p+q}{2}}}.\end{aligned} \tag{6.49}$$

Hence, plugging first $q = k = 0$ and then replacing d with $d+k$, we get

$$\mathcal{J}_{(d+k)p}^{(0)}[1] = \frac{\gamma_{d+k-p}}{(2\pi)^{\frac{(d+k-p)(p+1)}{2}}(p+1)^{\frac{d+k-p}{2}}}. \tag{6.50}$$

Dividing these two relations, we get

$$\frac{\mathcal{J}_{dp}^{(k)}[h^q]}{\mathcal{J}_{(d+k)p}^{(0)}[1]} = \frac{\gamma_{d-p+q}}{\gamma_{d-p+k}}(2\pi)^{\frac{k(p+1)}{2}}(p+1)^{\frac{k-q}{2}}. \tag{6.51}$$

As a consequence of Corollary 247.1 with $q = 0$, we get

$$v_p^{(k)}(\mathsf{N}_d) = \frac{(p+1)^{k/2}}{p!^k}\frac{\gamma_{d-p}}{\gamma_{d-p+k}}\prod_{j=0}^{p}\frac{\gamma_{d+k-j}}{\gamma_{d-j}} = \frac{(p+1)^{k/2}}{p!^k}\prod_{j=0}^{p-1}\frac{\gamma_{d+k-j}}{\gamma_{d-j}}. \tag{6.52}$$

Similarly, with $q = k$, we get

$$\eta_{p+1}^{(k)}(\mathsf{D}_d) = \frac{1}{(p+1)!^k}\prod_{j=0}^{p}\frac{\gamma_{d+k-j}}{\gamma_{d-j}}, \tag{6.53}$$

which concludes the proof. ■

Remark 255. Note that when $k = 2m$ for some m integer, $\frac{1}{2}(d+2m-j)$ and $\frac{1}{2}(d-j)$ are either both integers or both half-integers. Using Remark 253,

$$\eta_p^{(2m)}(\mathsf{N}_d) = \frac{v_p^{(2m)}(\mathsf{N}_d)}{(p+1)^m} = \frac{1}{p!^{2m}}\prod_{j=0}^{p-1}\frac{(d+2m-j-2)!!}{(d-j-2)!!} = \frac{1}{p!^{2m}}\prod_{i=0}^{m-1}\frac{(d+2i)!}{(d-p+2i)!}. \tag{6.54}$$

The last equality is a consequence of the following identity

$$\prod_{j=0}^{p-1}\frac{(d+2m-j-2)!!}{(d-j-2)!!} = \prod_{i=0}^{m-1}\frac{(d+2i)!}{(d-p+2i)!}, \tag{6.55}$$

the proof of which is left as an easy exercise.

6.2.2 Conditional simplices

Proposition 256. *Let $\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p$ be a standard normal sample in $\mathbb{R}^d$ (that is, $\mathbf{Z}_j \sim \mathsf{N}_d$) and $\mathbf{z}_0 \in \mathbb{R}^d$ be some fixed point. We assume that $p \le d$, so the convex hull of $\mathbf{z}_0, \mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_p$ is almost surely a $p-$dimensional simplex formed by those vertices. For the moments of its $p-$volume $\Delta_p(\mathsf{N}_d \mid \mathbf{z}_0)$ and m natural,*

$$v_p^{(2m)}(\mathsf{N}_d \mid \mathbf{z}_0) = \frac{1}{p!^{2m}} \left(\prod_{r=0}^{m-1} \frac{(d+2r)!}{(d-p+2r)!} \right) \sum_{s=0}^{m} \binom{m}{s} \frac{(d-2)!!}{(d+2s-2)!!} p^s \|\mathbf{z}_0\|^{2s}. \tag{6.56}$$

Proof. We already know that N_d is shape-preserving. By Equation (6.49), we get

$$\frac{\mathcal{J}_{dp}^{(2m)}[h^{2m-2s}]}{\mathcal{J}_{dp}^{(2m)}[h^{2m}]} = (p+1)^s \frac{\gamma_{d-p+2m-2s}}{\gamma_{d-p+2m}}, \tag{6.57}$$

so by Theorem 250,

$$v_{p+1}^{(2m)}(\mathsf{N}_d \mid \mathbf{x}_0) = \eta_{p+1}^{(2m)}(\mathsf{N}_d) \sum_{s=0}^{m} \frac{\gamma_d}{\gamma_{d+2s}} \binom{m}{s} (p+1)^s \|\mathbf{z}_0\|^{2s} \tag{6.58}$$

as desired since $\frac{\gamma_d}{\gamma_{d+2s}} = \frac{\Gamma\left(\frac{d}{2}\right)}{2^s \Gamma\left(\frac{d}{2}+s\right)} = \frac{(d-2)!!}{(d+2s-2)!!}$. ∎

Remark 257. The proposition is equivalent to Theorem 181. To see this, define random points $\mathbf{X}_j = (X_{1j}, X_{2j}, \dots, X_{nj})^\top$ with $X_{ij} \sim \mathsf{N}(\mu, \sigma^2)$ and write $U = (X_{ij})_{n\times p}$. On one hand, $\mathbb{E}\,(\det(U^\top U))^{k/2} = f_k(n,p)$. On the other hand, note that the Gram determinant $\sqrt{\det(U^\top U)}$ is equal to the volume of a parallelotope $[\mathbf{0}, \mathbf{X}_1] + \ldots + [\mathbf{0}, \mathbf{X}_p]$. Equivalently, we can relate the Gram determinant with the volume of a random polytope

$$\sqrt{\det(U^\top U)} = p!\,\mathrm{vol}_p \operatorname{conv}(\mathbf{0}, \mathbf{X}_1, \dots, \mathbf{X}_n) \tag{6.59}$$

Since $X_{ij} \sim \mathsf{N}(\mu, \sigma^2)$, we can shift the points by the point $\mathbf{x}_0 = (\mu, \mu, \dots, \mu)$, so

$$\mathrm{vol}_p \operatorname{conv}(\mathbf{0}, \mathbf{X}_1, \dots, \mathbf{X}_p) = \mathrm{vol}_p \operatorname{conv}(-\mathbf{x}_0, \mathbf{X}_1 - \mathbf{x}_0, \dots, \mathbf{X}_n - \mathbf{x}_0). \tag{6.60}$$

Define a new set of points $\mathbf{Z}_i = (\mathbf{X}_i - \mathbf{x}_0)/\sigma$ and $\mathbf{z}_0 = -\mathbf{x}_0/\sigma$. Now, the points $\mathbf{Z}_i \sim \mathsf{N}_d$ form a standard normal sample. For the volumes, we immediately get

$$\mathrm{vol}_p \operatorname{conv}(\mathbf{0}, \mathbf{X}_1, \dots, \mathbf{X}_p) = \sigma^n \,\mathrm{vol}_p \operatorname{conv}(\mathbf{z}_0, \mathbf{Z}_1, \dots, \mathbf{Z}_p), \tag{6.61}$$

from which immediately, for any even k,

$$f_k(n,p) = p!^k \sigma^{pk} \mathbb{E}\,\Delta_p^k(\mathsf{N}_d \mid \mathbf{z}_0) = p!^k \sigma^{pk} v_p^{(k)}(\mathsf{N}_d \mid \mathbf{z}_0). \tag{6.62}$$

Finally, by spherically symmetry, the point $\mathbf{z}_0$ can be chosen arbitrarily, the result is a function of $\|\mathbf{z}_0\|$ only.

Note that the special case $\mathbf{z}_0 = \mathbf{0}$ gives

$$v_p^{(2m)}(\mathsf{N}_d \mid \mathbf{0}) = \eta_p^{(2m)}(\mathsf{N}_d) = \frac{1}{p!^{2m}} \prod_{r=0}^{m-1} \frac{(d+2r)!}{(d-p+2r)!}, \tag{6.63}$$

Similarly, by Remark 244, by integrating $\|\mathbf{z}_0\|$ out from (6.56) with respect to $\mathsf{N}_d(\mathrm{d}\mathbf{z}_0)$, we recover the special case Miles' formula for even moments. That is the Proposition 254 with $k = 2m$, where m is an integer, which is equivalent to

Corollary 257.1 (Miles, 1971). *Let $p \le d$, then for any $k = 2m$ with m integer,*

$$v_p^{(2m)}(\mathsf{N}_d) = (p+1)^m \eta_p^{(2m)}(\mathsf{N}_d) = \frac{(p+1)^m}{p!^{2m}} \prod_{r=0}^{m-1} \frac{(d+2r)!}{(d-p+2r)!}. \tag{6.64}$$

Proof. To see that, we have $v_p^{(2m)}(\mathsf{N}_d) = \mathbb{E}\, v_p^{(2m)}(\mathsf{N}_d \mid \mathbf{Z}_0)$. For the moments, we have

$$\mathbb{E}\,\|\mathbf{Z}_0\|^{2s} = \frac{\gamma_{d+2s}}{\gamma_d} = \frac{2^s \Gamma\left(\frac{d}{2}+s\right)}{\Gamma\left(\frac{d}{2}\right)} = \frac{(d+2s-2)!!}{(d-2)!!}, \tag{6.65}$$

then we use the Binomial formula. ■

6.3 Beta and Beta' simplices

Miles and Ruben [62] studied volumetric moments of random Beta and Beta′ simplices formed by vertices drawn independently from distribution $\mathsf{Beta}_d(a)$ or $\mathsf{Beta}'_d(a)$, respectively. Their method is to decompose certain class of distributions into part dependent on H and points on $\boldsymbol{\gamma}$. This is essentially the method we are using in this thesis. Moreover, they showed that the only distributions which can form a shape-preserving families are either Gaussian, Beta or Beta′ (with degenerate subcase of uniform distribution on a sphere).

6.3.1 Radial volumetric moments

While $\eta_p^{(k)}(\,\cdot\,)$ already appeared in Miles ([48]), the values of $v_p^{(k)}(\,\cdot\,)$ have only been expressed recently by Kabluchko and Steigenberger [38].

Proposition 258. *Let $\mathbb{B} = (\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_p)$ be a sample drawn form $\mathsf{Beta}_d(a)$ distribution and let $\mathbb{B}' = (\mathbf{B}'_1, \mathbf{B}'_2, \dots, \mathbf{B}'_p)$ be a sample drawn form $\mathsf{Beta}'_d(a)$ distribution. Assuming $p \le d$, we get for moments of the $p-$volume $\nabla_p(\mathsf{Beta}_d(a)) = \mathrm{vol}_p \operatorname{conv}(\mathbf{0}, \mathbf{B}_1, \dots, \mathbf{B}_p)$ and $\nabla_p(\mathsf{Beta}'_d(a)) = \mathrm{vol}_p \operatorname{conv}(\mathbf{0}, \mathbf{B}'_1, \dots, \mathbf{B}'_p)$ of simplices formed by a convex hull of those points*

$$\eta_p^{(k)}(\mathsf{Beta}_d(a)) = \frac{1}{p!^k} \prod_{j=0}^{p-1} \frac{\gamma_{d+k-j}\gamma_{d-a+2}}{\gamma_{d-j}\gamma_{d-a+2+k}}, \quad \eta_p^{(k)}(\mathsf{Beta}'_d(a)) = \frac{1}{p!^k} \prod_{j=0}^{p-1} \frac{\gamma_{d+k-j}\gamma_{a-d-k}}{\gamma_{d-j}\gamma_{a-d}}.$$

Proof. By Lemma 267, there exists a set of random variables $V_j \sim \chi_{d-a+2}$ independent of $\mathbf{B}_j$, $j = 1, \dots p$, such that $\mathbf{B}_j V_j \overset{d}{=} \mathbf{Z}_j \sim \mathsf{N}_d$ for each j. By linearity of determinants,

$$\mathrm{vol}_p \operatorname{conv}(\mathbf{0}, \mathbf{Z}_1, \dots, \mathbf{Z}_p) = V_1 \cdots V_p \, \mathrm{vol}_p \operatorname{conv}(\mathbf{0}, \mathbf{B}_1, \dots, \mathbf{B}_p), \tag{6.66}$$

from which, immediately,

$$\eta_p^{(k)}(\mathsf{N}_d) = \mathbb{E}\,\nabla_p^k(\mathsf{N}_d) = (\mathbb{E}\,V_1^k)^p \mathbb{E}\,\nabla_p^k(\mathsf{Beta}_d(a)) = \left(\frac{\gamma_{d-a+2+k}}{\gamma_{d-a+2}}\right)^p \eta_p^{(k)}(\mathsf{Beta}_d(a)). \tag{6.67}$$

Similarly, there is a set of random variables $U_j \sim \chi_{a-d}$ independent of $\mathbf{Z}_j \sim \mathsf{N}_d$ such that $\mathbf{B}'_j \stackrel{d}{=} \mathbf{Z}_j/U_j \sim \mathsf{Beta}'_d(a)$ for each $j = 1, \ldots, p$. Hence

$$\operatorname{vol}_p \operatorname{conv}(\mathbf{0}, \mathbf{B}'_1, \ldots, \mathbf{B}'_p) = \frac{1}{U_1 \cdots U_p} \operatorname{vol}_p \operatorname{conv}(\mathbf{0}, \mathbf{Z}_1, \ldots, \mathbf{Z}_p), \tag{6.68}$$

from which, immediately,

$$\eta_p^{(k)}(\mathsf{Beta}'_d(a)) = \mathbb{E}\,\nabla_p^k(\mathsf{Beta}'_d(a)) = (\mathbb{E}\,U_1^{-k})^p \mathbb{E}\,\nabla_p^k(\mathsf{N}_d) = \left(\frac{\gamma_{a-d-k}}{\gamma_{a-d}}\right)^p \eta_p^{(k)}(\mathsf{N}_d). \tag{6.69}$$

■

Proposition 259. *Let $\mathbb{B} = (\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_p)$ be a sample drawn form $\mathsf{Beta}_d(a)$ distribution and let $\mathbb{B}' = (\mathbf{B}'_0, \mathbf{B}'_1, \ldots, \mathbf{B}'_p)$ be a sample drawn form $\mathsf{Beta}'_d(a)$ distribution. Assuming $p \le d$, we get for moments of the $p-$volume $\Delta_p(\mathsf{Beta}_d(a)) = \operatorname{vol}_p \operatorname{conv}(\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_p)$ and $\Delta_p(\mathsf{Beta}'_d(a)) = \operatorname{vol}_p \operatorname{conv}(\mathbf{B}'_0, \mathbf{B}'_1, \ldots, \mathbf{B}'_p)$ of simplices formed by a convex hull of those points*

$$\begin{aligned} \upsilon_p^{(k)}(\mathsf{Beta}_d(a)) &= \frac{1}{p!^k} \frac{\gamma_{(d-a+k)(p+1)+2}\gamma_{d-a+2}}{\gamma_{(d-a+k)(p+1)+2-k}\gamma_{d-a+2+k}} \prod_{j=0}^{p-1} \frac{\gamma_{d+k-j}\gamma_{d-a+2}}{\gamma_{d-j}\gamma_{d-a+2+k}}, \\ \upsilon_p^{(k)}(\mathsf{Beta}'_d(a)) &= \frac{1}{p!^k} \frac{\gamma_{(a-d-k)(p+1)+k}\gamma_{a-d-k}}{\gamma_{(a-d-k)(p+1)}\gamma_{a-d}} \prod_{j=0}^{p-1} \frac{\gamma_{d+k-j}\gamma_{a-d-k}}{\gamma_{d-j}\gamma_{a-d}}. \end{aligned} \tag{6.70}$$

Proof. Both Beta and Beta′ distributions are shape-preserving. As a consequence, we can extract the moments from Proposition 249 with $g(h) = 1$, which states, for any shape-preserving family of distributions D_p, $p = 0, \ldots, d$,

$$\upsilon_p^{(k)}(\mathsf{D}_d) = (p+1)^k \eta_{p+1}^{(k)}(\mathsf{D}_p) \frac{\mathcal{J}_{dp}^{(k)}[1]}{\mathcal{J}_{dp}^{(k)}[h^k]} \tag{6.71}$$

Let $\gamma \in \mathbb{G}(d,p)$, $\mathbf{u} \in \boldsymbol{\gamma}$, $s = \|\mathbf{u}\|$, $\mathbf{y} \in \boldsymbol{\gamma}_\perp$, $h = \|\mathbf{y}\|$ and $s = \|\mathbf{u}\|$. Let $r = \|\mathbf{x}\|$, $\mathbf{x} \in \mathbb{R}^d$, the density of the multivariate distributions $\mathsf{Beta}_d(a)$ and $\mathsf{Beta}'_d(a)$ are $\rho_d(\mathbf{x}) = c_{da}\mathbb{1}_{r<1}/(1-r^2)^{a/2}$ and $\rho'_d(\mathbf{x}) = c'_{da}/(1+r^2)^{a/2}$, respectively. Plugging $r = \sqrt{h^2+s^2}$,

$$\begin{aligned} \rho_d(\sqrt{h^2+s^2}) &= \frac{c_{da}\mathbb{1}_{h^2+s^2<1}}{(1-h^2-s^2)^{a/2}} = \frac{c_{da}}{c_{pa}} \frac{\mathbb{1}_{h<1}}{(1-h^2)^{a/2}} \frac{c_{pa}\mathbb{1}_{\frac{s}{\sqrt{1-h^2}}<1}}{\left(1-\left(\frac{s}{\sqrt{1-h^2}}\right)^2\right)^{a/2}}, \\ \rho'_d(\sqrt{h^2+s^2}) &= \frac{c'_{da}}{(1+h^2+s^2)^{a/2}} = \frac{c'_{da}}{c'_{pa}} \frac{1}{(1+h^2)^{a/2}} \frac{c'_{pa}}{\left(1+\left(\frac{s}{\sqrt{1+h^2}}\right)^2\right)^{a/2}}. \end{aligned} \tag{6.72}$$

Hence, we got a family of shape-preserving distributions $\mathsf{Beta}_p(a)$ with $\xi(h) = \sqrt{1-h^2}$ and $\varepsilon_{dp}(h) = c_{da}\mathbb{1}_{h<1}/(c_{pa}(1-h^2)^{a/2})$ and a family of shape-preserving distributions $\mathsf{Beta}'_p(a)$ with $\xi'(h) = \sqrt{1+h^2}$ and $\varepsilon'_{dp}(h) = c'_{da}/(c'_{pa}(1+h^2)^{a/2})$, respectively. Now, let us compute the corresponding functionals $\mathcal{J}^{(k)}_{dp}[h^q]$ and $\tilde{\mathcal{J}}^{(k)}_{dp}[h^q]$ (for $\mathsf{Beta}_p(a)$ and $\mathsf{Beta}'_p(a)$ families). We have

$$\begin{aligned}
\mathcal{J}^{(k)}_{dp}[h^q] &= \int_0^\infty h^{d+q-p-1}\xi(h)^{(d+k+1)p}\varepsilon^{p+1}_{dp}(h)\mathrm{d}h \\
&= \left(\frac{c_{da}}{c_{pa}}\right)^{p+1}\int_0^1 \frac{h^{d-p+q-1}}{(1-h^2)^{\frac{(p+1)a-(d+k+1)p}{2}}}\mathrm{d}h \\
&= \left(\frac{c_{da}}{c_{pa}}\right)^{p+1}\frac{1}{\omega_{d-p+q}c_{(d-p+q)((p+1)a-(d+k+1)p)}}, \\
\tilde{\mathcal{J}}^{(k)}_{dp}[h^q] &= \int_0^\infty h^{d+q-p-1}\xi'(h)^{(d+k+1)p}\varepsilon'^{p+1}_{dp}(h)\mathrm{d}h \\
&= \left(\frac{c'_{da}}{c'_{pa}}\right)^{p+1}\int_0^\infty \frac{h^{d-p+q-1}}{(1+h^2)^{\frac{(p+1)a-(d+k+1)p}{2}}}\mathrm{d}h \\
&= \left(\frac{c'_{da}}{c'_{pa}}\right)^{p+1}\frac{1}{\omega_{d-p+q}c'_{(d-p+q)((p+1)a-(d+k+1)p)}}.
\end{aligned} \tag{6.73}$$

Hence, plugging $q=0$ and $q=k$, we get

$$\begin{aligned}
\frac{\mathcal{J}^{(k)}_{dp}[1]}{\mathcal{J}^{(k)}_{dp}[h^k]} &= \frac{\omega_{d-p+k}c_{(d-p+k)((p+1)a-(d+k+1)p)}}{\omega_{d-p}c_{(d-p)((p+1)a-(d+k+1)p)}}, \\
\frac{\tilde{\mathcal{J}}^{(k)}_{dp}[1]}{\tilde{\mathcal{J}}^{(k)}_{dp}[h^k]} &= \frac{\omega_{d-p+k}c'_{(d-p+k)((p+1)a-(d+k+1)p)}}{\omega_{d-p}c'_{(d-p)((p+1)a-(d+k+1)p)}}.
\end{aligned} \tag{6.74}$$

Our proof is concluded by plugging those results into Equation (6.71) and by relations

$$\frac{\omega_{d-p+k}c_{(d-p+k)b}}{\omega_{d-p}c_{(d-p)b}} = \frac{\gamma_{d-p}\gamma_{d-p+k-b+2}}{\gamma_{d-p+k}\gamma_{d-p-b+2}}, \qquad \frac{\omega_{d-p+k}c'_{(d-p+k)b}}{\omega_{d-p}c'_{(d-p)b}} = \frac{\gamma_{d-p}\gamma_{b-d+p}}{\gamma_{d-p+k}\gamma_{b-d+p-k}}, \tag{6.75}$$

which follow from Equations (A.13) and (A.15) by replacing d with $d-p$ and a with b. ∎

Remark 260. A simple consequence of the proposition is Miles's formula for the volumetric moments in the unit d-ball (Equation (4.3)), that is

$$v^{(k)}_d(\mathbb{B}_d) = \left(\frac{\Gamma\left(\frac{d}{2}+1\right)}{\pi^{d/2}d!}\right)^k \left(\frac{d}{d+k}\right)^{d+1} \frac{\Gamma\left(\frac{(d+1)(d+k)}{2}+1\right)}{\Gamma\left(\frac{d(d+k+1)}{2}+1\right)} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+k}{2}\right)}\right)^d \prod_{l=1}^{d-1}\frac{\Gamma\left(\frac{k+l}{2}\right)}{\Gamma\left(\frac{l}{2}\right)}.$$

To see this, note that $\mathsf{Unif}(\mathbb{B}_d)$ is a special case of $\mathsf{Beta}_d(a)$ with $a=0$. Thus, by Proposition 259 with $p=d$ and $a=0$, we get

$$v^{(k)}_d(\mathsf{Unif}(\mathbb{B}_d)) = \frac{1}{d!^k}\frac{\gamma_{(d+k)(d+1)+2}\gamma_{d+2}}{\gamma_{(d+k)(d+1)+2-k}\gamma_{d+2+k}}\prod_{j=0}^{d-1}\frac{\gamma_{d+k-j}\gamma_{d+2}}{\gamma_{d-j}\gamma_{d+2+k}}. \tag{6.76}$$

Keeping in mind that $v^{(k)}_d(\cdot)$ is defined differently for bodies and distributions, we get that these two results are ideed equivalent after appropriate normalisation $v_d^{(k)}(\mathbb{B}_d) = v_d^{(k)}(\mathsf{Unif}(\mathbb{B}_d))/\kappa_d^k$.

