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Analysing the Moments of the Determinant of a Random Matrix Via Analytic Combinatorics of Permutation Tables

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Abstract. We consider the following natural question. Given a matrix A with i.i.d. random entries, what are the moments of the determinant of A? In other words, what is $\mathbb{E}[\det(A)^k]$? While there is a general expression for $\mathbb{E}[\det(A)^k]$ when the entries of A are Gaussian, much less is known when the entries of A have some other distribution.

In two recent papers, we answered this question for $k = 4$ when the entries of A are drawn from an arbitrary distribution and for $k = 6$ when the entries of A are drawn from a distribution which has mean 0. These analyses used recurrence relations and were highly intricate. In this paper, we show how these analyses can be simplified considerably by using analytic combinatorics on permutation tables.

Keywords: random determinants, permutation tables, generating functions, analytic combinatorics

1 Introduction

The determinant of a matrix is a fundamental quantity which is ubiquitous in linear algebra. A natural question about the determinant which is only partially understood is to determine the moments of the determinant of a matrix A with i.i.d. random entries. In other words, given $k \in \mathbb{N}$, what is $\mathbb{E}[\det(A)^k]$?

When the entries of A are drawn from the normal distribution $\mathsf{N}(0, 1)$, there is a general formula for $\mathbb{E}[\det(A)^k]$. In particular, Forsythe and Tukey [6] and Nyquist, Rice, and Riordan [10] independently showed that when k is *even*, $\mathbb{E}[\det(A)^k] = \prod_{j=0}^{\frac{k}{2}-1} \frac{(n+2j)!}{(2j)!}$.

That said, only a few results were previously known about $\mathbb{E}[\det(A)^k]$ when the entries of A are drawn from a distribution which is not Gaussian.

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1.1 Prior work for non-Gaussian entries and our results

In order to describe prior work and our results, we need a few definitions.

Definition 1. *Given an $n \times n$ matrix A with i.i.d. entries drawn from some distribution Ω, we make the following definitions:*

1. *We define $m_r = \mathbb{E}[A_{ij}^r]$ to be the rth moment of the entries of A. When $m_1 \neq 0$, we define $B_{ij} = A_{ij} - m_1$ and $\mu_r = \mathbb{E}[B_{ij}^r]$ to be the rth centralized moment of the entries of A.*
2. *We define $f_k(n) = \mathbb{E}[\det(A)^k]$ to be the kth moment of the determinant.*
3. *We define $F_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} f_k(n)$ to be the generating function for the determinant.*

Example 1. When $n = 2$ and $k = 4$, we have $f_4(2) = \mathbb{E}\ \det(A)^4 = \mathbb{E}\ (A_{11}A_{22} - A_{12}A_{21})^4 = \mathbb{E}\ (A_{11}^4 A_{22}^4 - 4A_{11}^3 A_{22}^3 A_{12} A_{21} + 6A_{11}^2 A_{22}^2 A_{12}^2 A_{21}^2 - 4A_{11}A_{22}A_{12}^3 A_{21}^3 + A_{12}^4 A_{21}^4)$ $= m_4^2 - 4m_3^2 m_1^2 + 6m_2^4 - 4m_1^2 m_3^2 + m_4^2 = 2m_4^2 - 8m_3^2 m_1^2 + 6m_2^4$.

Example 2. For $k = 6$, when the entries of A are drawn from $\mathsf{N}(0,1)$, we have $f_6(n) = \frac{n!(n+2)!(n+4)!}{48}$ and thus $F_6(t) = \sum_{n=0}^{\infty} \frac{f_6(n)t^n}{n!^2} = \sum_{n=0}^{\infty} (n+1)(n+2)(n+4)!t^n$. Note that $F_6(t)$ diverges everywhere except $t = 0$. That said, we can still treat it as a formal power series.

We now describe prior work on the moments of the determinant of a random matrix with i.i.d. entries which are not Gaussian and our results. For $k = 2$, Turán observed that when $m_1 = 0$ and $m_2 = 1$, $f_2(n) = n!$. More generally, Fortet [7] showed that $f_2(n) = n!(m_2 + m_1^2(n-1))(m_2 - m_1^2)^{n-1}$ and $F_2(t) = (1 + m_1^2 t)e^{(m_2 - m_1^2)t}$. For $k = 4$, Nyquist, Rice, and Riordan [10] showed that when $m_1 = 0$, $F_4(t) = e^{t(m_4 - 3m_2^2)}/(1 - m_2^2 t)^3$, which implies that when $m_1 = 0$, $f_4(n) = n!^2 \sum_{j=0}^{n} \frac{1}{j!} \left(m_4 - 3m_2^2\right)^j m_2^{2n-2j} \binom{n-j+2}{2}$.

Recently, the authors made progress on analyzing $\mathbb{E}[\det(A)^k]$ in two different ways. First, the first author [2] generalised Nyquist, Rice, and Riordan's result for $k = 4$ to the setting where m_1 is arbitrary [2], showing that

$$f_4(n) = n!^2 \sum_{w=0}^{2} \sum_{s=0}^{4-2w} \sum_{c=0}^{n-s} \binom{4-2w}{s} \frac{(1+c)m_1^{s+2w} \mu_2^{2c-w} \mu_3^s \left(\mu_4 - 3\mu_2^2\right)^{n-c-s}}{(n-c-s)!(2-w)!w!} d_w(c),$$

where $d_0(c) = 2 + c$, $d_1(c) = c(2 + c)$ and $d_2(c) = c^3$.

Second, we solved the case when $k = 6$ and $m_1 = 0$ [1]. In particular, we showed that for any distribution Ω over $\mathbb{R}$ such that $m_1 = 0$ and $m_2 = 1$,

$$f_6(n) = n!^2 \sum_{j=0}^{n} \sum_{i=0}^{j} \sum_{k=0}^{n-j} \frac{(1+i)(2+i)(4+i)!}{48(n-j-k)!} \binom{10}{k} \binom{14+j+2i}{j-i} q_6^{n-j-k} q_4^{j-i} q_3^k,$$

where $q_6 = m_6 - 10m_3^2 - 15m_4 + 30$, $q_4 = m_4 - 3$, and $q_3 = m_3^2$.

Both of these analyses were technical and involved intricate recurrence relations. In this paper, we give a considerably simpler derivation of these results by using analytic combinatorics on permutation tables.

1.2 Related work

There are two natural variants of the question of determining the moments of the determinant of a random matrix which have been analysed. First, for $\mathbb{E}[\det(A)^k]$ when A is symmetric or Hermitian, Zhurbenko solved the $k = 2$ case for i.i.d. entries with mean 0 [11], and later works extended this to Wigner matrices [8, 9]. Recently, we generalised these results for $k = 2$ to Hermitian A with i.i.d. entries above the diagonal with real expected values and i.i.d. diagonal entries [3]. Second, for $\mathbb{E}[\det(U^\top U)^{k/2}]$ when U is an $n \times p$ matrix with i.i.d. entries ($p \le n$), Dembo solved the $k = 4$ case for mean-0 entries [4]. This was recently generalised to U with i.i.d. entries from an arbitrary distribution by the first author [2].

2 Techniques

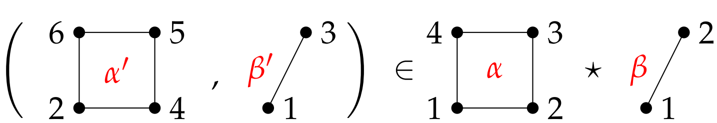
2.1 Analytic combinatorics

We now give an overview of the analytic combinatorics techniques we use for our analyses. This overview is taken from our paper "On the second moment of the determinant of random symmetric, Wigner, and Hermitian matrices" [3]. We follow the notation of the textbook *Analytic combinatorics* [5] by Flajolet and Sedgewick.

Let $\mathcal{A}$ be a set of objects with a given structure where each $\alpha \in \mathcal{A}$ has a size $|\alpha| \in \mathbb{N} \cup \{0\}$ and a weight $w(\alpha) \in \mathbb{C}$ (may be negative or even complex). We call $\mathcal{A}$ a combinatorial strucuture and view it in terms of the structure that its elements satisfy.

We say that $\mathcal{A}$ is labeled if each $\alpha \in \mathcal{A}$ is composed of atoms labeled by $[|\alpha|] = \{1, 2, 3, 4, \dots, |\alpha|\}$. Moreover, we assume that $\mathcal{A}_n = \{\alpha \in \mathcal{A} : |\alpha| = n\}$ is finite for all $n \ge 0$. We define $a_n = \sum_{\alpha \in \mathcal{A}:|\alpha|=n} w(\alpha)$ to be the total weight of the objects with size n.

Combinatorial structures can be composed together. One common composition is the *star product*. Note that a tuple $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$ cannot represent a labeled object of any structure as the atoms of α and β are labeled by $[|\alpha|]$ and $[|\beta|]$, respectively. Relabeling our α, β as α', β' so that every number from 1 to $|\alpha| + |\beta|$ appears once, we get a correctly labeled object. There are of course many ways how to relabel the objects. The canonical way is to use the *star product*. We say $(\alpha', \beta') \in \alpha \star \beta$ if the new labels in both α' and β' increase in the same order as in α and β separately. An example is illustrated below.