6.3.2 Conditional simplices

We extend the work of Steigenberger and Kabluchko to conditional simplices.

Proposition 261. *Let $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_p$ be a sample drawn form $\mathsf{Beta}_d(a)$ distribution and let $\mathbf{B}'_1, \mathbf{B}'_2, \ldots, \mathbf{B}'_p$ be a sample drawn form $\mathsf{Beta}'_d(a)$ distribution and $\mathbf{b}_0, \mathbf{b}'_0 \in \mathbb{R}^d$ be some fixed points. Assuming $p \le d$, we get for moments of the $p-$volume $\Delta_p(\mathsf{Beta}_d(a) \mid \mathbf{b}_0) = \mathrm{vol}_p \operatorname{conv}(\mathbf{b}_0, \mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_p)$ and $\Delta_p(\mathsf{Beta}'_d(a) \mid \mathbf{b}'_0) = \mathrm{vol}_p \operatorname{conv}(\mathbf{b}'_0, \mathbf{B}'_1, \mathbf{B}'_2, \ldots, \mathbf{B}'_p)$ of simplices formed by a convex hull of those points and for m natural,*

$$\begin{aligned}
v_p^{(2m)}(\mathsf{Beta}_d(a) \mid \mathbf{b}_0) &= \frac{1}{p!^{2m}} \left(\prod_{j=0}^{p-1} \frac{\gamma_{d+2m-j}\gamma_{d-a+2}}{\gamma_{d-j}\gamma_{d-a+2+2m}} \right) \sum_{s=0}^{m} \binom{m}{s} \frac{\gamma_d \gamma_{(d-a+2m)p+2}}{\gamma_{d+2s}\gamma_{(d-a+2m)p+2-2s}} \|\mathbf{b}_0\|^{2s}, \\
v_p^{(2m)}(\mathsf{Beta}'_d(a) \mid \mathbf{b}'_0) &= \frac{1}{p!^{2m}} \left(\prod_{j=0}^{p-1} \frac{\gamma_{d+2m-j}\gamma_{a-d-2m}}{\gamma_{d-j}\gamma_{a-d}} \right) \sum_{s=0}^{m} \binom{m}{s} \frac{\gamma_d \gamma_{(a-d-2m)(p+1)-2s}}{\gamma_{d+2s}\gamma_{(a-d-2m)(p+1)}} \|\mathbf{b}'_0\|^{2s}.
\end{aligned} \tag{6.77}$$

Proof. We already know that $\mathsf{Beta}_d(a)$ and $\mathsf{Beta}'_d(a)$ are shape-preserving. By Equation (6.49), we get

$$\begin{aligned}
\frac{\mathcal{J}_{dp}^{(2m)}[h^{2m-2s}]}{\mathcal{J}_{dp}^{(2m)}[h^{2m}]} &= \frac{\omega_{d-p+2m} c_{(d-p+2m)((p+1)a-(d+2m+1)p)}}{\omega_{d-p+2m-2s} c_{(d-p+2m-2s)((p+1)a-(d+2m+1)p)}}, \\
\frac{\tilde{\mathcal{J}}_{dp}^{(2m)}[h^{2m-2s}]}{\tilde{\mathcal{J}}_{dp}^{(2m)}[h^{2m}]} &= \frac{\omega_{d-p+2m} c'_{(d-p+2m)((p+1)a-(d+2m+1)p)}}{\omega_{d-p+2m-2s} c'_{(d-p+2m-2s)((p+1)a-(d+2m+1)p)}}.
\end{aligned} \tag{6.78}$$

From which, by relations

$$\frac{\omega_{d-p+k} c_{(d-p+k)b}}{\omega_{d-p+l} c_{(d-p+l)b}} = \frac{\gamma_{d-p+k-b+2}\gamma_{d-p+l}}{\gamma_{d-p+k}\gamma_{d-p+l-b+2}}, \qquad \frac{\omega_{d-p+k} c'_{(d-p+k)b}}{\omega_{d-p+l} c'_{(d-p+l)b}} = \frac{\gamma_{d-p+l}\gamma_{b-d+p-l}}{\gamma_{d-p+k}\gamma_{b-d+p-k}}, \tag{6.79}$$

we get

$$\frac{\mathcal{J}_{dp}^{(2m)}[h^{2m-2s}]}{\mathcal{J}_{dp}^{(2m)}[h^{2m}]} = \frac{\gamma_{d-p+2m-2s}\gamma_{(d-a+2m)(p+1)+2}}{\gamma_{d-p+2m}\gamma_{(d-a+2m)(p+1)+2-2s}}, \tag{6.80}$$

$$\frac{\tilde{\mathcal{J}}_{dp}^{(2m)}[h^{2m-2s}]}{\tilde{\mathcal{J}}_{dp}^{(2m)}[h^{2m}]} = \frac{\gamma_{d-p+2m-2s}\gamma_{(a-d-2m)(p+1)-2s}}{\gamma_{d-p+2m}\gamma_{(a-d-2m)(p+1)}} \tag{6.81}$$

and finally, by Theorem 250,

$$\begin{aligned}
v_{p+1}^{(2m)}(\mathsf{Beta}_d(a) \mid \mathbf{b}_0) &= \eta_{p+1}^{(2m)}(\mathsf{Beta}_d(a)) \sum_{s=0}^{m} \frac{\gamma_d \gamma_{(d-a+2m)(p+1)+2}}{\gamma_{d+2s}\gamma_{(d-a+2m)(p+1)+2-2s}} \binom{m}{s} \|\mathbf{b}_0\|^{2s}, \\
v_{p+1}^{(2m)}(\mathsf{Beta}'_d(a) \mid \mathbf{b}'_0) &= \eta_{p+1}^{(2m)}(\mathsf{Beta}'_d(a)) \sum_{s=0}^{m} \frac{\gamma_d \gamma_{(a-d-2m)(p+1)-2s}}{\gamma_{d+2s}\gamma_{(a-d-2m)(p+1)}} \binom{m}{s} \|\mathbf{b}'_0\|^{2s}
\end{aligned} \tag{6.82}$$

as desired. ■

6.4 Spherical shell simplices

Let us call random simplices whose verices are drawn independently and uniformly from the surface of the unit $(d-1)$sphere as *spherical shell simplices*. The mean d-volume and the corresponding moments of those simplices were already derived by Forsythe and Tukey [31]. However, note that our Beta simplices inherently contain the special case in which the vertices are drawn from the uniform distribution on the unit $(d-1)$sphere $\mathbb{S}^{d-1}$. This is a consequence of the following equivalence in distributions

$$\mathsf{Unif}(\mathbb{S}^{d-1}) = \lim_{a\to 2^-} \mathsf{Beta}_d(a). \tag{6.83}$$

6.4.1 Radial volumetric moments

By simply plugging $a = 2$ in Propositions 258 and 259, we get

Proposition 262. *Let* $\mathbb{S} = (\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_p)$ *be a sample drawn form* $\mathsf{Unif}(\mathbb{S}^{d-1})$ *distribution. Assuming* $p \le d$*, we get for moments of the* $p-$*volume* $\nabla_p(\mathsf{Unif}(\mathbb{S}^{d-1}))) = \mathrm{vol}_p\,\mathrm{conv}(\mathbf{0}, \mathbf{S}_1, \ldots, \mathbf{S}_p)$ *and* $\Delta_p(\mathsf{Unif}(\mathbb{S}^{d-1})) = \mathrm{vol}_p\,\mathrm{conv}(\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_p)$ *of simplices formed by a convex hull of those points*

$$\eta_p^{(k)}(\mathbb{S}^{d-1}) = \frac{1}{p!^k}\prod_{j=0}^{p-1}\frac{\gamma_{d+k-j}\gamma_d}{\gamma_{d-j}\gamma_{d+k}}, \quad v_p^{(k)}(\mathbb{S}^{d-1}) = \frac{1}{p!^k}\frac{\gamma_{(d-2+k)(p+1)+2}\gamma_d}{\gamma_{(d-2+k)(p+1)+2-k}\gamma_{d+k}}\prod_{j=0}^{p-1}\frac{\gamma_{d+k-j}\gamma_d}{\gamma_{d-j}\gamma_{d+k}}.$$

6.4.2 Conditional simplices

Similarly, Proposition 261 with $a = 2$ becomes

Proposition 263. *Let* $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_p$ *be a sample drawn form* $\mathsf{Unif}(\mathbb{S}^{d-1})$ *distribution and* $\mathbf{s}_0 \in \mathbb{R}^d$ *be some fixed point. Assuming* $p \le d$*, we get for moments of the* $p-$*volume* $\Delta_p(\mathsf{Unif}(\mathbb{S}^{d-1}) \mid \mathbf{s}_0) = \mathrm{vol}_p\,\mathrm{conv}(\mathbf{s}_0, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_p)$ *of simplices formed by a convex hull of those points and for* m *natural,*

$$v_p^{(2m)}(\mathsf{Unif}(\mathbb{S}^{d-1}) \,|\, \mathbf{s}_0) = \frac{1}{p!^{2m}}\left(\prod_{j=0}^{p-1}\frac{\gamma_{d+2m-j}\gamma_d}{\gamma_{d-j}\gamma_{d+2m}}\right)\sum_{s=0}^{m}\binom{m}{s}\frac{\gamma_d\gamma_{(d-2+2m)p+2}}{\gamma_{d+2s}\gamma_{(d-2+2m)p+2-2s}}\|\mathbf{s}_0\|^{2s}.$$

Appendices

A Probability distributions and their stochastic decomposition

A.1 Common 1-dimensional distributions

The following Table A.1 enlists common distributions $\mathbb{P}_X$ of a real random variable X, its probability density function (PDF) $f_X(x)$ and (non-central) moments $m_q = \mathbb{E}\, X^q$ for q possibly being a real number. As usual, $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the *Beta function*.

distribution	$\mathbb{P}_X$	$f_X(x)$	$m_q = \mathbb{E}X^q$	notes
standard normal	$\mathsf{N}(0,1)$	$\frac{e^{-x^2/2}}{\sqrt{2\pi}}$	$\begin{cases}(q-1)!!, & q \text{ even}\\ 0, & q \text{ odd}\end{cases}$	
chi	χ_d	$\frac{x^{d-1}e^{-x^2/2}}{2^{\frac{d}{2}-1}\,\Gamma(\frac{d}{2})}\mathbb{1}_{(0,\infty)}(x)$	$\frac{2^{q/2}\Gamma\left(\frac{d+q}{2}\right)}{\Gamma(\frac{d}{2})}$	$q > -d$
chi-square	χ_d^2	$\frac{x^{d/2-1}e^{-x/2}}{2^{\frac{d}{2}}\Gamma(\frac{d}{2})}\mathbb{1}_{(0,\infty)}(x)$	$\frac{2^r\Gamma\left(\frac{d}{2}+q\right)}{\Gamma(\frac{d}{2})}$	$q > -d/2$
Gamma	$\Gamma(d)$	$\frac{x^{d-1}e^{-x}}{\Gamma(d)}\mathbb{1}_{(0,\infty)}(x)$	$\frac{\Gamma(d+q)}{\Gamma(d)}$	$q > -d$
Beta	$\mathsf{Beta}(d,p)$	$\frac{x^{d-1}(1-x)^{p-1}}{\mathrm{B}(d,p)}\mathbb{1}_{(0,1)}(x)$	$\frac{\mathrm{B}(d+q,p)}{\mathrm{B}(d,p)}$	$q > -d$
exponential	$\mathsf{Exp}(\lambda)$	$\lambda e^{-\lambda x}\mathbb{1}_{(0,\infty)}(x)$	$\frac{\Gamma(q+1)}{\lambda^q}$	$q > -1$
uniform	$\mathsf{Unif}(a,b)$	$\frac{1}{b-a}\mathbb{1}_{(a,b)}(x)$	$\frac{b^{q+1}-a^{q+1}}{(q+1)(b-a)}$	$b > a$
Dirac	δ_a	$\delta(x-a)$	a^q	

Table A.1: Common distributions and their properties

Some remarks: Let X_i be i.i.d. random variables with distribution $\mathsf{N}(0,1)$. Then

$$\sqrt{\textstyle\sum_{i=1}^d X_i^2} \sim \chi_d.$$

Let $X \sim \chi_d$, then $X^2 \sim \chi_d^2$ and $X^2/2 \sim \Gamma(d/2)$, from which we get the relation between moments and density functions. It is convenient to denote

$$\gamma_d = \int_0^\infty r^{d-1}e^{-\frac{1}{2}r^2}\mathrm{d}r = 2^{\frac{d}{2}-1}\Gamma\left(\tfrac{d}{2}\right), \tag{A.1}$$

using which we can write for the probability density of $X \sim \chi_d$

$$f_X(x) = \frac{x^{d-1}}{\gamma_p} e^{-\frac{x^2}{2}} \mathbb{1}_{(0,\infty)} \tag{A.2}$$

and for its moments $\mathbb{E}\, X^q = \gamma_{d+q}/\gamma_d$.

A.2 Langford and related distributions

Let $U, U', U'' \sim \mathsf{Unif}(0,1)$ (independent), we define four random variables

$$\Lambda = (U' - U)(U'' - U), \quad \Sigma = (U - U')U, \quad \Xi = UU', \quad \Omega = U(1-U). \tag{A.3}$$

The equalities between $\Lambda, \Sigma, \Xi, \Omega$ with U, U', U'' have to be interpreted only in terms of distributions. That means, we will assume $\Lambda, \Sigma, \Xi, \Omega$ to be in fact independent. We say Λ follows the *Langford distribution* ($\Lambda \sim \mathsf{Lang}$) [42]. We call those variables as our thesis' *auxiliary Langford random variables*. The probability density functions (PDFs) and the cumulative density functions (CDFs) of those are shown in Table A.2 below. Trivially, PDF of U is $f_U(u) = 1$ when $0 < u < 1$ and zero otherwise.

X	x	PDF: $f_X(x) = \frac{d}{dx} F_X(x)$	CDF: $F_X(x) = \mathbb{P}[X \le x]$
Λ	λ	$f_\Lambda(\lambda) = \begin{cases} 4 \operatorname{argtanh} \sqrt{1+4\lambda} - 4\sqrt{1+4\lambda}, & -\frac{1}{4} \le \lambda < 0, \\ 4\sqrt{\lambda} - 2\ln\lambda - 4, & 0 < \lambda \le 1, \\ 0, & \text{otherwise} \end{cases}$	$F_\Lambda(\lambda) = \begin{cases} 0, & \lambda < -\frac{1}{4} \\ \frac{1}{3}(1-8\lambda)\sqrt{1+4\lambda} + 4\lambda \operatorname{argtanh} \sqrt{1+4\lambda}, & -\frac{1}{4} \le \lambda < 0 \\ \frac{1}{3}, & \lambda = 0, \\ \frac{1}{3}\left(1 - 6\lambda + 8\lambda^{3/2}\right) - 2\lambda\ln\lambda, & 0 < \lambda < 1, \\ 1, & \lambda \ge 1. \end{cases}$
Σ	σ	$f_\Sigma(\sigma) = \begin{cases} 2 \operatorname{argtanh} \sqrt{1+4\sigma}, & -\frac{1}{4} \le \sigma < 0, \\ -\frac{1}{2}\ln\sigma, & 0 < \sigma \le 1, \\ 0, & \text{otherwise} \end{cases}$	$F_\Sigma(\sigma) = \begin{cases} 0, & \sigma < -\frac{1}{4} \\ \frac{1}{2}\sqrt{1+4\sigma} + 2\sigma \operatorname{argtanh} \sqrt{1+4\sigma}, & -\frac{1}{4} \le \sigma < 0 \\ \frac{1}{2}, & \sigma = 0, \\ \frac{1}{2}(1 + \sigma - \sigma\ln\sigma), & 0 \le \sigma < 1, \\ 1, & \sigma \ge 1. \end{cases}$
Ω	ω	$f_\Omega(\omega) = \begin{cases} \frac{2}{\sqrt{1-4\omega}}, & 0 \le \omega < 1/4, \\ 0, & \text{otherwise} \end{cases}$	$F_\Omega(\omega) = \begin{cases} 0, & \omega \ge 0 \\ 1 - \sqrt{1-4\omega}, & 0 \le \omega < 1/4, \\ 1, & \omega \ge 1/4. \end{cases}$
Ξ	ξ	$f_\Xi(\xi) = \begin{cases} -\ln\xi, & 0 \le \xi < 1, \\ 0, & \text{otherwise} \end{cases}$	$F_\Xi(\xi) = \begin{cases} 0, & \xi \le 0 \\ \xi(1 - \ln\xi), & 0 < \xi < 1, \\ 1, & \xi \ge 1. \end{cases}$

Table A.2: PDFs and CDFs of auxiliary Langford variables

A step-by-step derivation of $f_\Lambda(\lambda)$ is shown in Example 284. Similar derivation of $f_\Sigma(\sigma)$ is shown in Example 283 and finally, derivation of $f_\Omega(\omega)$ is shown in Example 278.

A.3 Radial multi-dimensional distributions

The following Table A.3 enlists common radially symmetric distributions $\mathbb{P}_{\mathbf{X}}$ of a real random vector $\mathbf{X} \in \mathbb{R}^d$ and its density function $f_{\mathbf{X}}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$. Furthermore, to emphasize these distributions are intimately related, distributions of the their projections onto p-planes $\boldsymbol{\gamma} \in \mathbb{G}(d,p)$, $p \leq d$ are given.

distribution	$\mathbb{P}_{\mathbf{X}}$	$f_{\mathbf{X}}(\mathbf{x})$	$\mathbb{P}_{\mathrm{proj}_\gamma \mathbf{X}}$
standard multi-normal	N_d	$\frac{1}{\sqrt{2\pi}^d} e^{-\|\mathbf{x}\|^2/2}$	N_p
uniform on d-ball	$\mathsf{Unif}(\mathbb{B}_d)$	$\frac{1}{\kappa_d} \mathbb{1}_{\|\mathbf{x}\|<1}$	$\mathsf{Beta}_p(p-d)$
uniform on $(d-1)$-sphere	$\mathsf{Unif}(\mathbb{S}^{d-1})$	$\frac{1}{\omega_d}\delta(1-\|\mathbf{x}\|)$	$\mathsf{Beta}_p(2-d+p)$
multi-Beta	$\mathsf{Beta}_d(a)$	$\dfrac{c_{da}\mathbb{1}_{\|\mathbf{x}\|<1}}{(1-\|\mathbf{x}\|^2)^{a/2}}$	$\mathsf{Beta}_p(a-d+p)$
multi-Beta′	$\mathsf{Beta}'_d(a)$	$\dfrac{c'_{da}}{(1+\|\mathbf{x}\|^2)^{a/2}}$	$\mathsf{Beta}'_p(a-d+p)$

Table A.3: Common multi-dimensional distributions of radial random vectors and their properties

Normalisation

In the table above, ω_d is the surface of the unit $\mathbb{S}^{d-1}$ sphere and $\kappa_d = \omega_d/d$ is the volume of the corresponding unit d-ball $\mathbb{B}_d$. For their exact value, we have the following result

Lemma 264. $$\omega_d = \sigma_d(\mathbb{S}^{d-1}) = \int_{\mathbb{S}^{d-1}} \sigma_d(d\hat{\mathbf{n}}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

Proof. Note that the standard multi-normal distribution must normalise to one, integrating over $\mathbb{R}^d$ and realising that for radial functions $\lambda_d(\mathrm{d}\mathbf{x}) = \omega_d r^{d-1}\mathrm{d}r$ where $r = \|\mathbf{x}\|$,

$$1 = \int_{\mathbb{R}^d} \frac{e^{-\frac{1}{2}\|\mathbf{x}\|^2}}{\sqrt{2\pi}^d} \lambda_d(\mathrm{d}\mathbf{x}) = \omega_d \int_0^\infty \frac{r^{d-1}e^{-\frac{1}{2}r^2}}{\sqrt{2\pi}^d}\mathrm{d}r = \frac{\omega_d \gamma_d}{\sqrt{2\pi}^d}, \tag{A.4}$$

where $\gamma_d = \int_0^\infty r^{d-1} e^{-r^2/2} \mathrm{d}r = 2^{\frac{d}{2}-1} \Gamma\left(\frac{d}{2}\right)$ (see Equation (A.1)). ∎

Similarly, we can express c_{da} and c'_{da} in the exact form.

Lemma 265. $$\omega_d c_{da} = \frac{\gamma_{d-a+2}}{\gamma_d \gamma_{2-a}}, \qquad \omega_d c'_{da} = \frac{\gamma_a}{\gamma_d \gamma_{a-d}}.$$

Proof. First, let us write down how the normalisation conditions look like

$$1 = c_{da}\omega_d \int_0^1 \frac{r^{d-1}}{(1-r^2)^{a/2}}\, \mathrm{d}r, \qquad 1 = c'_{da}\omega_d \int_0^\infty \frac{r^{d-1}}{(1+r^2)^{a/2}}\, \mathrm{d}r. \tag{A.5}$$

The remaining integrals can be solved in terms of the Beta function. Instead, we show somewhat more elementary approach. Consider the following integral

$$\int_0^\infty r^{d-1}\lambda^{a-1} e^{-\sigma^2\lambda^2/2}\, \mathrm{d}\lambda = \gamma_a \frac{r^{d-1}}{\sigma^a}. \tag{A.6}$$

First, by plugging $\sigma = \sqrt{1+r^2}$ and integrating out r over $(0,\infty)$, we get

$$\gamma_d \int_0^\infty \lambda^{a-d-1} e^{-\lambda^2/2}\, \mathrm{d}\lambda = \gamma_a \int_0^\infty \frac{r^{d-1}}{(1+r^2)^{a/2}}\, \mathrm{d}r, \tag{A.7}$$

from which immediately $\gamma_d\gamma_{a-d} = \gamma_a/(\omega_d c'_{da})$. For the first integral, we substitute $r = s/\sqrt{1+s^2}$, which gives

$$\frac{1}{\omega_d c_{da}} = \int_0^1 \frac{r^{d-1}}{(1-r^2)^{a/2}}\, \mathrm{d}r = \int_0^\infty \frac{s^{d-1}}{(1+s^2)^{(d-a+2)/2}}\, \mathrm{d}s = \frac{1}{\omega_d c'_{d(d-a+2)}}, \tag{A.8}$$

from which $c_{da} = c'_{d(d-a+2)}$. ∎

Radial moments and spherical representation

Note that any radially symmetric d-dimensional random vector $\mathbf{X}$ can be represented as a product of two *independent* random variables R and $\mathbf{S}$,

$$\mathbf{X} \overset{d}{=} R\mathbf{S}, \tag{A.9}$$

where $\mathbf{S} \sim \mathsf{Unif}(\mathbb{S}^{d-1})$ and R has the distribution of $\|\mathbf{X}\|$.