A key concept for labeled combinatorial structures is their exponential generating function (EGF for short) defined as $A(t) = \sum_{n=0}^{\infty} \sum_{\alpha \in \mathcal{A}_n} \frac{w(\alpha)t^n}{n!} = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$. These exponential generating functions encode the relationships between combinatorial structures (i.e., how they are composed). We can write the following relations.

$\mathcal{C}$	representation of $\mathcal{C}$	$w_{\mathcal{C}}(\gamma), \gamma \in \mathcal{C}$	$C(t)$
$\mathcal{A}+\mathcal{B}$	$\mathcal{A} \cup \mathcal{B}$	$\begin{cases} w_{\mathcal{A}}(\gamma) & \text{if } \gamma \in \mathcal{A}, \\ w_{\mathcal{B}}(\gamma) & \text{if } \gamma \in \mathcal{B} \end{cases}$	$A(t)+B(t)$
$\mathcal{A} \star \mathcal{B}$	$\{\gamma \mid \gamma \in \alpha \star \beta, \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$	$w_{\mathcal{A}}(\alpha) w_{\mathcal{B}}(\beta)$	$A(t)B(t)$
$\textsc{Seq}(\mathcal{A})$	$\sum_{k=0}^{\infty} \textsc{Seq}_k(\mathcal{A})$		$\frac{1}{1-A(t)}$
$\textsc{Set}(\mathcal{A})$	$\sum_{k=0}^{\infty} \textsc{Set}_k(\mathcal{A})$		$\exp(A(t))$
$\textsc{Cyc}(\mathcal{A})$	$\sum_{k=1}^{\infty} \textsc{Cyc}_k(\mathcal{A})$		$\ln\left(\frac{1}{1-A(t)}\right)$

The meaning of $\textsc{Seq}_k(\mathcal{A})$, $\textsc{Set}_k(\mathcal{A})$, and $\textsc{Seq}_k(\mathcal{A})$ is as follows.

- $\textsc{Seq}_k(\mathcal{A})$ is shorthand for a *sequence* and indeed it can be represented as (relabeled) k-tuples of objects taken from $\mathcal{A}$. Note that since everything is relabeled, even though α_i, α_j might be the same for different i, j, the corresponding α'_i, α'_j are always distinct. Formally, $\textsc{Seq}_k(\mathcal{A}) = \{(\alpha'_1, \ldots, \alpha'_k) \mid \alpha_i \in \mathcal{A}, i \in [k]\}$, where $(\alpha'_1, \ldots, \alpha'_k) \in \alpha_1 \star \cdots \star \alpha_k$.

- $\textsc{Set}_k(\mathcal{A})$ is a structure of *sets* of k relabeled elements, that is, the order of objects α'_i is irrelevant. Formally, $\textsc{Set}_k(\mathcal{A}) = \{\{\alpha'_1, \ldots, \alpha'_k\} \mid \alpha_i \in \mathcal{A}, i \in [k]\}$. Alternatively, $\textsc{Set}_k(\mathcal{A})$ can be represented as the structure of classes of k-tuples in $\textsc{Seq}_k(\mathcal{A})$ which differ up to some permutation.

- $\textsc{Cyc}_k(\mathcal{A})$ represents the structure of classes of k-tuples in $\textsc{Seq}_k(\mathcal{A})$ which differ up to some cyclical permutation.

For completeness, we briefly explain these results. To see that the exponential generating function for $\mathcal{A} \star \mathcal{B}$ is $A(t)B(t)$, let a_n, b_n, and c_n be the total weight of the objects of size n in $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A} \star \mathcal{B}$ respectively. We have that $c_n = \sum_{j=0}^{n} \binom{n}{j} a_j b_{n-j}$, so $C(t) = \sum_{n=0}^{\infty} \frac{c_n t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{a_j t^j}{j!} \cdot \frac{b_{n-j} t^{n-j}}{(n-j)!} = A(t)B(t)$. The generating functions for $\textsc{Seq}(\mathcal{A})$, $\textsc{Set}(\mathcal{A})$, and $\textsc{Cyc}(\mathcal{A})$ come from the Taylor series $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, and $-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$ respectively.

Example 3. Let $\mathcal{D}$ be the combinatorial structure of all derangements (i.e., permutations of $[n]$ where no element is mapped to itself). Any derangement can be decomposed into cycles of length at least two. Attaching a tag u to each cycle so that the weight of each derangement is equal to $u^{\#\text{ of cycles}}$, we get a structure $\mathcal{D}_u$ which can be also constructed as $\mathcal{D}_u = \textsc{Set}\big(u\,\textsc{Cyc}_{\geq 2}(\textcircled{1})\big)$. In terms of generating functions, this translates to $D_u(t) = \exp(-ut - u\ln(1-t)) = e^{-ut}/(1-t)^u$.

2.2 Permutation tables

Definition 2. *We say τ is a permutation (k-)table with $n = \text{col}(\tau)$ columns, if its rows are permutations $\pi_j, j = 1, \ldots, k$ of order n. We denote $F_{k,n}$ the set of all such k-tables with n columns and by F_k we denote k-tables with any number of columns. That is, structurally, $F_k = \sum_{n=0}^{\infty} F_{k,n}$.*

Definition 3. *We define the sign of a table as the product of signs of permutations in its rows.*

Definition 4. *We define the weight of the i-th column of $\tau \in F_k$ as the expectation $\mathbb{E} \prod_{j=1}^{k} A_{i\pi_j(i)}$. Then we define the weight $w(\tau)$ of the whole table τ as the product of the weights of its columns.*

Example 4. The following example in Figure 1 shows a permutation table $\tau \in F_{4,9}$ with weight $w(\tau) = m_1^{12} m_2^7 m_3^2 m_4$. The weight of each individual column is shown below each column. For instance, the second column corresponds to term $A_{26}A_{22}A_{26}A_{23}$, whose expectation is obviously $m_1^2 m_2$.

1	6	3	9	5	2	7	8	4	+
3	2	1	9	4	6	7	5	8	+
4	6	1	9	3	2	7	5	8	+
2	3	1	5	4	6	7	8	9	−
m_1^4	$m_1^2 m_2$	$m_1 m_3$	$m_1 m_3$	$m_1^2 m_2$	m_2^2	m_4	m_2^2	$m_1^2 m_2$	

Figure 1: A permutation table $\tau \in F_{4,9}$ with $w(\tau) = m_1^{12} m_2^7 m_3^2 m_4$ and $\text{sgn}(\tau) = -1$.

Proposition 5. *For any distribution Ω, we have $f_k(n) = \sum_{\tau \in F_{k,n}} w(\tau)\, \text{sgn}(\tau)$.*

Proof. This follows directly from the expansion $\det A = \sum_{\pi \in F_n} \text{sgn}(\pi) \prod_{i \in [n]} A_{i\pi(i)}$ raised to the k-th power and by taking expectation as $f_k(n) = \mathbb{E}\,(\det A)^k$. ■

Example 6. The correspondence between $f_k(n)$ and permutation tables is shown below for $n = 2$ and $k = 2$ showing $f_2(2) = 2m_2^2 - 2m_1^4 = 2(m_2 - m_1^2)(m_2 + m_1^2)$.

$(\det A)^2 =$	$A_{\mathbf{11}}^2 A_{\mathbf{22}}^2$	$- A_{\mathbf{11}} A_{\mathbf{22}} A_{12} A_{21}$	$- A_{12} A_{21} A_{\mathbf{11}} A_{\mathbf{22}}$	$+ A_{12}^2 A_{21}^2$
$F_{2,2}$:	$\begin{array}{\|c\|c\|}\hline \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline \mathbf{1} & \mathbf{2} \\ 2 & 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 1 \\ \mathbf{1} & \mathbf{2} \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 1 \\ 2 & 1 \\ \hline\end{array}$
Weight:	$m_2 m_2$	$m_1^2 m_1^2$	$m_1^2 m_1^2$	$m_2 m_2$
Sign:	+	−	−	+

2.3 Marked permutations and tables

Definition 5. *We say σ is a* marked permutation *if it was formed from some $\pi \in F_n$ in which we replaced at most one element by the mark "$\times$". We define $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\pi)$ and $B^{\times}_{i\sigma(i)} = m_1$ if i is marked and $B^{\times}_{i\sigma(i)} = B_{i\pi(i)}$ otherwise. We write $G^{\times}_n$ for the set of all marked permutations.*

Proposition 7. *In terms of marked permutations, $\det(A) = \sum_{\sigma \in G^{\times}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} B^{\times}_{i\sigma(i)}$.*

Definition 6. *We say τ is a marked k-table with n columns if its rows are marked permutations $\sigma_j, j = 1, \dots, k$ of order n. We define $G^{\times}_{k,n}$ to be the set of all such tables and let $G^{\times}_k = \sum_{n=0}^{\infty} G^{\times}_{k,n}$. We define the marked weight w of the i-th column of $\tau \in G^{\times}_{k,n}$ as the expectation $\mathbb{E} \prod_{j=1}^{k} B^{\times}_{i\sigma_j(i)}$. Similarly, we define the sign $\operatorname{sgn}(\tau)$ of τ to be the product of the signs of $\sigma_j, j = 1, \dots, k$ and we define the marked weight $w(\tau)$ of τ to be the product of the weights of its individual columns.*

Example 8. Figures 2 and 3 show two examples of marked permutation tables.