Lemma 266. *Let* $\mathbf{Z} \sim \mathsf{N}_d$*, then* $\|\mathbf{Z}\| \sim \chi_d$.

Proof. We only need to check for the radial moments, we have

$$\mathbb{E}\|\mathbf{Z}\|^k = \int_{\mathbb{R}^d} \|\mathbf{z}\|^k \frac{e^{-\frac{1}{2}\|\mathbf{z}\|^2}}{\sqrt{2\pi}^d} \lambda_d(\mathrm{d}\mathbf{z}) = \omega_d \int_0^\infty \frac{r^{k+d-1} e^{-\frac{1}{2}r^2}}{\sqrt{2\pi}^d} \mathrm{d}r = \frac{\omega_d \gamma_{d+k}}{\sqrt{2\pi}^d} = \frac{\gamma_{d+k}}{\gamma_d}, \tag{A.10}$$

which matches $\|\mathbf{Z}\| \sim \chi_d$. ∎

Lemma 267. *Let* $\mathbf{B} \sim \mathsf{Beta}_d(a)$, $\mathbf{B}' \sim \mathsf{Beta}'_d(a)$ *and* $\mathbf{Z} \sim \mathsf{N}_d$,

$$\mathbf{B}V \stackrel{d}{=} \mathbf{Z}, \qquad \mathbf{B}' \stackrel{d}{=} \frac{\mathbf{Z}}{U}, \tag{A.11}$$

where $\mathbf{B}$ *is independent with* $V \sim \chi_{d-a+2}$ *and* $\mathbf{Z}$ *is independent with* $U \sim \chi_{a-d}$. *We may write this as*

$$\mathsf{Beta}_d(a)\chi_{d-a+2} \stackrel{d}{=} \mathsf{N}_d, \qquad \mathsf{Beta}'_d(a) \stackrel{d}{=} \frac{\mathsf{N}_d}{\chi_{a-d}}. \tag{A.12}$$

Proof. Let $R = \|\mathbf{Z}\| \sim \chi_d$. For the radial moments, we have

$$\mathbb{E}\,\|\mathbf{B}\|^k = \omega_d c_{da} \int_0^1 \frac{r^{k+d-1}}{(1-r^2)^{a/2}} = \frac{\omega_d c_{da}}{\omega_{d+k} c_{(d+k)a}} = \frac{\gamma_{d+k}\gamma_{d-a+2}}{\gamma_d \gamma_{d+k-a+2}}. \tag{A.13}$$

We can write this as

$$\mathbb{E}\,\|\mathbf{B}\|^k \frac{\gamma_{d-a+2+k}}{\gamma_{d-a+2}} = \frac{\gamma_{d+k}}{\gamma_d} \tag{A.14}$$

which is $\mathbb{E}\|\mathbf{B}\|^k\,\mathbb{E}V^k = \mathbb{E}R^k$ with some $V \sim \chi_{d-a+2}$ independent of $\mathbf{B}$. Similarly,

$$\mathbb{E}\,\|\mathbf{B}'\|^k = \omega_d c'_{da} \int_0^\infty \frac{r^{k+d-1}}{(1+r^2)^{a/2}} = \frac{\omega_d c'_{da}}{\omega_{d+k} c'_{(d+k)a}} = \frac{\gamma_{d+k}\gamma_{a-d-k}}{\gamma_d \gamma_{a-d}}. \tag{A.15}$$

which is $\mathbb{E}\|\mathbf{B}'\|^k = \mathbb{E}R^k\,\mathbb{E}U^{-k}$ with $U \sim \chi_{a-d}$ independent of R. ■

Projections

Let $\boldsymbol{\gamma} \in \mathbb{G}(d,p)$ be a p-plane. Note that the orthogonal projection $\operatorname{proj}_{\boldsymbol{\gamma}} \mathbf{X}$ of any radial random vector $\mathbf{X}$ is again a radial random vector (on $\boldsymbol{\gamma}$). Moreover, by symmetry, the distribution of this projection on $\boldsymbol{\gamma}$ is the same for any $\boldsymbol{\gamma} \in \mathbb{G}(d,p)$.

Lemma 268. *Let* $\mathbf{Z} \sim \mathsf{N}_d$ *and* $\boldsymbol{\gamma} \in \mathbb{G}(d,p)$ *any, then* $\operatorname{proj}_{\boldsymbol{\gamma}} \mathbf{Z} \sim \mathsf{N}_p$ *on* $\boldsymbol{\gamma}$.

Proof. Let $\mathbf{Z} \sim \mathsf{N}_d$, then there is a well known representation $\mathbf{Z} \stackrel{d}{=} (Z_1, \ldots, Z_d)^\top$, where Z_j are identically and independently distributed according to the standard normal distribution $\mathsf{N}(0,1)$. Let us select $\boldsymbol{\gamma}_0 = \operatorname{span}(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_p)$, then

$$\operatorname{proj}_{\boldsymbol{\gamma}_0} \mathbf{Z} = (Z_1, Z_2, Z_3, \ldots, Z_{p-1}, Z_p, 0, 0, \ldots, 0)^\top \sim \mathsf{N}_p \tag{A.16}$$

on $\boldsymbol{\gamma}_0$. By radial symmetry, the result applies for any $\boldsymbol{\gamma} \in \mathbb{G}(d,p)$. ■

Lemma 269. *Let* $\mathbf{S} \sim \mathsf{Unif}(\mathbb{S}^{d-1})$ *and* $\boldsymbol{\gamma} \in \mathbb{G}(d,p)$, *then* $\operatorname{proj}_{\boldsymbol{\gamma}} \mathbf{S} \sim \mathsf{Beta}_p(2-d+p)$ *on* $\boldsymbol{\gamma}$.

Proof. Let $\mathbf{Z} \sim \mathsf{N}_d$, then $\mathbf{Z} \stackrel{d}{=} R\mathbf{S}$, where $R \sim \chi_d$ is independent with $\mathbf{S} \sim \mathsf{Unif}(\mathbb{S}^{d-1})$. Taking the projection, we get

$$\operatorname{proj}_{\boldsymbol{\gamma}} \mathbf{Z} \stackrel{d}{=} R \operatorname{proj}_{\boldsymbol{\gamma}} \mathbf{S}. \tag{A.17}$$

Note that $\operatorname{proj}_{\gamma} \mathbf{S}$ is again radial on $\boldsymbol{\gamma}$. Since $\operatorname{proj}_{\gamma} \mathbf{Z} \sim N_p$, we get in distributions,

$$\mathsf{N}_p \overset{d}{=} \chi_d \operatorname{proj}_{\gamma} \mathbf{S}. \tag{A.18}$$

Comparing with Lemma 267, we get $\operatorname{proj}_{\gamma} \mathbf{S} \sim \mathsf{Beta}_p(2-d+p)$. ■

Lemma 270. *Let* $\mathbf{B} \sim \mathsf{Beta}_d(a)$ *and* $\boldsymbol{\gamma} \in \mathbb{G}(d,p)$*, then* $\operatorname{proj}_{\gamma} \mathbf{B} \sim \mathsf{Beta}_p(a-d+p)$ *on* $\boldsymbol{\gamma}$*.*

Proof. Let $V \sim \chi_{d-a+2}$ be a random variable independent on $\mathbf{B} \sim \mathsf{Beta}_d(a)$. By Lemma 267, $\mathbf{B}V \overset{d}{=} \mathbf{Z} \sim \mathsf{N}_d$. Taking the projection onto $\boldsymbol{\gamma}$, we get

$$(\operatorname{proj}_{\gamma} \mathbf{B})V \overset{d}{=} \operatorname{proj}_{\gamma} \mathbf{Z} \sim \mathsf{N}_p. \tag{A.19}$$

By using Lemma 267 again, we get $\operatorname{proj}_{\gamma} \mathbf{B} \sim \mathsf{Beta}_p(a-d+p)$. ■

Corollary 270.1. *Let* $\mathbf{W} \sim \mathsf{Unif}(\mathbb{B}_d)$ *and* $\boldsymbol{\gamma} \in \mathbb{G}(d,p)$*, then* $\operatorname{proj}_{\gamma} \mathbf{W} \sim \mathsf{Beta}_p(p-d)$ *on* $\boldsymbol{\gamma}$*.*

Proof. Follows trivially from Lemma 270 with $a = 0$ as $\mathsf{Unif}(\mathbb{B}_d) = \mathsf{Beta}_d(0)$. ■

Remark 271. Note that, actually, also the uniform distribution $\mathsf{Unif}(\mathbb{S}^{d-1})$ on the unit $(d-1)$-sphere may be viewed as a singular case of multi-Beta distribution with $a = 2$. The projection formula in Lemma 269 then follows also from Lemma 270 by putting $a = 2$.

Lemma 272. *Let* $\mathbf{B}' \sim \mathsf{Beta}'_d(a)$ *and* $\boldsymbol{\gamma} \in \mathbb{G}(d,p)$*, then* $\operatorname{proj}_{\gamma} \mathbf{B}' \sim \mathsf{Beta}'_p(a-d+p)$ *on* $\boldsymbol{\gamma}$*.*

Proof. Let $U \sim \chi_{a-d}$ be independent with $\mathbf{Z} \sim \mathsf{N}_d$. By Lemma 267, $\mathbf{B}' \overset{d}{=} \mathbf{Z}/U \sim \mathsf{Beta}'_d(a)$. Taking the projection onto $\boldsymbol{\gamma}$, we get

$$\operatorname{proj}_{\gamma} \mathbf{B}' \overset{d}{=} \frac{\operatorname{proj}_{\gamma} \mathbf{Z}}{U} \sim \frac{\mathsf{N}_p}{\chi_{a-d}}. \tag{A.20}$$

Hence, Lemma 267 gives $\operatorname{proj}_{\gamma} \mathbf{B}' \sim \mathsf{Beta}'_p(a-d+p)$ ■

A.4 Dirichlet distribution

The following Table A.4 enlists one remaining distribution $\mathbb{P}_{\mathbf{Y}}$ of a real random vector $\mathbf{Y} = (Y_0, Y_1, \ldots, Y_d)^\top \in \mathbb{R}^{d+1}$ and its probability measure $\mathbb{P}_{\mathbf{Y}}(\mathrm{d}\mathbf{y})$, $\mathbf{y} \in \mathbb{R}^{d+1}$ used in this thesis, namely the *Dirichlet distribution.*

distribution	$\mathbb{P}_{\mathbf{Y}}$	$\mathbb{P}_{\mathbf{Y}}(\mathrm{d}\mathbf{y})$
Dirichlet	$\mathsf{Dir}(\alpha_0, \ldots, \alpha_d)$	$\frac{\Gamma(\sum_{i=0}^d \alpha_i)}{\prod_{i=0}^d \Gamma(\alpha_i)} \left(\prod_{i=0}^d y_i^{\alpha_i - 1} \right) \frac{\tau_d(\mathrm{d}\mathbf{y})}{\sqrt{d+1}}$

Table A.4: Dirichlet distribution of a random vector

In the table above, τ_d is the surface measure on a regular d-dimensional simplex $T_d^* = \{(y_0, \ldots, y_d)^\top \in [0,1]^{d+1} \,|\, \sum_{i=0}^d y_i = 1\}$ embedded in $\mathbb{R}^{d+1}$ (as a d-dimensional surface). Dirichlet distribution is said to be symmetric if there exists a single *concentration parameter* α such that $\alpha_i = \alpha$ for each $i = 0, \ldots, d$.

Remark 273. Often, the probability measure of the Dirichlet distribution is written in terms of $y_1, \ldots, y_d$. This form can be recovered from the probability measure by writing $y_0 = 1 - \sum_{i=1}^d y_i$ and by realizing that by the projection onto $y_0 = 0$ plane, we have the following transformation $\tau_d(\mathrm{d}\mathbf{y}) = \sqrt{d+1}\,\mathrm{d}y_1\mathrm{d}y_2 \cdots \mathrm{d}y_d$.

We will see that the normalisation constant is correct using the following stochastic decomposition argument:

Lemma 274. *Let $\mathbf{X} = (X_0, X_1, \ldots, X_d)^\top$ be a random vector with $X_i \sim \Gamma(\alpha_i)$ being independent and let $S = \sum_{i=0}^d X_i$, then $\mathbf{X}/S \sim \mathsf{Dir}(\alpha_0, \alpha_1, \ldots, \alpha_d)$ and $S \sim \Gamma(\alpha_0 + \cdots + \alpha_d)$. Moreover, $\mathbf{X}/S$ and S are stochastically independent.*

Proof. Note that since $X_i \sim \Gamma(\alpha_i)$ are independent,

$$\mathbb{P}_\mathbf{X}(\mathrm{d}\mathbf{x}) = \left(\prod_{i=0}^d \frac{x_i^{\alpha_i - 1}}{\Gamma(\alpha_i)} \mathbb{1}_{(0,\infty)}(x_i)\right) e^{-\sum_{i=0}^d x_i} \lambda_{d+1}(\mathrm{d}\mathbf{x}) \tag{A.21}$$

Let us perform the change of variables $x_i = sy_i$ such that $\sum_{i=0}^d y_i = 1$ and $s > 0$. Note that now $\mathbf{y} = (y_0, \ldots, y_d)^\top$ lies on T_d^* and thus the Lebesgue measure splits as $\lambda_{d+1}(\mathrm{d}\mathbf{x}) = s^d \tau_d(\mathrm{d}\mathbf{y})\mathrm{d}s/\sqrt{d+1}$. The additional factor $\sqrt{d+1}$ comes from the projection of s into the direction of the vector $(1, 1, \ldots, 1)$. In total,

$$\mathbb{P}_\mathbf{X}(\mathrm{d}\mathbf{x}) = \left(\prod_{i=0}^d \frac{y_i^{\alpha_i - 1}}{\Gamma(\alpha_i)}\right) \frac{s^{\alpha_0 + \cdots + \alpha_d - 1} e^{-s}}{\sqrt{d+1}} \mathbb{1}_{(0,\infty)}(s) \tau_d(\mathrm{d}\mathbf{y})\mathrm{d}s = \mathbb{P}_\mathbf{Y}(\mathrm{d}\mathbf{y})\mathbb{P}_S(\mathrm{d}s), \tag{A.22}$$

with $S \sim \Gamma(\alpha_0 + \cdots + \alpha_d)$ since $f_S(s) = s^{\alpha_0 + \cdots + \alpha_d - 1} e^{-s} \mathbb{1}_{(0,\infty)}(s)/\Gamma(\alpha_0 + \cdots + \alpha_d)$. Independence follows from the fact that we factorised the probability measure into a product of two measures. ■

The proof of Lemma 274 is somewhat standard. The version shown here is an adaptation taken from Ranošová [58].

Corollary 274.1.

$$\mathrm{vol}_d\, T_d^* = \frac{\sqrt{d+1}}{d!}. \tag{A.23}$$

Proof. Let $\alpha_i = 1$ for all i in $\mathsf{Dir}(\alpha_0, \ldots, \alpha_d)$. Then, since $\mathbb{E}\,1 = 1$,

$$\mathrm{vol}_d\, T_d^* = \int_{T_d^*} \tau_d(\mathrm{d}\mathbf{y}) = (\sqrt{d+1}) \frac{\prod_{i=0}^d \Gamma(\alpha_i)}{\Gamma(\sum_{i=0}^d \alpha_i)} \mathbb{E}\,1 = \frac{\sqrt{d+1}}{\Gamma(d+1)}. \tag{A.24}$$

■

Definition 275. We define yet another set of simplices $\mathbb{T}_d$ which we call *canonical* simplices as

$$\mathbb{T}_d = \mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d). \tag{A.25}$$

Proposition 276.

$$\mathrm{vol}_d\, \mathbb{T}_d = \frac{1}{d!}. \tag{A.26}$$

Proof. Let $\mathbf{x} = (x_0, \dots, x_d)^\top$. The distance to the plane $(1, \dots, 1)^\top \mathbf{x} = 1$ is equal to $1/\sqrt{d+1}$. By base-height decomposition, we have

$$\mathrm{vol}_{d+1}\, \mathbb{T}_{d+1} = \frac{1}{d+1} \frac{1}{\sqrt{d+1}} \mathrm{vol}_d\, T_d^* = \frac{1}{(d+1)!}. \tag{A.27}$$

■

Remark 277. As a consequence of Remark 273, we get the following formula

$$\int_{\mathbb{T}_d} \left(1 - \textstyle\sum_{i=1}^d y_i\right)^{\alpha_0 - 1} \prod_{i=1}^d y_i^{\alpha_i - 1} \lambda_d(\mathrm{d}\mathbf{y}) = \frac{\prod_{i=0}^d \Gamma(\alpha_i)}{\Gamma(\sum_{i=0}^d \alpha_i)}, \tag{A.28}$$

where $\mathbf{y} = (y_1, \dots, y_d)^\top \in \mathbb{R}^d$.

A.5 Reconstruction of density of random variables from moments via Inverse Mellin Transform

Positive random variables

Recall the definition of the Mellin transform of a function $f : \mathbb{R}^+ \to \mathbb{R}$,

$$\mathcal{M}[f] = \mathcal{M}[f(s)](k) = \int_0^\infty s^{k-1} f(s)\, \mathrm{d}s. \tag{A.29}$$

For example, for any $\alpha > 0$, we have

$$\mathcal{M}[\delta(s - \alpha)] = \alpha^{k-1}. \tag{A.30}$$

If we know the moments of a positive real random variable S, say $m_k = \mathbb{E}\, S^k$, that is $m_k = \int_0^\infty s^k f(s) \mathrm{d}s$, we can then recover its density $f(s)$ by the inverse Mellin transform since $m_{k-1} = \mathcal{M}[f(s)](k)$. Formally,

$$f(s) = \mathcal{M}^{-1}[m_{k-1}](s). \tag{A.31}$$

Example 278. Let us derive the PDF $f_\Omega(\omega)$ of a positive random variable $\Omega = X(1 - X)$, where $X \sim \mathsf{Unif}(0, 1)$. Let us write down an integral for moments of Ω for k being a positive integer,

$$m_k = \mathbb{E}\left[\Omega^k\right] = \int_0^1 x^k (1 - x)^k\, \mathrm{d}x. \tag{A.32}$$

Taking the inverse Mellin Transform, we get, rather unsurprisingly, the *Dirac kernel method* (which works not only for positive random variables)

$$f_\Omega(\omega) = \mathcal{M}^{-1}[m_{k-1}] = \mathcal{M}^{-1}\left[\int_0^1 x^{k-1}(1-x)^{k-1}\, \mathrm{d}x\right] = \int_0^1 \delta(x(1-x) - \omega) \mathrm{d}x. \tag{A.33}$$

This integral is trivial and can be solved via the known formula

$$\delta(g(x)) = \sum_{y, g(y)=0} \delta(x-y)/|g'(y)| \tag{A.34}$$

valid for any suitable real function g, The roots of $g(x) = x(1-x) - \omega = 0$ are $y_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4\omega}$ and thus $g'(y_\pm) = 1 - 2y_\pm = \mp\sqrt{1-4\omega}$, from which

$$f_\Omega(\omega) = \frac{2}{\sqrt{1-4\omega}} \mathbb{1}_{0<\omega<1/4}. \tag{A.35}$$

Integral operators

Let $r \geq 0$. Integrating by parts,

$$\mathcal{M}[f] = -\int_0^\infty \frac{s^{k+r}}{k+r}(s^{-r}f(s))'\mathrm{d}s = -\frac{1}{k+r}\mathcal{M}[s^{r+1}(s^{-r}f(s))'], \tag{A.36}$$

so

$$s^{r+1}(s^{-r}f(s))' = -\mathcal{M}^{-1}[(k+r)\mathcal{M}[f]] \tag{A.37}$$

We define a (commutative) integral operator $\mathcal{I}_r$ with $r \geq 0$, such that

$$\mathcal{I}_r f(s) = s^r \int_s^\infty t^{-r-1} f(t)\ \mathrm{d}t. \tag{A.38}$$

We can invert Equation (A.37) as follows,

$$f(s) = \mathcal{I}_r \mathcal{M}^{-1}[(k+r)\mathcal{M}[f]]. \tag{A.39}$$

Iterating the process,

$$f(s) = \mathcal{I}_{r_1} \ldots \mathcal{I}_{r_n} \mathcal{M}^{-1}[(k+r_1)\cdots(k+r_n)\mathcal{M}[f]]. \tag{A.40}$$

Example 279. Assuming $\alpha, r > 0$, we have

$$\mathcal{I}_r \delta(s-\alpha) = s^r \int_s^\infty t^{-r-1}\delta(s-\alpha)\ \mathrm{d}t = s^r \alpha^{-r-1} \mathbb{1}_{s<\alpha}. \tag{A.41}$$

Example 280. Similarly, for $q \neq r$ non-negative,

$$\mathcal{I}_q \mathcal{I}_r \delta(s-\alpha) = s^q\, \alpha^{-r-1} \int_s^\infty t^{r-q-1} \mathbb{1}_{t<\alpha} \mathrm{d}t = \frac{s^q \alpha^{-q-1} - s^r \alpha^{-r-1}}{r-q} \mathbb{1}_{s<\alpha}. \tag{A.42}$$

Note that the result is only some linear combination of $\mathcal{I}_r\delta(s-\alpha)$ and $\mathcal{I}_q\delta(s-\alpha)$. This pattern is general and arises from the partial fraction decomposition. To see this, note that, by Equation (A.39),

$$\mathcal{M}[\mathcal{I}_r f] = \frac{\mathcal{M}[f]}{k+r}, \tag{A.43}$$

so

$$\mathcal{M}[\mathcal{I}_{r_1} \ldots \mathcal{I}_{r_n} f] = \frac{\mathcal{M}[f]}{(k+r_1)\cdots(k+r_n)} = \sum_{l=1}^n \frac{\beta_l}{k+r_l}\mathcal{M}[f] = \sum_{l=1}^n \beta_l \mathcal{M}[\mathcal{I}_{r_l} f], \tag{A.44}$$

where $\beta_l = 1/\prod_{j\neq l}(r_j - r_l)$. Hence, taking the inverse Mellin transform,

$$\mathcal{I}_{r_1} \ldots \mathcal{I}_{r_n} = \sum_{l=1}^n \beta_l \mathcal{I}_{r_l}. \tag{A.45}$$

Example 281. Since $\frac{1}{(1+k)(2+k)} = \frac{1}{1+k} - \frac{1}{2+k}$, we get $\mathcal{I}_1\mathcal{I}_2 = \mathcal{I}_1 - \mathcal{I}_2$ and therefore, by $\mathcal{I}_r\delta(s-\alpha) = s^r\alpha^{-r-1}\mathbb{1}_{s<\alpha}$ (Equation (A.41)),

$$\mathcal{I}_1\mathcal{I}_2\delta(s-\alpha) = s\alpha^{-3}(\alpha - s)\mathbb{1}_{s<\alpha}. \tag{A.46}$$

Since the operator $\mathcal{I}_r$ is continuous, we can take the limit $q \to r$ in Equation (A.42) to get $\mathcal{I}_r^2\delta(s-\alpha)$ for $r \geq 0$ and $\alpha > 0$. Alternatively, differentiating Equation (A.43) by r, we get $\mathcal{I}_r^2 = -\frac{\partial}{\partial r}\mathcal{I}_r$. Either way, we obtain

$$\mathcal{I}_r^2\delta(s-\alpha) = \frac{s^r}{\alpha^{r+1}}\ln\frac{\alpha}{s}\mathbb{1}_{s<\alpha}. \tag{A.47}$$

Example 282. Since $\frac{1}{k(1+k)^2(2+k)} = \frac{1}{2k} - \frac{1}{(1+k)^2} - \frac{1}{2(2+k)}$, we get $\mathcal{I}_0\mathcal{I}_1^2\mathcal{I}_2 = \frac{1}{2}\mathcal{I}_0 - \mathcal{I}_1^2 - \frac{1}{2}\mathcal{I}_2$ and thus, after some simplifications

$$\mathcal{I}_0\mathcal{I}_1^2\mathcal{I}_2\delta(s-\alpha) = \frac{\alpha^2 - s^2 - 2\alpha s\ln\frac{\alpha}{s}}{2\alpha^3}\mathbb{1}_{s<\alpha}. \tag{A.48}$$

Differentiating Equation (A.43) by r twice, we get $\mathcal{I}_r^3 = \frac{1}{2}\frac{\partial^2}{\partial r^2}\mathcal{I}_r$. Hence, from $\mathcal{I}_r\delta(s-\alpha) = s^r\alpha^{-r-1}\mathbb{1}_{s<\alpha}$ (Equation (A.41)), we obtain

$$\mathcal{I}_r^3\delta(s-\alpha) = \frac{s^r}{2\alpha^{r+1}}\ln^2\frac{\alpha}{s}\mathbb{1}_{s<\alpha}. \tag{A.49}$$

More generally, for any non-negative integer k, we have $\mathcal{I}_r^{k+1} = \frac{(-1)^k}{k!}\frac{\partial^k}{\partial r^k}\mathcal{I}_r$ so

$$\mathcal{I}_r^{k+1}\delta(s-\alpha) = \frac{1}{k!}\frac{s^r}{\alpha^{r+1}}\ln^k\frac{\alpha}{s}\mathbb{1}_{s<\alpha}. \tag{A.50}$$

Table A.5 shows selected products $\mathcal{I}_\Pi$ of integral operators $\mathcal{I}_r$, their decomposition into sum of individual operators (done via partial fractions decomposition) and their action on the Dirac kernel $\delta(s-\alpha)$, $\alpha > 0$.