1	$\times$	3	4	5	2	7	8	9
3	2	1	9	4	6	7	5	8
1	$\times$	3	9	4	2	7	5	8
3	2	1	4	5	6	7	8	9

Figure 2: $\tau \in G^2_{4,9}$, $w(\tau) = m_1^2 \mu_2^{15} \mu_4$.

$\times$	2	3	4	5	6	7	8	9
$\times$	2	1	9	4	6	7	5	8
2	$\times$	1	9	4	6	7	5	8
2	$\times$	3	4	5	6	7	8	9

Figure 3: $\tau \in G^4_{4,9}$, $w(\tau) = m_1^4 \mu_2^{12} \mu_4^2$.

Since $\mu_1 = 0$, it turns out that most tables in $G^{\times}_{k,n}$ have $w(\tau) = 0$.

Definition 7. *We say a table $\tau \in G^{\times}_{k,n}$ is trivial if its weight vanishes, otherwise the table is nontrivial. The set all all nontrivial tables form a subset $T^{\times}_{k,n} \subseteq G^{\times}_{k,n}$.*

Proposition 9. *For any distribution Ω, we have $f_k(n) = \sum_{\tau \in T^{\times}_{k,n}} w(t) \operatorname{sgn}(\tau)$.*

Example 10. The correspondence between $f_k(n)$ and marked permutation tables is shown below for $n = 2$ and $k = 2$. By summing up the contribution from all nontrivial marked tables and since $\mu_2 = m_2 - m_1^2$, we again get $f_2(2) = 2\mu_2^2 + 4m_1^2\mu_2 = 2(m_2 - m_1^2)(m_2 + m_1^2)$.

$(\det A)^2 =$	$B_{11}^2 B_{22}^2$	$+ \; B_{12}^2 B_{21}^2$	$+ \; m_1^2 B_{22}^2$	$+ \; m_1^2 B_{21}^2$	$+ \; B_{11}^2 m_1^2$	$+ \; B_{12}^2 m_1^2$
$T^{\times}_{2,2}$:	1 2 1 2	2 1 2 1	$\times$ 2 $\times$ 2	$\times$ 1 $\times$ 1	1 $\times$ 1 $\times$	2 $\times$ 2 $\times$
w:	$\mu_2\mu_2$	$\mu_2\mu_2$	$m_1^2\mu_2$	$m_1^2\mu_2$	$\mu_2 m_1^2$	$\mu_2 m_1^2$
Sign:	$+$	$+$	$+$	$+$	$+$	$+$

Definition 8. *We denote $\mathcal{F}_{k,n}$ as $F_{k,n}$ tables with irrelevant column order and $\mathcal{F}_k = \sum_{n=0}^{\infty} \mathcal{F}_{k,n}$. We define $\mathcal{G}^{\times}_{k,n}$ and $\mathcal{T}^{\times}_{k,n}$ in the same way. These tables now have exponential generating functions and can thus be analysed by tools of analytic combinatorics for labeled combinatorial structures.*

3 General fourth moment

In this section, we generalise the result for $F_4(t)|_{m_1=0}$ by Nyquist, Rice and Riordan.

Theorem 11 (Beck 2023 [2])**.** *For any distribution of* Ω*,*

$$F_4(t) = \frac{e^{t(\mu_4 - 3\mu_2^2)}}{(1-\mu_2^2 t)^3}\left((1+m_1\mu_3 t)^4 + 6m_1^2\mu_2 t\frac{(1+m_1\mu_3 t)^2}{1-\mu_2^2 t} + m_1^4 t\frac{1+7\mu_2^2 t + 4\mu_2^4 t^2}{(1-\mu_2^2 t)^2}\right).$$

Corollary 11.1. *Furthermore, defining* $d_0(c) = 2 + c$*,* $d_1(c) = c(2+c)$ *and* $d_2(c) = c^3$*,*

$$f_4(n) = n!^2 \sum_{w=0}^{2}\sum_{s=0}^{4-2w}\sum_{c=0}^{n-s}\binom{4-2w}{s}\frac{(1+c)m_1^{s+2w}\mu_2^{2c-w}\mu_3^s\left(\mu_4 - 3\mu_2^2\right)^{n-c-s}}{(n-c-s)!(2-w)!w!}d_w(c).$$

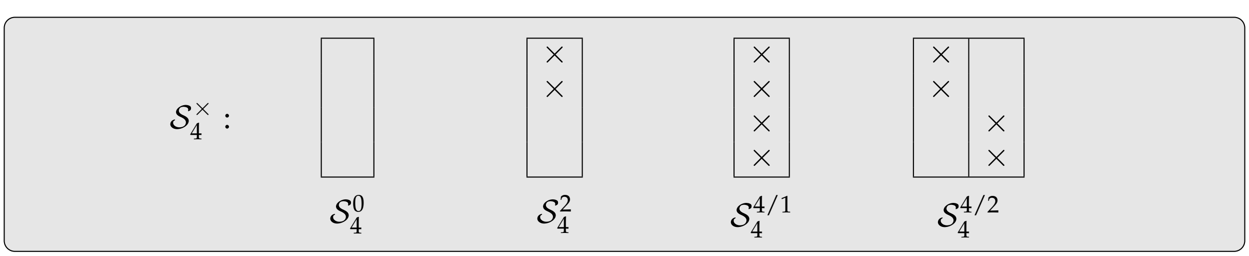
Proof. Without the loss of generality, we set $\mu_2 = 1$ (the general case is obtained by the scaling property of determinants). Let a, b denote different numbers selected from $[n] = \{1, 2, 3, \ldots, n\}$. Up to a permutation of rows, the only way how the columns of 4 by n tables with nonzero weight could look like is the following:

Type:	4-column	2-column	$\times^1$-column	$\times^2$-column	$\times^4$-column
$\mathcal{T}_4^\times$:	a a a a	a a b b	$\times$ a a a	$\times$ $\times$ a a	$\times$ $\times$ $\times$ $\times$
Weight w:	μ_4	1	$m_1\mu_3$	m_1^2	m_1^4

The $\times^1$ columns contain a single element a, one instance of which is covered by $\times$, hence they are **disjoint** from the rest of a table. As a result, we can consider only tables $\mathcal{S}_4^\times \subset \mathcal{T}_4^\times$ which do not contain $\times^1$-columns. In any given table $\tau \in \mathcal{T}_4^\times$, there could be either four, two or no $\times^1$-columns. In terms generating functions, this corresponds to

$$F_4(t) = (1 + m_1\mu_3 t)^4 S_4^0(t) + (1 + m_1\mu_3 t)^2 S_4^2(t) + S_4^4(t), \tag{3.1}$$

where $S_4^r(t)$ denotes EGF of tables $\mathcal{S}_4^r \subset \mathcal{S}_4^\times$ with r marks containing no $\times^1$ columns. It is convenient to denote $\mathcal{S}_4^{r/s} \subseteq \mathcal{S}_4^r$ as tables whose marks are distributed in exactly s different columns. Since there is at most one mark per row, $\mathcal{S}_4^{r/s}$ tables contain only few marked columns (see below). Structurally, $\mathcal{S}_4^\times = \mathcal{S}_4^0 + \mathcal{S}_4^2 + \mathcal{S}_4^4$ with $\mathcal{S}_4^4 = \mathcal{S}_4^{4/1} + \mathcal{S}_4^{4/2}$.



Proposition 12 (Nyquist, Rice and Riordan 1954 [10]). $S_4^0(t) = F_4(t)|_{m_1=0, m_4 \to \mu_4} = \frac{e^{t(\mu_4-3)}}{(1-t)^3}$.

Proof. Since μ_4 equals m_4 when $m_1 = 0$, $S_4^0(t)$ coincides with the expression for $F_4(t)|_{m_1=0}$ obtained by Nyquist, Rice and Riordan. In $\mathcal{S}_4^0$ tables, the 4-columns are disjoint from the remaining 2-columns. Furthermore, the 2-columns can be further divided into disjoint components. To a given table of 2-columns, we can associate a derangement π whose cycles correspond to disjoint sub-tables into which this table decomposes. To make this association a bijection, there are 3 ways how the remaining elements in a given sub-table can be arranged (see Figure 4, each arrangement is represented by a vertical box with four slots filled with two dots representing in which rows the number in the first row appears). Since each row of appears twice in any sub-table, the overall sign of those sub-tables is always positive.