$\mathcal{I}_\Pi$	decomposition	$\mathcal{I}_\Pi\delta(s-\alpha)$
$\mathcal{I}_1\mathcal{I}_2$	$\mathcal{I}_1 - \mathcal{I}_2$	$s\alpha^{-3}(\alpha-s)\mathbb{1}_{s<\alpha}$
$\mathcal{I}_1\mathcal{I}_3$	$\frac{1}{2}\mathcal{I}_1 - \frac{1}{2}\mathcal{I}_3$	$\frac{s}{2\alpha^4}(\alpha^2 - s^2)\mathbb{1}_{s<\alpha}$
$\mathcal{I}_0\mathcal{I}_1\mathcal{I}_2$	$\frac{1}{2}\mathcal{I}_0 - \mathcal{I}_1 + \frac{1}{2}\mathcal{I}_2$	$\frac{1}{2\alpha^3}(\alpha-s)^2\mathbb{1}_{s<\alpha}$
$\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3$	$\frac{1}{2}\mathcal{I}_1 - \mathcal{I}_2 + \frac{1}{2}\mathcal{I}_3$	$\frac{s}{2\alpha^4}(\alpha-s)^2\mathbb{1}_{s<\alpha}$
$\mathcal{I}_2\mathcal{I}_3\mathcal{I}_5$	$\frac{1}{3}\mathcal{I}_2 - \frac{1}{2}\mathcal{I}_3 + \frac{1}{6}\mathcal{I}_5$	$\frac{s^2}{6\alpha^6}(\alpha-s)^2(2\alpha+s)\mathbb{1}_{s<\alpha}$
$\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3\mathcal{I}_4\mathcal{I}_5$	$\frac{1}{24}\mathcal{I}_1 - \frac{1}{6}\mathcal{I}_2 + \frac{1}{4}\mathcal{I}_3 - \frac{1}{6}\mathcal{I}_4 + \frac{1}{24}\mathcal{I}_5$	$\frac{s}{24\alpha^6}(\alpha-s)^4\mathbb{1}_{s<\alpha}$
$\mathcal{I}_0\mathcal{I}_1^2\mathcal{I}_2$	$\frac{1}{2}\mathcal{I}_0 - \mathcal{I}_1^2 - \frac{1}{2}\mathcal{I}_2$	$\frac{1}{2\alpha^3}\left[\alpha^2 - s^2 - 2\alpha s\ln\frac{\alpha}{s}\right]\mathbb{1}_{s<\alpha}$
$\mathcal{I}_0\mathcal{I}_1^2\mathcal{I}_2^2$	$\frac{1}{4}\mathcal{I}_0 + \mathcal{I}_1 - \mathcal{I}_1^2 - \frac{5}{4}\mathcal{I}_2 - \frac{1}{2}\mathcal{I}_2^2$	$\frac{1}{4\alpha^3}\left[(\alpha-s)(\alpha+5s) - 2s(2\alpha+s)\ln\frac{\alpha}{s}\right]\mathbb{1}_{s<\alpha}$

Table A.5: Decomposition of the product of integral operators and their action on the Dirac kernel

Real random variables

Let us slightly generalise the method of reconstructing the PDFs from moments. Let S be a real random variable. Until now, we assumed that S is positive. However, we split any real random variable S into its positive and negative part and treat those cases separately by the technique introduced in the previous section. We have

$$S = S_+ - S_-, \tag{A.51}$$

where $S_+ = \max\{S, 0\}$ and $S_- = \max\{-S, 0\}$. For moments in general,

$$m_q = \mathbb{E}\left[S^q\right] = \mathbb{E}\left[S_+^q\right] + (-1)^q \mathbb{E}\left[S_-^q\right] = m_q^+ + e^{\pi i q} m_q^-. \tag{A.52}$$

Moreover, when $q = k$ is an integer, we get

$$m_k = m_k^+ + (-1)^k m_k^-. \tag{A.53}$$

We can split the PDF $f_S(s)$ of S as

$$f_S(s) = \begin{cases} f_S^+(s), & s \geq 0 \\ f_S^-(-s), & s < 0, \end{cases} \tag{A.54}$$

where $f_S^+(s)$ and $f_S^-(s)$ are now conditional, that is apart from a constant multiple) PDFs of *positive* random variables S_+ and S_-, respectively. Hence, by Mellin Inverse transform, we can reconstruct those functions as

$$f_S^+(s) = \mathcal{M}^{-1}[m_{k-1}^+](s), \qquad f_S^+(s) = \mathcal{M}^{-1}[m_{k-1}^-](s). \tag{A.55}$$

Example 283. Let us derive PDF $f_\Sigma(\sigma)$ of a random variable $\Sigma = (X - Y)X$, where $X, Y \sim \mathsf{Unif}(0, 1)$. First, for the moments of Σ for k a positive integer,

$$m_k = \mathbb{E}\left[\Sigma^k\right] = \int_0^1 \int_0^1 (x - y)^k x^k \, \mathrm{d}x\mathrm{d}y. \tag{A.56}$$

Integrating out y,

$$m_k = \int_0^1 \frac{x^{k+1} - (x-1)^{k+1}}{1+k} x^k \, \mathrm{d}x = \int_0^1 \frac{x^{2k+1}}{1+k} \, \mathrm{d}x + (-1)^k \int_0^1 \frac{x^k (1-x)^{k+1}}{1+k} \, \mathrm{d}x, \tag{A.57}$$

from which we immediately identify,

$$m_k^+ = \int_0^1 \frac{x^{2k+1} \, \mathrm{d}x}{1+k}, \qquad m_k^- = \int_0^1 \frac{x^k (1-x)^{k+1}}{1+k} \mathrm{d}x. \tag{A.58}$$

Taking the inverse Mellin Transform, we get for the positive part Σ_+,

$$f_\Sigma^+(\sigma) = \mathcal{I}_0 \mathcal{M}^{-1}\left[k m_{k-1}^+\right] = \mathcal{I}_0 \mathcal{M}^{-1}\left[\int_0^1 x^{2k-1} \, \mathrm{d}x\right] = \mathcal{I}_0 \int_0^1 x \, \delta(x^2 - \sigma) \, \mathrm{d}x \tag{A.59}$$

and for the negative part Σ_-, similarly,

$$f_\Sigma^-(\sigma) = \mathcal{I}_0 \mathcal{M}^{-1}\left[\int_0^1 x^k (1-x)^{k+1} \, \mathrm{d}x\right] = \mathcal{I}_0 \int_0^1 (1-x) \, \delta(x(1-x) - \sigma) \mathrm{d}x. \tag{A.60}$$

Equation (A.41) with $r = 0$ yields $\mathcal{I}_0\delta(\sigma - \alpha) = \frac{1}{\alpha}\mathbb{1}_{\sigma<\alpha}$, so

$$\begin{aligned} f_\Sigma^+(\sigma) &= \int_0^1 \frac{1}{x}\mathbb{1}_{\sigma<x^2}\ \mathrm{d}x = -\tfrac{1}{2}\ln\sigma\ \mathbb{1}_{\sigma<1} \\ f_\Sigma^-(\sigma) &= \int_0^1 \frac{1}{x}\mathbb{1}_{\sigma<x(1-x)}\mathrm{d}x = 2\,\mathrm{argtanh}(\sqrt{1-4\sigma})\ \mathbb{1}_{\sigma<1/4}. \end{aligned} \tag{A.61}$$

Overall, by Equation (A.54), we get for the full density of $\Sigma = \Sigma_+ - \Sigma_-$,

$$f_\Sigma(\sigma) = \begin{cases} 2\,\mathrm{argtanh}\sqrt{1+4\sigma}, & -\frac{1}{4} \le \sigma < 0 \\ -\frac{1}{2}\ln\sigma, & 0 < \sigma \le 1 \end{cases} \tag{A.62}$$

when $\sigma \in [-1/4, 1]$ and $f_\Sigma(\sigma) = 0$ otherwise. Integrating, we get the CDF of Σ,

$$F_\Sigma(\sigma) = \begin{cases} 0, & \sigma < -\frac{1}{4} \\ \frac{1}{2}\sqrt{1+4\sigma} + 2\sigma\,\mathrm{argtanh}\sqrt{1+4\sigma}, & -\frac{1}{4} \le \sigma < 0 \\ \frac{1}{2}(1+\sigma-\sigma\ln\sigma), & 0 \le \sigma < 1, \\ 1, & \sigma \ge 1. \end{cases} \tag{A.63}$$

Example 284. Let us derive PDF $f_\Lambda(\lambda)$ of a random variable $\Lambda \sim \mathsf{Lang}$. By definition, we can write $\Lambda = (Y - X)(Z - X)$, where $X, Y, Z \sim \mathsf{Unif}(0, 1)$. Let us calculate the moments of Λ for k being a positive integer. We have

$$m_k = \mathbb{E}[\Lambda^k] = \int_0^1\int_0^1\int_0^1 (y-x)^k(z-x)^k\ \mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{A.64}$$

Integrating out y and z and by symmetry,

$$m_k = \int_0^1\left(\frac{(1-x)^{k+1}-(-x)^{k+1}}{1+k}\right)^2\mathrm{d}x = \int_0^1\frac{2x^{2k+2}\ \mathrm{d}x}{(1+k)^2} + (-1)^k\int_0^1\frac{2(x(1-x))^{k+1}}{(1+k)^2}\mathrm{d}x, \tag{A.65}$$

from which we immediately identify,

$$m_k^+ = \int_0^1\frac{2x^{2k+2}\ \mathrm{d}x}{(1+k)^2}, \qquad m_k^- = \int_0^1\frac{2(x(1-x))^{k+1}}{(1+k)^2}\mathrm{d}x. \tag{A.66}$$

Taking the inverse Mellin Transform, we get for the positive part Λ_+,

$$f_\Lambda^+(\lambda) = 2\mathcal{I}_0^2\mathcal{M}^{-1}\left[k^2 m_{k-1}^+\right] = 2\mathcal{I}_0^2\mathcal{M}^{-1}\left[\int_0^1 x^{2k}\mathrm{d}x\right] = 2\mathcal{I}_0^2\int_0^1 x^2\delta(x^2-\lambda)\ \mathrm{d}x \tag{A.67}$$

and for the negative part Λ_-, similarly,

$$f_\Lambda^-(\lambda) = 2\mathcal{I}_0^2\mathcal{M}^{-1}\left[\int_0^1 x^k(1-x)^k\ \mathrm{d}x\right] = 2\mathcal{I}_0^2\int_0^1 x(1-x)\,\delta(x(1-x)-\lambda)\mathrm{d}x. \tag{A.68}$$

Equation (A.43) with $r = 0$ yields $\mathcal{I}_0^2\delta(\lambda - \alpha) = \frac{1}{\alpha}\ln\frac{\alpha}{\lambda}\mathbb{1}_{\lambda<\alpha}$, so

$$\begin{aligned} f_\Lambda^+(\lambda) &= 2\int_0^1 \ln\frac{x^2}{\lambda}\mathbb{1}_{\lambda<x^2}\ \mathrm{d}x = \left[4\sqrt{\lambda} - 2\ln\lambda - 4\right]\mathbb{1}_{\lambda<1} \\ f_\Lambda^-(\lambda) &= 2\int_0^1 \ln\frac{x(1-x)}{\lambda}\mathbb{1}_{\lambda<x(1-x)}\mathrm{d}x = 4\left[\mathrm{argtanh}\sqrt{1-4\lambda} - \sqrt{1-4\lambda}\right]\mathbb{1}_{\lambda<1/4}. \end{aligned} \tag{A.69}$$

Overall, by Equation (A.54), we get for the full density of $\Lambda = \Lambda_+ - \Lambda_-$,

$$f_\Lambda(\lambda) = \begin{cases} 4\,\mathrm{argtanh}\sqrt{1+4\lambda} - 4\sqrt{1+4\lambda}, & -\frac{1}{4} \le \lambda < 0 \\ 4\sqrt{\lambda} - 2\ln\lambda - 4, & 0 < \lambda \le 1 \end{cases} \tag{A.70}$$

when $\lambda \in [-1/4, 1]$ and $f_\Lambda(\lambda) = 0$ otherwise. Integrating, we get the CDF of Λ.

B Integral calculus on real affine subspaces

First, we shall discuss the common techniques of multidimensional integration. The notation used in this section is borrowed from the textbook *Lectures on convex geometry* by Hug and Weil [37]. Once again, let us recall some basic facts and definitions.

Definition 285 ($\mathbb{S}^{d-1}, \omega_d$)**.** Let $\mathbb{S}^{d-1}$ be a unit sphere in $\mathbb{R}^d$ with the usual surface area measure $\sigma_d(\cdot)$. That is, for the surface area of $\mathbb{S}^{d-1}$, we get

$$\omega_d = \int_{\mathbb{S}^{d-1}} \sigma_d(\mathrm{d}\mathbf{u}) = \sigma_d(\mathbb{S}^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{B.71}$$

as shown in the proof of Lemma 264. Also note that we can decompose $\mathbf{x} = r\mathbf{u}$, where $\mathbf{u} \in \mathbb{S}^{d-1}$ and $r \in (0, \infty)$, the usual Lebesgue measure λ_d splits into radial and angular part as $\lambda_d(\mathrm{d}\mathbf{x}) = r^{d-1}\mathrm{d}r\sigma_d(\mathrm{d}\mathbf{u})$.

Definition 286 ($\mathbb{B}_d, \kappa_d$)**.** We write $\mathbb{B}_d \subset \mathbb{R}^d$ for the unit ball (with unit radius) and κ_d for its volume. Splitting the Lebesgue measure into radial and angular part,

$$\kappa_d = \mathrm{vol}_d\, \mathbb{B}_d = \int_{\mathbb{B}_d} \lambda_d(\mathrm{d}\mathbf{x}) = \omega_d \int_0^1 r^{d-1}\ \mathrm{d}r = \omega_d/d = \frac{\pi^{d/2}}{\Gamma(\frac{d+2}{2})}. \tag{B.72}$$

Definition 287. We denote $\mathbb{G}(d,p)$ as the set of all linear p-dimensional subspaces of $\mathbb{R}^d$, this set is often called the (linear) Grassmannian. More generally, we denote $\mathbb{A}(d,p)$ as the set of all p-dimensional affine subspaces of $\mathbb{R}^d$ (p-planes), this set is called the affine Grasmannian.

Remark 288. Both spaces $\mathbb{G}(d,p)$ and $\mathbb{A}(d,p)$ smooth finite-dimensional manifolds. More concretely, we have $\dim \mathbb{G}(d,p) = (d-p)p$ and $\dim \mathbb{A}(d,p) = (d-p)(p+1)$.

Definition 289. Let $K_d \subset \mathbb{R}^d$. We define $\mathbb{G}_{K_d}(d,p) = \{\boldsymbol{\gamma} \in \mathbb{G}(d,p) \,|\, \boldsymbol{\gamma} \cap K_d \neq \emptyset\}$ and analogously, $\mathbb{A}_{K_d}(d,p) = \{\boldsymbol{\sigma} \in \mathbb{A}(d,p) \,|\, \boldsymbol{\sigma} \cap K_d \neq \emptyset\}$.

Definition 290. Let ν_p be the probability Haar measure on $\mathbb{G}(d,p)$. That is, ν_p is invariant under action of the group of proper rigid sphere transformations $\mathcal{SO}(n)$ and $\nu_p(\mathbb{G}(d,p)) = 1$.

Definition 291. We define the standard Haar measure μ_p on $\mathbb{A}(d,p)$ by

$$\mu_p(\cdot) = \int_{\mathbb{G}(d,p)} \int_{\boldsymbol{\gamma}_\perp} \mathbb{1}\{\boldsymbol{\gamma} + \mathbf{y} \in \cdot\}\lambda_{d-p}(\mathrm{d}\mathbf{y})\nu_p(\mathrm{d}\boldsymbol{\gamma}), \tag{B.73}$$

where $\boldsymbol{\gamma}_\perp \in \mathbb{G}(d, d-p)$ is the linear space orthogonal to $\boldsymbol{\gamma}$. That is, $\boldsymbol{\gamma}_\perp \oplus \boldsymbol{\gamma} = \mathbb{R}^d$.

Lemma 292. $\mu_p(\mathbb{A}_{\mathbb{B}_d}(d,p)) = \kappa_{d-p} = \omega_{d-p}/(d-p)$.

Proof. By symmetry, we have for any $\gamma_0 \in \mathbb{G}(d,p)$,

$$\mu_p(\mathbb{A}_{\mathbb{B}_d}(d,p)) = \int_{\gamma_\perp} \mathbb{1}\{\boldsymbol{\gamma} + \mathbf{y} \in \mathbb{A}_{\mathbb{B}_d}(d,p)\}\lambda_{d-p}(\mathrm{d}\mathbf{y}) = \mathrm{vol}_{d-p}(\mathbb{B}_d \cap \boldsymbol{\gamma}_\perp) = \kappa_{d-p}. \tag{B.74}$$

■

B.1 Cartesian parametrisation

In the case of $p = d-1$, the affine Grassmannian $\mathbb{A}(d, d-1)$ consists of hyperplanes of dimension $d-1$. Note that $\dim \mathbb{A}(d, d-1) = d$ so in order to parametrise the space of all affine planes, we need exactly d parameters. One choice of those parameters are the coordinates of the closest point to a given hyperplane, we write $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)^\top$ to be the vector from the origin to the closest point on the hyperplane $\boldsymbol{\sigma}$. Another choice of parametrisation is by using spherical inversion of $\boldsymbol{\xi}$. Namely,

$$\boldsymbol{\eta} = \frac{\boldsymbol{\xi}}{\boldsymbol{\xi}^\top \boldsymbol{\xi}} = \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}. \tag{B.75}$$

so $\|\boldsymbol{\xi}\| = 1/\|\boldsymbol{\eta}\|$. There is a nice interpretation of $\boldsymbol{\eta}$. Namely, a plane $\boldsymbol{\sigma}$ defined uniquely by the vector $\boldsymbol{\eta}$ has a nonempty intersection with convex body $K_d \subset \mathbb{R}^d$ if and only if $\boldsymbol{\eta}$ does not lie in the *polar body* K_d° defined as

$$K_d^\circ = \{\mathbf{x} \in \mathbb{R}^d \,|\, \mathbf{x}^\top \mathbf{y} \leq 1, \mathbf{y} \in K_d\}. \tag{B.76}$$

This follows from the fact that the points $\mathbf{x}$ on the hyperplane $\boldsymbol{\sigma} \in \mathbb{A}(d, d-1)$ satisfy $\boldsymbol{\eta}^\top \mathbf{x} = 1$. Therefore, we must remember that

$$\boldsymbol{\sigma} \cap K_d \neq \emptyset \iff \boldsymbol{\eta} \in \mathbb{R}^d \setminus K_d^\circ. \tag{B.77}$$

Finally, the following lemma gives us then the Jacobian of transformation between the standard Haar measure on a Grasmannian of hyperplanes and the Lebesgue measure of the closest point intercepts:

Lemma 293. *Let $\boldsymbol{\sigma} \in \mathbb{A}(d, d-1)$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_d)^\top$ be the plane vector associated to $\boldsymbol{\sigma}$ such that $\mathbf{x} \in \boldsymbol{\sigma} \Leftrightarrow \boldsymbol{\eta}^\top \mathbf{x} = 1$, then*

$$\mu_{d-1}(\mathrm{d}\boldsymbol{\sigma}) = \frac{2}{\omega_d} \frac{1}{\|\boldsymbol{\eta}\|^{1+d}} \lambda_d(\mathrm{d}\boldsymbol{\eta}). \tag{B.78}$$