1	3	2	6	7	4	5	8	9
1	3	6	7	2	9	4	5	8
3	1	2	6	7	9	4	5	8
3	1	6	7	2	4	5	8	9

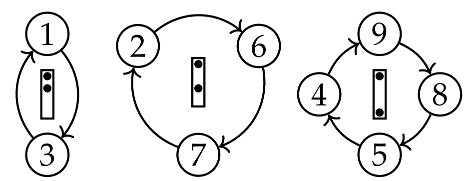
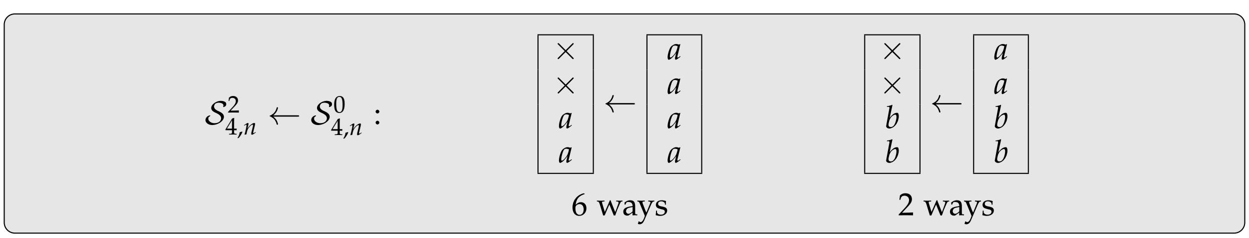


Figure 4: One-to-one correspondence between a table τ with nine 2-columns decomposable into three disjoint sub-tables, and its associated derangement π with cycles labeled according to the repetitions of the number in the first row of τ

Hence, structurally, $\mathcal{S}_4^0 = \textsc{Set}\big(m_4 \,\textcircled{1}\big) \star \textsc{Set}\big(+3\ \textsc{Cyc}_{\geq 2}\big(+\,\textcircled{1}\big)\big)$ and thus in terms of generating functions, $S_4^0(t) = \exp(\mu_4 t)\exp(-3t - \ln(1-t))$. ■

Proposition 13. $S_4^2(t) = m_1^2(6 - 2\mu_4)\frac{\partial S_4^0(t)}{\partial \mu_4} + 2m_1^2 t\,\frac{\partial S_4^0(t)}{\partial t} = \frac{6m_1^2 t e^{t(\mu_4-3)}}{(1-t)^4}$.

Proof. Let $\tau' \in \mathcal{S}_{4,n}^0$ have c 4-columns and thus $n-c$ 2-columns. Its weight is μ_4^c. From this τ', we create $\tau \in \mathcal{S}_{4,n}^2$ by covering one pair of identical elements by marks in either 4-column or 2-column. The contribution of τ' to $\sum_{\tau \in \mathcal{S}_{4,n}^2} w(\tau)\operatorname{sgn}\tau$ is then $6cm_1^2\mu_4^{c-1} + 2(n-c)m_1^2\mu_4^c = m_1^2(6-2\mu_4)\frac{\partial \mu_4^c}{\partial \mu_4} + 2m_1^2 n\mu_4^c$.



■

Proposition 14. $S_4^{4/1}(t) = m_1^4(1-\mu_4)\frac{\partial S_4^0(t)}{\partial \mu_4} + m_1^4 t\,\frac{\partial S_4^0(t)}{\partial t} = \frac{m_1^4 t(1+2t)}{(1-t)^4}e^{t(\mu_4-3)}$.

Proof. Let $\tau' \in \mathcal{S}_{4,n}^0$ have c 4-columns and thus $n-c$ 2-columns, $w(\tau') = \mu_4^c$. From this τ', we create $\tau \in \mathcal{S}_{4,n}^{4/1}$ by covering an entire 4-column or 2-column. The contribution of τ' to $\sum_{\tau\in\mathcal{S}_{4,n}^{4/1}} w(\tau)\operatorname{sgn}\tau$ is $cm_1^4\mu_4^{c-1}+(n-c)m_1^4\mu_4^c = m_1^4(1-\mu_4)\frac{\partial \mu_4^c}{\partial \mu_4}+m_1^4 n\mu_4^c$.

$$\mathcal{S}_{4,n}^{4/1} \leftarrow \mathcal{S}_{4,n}^{0} : \quad \begin{array}{|c|}\hline \times\\ \times\\ \times\\ \times\\ \hline\end{array} \leftarrow \begin{array}{|c|}\hline a\\ a\\ a\\ a\\ \hline\end{array} \qquad \begin{array}{|c|}\hline \times\\ \times\\ \times\\ \times\\ \hline\end{array} \leftarrow \begin{array}{|c|}\hline a\\ a\\ b\\ b\\ \hline\end{array}$$

■

Proposition 15. $S_4^{4/2}(t) = (3-\mu_4)\frac{\partial S_4^{4/1}(t)}{\partial \mu_4} + t\,\frac{\partial S_4^{4/1}(t)}{\partial t} - S_4^{4/1}(t) = \frac{6m_1^4 t^2(1+t)}{(1-t)^5}e^{t(\mu_4-3)}$.