Proof. First, we show that our measure on the right hand side is invariant with respect to action of the group $\mathcal{G}(d)$ of all proper rigid motions in $\mathbb{R}^d$. We may view any $g(M, \mathbf{b}) \in \mathcal{G}(d)$ by its corresponding action on points $\mathbf{x} \in \mathbb{R}^d$. That is,

$$\mathbf{x}' = g(M, \mathbf{b}) \circ \mathbf{x} = M\mathbf{x} + \mathbf{b} \tag{B.79}$$

where $\mathbf{b}$ is a translation vector and the matrix M corresponds to (proper) rotations, hence M satisfies

$$\det M = 1 \text{ and } M^\top M = M M^\top = I_d, \tag{B.80}$$

where I_d is the $d \times d$ identity matrix. Let us find $\boldsymbol{\sigma}' = g^{-1}(M, \mathbf{b}) \circ (\boldsymbol{\sigma})$ onto which $\boldsymbol{\sigma}$ is mapped by applying $g^{-1}(M, \mathbf{b})$. Its associated plane vector $\boldsymbol{\eta}'$ must satisfy $\boldsymbol{\eta}'^\top \mathbf{x}' = 1$. By multiplying Equation (B.79) by $M^\top$ from the left, we obtain $M^\top \mathbf{x}' = \mathbf{x} + M^\top \mathbf{b}$. Further multiplying by $\boldsymbol{\eta}^\top$ from the left, we get $\boldsymbol{\eta}^\top M^\top \mathbf{x}' = 1 + \boldsymbol{\eta}^\top M^\top \mathbf{b}$, from which we identify

$$\boldsymbol{\eta}' = g^{-1}(M, \mathbf{b}) \circ \boldsymbol{\eta} = \frac{M\boldsymbol{\eta}}{1 + \mathbf{b}^\top M \boldsymbol{\eta}}. \tag{B.81}$$

For the norm, we have by using $M^\top M = I_d$,

$$\|\boldsymbol{\eta}'\| = \frac{\|\boldsymbol{\eta}\|}{|1 + \mathbf{b}^\top M \boldsymbol{\eta}|}. \tag{B.82}$$

Let us calculate the Jacobian of transformation from $\boldsymbol{\eta}'$ to $\boldsymbol{\eta}$. We have

$$\frac{\partial \boldsymbol{\eta}'}{\partial \boldsymbol{\eta}} = \frac{M(1 + \mathbf{b}^\top M\boldsymbol{\eta}) - M\boldsymbol{\eta}\mathbf{b}^\top M}{(1 + \mathbf{b}^\top M\boldsymbol{\eta})^2} = M\frac{I_d - \frac{\boldsymbol{\eta}\mathbf{b}^\top M}{1+\mathbf{b}^\top M\boldsymbol{\eta}}}{1 + \mathbf{b}^\top M\boldsymbol{\eta}}, \tag{B.83}$$

By Matrix Determinant Lemma 119,

$$\det\left(\frac{\partial \boldsymbol{\eta}'}{\partial \boldsymbol{\eta}}\right) = \frac{\det M}{(1 + \mathbf{b}^\top M\boldsymbol{\eta})^d}\left(1 - \frac{\mathbf{b}^\top M\boldsymbol{\eta}}{1 + \mathbf{b}^\top M\boldsymbol{\eta}}\right) = \frac{\det M}{(1 + \mathbf{b}^\top M\boldsymbol{\eta})^{1+d}}. \tag{B.84}$$

In total,

$$\frac{1}{\|\boldsymbol{\eta}'\|^{1+d}}\lambda_d(\mathrm{d}\boldsymbol{\eta}') = \frac{|1+\mathbf{b}^\top M\boldsymbol{\eta}|^{1+d}}{\|\boldsymbol{\eta}\|^{1+d}}\frac{\det M}{|1+\mathbf{b}^\top M\boldsymbol{\eta}|^{1+d}}\lambda_d(\mathrm{d}\boldsymbol{\eta}) = \frac{1}{\|\boldsymbol{\eta}\|^{1+d}}\lambda_d(\mathrm{d}\boldsymbol{\eta}) \tag{B.85}$$

for any M and $\mathbf{b}$. Therefore, $\|\boldsymbol{\eta}\|^{-1-d}\lambda_d(\mathrm{d}\boldsymbol{\eta})$ is a Haar measure on $\mathbb{A}(d, d-1)$ and as such, it must differ from $\mu_{d-1}(\mathrm{d}\boldsymbol{\sigma})$ by a constant multiple [37, Theorem 5.4], say

$$\mu_{d-1}(\mathrm{d}\boldsymbol{\sigma}) = \frac{c}{\|\boldsymbol{\eta}\|^{1+d}}\lambda_d(\mathrm{d}\boldsymbol{\eta}) \tag{B.86}$$

for some c. To check this constant is indeed $c = 2/\omega_d$, let us calculate the μ_{d-1} measure over planes which pass trough $\mathbb{B}_d$ (the unit ball with radius one). On one hand, by definition, we already know that $\mu_{d-1}(\mathbb{A}_{\mathbb{B}_d}(d, d-1)) = \omega_1 = 2$. On the other, let us characterise the condition under which a $(d-1)$ hyperplane $\boldsymbol{\sigma}$ intercepts $\mathbb{B}_d$. This happens exactly when the closest point on $\boldsymbol{\sigma}$ lies inside of $\mathbb{B}_d$. That is, $\|\boldsymbol{\xi}\| < 1$, or equivalently $\|\boldsymbol{\eta}\| > 1$. Hence, by using spherical coordinates and symmetry, $\lambda_d(\mathrm{d}\boldsymbol{\eta}) = \omega_d r^{d-1}\,\mathrm{d}r$, where $r = \|\boldsymbol{\eta}\|$, and therefore

$$\mu_{d-1}(\mathbb{A}_{\mathbb{B}_d}(d, d-1)) = \int_{\mathbb{R}^d \setminus \mathbb{B}_d}\frac{c}{\|\boldsymbol{\eta}\|^{1+d}}\lambda_d(\mathrm{d}\boldsymbol{\eta}) = \omega_d\int_1^\infty c\frac{r^{d-1}}{r^{1+d}}\mathrm{d}r = \omega_d c, \tag{B.87}$$

so $c = 2/\omega_d$ indeed. ■

Remark 294. Simple calculation of Jacobian of transformation between $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ (only the radial part is affected) reveals that

$$\mu_{d-1}(\mathrm{d}\boldsymbol{\sigma}) = \frac{2}{\omega_d}\|\boldsymbol{\xi}\|^{1-d}\lambda_d(\mathrm{d}\boldsymbol{\xi}). \tag{B.88}$$

B.2 Section integral

In this section, we show an important result of integral geometry, namely the *section integral*. Let $\mathbb{x} = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{d-1})$ be a collection of d points in K_d, let $\Delta_{d-1} = \mathrm{vol}_{d-1} \operatorname{conv} \mathbb{x}$ be the $(d-1)$-volume of their convex hull and $\boldsymbol{\sigma} = \mathcal{A}(\mathbb{x})$ be a section plane passing through those points. Section plane $\boldsymbol{\sigma}$ is parametrised by $\boldsymbol{\eta} \in \mathbb{R}^d$ such that for any $\mathbf{x} \in \boldsymbol{\sigma}$, we have $\boldsymbol{\eta}^\top \mathbf{x} = 1$. We wish to integrate an integrable functional $f : K_d^d \to \mathbb{R}$ over all points in K_d. Normally, this would be a d^2-dimensional integral (d degrees of freedom for each point). However, often $f(\mathbb{x})$ posseses some simple form (a function of Δ_{d-1}, $\boldsymbol{\sigma}$ only or combination of both). In that case, this multidimensional integral can be drastically simplified. The overall idea is simple: Instead of integrating over individual points in a collection, we may first fix the plane on which the points lie and then integrate over all planes. See an illustration in Figure B.1 below.

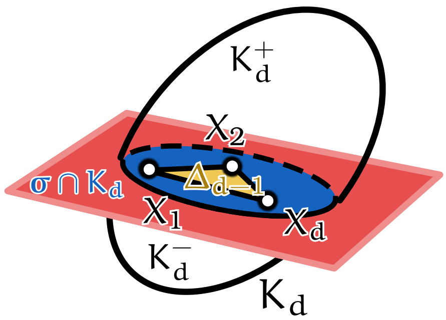


Figure B.1: The section integral replaces integration over space by integration over sections planes diving K_d into $K_d^- \sqcup K_d^+$

The only remaining question is then to correctly write down the Jacobian of this transformation. This leads to the *section integral* formula below

$$\int_{K_d^d} f(\mathbb{x}) \lambda_d^d(\mathrm{d}\mathbb{x}) = (d-1)! \int_{\mathbb{R}^d \setminus K_d^\circ} \int_{(\boldsymbol{\sigma} \cap K_d)^d} f(\mathbb{x}) \Delta_{d-1} \|\boldsymbol{\eta}\|^{-1-d} \lambda_{d-1}^d(\mathrm{d}\mathbb{x}) \lambda_d(\mathrm{d}\boldsymbol{\eta}).$$

B.3 Blaschke-Petkantschin formula

Apart from the already discussed Cartesian parametrisation, the section integral is a direct consequence of the famous Blaschke-Petkantschin formula which enables us to reparametrise an integral over set of points $\mathbb{x} = (\mathbf{x}_0, \ldots, \mathbf{x}_p)$ for any $p \leq d$ as an integral over q-planes in $\mathbb{A}(d, q)$ for any $q \geq p$, on which these points lie.

Lemma 295 (Blaschke-Petkantschin formula). *Let* $f : (\mathbb{R}^d)^{p+1} \to \mathbb{R}$ *be a Lebesgue integrable function of a collection* $\mathbb{x} = (\mathbf{x}_0, \dots, \mathbf{x}_p)$ *of points* $\mathbf{x}_j \in \mathbb{R}^d$, $j = 0, \dots, p$. *Denote* $\mathbb{H}_p = \operatorname{conv}(\mathbb{x})$ *and* $\Delta_p = \operatorname{vol}_p \mathbb{H}_p$, *then for any integer* q *such that* $0 \leq p \leq q \leq d$,

$$\int_{(\mathbb{R}^d)^{p+1}} f(\mathbb{x}) \lambda_d^{p+1}(\mathrm{d}\mathbb{x}) = \beta_{dqp} \int_{\mathbb{A}(d,q)} \int_{\boldsymbol{\sigma}^{p+1}} f(\mathbb{x}) \Delta_p^{d-q} \lambda_q^{p+1}(\mathrm{d}\mathbb{x}) \mu_q(\mathrm{d}\boldsymbol{\sigma}), \quad \text{(B.89)}$$

where

$$\beta_{dqp} = (p!)^{d-q} \pi^{\frac{1}{2}p(d-q)} \prod_{j=0}^{p-1} \frac{\Gamma\left(\frac{q-j}{2}\right)}{\Gamma\left(\frac{d-j}{2}\right)}, \quad \text{(B.90)}$$

$\lambda_d^{p+1}(\mathrm{d}\mathbb{x}) = \prod_{j=0}^{p} \lambda_d(\mathrm{d}\mathbf{x}_j)$ *and* $\lambda_q^{p+1}(\mathrm{d}\mathbb{x}) = \prod_{j=0}^{p} \lambda_q(\mathrm{d}\mathbf{x}_j)$ *are the Lebesgue measures on* $(\mathbb{R}^d)^{p+1}$ *and* $\boldsymbol{\sigma}^{p+1}$, *respectively.*

Proof. See Rubin [63] for an elementary proof. ∎

Remark 296. Denote $\gamma_d = \int_0^\infty r^{d-1} e^{-r^2/2}\, \mathrm{d}r = 2^{\frac{d}{2}-1}\Gamma(\frac{d}{2})$ as before. We have $\omega_d \gamma_d = \sqrt{2\pi}^d$ (see Equation (A.1)). We can express β_{dqp} in terms of γ's and ω's as follows:

$$\beta_{dqp} = (p!)^{d-q} \sqrt{2\pi}^{p(d-q)} \prod_{j=0}^{p-1} \frac{\gamma_{q-j}}{\gamma_{d-j}} = (p!)^{d-q} \prod_{j=0}^{p-1} \frac{\omega_{d-j}}{\omega_{q-j}}. \quad \text{(B.91)}$$

The statement of the Blaschke-Petkantschin formula is way too general for our purposes. We will only need its special cases. First, often we assume that the affine plane on which the points lie is exactly their affine hull almost surely. This corresponds to the case $p = q$. Another special case is got by restricting the domain of integration using the following choice of f: Let $K_d \subset \mathbb{R}^d$ be a compact convex body with $\dim K_d = d$ and let $f(\mathbb{x}) = \tilde{f}(\mathbb{x}) \prod_{0 \leq i \leq k} \mathbb{1}_{K_d}(\mathbf{x}_i)$ for some $\tilde{f} : K_d^{p+1} \to \mathbb{R}$ suitably integrable, then

$$\int_{K_d^{p+1}} \tilde{f}(\mathbb{x}) \lambda_d^{p+1}(\mathrm{d}\mathbb{x}) = \beta_{dqp} \int_{\mathbb{A}_{K_d}(d,q)} \int_{(K_d \cap \boldsymbol{\sigma})^{p+1}} \tilde{f}(\mathbb{x}) \Delta_p^{d-q} \lambda_q^{p+1}(\mathrm{d}\mathbb{x}) \mu_q(\mathrm{d}\boldsymbol{\sigma}). \quad \text{(B.92)}$$

In this thesis, mostly we use the special case with $\tilde{f}(\mathbb{x}) = g(\mathcal{A}(\mathbb{x}))\Delta_p^k$, where $g(\cdot)$ is a function of the cutting plane $\boldsymbol{\sigma} = \mathcal{A}(\mathbb{x}) \in \mathbb{A}(d,q)$ only. In this case, the Blaschke-Petkantschin formula restricted on K_d as in Equation (B.92) becomes, using definition of $v_p^{(n)}(\cdot)$ and denoting $\boldsymbol{\sigma}_{K_d} = K_d \cap \boldsymbol{\sigma}$ ($\dim \boldsymbol{\sigma}_{K_d} = q$ almost surely),

$$\int_{K_d^{p+1}} g(\boldsymbol{\sigma}) \Delta_p^k \lambda_d^{p+1}(\mathrm{d}\mathbb{x}) = \beta_{dqp} \int_{\mathbb{A}_{K_d}(d,q)} v_p^{(d-q+k)}(\boldsymbol{\sigma}_{K_d}) (\operatorname{vol}_q \boldsymbol{\sigma}_{K_d})^{1+(d+k)\frac{p}{q}} g(\boldsymbol{\sigma}) \mu_q(\mathrm{d}\boldsymbol{\sigma}). \quad \text{(B.93)}$$

This still very general relation can be further reformulated in terms of expected values. Let us select the collection $\mathbb{X} = (\mathbf{X}_0, \mathbf{X}_1, \dots, \mathbf{X}_p)$ of $(p+1)$ random points $\mathbf{X}_i$ independently from the same distribution $\mathsf{Unif}(K_d)$. Then

Corollary 296.1. *With respect to the uniform probability measure* $\mathsf{Unif}(K_d)$,

$$\mathbb{E}\left[g(\boldsymbol{\sigma})\Delta_p^k\right] = \frac{\beta_{dqp}}{(\operatorname{vol}_d K_d)^{p+1}} \int_{\mathbb{A}_{K_d}(d,q)} v_p^{(d-q+k)}(\boldsymbol{\sigma}_{K_d}) (\operatorname{vol}_q \boldsymbol{\sigma}_{K_d})^{1+(d+k)\frac{p}{q}} g(\boldsymbol{\sigma}) \mu_q(\mathrm{d}\boldsymbol{\sigma}). \quad \text{(B.94)}$$

If moreover $q = d - 1$, for which, by using Remark 296,

$$\beta_{d(d-1)p} = p! \prod_{j=0}^{p-1} \frac{\omega_{d-j}}{\omega_{d-1-j}} = \frac{p!\omega_d}{\omega_{d-p}}, \tag{B.95}$$

we may use the Cartesian parametrisation $\mathbf{x} \in \boldsymbol{\sigma} \Leftrightarrow \boldsymbol{\eta}^\top \mathbf{x} = 1$, to get

$$\mathbb{E}\left[g(\boldsymbol{\sigma})\Delta_p^k\right] = \frac{2\,p!}{\omega_{d-p}(\mathrm{vol}_d\, K_d)^{p+1}} \int_{\mathbb{R}^d \setminus K_d^\circ} v_p^{(k+1)}(\boldsymbol{\sigma}_{K_d})(\mathrm{vol}_{d-1}\,\boldsymbol{\sigma}_{K_d})^{1+\frac{(d+k)p}{d-1}} g(\boldsymbol{\sigma}) \frac{\lambda_d(\mathrm{d}\boldsymbol{\eta})}{\|\boldsymbol{\eta}\|^{1+d}}, \tag{B.96}$$

where $\boldsymbol{\sigma}$ is now a function of $\boldsymbol{\eta}$. If moreover $p = q = d - 1$, we get

$$\mathbb{E}\left[g(\boldsymbol{\sigma})\Delta_{d-1}^k\right] = \frac{(d-1)!}{(\mathrm{vol}_d\, K_d)^d} \int_{\mathbb{R}^d \setminus K_d^\circ} v_{d-1}^{(k+1)}(\boldsymbol{\sigma}_{K_d})(\mathrm{vol}_{d-1}\,\boldsymbol{\sigma}_{K_d})^{d+k+1} g(\boldsymbol{\sigma}) \frac{\lambda_d(\mathrm{d}\boldsymbol{\eta})}{\|\boldsymbol{\eta}\|^{1+d}}. \tag{B.97}$$

We may write this relation in the form of the following corollary

Corollary 296.2. *With respect to the uniform probability measure* $\mathsf{Unif}(K_d)$*,*

$$\mathbb{E}\left[g(\boldsymbol{\sigma})\Delta_{d-1}^k\right] = (d-1)!(\mathrm{vol}_d\, K_d)^{k+1} \int_{\mathbb{R}^d \setminus K_d^\circ} v_{d-1}^{(k+1)}(\boldsymbol{\sigma}_{K_d}) \zeta_d^{d+k+1}(\boldsymbol{\sigma}) g(\boldsymbol{\sigma}) \|\boldsymbol{\eta}\|^k \lambda_d(\mathrm{d}\boldsymbol{\eta})$$

where we defined the zeta section function

$$\zeta_d(\boldsymbol{\sigma}) = \frac{\mathrm{vol}_{d-1}(\boldsymbol{\sigma} \cap K_d)}{\|\boldsymbol{\eta}\|\,\mathrm{vol}_d\, K_d} \tag{B.98}$$

Especially, for $d = 3$ and $p = q = d - 1 = 2$, we get

$$\int_{K_3^3} g(\boldsymbol{\sigma})\Delta_2^k \lambda_3^3(\mathrm{d}\mathbf{x}) = 2\int_{\mathbb{R}^d \setminus K_3^\circ} v_2^{(k+1)}(\boldsymbol{\sigma}_{K_3})(\mathrm{vol}_2\,\boldsymbol{\sigma}_{K_3})^{k+4} g(\boldsymbol{\sigma}) \|\boldsymbol{\eta}\|^{-4} \lambda_3(\mathrm{d}\boldsymbol{\eta}). \tag{B.99}$$

Remark 297. We show that $\|\boldsymbol{\eta}\|$ always cancels out in $\zeta_d(\boldsymbol{\sigma})$. First, note that $\boldsymbol{\sigma}$ always separates K_d into disjoint union $K_d^+ \sqcup K_d^-$, where

$$K_d^+ = \{\mathbf{x} \in K_d \,|\, \boldsymbol{\eta}^\top \mathbf{x} < 1\}, \qquad K_d^- = \{\mathbf{x} \in K_d \,|\, \boldsymbol{\eta}^\top \mathbf{x} > 1\}. \tag{B.100}$$

From homogeneity of d-volume, $\frac{\mathrm{vol}_d(\boldsymbol{\sigma} \cap K_d)}{\|\boldsymbol{\eta}\|} = -\sum_{j=1}^d \eta_j \frac{\partial\,\mathrm{vol}_d\, K_d^+}{\partial \eta_j}$ and thus

$$\zeta_d(\boldsymbol{\sigma}) = -\frac{1}{\mathrm{vol}_d\, K_d} \sum_{j=1}^d \eta_j \frac{\partial\,\mathrm{vol}_d\, K_d^+}{\partial \eta_j} = \frac{1}{\mathrm{vol}_d\, K_d} \sum_{j=1}^d \eta_j \frac{\partial\,\mathrm{vol}_d\, K_d^-}{\partial \eta_j} \tag{B.101}$$

does not depend on $\|\boldsymbol{\eta}\|$.

B.4 Spherical parametrization

Lastly, let $\boldsymbol{\sigma} \in \mathbb{A}(d, 1)$ be a line in $\mathbb{R}^d$. Any $\boldsymbol{\sigma}$ can be decomposed as

$$\boldsymbol{\sigma} = \boldsymbol{\gamma} + \mathbf{y}, \tag{B.102}$$

where $\boldsymbol{\gamma} \in \mathbb{G}(d, 1)$ (lines passing through the origin) and $\mathbf{y} \in \boldsymbol{\gamma}_\perp$. Note that $\mathbb{G}(d, 1)$ is isomorphic to a *half-sphere* $\mathbb{S}_+^{d-1}$, since any $\boldsymbol{\gamma}$ can be associated with its *unit tangent vector* $\hat{\mathbf{n}}$ as

$$\boldsymbol{\gamma} = \{t\hat{\mathbf{n}} \mid t \in \mathbb{R}, \hat{\mathbf{n}} \in \mathbb{S}^{d-1}\}. \tag{B.103}$$

This isomorphism is a bijection as long as $\hat{\mathbf{n}}$ is taken from half-spheres, since for a given $\boldsymbol{\gamma} \in \mathbb{G}(d,1)$, both $\pm\hat{\mathbf{n}}$ would be its possible unit tangent vectors. By the definition of the invariant measures, we may write the *(half-)spherical parametrization* of $\mu_1(\mathrm{d}\boldsymbol{\sigma})$ as

$$\mu_1(\mathrm{d}\boldsymbol{\sigma}) = \frac{2}{\omega_d}\lambda_{d-1}(\mathrm{d}\mathbf{y})\sigma_d(\mathrm{d}\hat{\mathbf{n}}). \tag{B.104}$$

Moreover, it is convenient to denote $\hat{\mathbf{n}}_\perp = \boldsymbol{\gamma}_\perp$ (hyperplane perpendicular to $\hat{\mathbf{n}}$), so $\mathbf{y} \in \hat{\mathbf{n}}_\perp$, where $\hat{\mathbf{n}} \in \mathbb{S}_+^{d-1}$.

Blaschke-Petkantschin formula

Inserting the relation between measures into Blaschke-Petkantschin formula,

$$\int_{K_d^2} f(\mathbb{x})\lambda_d^2(\mathrm{d}\mathbb{x}) = \int_{\mathbb{S}_+^{d-1}}\int_{\hat{\mathbf{n}}_\perp}\int_{(\sigma\cap K_d)^2} f(\mathbb{x})\Delta_1^{d-1}\lambda_1^2(\mathrm{d}\mathbb{x})\lambda_{d-1}(\mathrm{d}y)\sigma_d(\mathrm{d}\hat{\mathbf{n}}),$$

where $\mathbb{x} = (\mathbf{x}_0, \mathbf{x}_1)$ and $\lambda_d^2(\mathrm{d}\mathbb{x}) = \mathrm{d}\mathbf{x}_0\mathrm{d}\mathbf{x}_1$. For arguments in the special form $f(\mathbb{x}) = g(\boldsymbol{\sigma})\Delta_1^k$, we may integrate out $\mathbf{x}_0, \mathbf{x}_1$ to obtain

Proposition 298. *Let $K_d \subset \mathbb{R}^d$ be a convex d-body and $\mathbf{X}_0, \mathbf{X}_1 \sim \mathsf{Unif}(K_d)$ be random points uniformly and independently selected from K_d. Denote, as usual, $\Delta_1 = \|\mathbf{X}_1 - \mathbf{X}_0\|$ the (random) distance between them and $\boldsymbol{\sigma} = \mathcal{A}(\mathbf{X}_0\mathbf{X}_1)$ the line passing through them. Then, for any integrable function $g(\boldsymbol{\sigma})$, any $d \geq 1$ and any (real) $k > -d$,*

$$\mathbb{E}\left[g(\boldsymbol{\sigma})\Delta_1^k\right] = \frac{2/(\mathrm{vol}_d\, K_d)^2}{(d+k)(d+k+1)}\int_{\mathbb{S}_+^{d-1}}\int_{\hat{\mathbf{n}}_\perp} g(\boldsymbol{\sigma})\,\mathrm{vol}_1(\boldsymbol{\sigma}\cap K_d)^{d+k+1}\lambda_{d-1}(\mathrm{d}\mathbf{y})\sigma_d(\mathrm{d}\hat{\mathbf{n}}) \tag{B.105}$$

where $\boldsymbol{\sigma} = \{\mathbf{y} + t\hat{\mathbf{n}} \mid t \in \mathbb{R}\} \in \mathbb{A}(d,1)$ (implicitly dependent on $\mathbf{y}$ and $\hat{\mathbf{n}}$).

Proof. By Blaschke-Petkantchin formula in form of Corollary 296.1 with $p = q = 1$,

$$\mathbb{E}\left[g(\boldsymbol{\sigma})\Delta_1^k\right] = \frac{\beta_{d11}}{(\mathrm{vol}_d\, K_d)^2}\int_{\mathbb{A}_{K_d}(d,1)} g(\boldsymbol{\sigma})v_1^{(d+k-1)}(\boldsymbol{\sigma}\cap K_d)\,\mathrm{vol}_1(\boldsymbol{\sigma}\cap K_d)^{d+k+1}\mu_q(\mathrm{d}\boldsymbol{\sigma}). \tag{B.106}$$

By Remark 296, we have $\beta_{d11} = \omega_d/\omega_1 = \omega_d/2$. Next, by affine invariancy, we get

$$v_1^{(d+k-1)}(\boldsymbol{\sigma}\cap K_d) = v_1^{(d+k-1)}(T_1) = \frac{2}{(d+k)(d+k+1)} \tag{B.107}$$

by Equation (4.39) (T_1 is the line segment $(0,1)$). Hence,

$$\mathbb{E}\left[g(\boldsymbol{\sigma})\Delta_1^k\right] = \frac{\omega_d/(\mathrm{vol}_d\, K_d)^2}{(d+k)(d+k+1)}\int_{\mathbb{A}_{K_d}(d,1)} g(\boldsymbol{\sigma})\,\mathrm{vol}_1(\boldsymbol{\sigma}\cap K_d)^{d+k+1}\mu_1(\mathrm{d}\boldsymbol{\sigma}). \tag{B.108}$$

The statement of the original proposition follows immediately from the spherical parametrization of $\mu_1(\mathrm{d}\boldsymbol{\sigma})$ (Equation (B.104)). ■

Distance moments

As a direct consequence of the previous proposition with $g(\boldsymbol{\sigma}) = 1$, we get the following formula on distance moments (Kingman [40, Eq. 34]).

Proposition 299. *Let $K_d \subset \mathbb{R}^d$ be a convex d-body and $\mathbf{X}_0, \mathbf{X}_1 \sim \mathsf{Unif}(K_d)$ be random points uniformly and independently selected from K_d. Denote, as usual, $\Delta_1 = \|\mathbf{X}_1 - \mathbf{X}_0\|$ the (random) distance between them. Then, for any $d \geq 1$ and any (real) $k > -d$,*

$$\mathbb{E}\left[\Delta_1^k\right] = \frac{2/(\mathrm{vol}_d\, K_d)^2}{(d+k)(d+k+1)} \int_{\mathbb{S}_+^{d-1}} \int_{\hat{\mathbf{n}}_\perp} \mathrm{vol}_1(\boldsymbol{\sigma} \cap K_d)^{d+k+1} \lambda_{d-1}(\mathrm{d}\mathbf{y}) \sigma_d(\mathrm{d}\hat{\mathbf{n}}) \quad \text{(B.109)}$$

where $\boldsymbol{\sigma} = \{\mathbf{y} + t\hat{\mathbf{n}} \mid t \in \mathbb{R}\} \in \mathbb{A}(d,1)$ (implicitly dependent on $\mathbf{y}$ and $\hat{\mathbf{n}}$).

As a special case, we get the following interesting corollary:

Corollary 299.1. *For any convex d-body $K_d \subset \mathbb{R}^d$ and any $d \geq 1$,*

$$\lim_{k \to -d^+} (d+k)\mathbb{E}\left[\Delta_1^k\right] = \frac{\omega_d}{\mathrm{vol}_d\, K_d}. \quad \text{(B.110)}$$

Proof. By Proposition 299 above,

$$\lim_{k \to -d^+} (d+k)\mathbb{E}\left[\Delta_1^k\right] = \frac{2}{(\mathrm{vol}_d\, K_d)^2} \int_{\mathbb{S}_+^{d-1}} \int_{\hat{\mathbf{n}}_\perp} \mathrm{vol}_1(\boldsymbol{\sigma} \cap K_d) \lambda_{d-1}(\mathrm{d}\mathbf{y}) \sigma_d(\mathrm{d}\hat{\mathbf{n}}). \quad \text{(B.111)}$$

The statement follows immediately from the trivial fact that $\int_{\mathbb{S}_+^{d-1}} \sigma_d(\mathrm{d}\hat{\mathbf{n}}) = \frac{\omega_d}{2}$ coupled with Fubini's theorem: For any fixed $\hat{\mathbf{n}} \in \mathbb{S}_+^{d-1}$,

$$\mathrm{vol}_d\, K_d = \int_{K_d} \lambda_d(\mathrm{d}\mathbf{x}) = \int_{\hat{\mathbf{n}}_\perp} \int_{\boldsymbol{\sigma} \cap K_d} \lambda_1(\mathrm{d}t) \lambda_{d-1}(\mathrm{d}\mathbf{y}) = \int_{\hat{\mathbf{n}}_\perp} \mathrm{vol}_1(\boldsymbol{\sigma} \cap K_d) \lambda_{d-1}(\mathrm{d}\mathbf{y}), \quad \text{(B.112)}$$

where $\boldsymbol{\sigma} = \{\mathbf{y} + t\hat{\mathbf{n}} \mid t \in \mathbb{R}\} \in \mathbb{A}(d,1)$ as usual. ■

C Symmetries and genealogic decomposition

C.1 Configurations

Let $\mathcal{G}(P_d)$ be the group of all isometries of P_d (the *symmetric group* of P_d). That is, $\mathcal{G}(P_d)$ is isomorphic to the group of permutations of vertices of P_d such that it leaves P_d unchanged upto rigid transformations (including reflections). Note that in $d = 3$, $\mathcal{G}(P_d)$ only consists of *rotations, reflections and improper rotations*. In **Schoenflies notation**, they are denoted C_n, σ, S_n, respectively (together with inversion I and identity E).

Example 300. All isometries of a regular octahedron are given as

$$\mathcal{G}(O_3) = \{E, 6C_2, 8C_3, 6C_4, 3C_4^2, I, 3\sigma_h, 6\sigma_d, 6S_4, 8S_6\}. \tag{C.113}$$

Let us select some subset S of vertices V of P_d. We can imagine the selected vertices are coloured (black/white), this way we get a polytope $P_d(S)$ with coloured vertices. We denote $\mathcal{P}_d$ as the set of all those polytopes with pre-selected (coloured) vertices. We say two $P_d(S_1), P_d(S_2) \in \mathcal{P}_d$ are equivalent if there is $g \in \mathcal{G}(P_d)$ such that $gP_d(S_2) = P_d(S_1)$. Moreover we say they are section equivalent if $gP_d(S_2) = P_d(S_1)$ or $gP_d(S_2) = P_d(V \setminus S_1)$. We see that the first condition is more strict since in the latter case, since in the section equivalent case we also identify two coloured polytopes with switched colours. We call the representants of all equivalent classes of coloured polytopes as **configurations**.

C.2 Weights and orders

The size of an orbit of some configuration $C = P_d(S)$ with selected representant vertices S is by definition $o_{\mathrm{C}} = |\mathcal{G}(P_d)C|$, where $\mathcal{G}(P_d)\mathrm{C} = \{g\mathrm{C} \mid g \in \mathcal{G}(P_d)\}$ is the **orbit** of C. By *orbit-stabiliser lemma*, $o_{\mathrm{C}} = |\mathcal{G}(P_d)|/|\mathcal{G}_{\mathrm{C}}(P_d)|$, where $\mathcal{G}_{\mathrm{C}}(P_d) = \{g \in \mathcal{G}(P_d) \mid g\mathrm{C} = \mathrm{C}\}$ is the **stabiliser** subgroup. The total number of equivalent configurations is given by Burnside's lemma as

$$\frac{1}{|\mathcal{G}(P_d)|} \sum_{g \in \mathcal{G}(P_d)} |\{\mathrm{C} \mid g\mathrm{C} = \mathrm{C}\}|, \tag{C.114}$$

where $\{\mathrm{C} \mid g\mathrm{C} = \mathrm{C}\}$ is the set of **fixed points** (that is the set of configurations that are unchanged by the action of the group element g). We can find those configurations via the help of computer, see GECRA (Code 9) in the appendix. The procedure is as follows: First, we represent $\mathcal{G}(P_d)$ as a subgroup of the symmetry group $\mathcal{S}_{|V|}$ with $|V|$ whose elements (permutations) which act of vertices of P_d we represent as permutation matrices. This representation is of course an isomorphism. Then, we can represent a selection (colouring) of vertices S as a vector $\mathbf{s}$ of length $|V|$ of ones and zeros. Let us denote the set of all such vectors as $\mathbb{S}$. There are $2^{|V|}$ such vectors. The set of all configurations is then simply the classes

$$\bigcup_{\mathbf{s} \in \mathbb{S}} |\mathcal{S}_{|V|}\mathbf{s}| = \bigcup_{\mathbf{s} \in \mathbb{S}} \{g\mathbf{s} \mid g \in \mathcal{S}_{|V|}\}. \tag{C.115}$$

So far, we have not employed the section equivalence $g\mathrm{C} \sim \mathrm{C}'$, where we write $\mathrm{C}' = P_d(V \setminus S)$. Therefore, for a given configuration C with S selected (coloured) vertices out of total n vertices of P_d, we define the section weight w_C as the size of the orbit of C with respect to the section equivalence, that is, by symmetry

$$w_\mathrm{C} = \begin{cases} o_\mathrm{C}, & |S| < n/2 \\ o_\mathrm{C}/2, & |S| = n/2. \end{cases} \tag{C.116}$$

Since $\boldsymbol{\sigma} \cap P_d$ is also a polytope (more precisely, a $(d-1)$-polytope), we define the *order* n_C of a configuration C as the number of vertices of $\boldsymbol{\sigma} \cap P_d$. We claim this number is well defined for a given configuration.

C.3 Realisable configurations

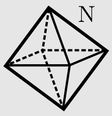
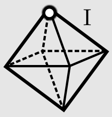
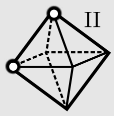
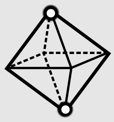
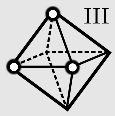
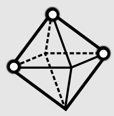
$\mathcal{G}_\mathrm{C}(O_3)$:	$\mathcal{G}(O_3)$	$\begin{Bmatrix} E,2C_4,C_4^2 \\ 2\sigma_h, 2\sigma_d \end{Bmatrix}$	$\begin{Bmatrix} E,C_2, \\ \sigma_h,\sigma_d \end{Bmatrix}$	$\begin{Bmatrix} E,2C_2,2C_4,3C_4^2, \\ I, 3\sigma_h, 2\sigma_d, 2S_4 \end{Bmatrix}$	$\begin{Bmatrix} E,2C_3, \\ 3\sigma_d \end{Bmatrix}$	$\begin{Bmatrix} E, C_2, \\ \sigma_h, \sigma_d \end{Bmatrix}$
C :	 N	 I	 II		 III	
o_C :	1	6	12	3	8	12
w_C :	1	6	12	3	4	6

Table C.6: Section equivalent configurations of O_3

In the example above shown in Table C.6, configurations N, I, II, III are those whose points can be separated by a plane. In general that is, there exists a $(d-1)$-plane $\boldsymbol{\sigma}$ such that all vertices in S lie on side of $\boldsymbol{\sigma}$ and all remaining vertices $V \setminus S$ lie on the other side. Those configurations are said to be *realisable*. We write $\boldsymbol{\sigma}/\mathrm{C}$, where C is a configuration and we write $\mathcal{C}(P_d)$ for the set of all realisable configurations. We can check whether a configuration is realisable by checking whether there is a nonempty subset of $\mathbb{R}^d$ satisfying Equations (4.29).

C.4 Genealogy

Realisable configurations have a unique property – assuming P_d is convex, we can obtain them from realisable configurations with fewer coloured vertices by successively adding (colouring) another neighbouring vertex. This corresponds to a continuous shift of $\boldsymbol{\sigma}$. The graph (in fact, a *Hasse diagam*) of such successions is called the **genealogy** of P_d configurations with section weights w_C. Generalogies for selected polyhedra are shown in Appendix D. For example, the genealogy of the octahedron section equivalent configurations from Table C.6 are shown in Figure D.3.

C.5 Decomposition of functionals

Let $P \subset \mathbb{R}^d$ be a convex d-polytope. Consider an affinely invariant functional

$$F(P) = \frac{1}{(\mathrm{vol}_d\, P)^d} \int_{P^d} f(\mathbb{x}) \lambda_d^d(\mathrm{d}\mathbb{x}). \tag{C.117}$$

By symmetry, we can decompose this functional as follows

$$F(P) = \sum_{C \in \mathcal{C}(P)} w_{\mathrm{C}} F(P)_{\mathrm{C}}, \tag{C.118}$$

where

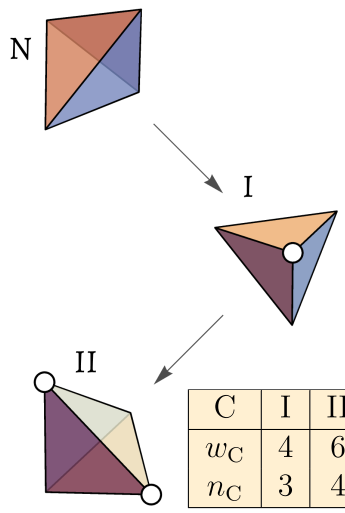
$$F(K)_{\mathrm{C}} = \frac{1}{(\mathrm{vol}_d\, P)^d} \int_{P^d} \mathbb{1}\{\mathcal{A}(\mathbb{x})/C\} f(\mathbb{x}) \lambda_d^d(\mathrm{d}\mathbb{x}). \tag{C.119}$$

Note that the property $\mathcal{A}(\mathbb{x})/\mathrm{C}$ is also affinelly invariant, since any affine transformation does not change the set of vertices S separated by $\boldsymbol{\sigma}$. As a consequence, also $F_{\mathrm{C}}(K)$ stays invariant under affine transformations of K. By defining $P_{\mathrm{C}} = \{\mathbf{x} \in P \,|\, \mathcal{A}(\mathbb{x})/\mathrm{C}\}$, we may also write

$$F(K)_{\mathrm{C}} = \frac{1}{(\mathrm{vol}_d\, P)^d} \int_{P_{\mathrm{C}}^d} f(\mathbb{x}) \lambda_d^d(\mathrm{d}\mathbb{x}). \tag{C.120}$$

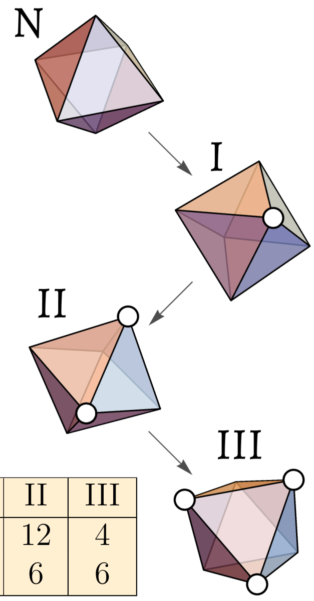
D Selected genealogies

Configurations $\mathcal{C}(P)$ derived from the empty configuration N (no points selected) by succesively adding an extra vertex (I, II, III, etc.). Genealogic decomposition is used to decompose affine functionals $F(K)$ as $\sum_{C\in\mathcal{C}(P)} w_{\mathrm{C}} F(K)_{\mathrm{C}}$. Each configuration is characterised by selection S of vertices (figures), by section equivalent weights w_{C} and the number of vertices of $\boldsymbol{\sigma} \cap P$, which is the order n_{C}.



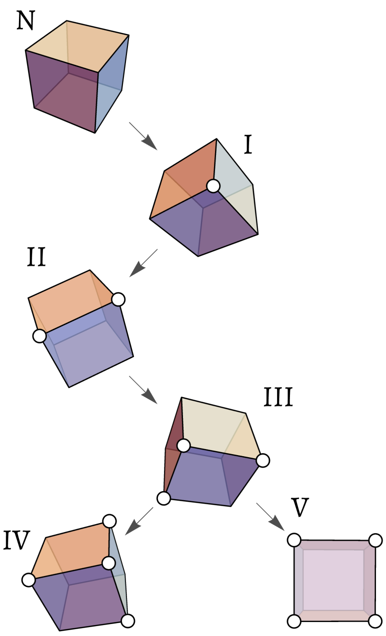
C	I	II
w_{C}	4	6
n_{C}	3	4

Figure D.2: Tetrahedron genealogy



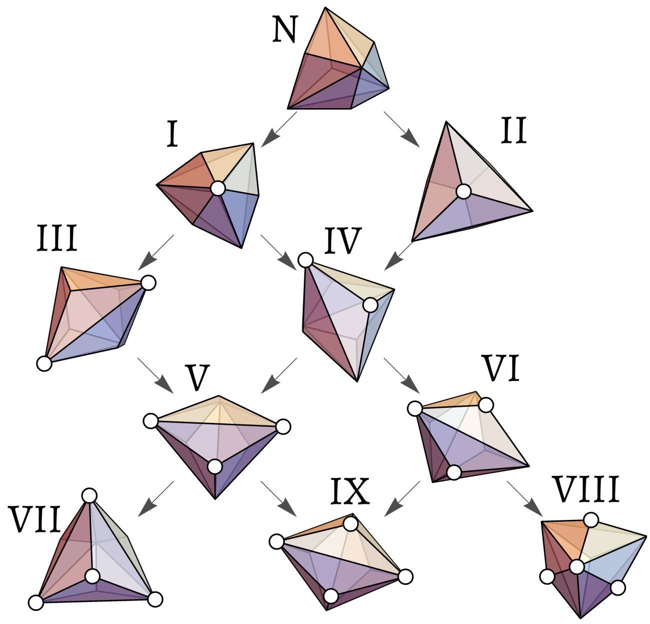
C	I	II	III
w_{C}	6	12	4
n_{C}	4	6	6

Figure D.3: Octahedron genealogy



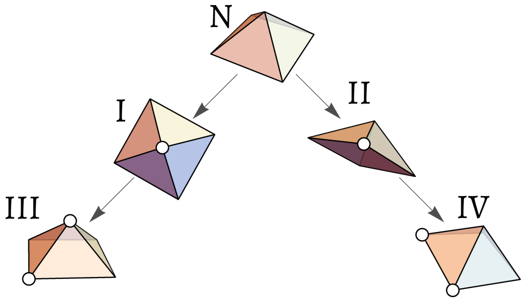
C	I	II	III	IV	V
w_{C}	8	12	24	4	3
n_{C}	3	4	5	6	4

Figure D.4: Cube genealogy



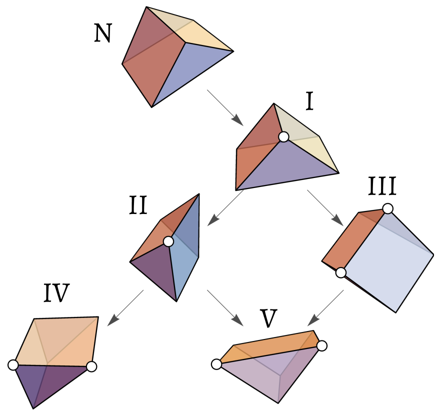
C	I	II	III	IV	V	VI	VII	VIII	IX
w_{C}	4	4	6	12	12	12	2	2	3
n_{C}	6	3	10	7	9	8	9	9	8

Figure D.5: Triakis tetrahedron genealogy



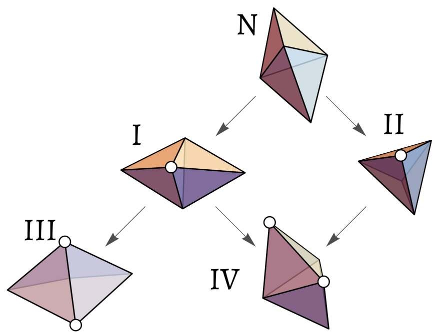
C	I	II	III	IV
w_C	1	4	4	4
n_C	4	3	5	4

Figure D.6: Square pyramid genealogy



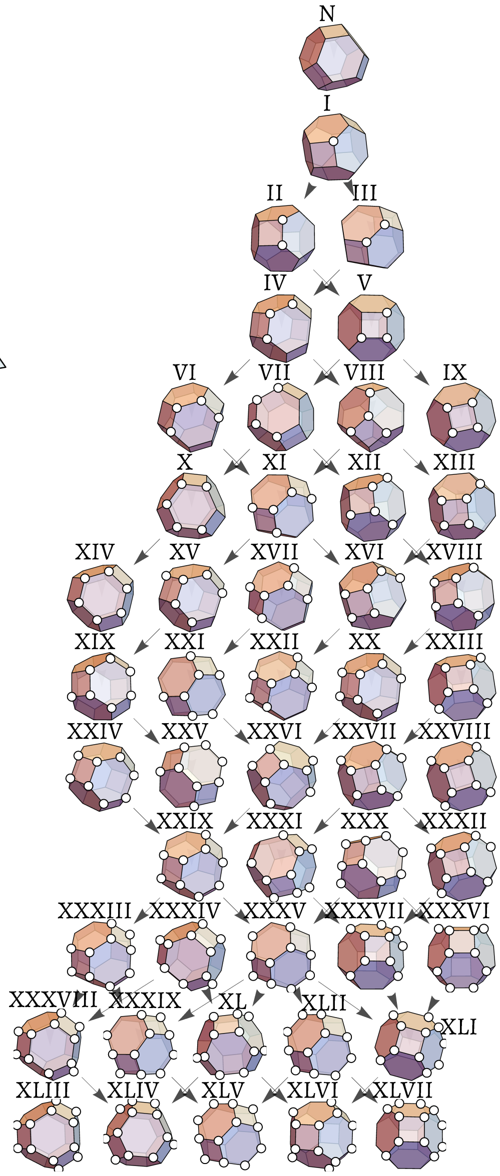
C	I	II	III	IV	V
w_C	6	6	3	1	6
n_C	3	4	4	3	5

Figure D.7: Triangular prism genealogy



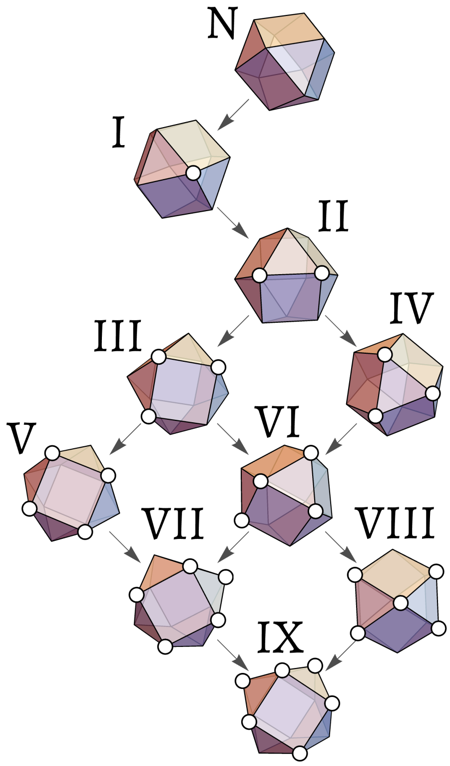
C	I	II	III	IV
w_C	3	2	3	6
n_C	4	3	6	5

Figure D.8: Triangular bipyramid genealogy



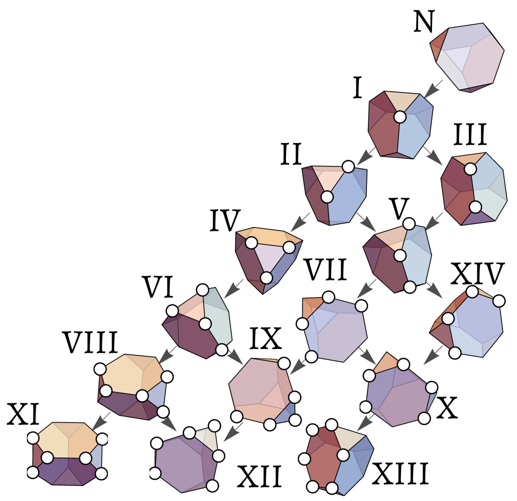
C	I	II	III	IV	V	VI	VII	VIII
w_C	24	24	12	48	24	24	24	24
n_C	3	4	4	5	5	6	6	6
C	IX	X	XI	XII	XIII	XIV	XV	XVI
w_C	6	48	48	48	24	8	48	48
n_C	4	7	7	7	5	6	8	6
C	XVII	XVIII	XIX	XX	XXI	XXII	XXIII	XXIV
w_C	12	24	48	48	48	24	24	24
n_C	8	6	7	7	9	7	7	6
C	XXV	XXVI	XXVII	XXVIII	XXIX	XXX	XXXI	XXXII
w_C	24	48	48	6	48	48	24	48
n_C	8	8	8	8	7	7	9	9
C	XXXIII	XXXIV	XXXV	XXXVI	XXXVII	XXXVIII	XXXIX	XL
w_C	24	24	48	24	24	48	48	48
n_C	6	8	8	8	10	7	7	9
C	XLI	XLII	XLIII	XLIV	XLV	XLVI	XLVII	
w_C	48	24	4	24	6	12	12	
n_C	9	7	6	8	6	10	8	

Figure D.9: Truncated octahedron genealogy



C	I	II	III	IV	V	VI	VII	VIII	IX
w_C	12	24	24	8	6	48	24	12	12
n_C	4	6	8	6	8	8	8	8	8

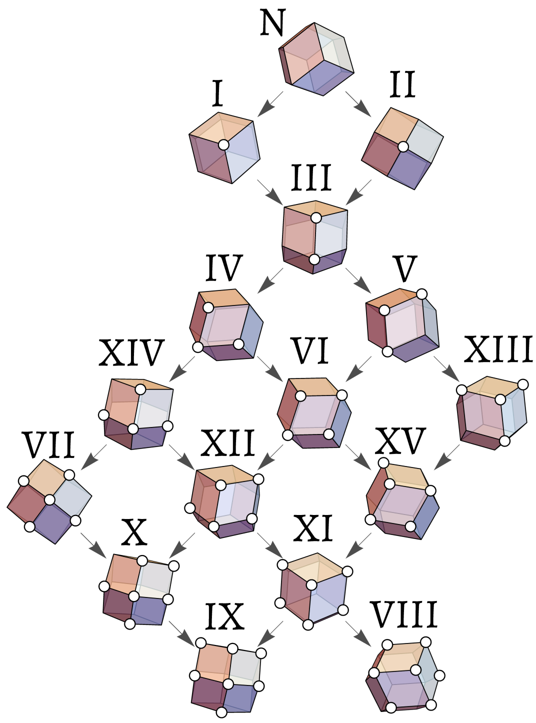
Figure D.10: Cuboctahedron genealogy



C	I	II	III	IV	V	VI	VII
w_C	12	12	6	4	24	12	12
n_C	3	4	4	3	5	4	6

C	I	II	III	IV	V	VI	VII
w_C	24	12	24	3	12	4	12
n_C	5	5	7	4	6	6	6

Figure D.11: Tuncated tetrahedron genealogy



C	I	II	III	IV	V
w_C	8	6	24	24	24
n_C	3	4	5	6	7

C	V	VII	VIII	IX	X
w_C	12	6	4	12	24
n_C	6	8	6	8	8

C	XI	XII	XIII	XIV	XV
w_C	24	48	8	24	24
n_C	7	7	9	7	8

Figure D.12: Rhombic dodecahedron genealogy

E Codes in Mathematica

E.1 General formulae

Code 1: Code to evaluate $e_d^{(k)}$ for general d and k

```
efun[d_,1]:= (d+1)*(a @@ ConstantArray[0,d])^(d+1);
efun[d_,k_] := Simplify[(d+1)!/(d!)^k
 Sum[Times @@ Array[Signature[p[#+1]] &, k-1]*
      (Times @@ (a @@ Table[Count[#,i], {i,1,d}] &
         /@ Table[Flatten[{i-1, Table[p[j][[i]]-1, {j,2,k}]}], {i,1,d+1}])), ##] &
   @@ Table[{p[i], Permutations[Range[d+1]]}, {i,2,k}]];
```

E.2 Tetrahedron area moments

Code 2: Code to evaluate $\iota_3^{(k)}(\boldsymbol{\sigma})$ in $\mathbb{T}_3$, configuration I

```
k = 1 (*desirable integer*);
Tcanon = Tetrahedron[{{0,0,0},{1,0,0},{0,1,0},{0,0,1}}];
Tabc = Tetrahedron[{{0,0,0},{1/a,0,0},{0,1/b,0},{0,0,1/c}}];
iotaint = Simplify[Integrate[(Dot[{a,b,c},x] - 1)^k, x \[Element] Tcanon] - (1 - (-1)^k) Integrate[(Dot[{a,b,c},x] - 1)^k, x \[Element] Tabc],Assumptions -> 1 < a && 1 < b && 1 < c]
```

Code 3: Code to evaluate $\iota_3^{(k)}(\boldsymbol{\sigma})$ in $\mathbb{T}_3$, configuration II

```
k = 1(*desirable integer*);
Tcanon = Tetrahedron[{{0,0,0},{1,0,0},{0,1,0},{0,0,1}}];
Tabc = Tetrahedron[{{0,0,0},{1/a,0,0},{0,1/b,0},{0,0,1/c}}];
Tstar = Tetrahedron[{{0,0,1},{(1 - c)/(a - c),0,(a - 1)/(a - c)},{0,(1 - c)/(b - c),(b - 1)/(b - c)},{0,0,1/c}}];
iotaint = Simplify[(Integrate[(Dot[{a,b,c},x] - 1)^k, x \[Element]
   Tcanon] - (1 - (-1)^k) Integrate[(Dot[{a,b,c},x] - 1)^k,
   x \[Element] Tabc] + (1 - (-1)^k) Integrate[(Dot[{a,b,c},x]
   - 1)^k, x \[Element] Tstar]),
 Assumptions -> 1 < a && 1 < b && 0 < c < 1]
```

Code 4: Code to evaluate $v_2^{(k+1)}(\mathbb{U}_2^{\alpha\beta})$, odd k

```
k = 1(*desirable odd integer*);
Delta = 1/2! Det[{x1 - x0, x2 - x0}];
```

```
trianab = Triangle[{{0, 0}, {\[Alpha], 0}, {0, \[Beta
   ]}}];
trianunit = Triangle[{{0, 0}, {1, 0}, {0, 1}}];
meancut =
 Simplify[(2/(1 - \[Alpha] \[Beta]))^(
  k + 4) (Integrate[Delta^(k + 1), x0 \[Element]
      trianunit,
     x1 \[Element] trianunit, x2 \[Element] trianunit] -
    3 Integrate[Delta^(k + 1), x0 \[Element] trianunit,
     x1 \[Element] trianunit, x2 \[Element] trianab] +
    3 Integrate[Delta^(k + 1), x0 \[Element] trianunit,
     x1 \[Element] trianab, x2 \[Element] trianab] -
    Integrate[Delta^(k + 1), x0 \[Element] trianab,
     x1 \[Element] trianab, x2 \[Element] trianab]),
  Assumptions -> 0 < \[Alpha] < 1 && 0 < \[Beta] < 1]
```

E.3 Pentachoron 4-volume moments

Code 5: Code to evaluate $\iota_4^{(k)}(\boldsymbol{\sigma})$ in $\mathbb{T}_4$, configuration I

```
k = 1 (*desirable integer*);
Tcanon =
  Simplex[{{0, 0, 0, 0}, {1, 0, 0, 0}, {0, 1, 0, 0}, {0,
     0, 1, 0}, {0, 0, 0, 1}}];
Tabcd =
  Simplex[{{0, 0, 0, 0}, {1/a, 0, 0, 0}, {0, 1/b, 0, 0},
     {0, 0, 1/c, 0}, {0, 0, 0, 1/d}}];
iotaint =
 Simplify[
  Integrate[(Dot[{a, b, c, d}, x] - 1)^k, x \[Element]
    Tcanon] - (1 - (-1)^k) Integrate[(Dot[{a, b, c, d}, x
      ] - 1)^
    k, x \[Element] Tabcd],
  Assumptions -> 1 < a && 1 < b && 1 < c && 1 < d]
```

Code 6: Code to evaluate $\iota_4^{(k)}(\boldsymbol{\sigma})$ in $\mathbb{T}_4$, configuration II

```
k = 1(*desirable integer*);
Tcanon = Simplex[{{0, 0, 0, 0}, {1, 0, 0, 0},
    {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}];
Tabcd = Simplex[{{0, 0, 0, 0}, {1/a, 0, 0, 0},
    {0, 1/b, 0, 0}, {0, 0, 1/c, 0}, {0, 0, 0, 1/d}}];
Tstar = Simplex[{{0, 0, 0, 1},
    {(1 - d)/(a - d), 0, 0, (a - 1)/(a - d)},
    {0, (1 - d)/(b - d), 0, (b - 1)/(b - d)},
    {0, 0, (1 - d)/(c - d), (c - 1)/(c - d)},
    {0, 0, 0, 1/d}}];
iotaint = Simplify[(Integrate[(Dot[{a, b, c, d}, x] - 1)^
   k,
```

```
   x \[Element] Tcanon] - 2 Integrate[(Dot[{a,b,c,d},x] -
      1)^k,
   x \[Element] Tabcd] + 2 Integrate[(Dot[{a,b,c,d},x] -
      1)^k, x
   \[Element] Tstar]),
  Assumptions -> 1 < a && 1 < b && 1 < c && 0 < d < 1]
```

Code 7: Code to evaluate $v_3^{(k+1)}(\mathbb{U}_3^{\alpha\beta\gamma})$, odd k

```
k = 1(*desirable odd integer*);
Delta = 1/3! Det[{x1 - x0, x2 - x0, x3 - x0}];
Tabc = Tetrahedron[{{0, 0, 0}, {\[Alpha], 0, 0}, {0, \[
   Beta], 0}, {0, 0, \[Gamma]}}];
Tcan = Tetrahedron[{{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0,
   0, 1}}];
meancut =
 Simplify[(6/(1 - \[Alpha] \[Beta] \[Gamma]))^(
  k + 5) (Integrate[Delta^(k + 1), x0 \[Element] Tcan,
   x1 \[Element] Tcan, x2 \[Element] Tcan,x3 \[Element]
      Tcan]
   - 4 Integrate[Delta^(k + 1), x0 \[Element] Tcan,
   x1 \[Element] Tcan, x2 \[Element] Tcan, x3 \[Element]
      Tabc]
   + 6 Integrate[Delta^(k + 1), x0 \[Element] Tcan,
   x1 \[Element] Tcan, x2 \[Element] Tabc, x3 \[Element]
      Tabc]
   - 4 Integrate[Delta^(k + 1), x0 \[Element] Tcan,
   x1 \[Element] Tabc, x2 \[Element] Tabc, x3 \[Element]
      Tabc]
   + Integrate[Delta^(k + 1), x0 \[Element] Tabc, x1 \[
      Element]
   Tabc, x2 \[Element] Tabc, x3 \[Element] Tabc]),
 Assumptions ->
  0 < \[Alpha] < 1 && 0 < \[Beta] < 1 && 0 < \[Gamma] <
     1]
```

E.4 GECRA: Genealogy creation algorithm

The following algorithm generates realisable configurations and their weights for any polytopes by exploiting their symmetries. The code works on iterating over nos (the number of selected vertices) and it has the following steps

- Step 0: initialise empty configuration N
- CYCLE
  - Step I: generate new configurations from old ones
  - Step II: group them into classes, select first configuration from each (the so called representant)
  - Step III: for each representant, determine if it is realisable, discard unrealisable

- repeat step I until nos reaches half the number of vertices,

The algorithm is initialised by inserting vertices of P_d into solid as a list of their coordinates and the symmetry group $\mathcal{G}(P_d)$ into symgroup as list of permutations on indexes of these vertices. In the code, dimen is the dimension d. Various inputs and results are stored in the library in the file GENERAL EFRON.nb (see Attachments). For example, Code 8 shows the input for $P_d = C_3$ (the three-dimensional unit cube). Note that we only store the generators of $\mathcal{G}(C_3)$ since the whole symmetric group can be obtain by successive composition of the elements with themselves.

Code 8: Input for GECRA for $P_d = C_3$

```
solid = {{0, 0, 0}, {1, 0, 0}, {0, 1, 0}, {0, 0, 1},
   {0, 1, 1}, {1, 0, 1}, {1, 1, 0}, {1, 1, 1}};
generators = {(*reflection*){4, 6, 5, 1, 3, 2, 8, 7},
 (*2fold rotation*){3, 7, 5, 1, 4, 2, 8, 6},
 (*3fold rotation*){7, 8, 3, 2, 1, 6, 5, 4}};
symgroup = FixedPoint[Union[Flatten[Table[
    PermutationProduct[#, p] & /@ #, {p, #}], 1]] &,
    generators];
```

The output of the GECRA program is the following

- alltypes:the list of cofigurations, each configuration is represented by a list of indices of vertices
- allweights: list of weights of configurations
- allgenealogy: the genealogy as a list of pairs $i \to j$, where i, j are indices of configurations in the list of configurations
- gengraph: the genealogy graph (a Hasse diagram)

Code 9: GECRA: Genealogies from symmetry groups

```
Clear[classreps, rawclassreps, oineqsel, isrealisable,
    orbitmaker,
 allsuccesors, weightsel, dimen];
dimen = 4;
ofvertices=Length[solid];
etaparams = Table[a[i], {i, dimen}];
orbitmaker[sel_] :=
 orbitmaker[sel] =
  Union[Table[(sel)[[#[[i]]]], {i, ofvertices}] & /@
      symgroup];
weightsel[sel_] :=
 If[Total[sel] < ofvertices/2, Length[orbitmaker[sel]],
  Length[orbitmaker[sel]]/2];
(*inequalities for etaparams a,b,c,d,... for a given 0,1
    selection of \vertices*)
oineqsel[sel_] :=
 oineqsel[sel] =
  With[{representant = Pick[solid, # == 1 & /@ sel]},
   Reduce[Or[
```

```
    And @@ Flatten[{Dot[etaparams, #] > 1 & /@
        representant,
       Dot[etaparams, #] < 1 & /@ Complement[solid,
          representant]}],
    And @@
     Flatten[{Dot[etaparams, #] < 1 & /@ representant,
       Dot[etaparams, #] > 1 & /@
        Complement[solid, representant]}]]]];
isrealisable[sel_] :=
  isrealisable[sel] = If[Length[oineqsel[sel]] == 0, 0,
     1];
(*step 0*)
classreps[0] = {ConstantArray[0, ofvertices]};
(*step I*)
allsuccesors[sel_] :=
  allsuccesors[sel] = ReplaceList[sel, {a___, 0, b___} :>
      {a, 1, b}];
(*step II*)
rawclassreps[i_] :=
  rawclassreps[i] =
   Map[Last,
    Union[orbitmaker[#] & /@
      Union[Flatten[allsuccesors[#] & /@ classreps[i - 1],
       1]]]]];
(*step III*)
classreps[nos_] :=
  classreps[nos] =(*sort by weight of a configuration*)
   SortBy[Select[rawclassreps[nos], isrealisable[#] == 1
      &], weightsel];

(*OUTPUT*)
allreps = Flatten[Table[classreps[i], {i, 1, Floor[
    ofvertices/2]}], 1];
(*01 representants as their index*)
repstoindexesrule =
  Flatten[{{ConstantArray[0, ofvertices] -> 0},
    Table[allreps[[i]] -> i, {i, Length[allreps]}]}, 1];
allgenealogy =
  Flatten[Table[(sel /. repstoindexesrule) -> (suc /.
      repstoindexesrule), {i, 0, Floor[ofvertices/2] - 1},
          {sel, classreps[i]}, {suc,
    Intersection[Flatten[(orbitmaker[#] & /@ allsuccesors
        [sel]), 1], classreps[i + 1]]}], 2];
gengraph =
  GraphPlot[
   RomanNumeral[#[[1]]] -> RomanNumeral[#[[2]]] & /@
      allgenealogy,
   VertexLabeling -> True, DirectedEdges -> True];
```

```
(*representants in coordinates*)allrepscoord =
 Pick[solid, # == 1 & /@ #] & /@ allreps;
alltypes = Pick[Range[ofvertices], # == 1 & /@ #] & /@
   allreps;
allweights = weightsel[#] & /@ allreps;
allnoofsel = Total[#] & /@ allreps;
```

F Auxiliary integrals

Recurrence relations for auxiliary integrals

Recall that $D(\zeta,\gamma)$ is the *fundamental triangle domain* with vertices $[0,0]$, $[\zeta,0]$, $[\zeta,\zeta\tan\gamma]$ ($\zeta>0$, $0<\gamma<\pi/2$). To express the integrals

$$I_{ij}^{(p)}(q,\gamma)=\int_{D(q,\gamma)} x^i y^j \left(1+x^2+y^2\right)^{p/2}\ \mathrm{d}x\mathrm{d}y, \tag{F.121}$$

we mainly employ recursive relations. However, $I_{11}^{(p)}(q,\gamma)$ can be expressed directly without recursions. We parametrise the domain $D(q,\gamma)$ as $y\in(0,x\tan\gamma)$, $x\in(0,q)$, by integrating out y and then x, we get

$$I_{11}^{(p)}(q,\gamma)=\frac{\sin^2\gamma+\cos^2\gamma\,(1+q^2\sec^2\gamma)^{2+\frac{p}{2}}-(1+q^2)^{2+\frac{p}{2}}}{(2+p)(4+p)}. \tag{F.122}$$

K's

In case of $I_{10}^{(p)}(q,\gamma)$ and $I_{10}^{(p)}(q,\gamma)$, we cannot integrate twice. To overcome this, we first define our first auxiliary integral

$$K^{(p)}(r)=\int_0^r \left(1+t^2\right)^{1+p/2}\ \mathrm{d}t \tag{F.123}$$

satisfying symmetry

$$K^{(p)}(-r)=-K^{(p)}(r) \tag{F.124}$$

and, via integration by parts, the recurrence relation

$$K^{(p)}(r)=\frac{2+p}{3+p}K^{(p-2)}(r)+\frac{r}{3+p}(1+r^2)^{1+p/2} \tag{F.125}$$

with boundary conditions

$$K^{(-2)}(r)=r,\quad K^{(-3)}(r)=\operatorname{argsinh} r. \tag{F.126}$$

We can then express our $I_{10}^{(p)}(q,\gamma)$ and $I_{10}^{(p)}(q,\gamma)$ in terms of $K's$ as

$$I_{10}^{(p)}(q,\gamma)=\frac{1}{2+p}\left[\left(1+q^2\right)^{\frac{3+p}{2}}K^{(p)}\left(\frac{q\tan(\gamma)}{\sqrt{1+q^2}}\right)-\sin\gamma\,K^{(p)}(q\sec\gamma)\right], \tag{F.127}$$

$$I_{01}^{(p)}(q,\gamma)=\frac{1}{2+p}\left[\cos\gamma K^{(p)}(q\sec\gamma)-K^{(p)}(q)\right]. \tag{F.128}$$

J's

We denote

$$J^{(p)}(q,\gamma)=-\gamma+\int_0^\gamma(1+q^2\sec^2\varphi)^{1+p/2}\ \mathrm{d}\varphi, \tag{F.129}$$

satisfying symmetry

$$J^{(p)}(q,-\gamma)=-J^{(p)}(q,\gamma) \tag{F.130}$$

and, via integration by parts, the recurrence relations

$$J^{(p)}(q,\gamma)=J^{(p-2)}(q,\gamma)+q\,(1+q)^{\frac{1+p}{2}}K^{(p-2)}\left(\frac{q\tan\gamma}{\sqrt{1+q^2}}\right), \tag{F.131}$$

with boundary conditions

$$J^{(-2)}(q,\gamma) = 0, \quad J^{(-3)}(q,\gamma) = -\gamma + \arcsin\frac{\sin\gamma}{\sqrt{1+q^2}}. \tag{F.132}$$

> *Remark* 301. Note that we can write $J^{(p)}(q,\gamma) = \int_0^\gamma \left((1+q^2\sec^2\varphi)^{1+p/2} - 1\right)\,\mathrm{d}\varphi$.

We transform $I_{00}^{(p)}(q,\gamma)$ by substitution into polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, our domain $D(\zeta,\gamma)$ becomes parametrised as $r \in (0, q\sec\varphi)$, $\varphi \in (0,\gamma)$ and thus

$$I_{00}^{(p)}(q,\gamma) = \int_0^\gamma \int_0^{q\sec\varphi} r\left(1+r^2\right)^{p/2}\,\mathrm{d}r\mathrm{d}\varphi, \tag{F.133}$$

Integrating out r, we get

$$I_{00}^{(p)}(q,\gamma) = \frac{1}{2+p}\int_0^\gamma (1+q^2\sec^2\varphi)^{1+p/2} - 1\,\mathrm{d}\varphi = \frac{1}{2+p}J^{(p)}(q,\gamma). \tag{F.134}$$

Note that, by this integral formula, we can extend the definition of $I_{00}^{(p)}(q,\gamma)$ for negative γ as well.

M's

The last set of auxiliary integrals we define is

$$M^{(p)}(q,\gamma) = \int_0^\gamma \cos^2\varphi\,\left[(1+q^2\sec^2\varphi)^{1+p/2} - 1\right]\,\mathrm{d}\varphi, \tag{F.135}$$

satisfying the recurrence relation

$$M^{(p)}(q,\gamma) = M^{(p-2)}(q,\gamma) + q^2\left(\gamma + J^{(p-2)}(q,\gamma)\right). \tag{F.136}$$

Using standard techniques of calculus it is not hard to derive their specific values for $p = -2$ and $p = -3$ are $M^{(-2)}(q,\gamma) = 0$ and

$$M^{(-3)}(q,\gamma) = \frac{1-q^2}{2}\arcsin\frac{\sin\gamma}{\sqrt{1+q^2}} - \frac{\gamma}{2} + \frac{\sin\gamma}{2}\left(\sqrt{\cos^2\gamma + q^2} - \cos\gamma\right). \tag{F.137}$$

Finally, we can express $I_{20}^{(p)}(q,\gamma)$ and $I_{02}^{(p)}(q,\gamma)$. Note that we only need to express the former as $I_{02}^{(p)}(q,\gamma)$ can be extracted from other integrals since

$$I_{00}^{(p)}(q,\gamma) + I_{20}^{(p)}(q,\gamma) + I_{02}^{(p)}(q,\gamma) = \int_{D(q,\gamma)} \left(1+x^2+y^2\right)^{1+p/2}\,\mathrm{d}x\mathrm{d}y = I_{00}^{(p+2)}(q,\gamma). \tag{F.138}$$

Again, by using the polar coordinates substitution, we transform the integral into

$$I_{20}^{(p)}(q,\gamma) = \int_0^\gamma \int_0^{q\sec\varphi} r^3\cos^2\varphi\left(1+r^2\right)^{p/2}\,\mathrm{d}r\mathrm{d}\varphi, \tag{F.139}$$

Integrating out r, we get

$$\begin{aligned} I_{20}^{(p)}(q,\gamma) &= \int_0^\gamma \frac{\cos^2\varphi\left[(1+q^2\sec^2\varphi)^{2+\frac{p}{2}} - 1\right]}{4+p} - \frac{\cos^2\varphi\left[(1+q^2\sec^2\varphi)^{1+\frac{p}{2}} - 1\right]}{2+p}\,\mathrm{d}\varphi \\ &= \frac{1}{4+p}M^{(p+2)}(q,\gamma) - \frac{1}{2+p}M^{(p)}(q,\gamma). \end{aligned} \tag{F.140}$$

Selected values of the auxiliary integrals $I_{ij}^{(p)}(q,\gamma)$ can be found below in the next section.

Special values of auxiliary integrals

The following Table F.1 lists some of the values of $I_{ij}^{(1)}(q,\gamma)$ used throughout our thesis.

$I_{00}^{(1)}\left(1,\frac{\pi}{4}\right)$	$\frac{1}{2\sqrt{3}}-\frac{\pi}{36}+\frac{2}{3}\operatorname{argcoth}\sqrt{3}$
$I_{00}^{(1)}\left(\frac{\sqrt{2}}{2},\frac{\pi}{3}\right)$	$\frac{1}{4}-\frac{\pi}{36}+\frac{7\operatorname{argcoth}\sqrt{2}}{12\sqrt{2}}$
$I_{00}^{(1)}\left(\frac{\sqrt{2}}{4},\frac{\pi}{3}\right)$	$\frac{1}{16\sqrt{2}}+\frac{\pi}{18}+\frac{25\ln 3}{192\sqrt{2}}-\frac{1}{3}\operatorname{arccot}\sqrt{2}$
$I_{00}^{(1)}\left(\frac{\sqrt{2}}{2},\frac{\pi}{4}\right)$	$\frac{1}{6\sqrt{2}}-\frac{\pi}{12}+\frac{7\ln 3}{24\sqrt{2}}+\frac{1}{3}\operatorname{arccot}\sqrt{2}$
$I_{00}^{(1)}\left(\frac{1}{2},\frac{\pi}{5}\right)$	$\frac{\sqrt{5}}{16}-\frac{5}{48}+\frac{\pi}{60}-\frac{13\ln 5}{192}-\frac{1}{6}\operatorname{arccot}2+\frac{13}{96}\operatorname{argsinh}2$
$I_{00}^{(1)}\left(\frac{1}{2},\frac{2\pi}{5}\right)$	$\frac{\sqrt{5}}{16}-\frac{5}{48}+\frac{\pi}{20}-\frac{13\ln 5}{192}-\frac{1}{6}\operatorname{arccot}2+\frac{13}{96}\operatorname{argsinh}2$
$I_{10}^{(1)}\left(1,\frac{\pi}{4}\right)$	$\frac{\sqrt{3}}{8}-\frac{\operatorname{argsinh}\sqrt{2}}{8\sqrt{2}}+\frac{1}{2}\operatorname{argcoth}\sqrt{3}$
$I_{10}^{(1)}\left(\frac{\sqrt{2}}{2},\frac{\pi}{3}\right)$	$\frac{3}{16\sqrt{2}}-\frac{\sqrt{3}}{16}\operatorname{argsinh}\sqrt{2}+\frac{9}{32}\operatorname{argcoth}\sqrt{2}$
$I_{10}^{(1)}\left(\frac{\sqrt{2}}{4},\frac{\pi}{3}\right)$	$\frac{3}{256}+\frac{81\ln 3}{1024}-\frac{\sqrt{3}}{16}\operatorname{argcoth}\sqrt{3}$
$I_{01}^{(1)}\left(1,\frac{\pi}{4}\right)$	$-\frac{7}{12\sqrt{2}}+\frac{3\sqrt{3}}{8}+\frac{\operatorname{argsinh}\sqrt{2}}{8\sqrt{2}}-\frac{1}{8}\operatorname{argcoth}\sqrt{2}$
$I_{01}^{(1)}\left(\frac{\sqrt{2}}{2},\frac{\pi}{3}\right)$	$\frac{3}{8}\sqrt{\frac{3}{2}}-\frac{\sqrt{3}}{8}-\frac{1}{8}\operatorname{argcoth}\sqrt{3}+\frac{1}{16}\operatorname{argsinh}\sqrt{2}$
$I_{11}^{(1)}\left(1,\frac{\pi}{4}\right)$	$\frac{1}{30}-\frac{4\sqrt{2}}{15}+\frac{3\sqrt{3}}{10}$
$I_{11}^{(1)}\left(\frac{\sqrt{2}}{2},\frac{\pi}{3}\right)$	$\frac{1}{20}-\frac{3}{20}\sqrt{\frac{3}{2}}+\frac{3\sqrt{3}}{20}$
$I_{20}^{(1)}\left(\frac{\sqrt{2}}{2},\frac{\pi}{3}\right)$	$\frac{1}{40}+\frac{1}{20\sqrt{3}}+\frac{\pi}{180}+\frac{13\operatorname{argcoth}\sqrt{2}}{120\sqrt{2}}$
$I_{20}^{(1)}\left(\frac{\sqrt{2}}{4},\frac{\pi}{3}\right)$	$\frac{1}{20\sqrt{3}}-\frac{29}{640\sqrt{2}}-\frac{\pi}{90}+\frac{43\ln 3}{7680\sqrt{2}}+\frac{1}{15}\operatorname{arccot}\sqrt{2}$

Table F.1: Selected values of $I_{ij}^{(1)}(q,\gamma)$ for various arguments

List of equivalent values

Note that the values in Table F.1 are get by not only recursions alone, but also with addition to the following rules (equivalent replacement rules). These rules are only aesthetic and have no effect on the correctness of our results.

$$\arcsin\frac{1}{\sqrt{3}} \to \frac{\pi}{2} - \arctan\sqrt{2}$$
$$\arcsin\sqrt{\frac{2}{3}} \to \frac{\pi}{2} - \operatorname{arccot}\sqrt{2}$$
$$\operatorname{argsinh} 1 \to \operatorname{argcoth}\sqrt{2}$$
$$\operatorname{argsinh}\frac{1}{\sqrt{3}} \to \frac{\ln 3}{2}$$
$$\operatorname{argsinh}\frac{1}{\sqrt{2}} \to \operatorname{argcoth}\sqrt{3}$$

Bibliography

[1] **Affentranger F.** 1988. Generalization of a formula of C. Buchta about the convex hull of random points. *Elemente der Mathematik* **43**.2:39–45 and 151–152.

[2] **Alagar V.** 1977. On the distribution of a random triangle. *Journal of Applied Probability*:284–297.

[3] **Badertscher E.** 1989. An explicit formula about the convex hull of random points. *Elemente der Mathematik* **44**.4:104–106.

[4] **Bäsel U.** 2021. The moments of the distance between two random points in a regular polygon. *arXiv preprint arXiv:2101.03815*.

[5] **Beck D, Lv Z, Potechin A.** 2023. The Sixth Moment of Random Determinants. *Journal of Integer Sequences* **26**.6.

[6] **Beck D.** 2023. Efron's Mean Volume Formula in Higher Dimensions. *arXiv preprint arXiv:2308.02854*.

[7] **Beck D.** 2023. *Mean distance in polyhedra*. eprint: `arXiv:2309.13177`.

[8] **Beck D.** 2023. On the fourth moment of a random determinant. *Random Matrices: Theory and Applications*. URL: `https://doi.org/10.1142/s2010326323500107`.

[9] **Beck D.** 2024. On Random Simplex Picking Beyond the Blashke Problem. *arXiv preprint arXiv:2412.07952*.

[10] **Beck D.** 2025. The Probability that a Random Triangle in a Cube is Obtuse. *arXiv preprint arXiv:2501.11611*.

[11] **Bhatnagar S.** *Proving* $Li_3\left(-\frac{1}{3}\right) - 2Li_3\left(\frac{1}{3}\right) = -\frac{\log^3 3}{6} + \frac{\pi^2}{6}\log 3 - \frac{13\zeta(3)}{6}$*?* Mathematics Stack Exchange. (version: 2014-02-15). URL: `https://math.stackexchange.com/q/676124`.

[12] **Bonnet G, Gusakova A, Thäle C, Zaporozhets D.** 2021. Sharp inequalities for the mean distance of random points in convex bodies. *Advances in Mathematics* **386**:107813.

[13] **Buchta C.** 1984. Zufallspolygone in konvexen Vielecken. *J. Reine Angew. Math.*

[14] **Buchta C.** 1986. On a conjecture of RE Miles about the convex hull of random points. *Monatshefte für Mathematik* **102**.2:91–102.

[15] **Buchta C.** 2005. An identity relating moments of functionals of convex hulls. *Discrete & Computational Geometry* **33**:125–142.

[16] **Buchta C.** 2023. The Duality of the Volumes and the Numbers of Vertices of Random Polytopes. *Discrete & Computational Geometry*:1–9.

[17] **Buchta C, Müller J.** 1984. Random polytopes in a ball. *Journal of applied probability* **21**.4:753–762.

[18] **Buchta C, Reitzner M.** 1992. What is the expected volume of a tetrahedron whose vertices are chosen at random from a given tetrahedron? *Anz. Österreich. Akad. Wiss. Math.-Natur. Kl* **129**:63–68.

[19] **Buchta C, Reitzner M.** 1997. Equiaffine inner parallel curves of a plane convex body and the convex hulls of randomly chosen points. *Probability Theory and Related Fields* **108**.3:385–415.

[20] **Buchta C, Reitzner M.** 2001. The convex hull of random points in a tetrahedron: Solution of Blaschke's problem and more general results. *Journal für die reine und angewandte Mathematik* **2001**.536:1–29.

[21] **Ciccariello S.** 2020. The chord-length distribution of a polyhedron. *Acta Crystallographica Section A: Foundations and Advances* **76**.4:474–488.

[22] **Cowan R.** 2010. Recurrence relationships for the mean number of faces and vertices for random convex hulls. *Discrete & Computational Geometry* **43**.2:209–220.

[23] **Deltheil R.** 1926. *Probabilités géométriques*. Vol. 2. Gauthier-Villars.

[24] **Dembo A.** 1989. On random determinants. *Quarterly of applied mathematics* **47**.2:185–195.

[25] **Dunbar SR.** 1997. The average distance between points in geometric figures. *The College Mathematics Journal* **28**.3:187–197.

[26] **Efron B.** 1965. The convex hull of a random set of points. *Biometrika* **52**.3-4:331–343.

[27] **Eisenberg B, Sullivan R.** 1996. Random triangles in n dimensions. *The American mathematical monthly* **103**.4:308–318.

[28] **Finch S.** 2010. Perimeter variance of uniform random triangles. *arXiv preprint arXiv:1007.0261*.

[29] **Finch SR.** 2018. *Mathematical Constants II*. CAMBRIDGE University Press.

[30] **Flajolet P, Sedgewick R.** 2009. *Analytic combinatorics*. cambridge University press.

[31] **Forsythe GE, Tukey JW.** 1952. The extent of n-random unit vectors. *Bulletin of the American Mathematical Society*. Vol. 58. 4:502–502.

[32] **Fortet R.** 1951. Random determinants. *J. Research Nat. Bur. Standards* **47**:465–470.

[33] **Gradshteyn IS, Ryzhik IM.** 2014. *Table of integrals, series, and products*. Academic press.

[34] **Hall GR.** 1982. Acute triangles in the n-ball. *Journal of Applied Probability* **19**.3:712–715.

[35] **Henze N.** 1983. Random triangles in convex regions. *Journal of Applied Probability* **20**.1:111–125.

[36] **Hostinský B.** 1925. *Sur les probabilités géométriques*. Přírodovědecká fakulta Masarykovy university.

[37] **Hug D, Weil W, et al.** 2020. *Lectures on convex geometry*. Vol. 286. Springer.

[38] **Kabluchko Z, Steigenberger DA.** 2025. Beta Polytopes and Beta Cones: An Exactly Solvable Model in Geometric Probability. *arXiv:2503.22488*.

[39] **Kendall DG.** 1985. Exact distributions for shapes of random triangles in convex sets. *Advances in Applied Probability* **17**.2:308–329.

[40] **Kingman JF.** 1969. Random secants of a convex body. *Journal of Applied Probability*:660–672.

[41] **Klee V.** 1969. What is the expected volume of a simplex whose vertices are chosen at random from a given convex body? *The American Mathematical Monthly* **76**.3:286–288.

[42] **Langford E.** 1969. The probability that a random triangle is obtuse. *Biometrika* **56**.3:689–690.

[43] **Maesumi A.** 2019. Triangle Inscribed-Triangle Picking. *The College Mathematics Journal* **50**.5:364–371.

[44] **Mannion D.** 1994. The volume of a tetrahedron whose vertices are chosen at random in the interior of a parent tetrahedron. *Advances in Applied Probability*:577–596.

[45] **Mathai AM et al.** 1982. On a conjecture in geometric probability regarding asymptotic normality of a random simplex. *The Annals of Probability* **10**.1:247–251.

[46] **Mathai AM.** 1999. *An introduction to geometrical probability: distributional aspects with applications*. Vol. 1. CRC Press.

[47] **Mathar RJ.** *Evaluating* $\int_1^{\sqrt{2}} \frac{\coth^{-1}(x)}{(1+x)\sqrt{2-x^2}}\, dx$. Mathematics Stack Exchange. (version: 2025-01-17). URL: https://math.stackexchange.com/q/5024288.

[48] **Miles RE.** 1971. Isotropic random simplices. *Advances in Applied Probability* **3**.2:353–382.

[49] **Muirhead RJ.** 1982. *Aspects of multivariate statistical theory*. John Wiley & Sons.

[50] **Nyquist H, Rice S, Riordan J.** 1954. The distribution of random determinants. *Quarterly of Applied mathematics* **12**.2:97–104.

[51] **Philip J.** 2005. *The Area of a Random Convex Polygon in a Triangle*. TRITA MAT 04 MA 07. URL: https://people.kth.se/~johanph/area.pdf.

[52] **Philip J.** 2006. *The average volume of a random tetrahedron in a tetrahedron.* TRITA MAT 06 MA 02. URL: https://people.kth.se/~johanph/ev.pdf.

[53] **Philip J.** 2007. *The Expected Volume of a Random Tetrahedron in a Cube.* Unpublished manuscript. URL: https://people.kth.se/~johanph/ETC.pdf.

[54] **Philip J.** 2010. *The area of a random triangle in a regular hexagon.* TRITA MAT 10 MA 12. URL: https://people.kth.se/~johanph/hexagon.pdf.

[55] **Philip J.** 2010. *The area of a random triangle in a square.* TRITA MAT 10 MA 01. URL: https://people.kth.se/~johanph/squaref.pdf.

[56] **Philip J.** 2012. *The area of a random triangle in a regular pentagon and the golden ratio.* TRITA MAT 12 MA 02. URL: https://people.kth.se/~johanph/pentagon.pdf.

[57] **Prékopa A.** 1967. On random determinants I. *Studia Sci. Math. Hungar* **2**.1-2:125–132.

[58] **Ranošová H.** 2021. Sféricky symetrické míry. Bakalářská práce. Praha: Univerzita Karlova, Matematicko-fyzikální fakulta, Katedra pravděpodobnosti a matematické statistiky.

[59] **Reed W.** 1974. Random points in a simplex. *Pacific Journal of Mathematics* **54**.2:183–198.

[60] **Robbins D, Bolis T.** 1978. Average Distance between Two Points in a Box. *Amer. Math. Monthly* **85**:278.

[61] **Ruben H, Reed W.** 1973. A more general form of a theorem of Crofton. *Journal of Applied Probability*:479–482.

[62] **Ruben H, Miles R.** 1980. A canonical decomposition of the probability measure of sets of isotropic random points in Rn. *Journal of Multivariate Analysis* **10**.1:1–18.

[63] **Rubin B.** 2018. A note on the Blaschke-Petkantschin formula, Riesz distributions, and Drury's identity. *Fractional Calculus and Applied Analysis* **21**.6:1641–1650.

[64] **Schläfli L.** 1950. Theorie der vielfachen Kontinuität. *Gesammelte Mathematische Abhandlungen: Band I*:167–387.

[65] **Small C.** 1982. Random uniform triangles and the alignment problem. *Mathematical Proceedings of the Cambridge Philosophical Society.* Vol. 91. 2. Cambridge University Press:315–322.

[66] **Sommerville DM.** 1927. The relations connecting the angle-sums and volume of a polytope in space of n dimensions. *Proceedings of the Royal Society of London. Series A, Containing papers of a mathematical and physical character* **115**.770:103–119.

[67] **Stanley RP.** *Expected size of determinant of AA^T for non-square random A.* MathOverflow. (version: 2015-07-02). URL: https://mathoverflow.net/q/210668.

[68] **Stanley RP, Fomin S.** 1999. *Enumerative Combinatorics.* Vol. 2. Cambridge Studies in Advanced Mathematics. Cambridge University Press:152.

[69] **Sullivan R.** 1996. *Crofton's theorem for parameterized families of convex polygons.* Lehigh University.

[70] **Tao T.** *Expected determinant of a random NxN matrix.* MathOverflow. (version: 2010-06-09). URL: https://mathoverflow.net/q/27616.

[71] **Toth CD, O'Rourke J, Goodman JE.** 2017. *Handbook of discrete and computational geometry.* CRC press.

[72] **Tu SJ, Fischbach E.** 2000. A new geometric probability technique for an N-dimensional sphere and its applications to physics. *arXiv preprint math-ph/0004021.*

[73] **Turán P.** 1955. On a problem in the theory of determinants. *Acta Math. Sinica* **5**.41:417–423.

[74] **user170231.** *Evaluating* $\int_1^{\sqrt{2}} \frac{\tanh^{-1}(\sqrt{2-x^2})}{1+x} dx$. Mathematics Stack Exchange. (version: 2023-12-27). URL: https://math.stackexchange.com/q/4834162.

[75] **Weisstein EW.** *Tetrahedron Line Picking. From MathWorld – A Wolfram Web Resource.* accessed 21/11/2020. URL: https://mathworld.wolfram.com/TetrahedronLinePicking.html.

[76] **Weisstein EW.** *Triangle Triangle Picking. From MathWorld—A Wolfram Web Resource.* accessed 5/11/2020. URL: %5Curl%7Bhttps://mathworld.wolfram.com/TriangleTrianglePicking.html%7D.

[77] **Woolhouse W.** 1867. Some additional observations on the four-point problem. *Mathematical Questions and their Solutions from the Educational Times* **7**:81.

[78] **Zinani A.** 2003. The expected volume of a tetrahedron whose vertices are chosen at random in the interior of a cube. *Monatshefte für Mathematik* **139**.4:341–348.

Index

List of Theorems

List of Figures

List of Tables

List of publications

Beck D. 2025. The Probability that a Random Triangle in a Cube is Obtuse. *arXiv preprint arXiv:2501.11611*

Beck D. 2024. On Random Simplex Picking Beyond the Blashke Problem. *arXiv preprint arXiv:2412.07952*

Beck D. 2023. Efron's Mean Volume Formula in Higher Dimensions. *arXiv preprint arXiv:2308.02854*

Beck D. 2023. *Mean distance in polyhedra.* eprint: `arXiv:2309.13177`

Beck D. 2023. On the fourth moment of a random determinant. *Random Matrices: Theory and Applications.* URL: `https://doi.org/10.1142/s2010326323500107`

Beck D, Lv Z, Potechin A. 2023. The Sixth Moment of Random Determinants. *Journal of Integer Sequences* **26**.6

Attachments

GENERAL EFRON.nb

A Mathematica worksheet containing all necessary codes and implementations, library of solid as well as integration algorithms and various simplifications subroutines.

simplex.f90

Fortran program which computes 1st and 2nd volume moments of random 4D simplex (pentachoron) for Monte-Carlo simulation, see section 4.5.1 on 4D simplex volumetric moments.