Proof. Let $\tau' \in \mathcal{S}_{4,n}^{4/1}$ have c 4-columns and thus $n-c-1$ 2-columns as now one column is a $\times^4$-column. The weight of τ' is $m_1^4\mu_4^c$. From this τ', we create $\tau \in \mathcal{S}_{4,n}^{4/2}$ by **swapping** its two $\times$ marks with a pair of numbers in a 4-column or 2-column. By symmetry, each table in $\mathcal{S}_{4,n}^{4/2}$ is counted twice. The contribution of τ' to $2\sum_{\tau\in\mathcal{S}_{4,n}^{4/2}} w(\tau)\operatorname{sgn}(\tau)$ is $6cm_1^4\mu_4^{c-1}+2(n-c-1)m_1^4\mu_4^c = (6-2\mu_4)\frac{\partial(m_1^4\mu_4^c)}{\partial \mu_4}+2nm_1^4\mu_4^c - 2m_1^4\mu_4^c$.

$$2\mathcal{S}_{4,n}^{4/2} \leftarrow \mathcal{S}_{4,n}^{4/1} : \quad \underbrace{\begin{array}{|c|c|}\hline \times & a\\ \times & a\\ a & \times\\ a & \times\\ \hline\end{array} \leftarrow \begin{array}{|c|c|}\hline \times & a\\ \times & a\\ \times & a\\ \times & a\\ \hline\end{array}}_{\text{6 ways}} \qquad \underbrace{\begin{array}{|c|c|}\hline \times & a\\ \times & a\\ b & \times\\ b & \times\\ \hline\end{array} \leftarrow \begin{array}{|c|c|}\hline \times & a\\ \times & a\\ \times & b\\ \times & b\\ \hline\end{array}}_{\text{2 ways}}$$

■

Corollary 15.1. $S_4^4(t) = S_4^{4/1}(t) + S_4^{4/2}(t) = \frac{m_1^4 t(1+7t+4t^2)}{(1-t)^5}e^{t(\mu_4-3)}$.

Corollary 15.2. $F_4(t) = \frac{e^{t(\mu_4-3)}}{(1-t)^3}\left((1+m_1\mu_3 t)^4 + 6m_1^2 t\frac{(1+m_1\mu_3 t)^2}{1-t} + m_1^4 t\frac{1+7t+4t^2}{(1-t)^2}\right)$.

■

4 Sixth moment when $m_1 = 0$

The proof of the following theorem was already established in our paper [1]. In this section, we provide a more compact version of the proof based on inclusion/exclusion and the fact we know the EGF for the special case where A_{ij} is normally distributed.

Theorem 16 (Beck, Lv, Potechin 2023 [1]). *For any distribution of Ω with $m_1 = 0$,*

$$F_6(t)|_{m_1=0} = (1+m_3^2t)^{10}\frac{e^{t\left(m_6-15m_4m_2-10m_3^2+30m_2^3\right)}}{\left(1+3m_2^3t-m_4m_2t\right)^{15}}N_6\left(\frac{m_2^3t}{\left(1+3m_2^3t-m_4m_2t\right)^3}\right).$$

Corollary 16.1. *Furthermore, defining $q_6 = m_6-10m_3^2-15m_4m_2+30m_2^3$ and $q_4 = m_4m_2-3m_2^3$,*

$$f_6(n)|_{m_1=0} = n!^2\sum_{j=0}^{n}\sum_{i=0}^{j}\sum_{k=0}^{n-j}\frac{(1+i)(2+i)(4+i)!}{48(n-j-k)!}\binom{10}{k}\binom{14+j+2i}{j-i}q_6^{n-j-k}q_4^{j-i}m_3^{2k}m_2^{3i}.$$

Proof. Without the loss of generality, we assume $m_2 = 1$ throughout the proof. The fact we have $m_1 = 0$ reduces the number of tables with nontrivial weight. It is convenient to denote $\mathcal{F}_6^{\text{cen}}$ as the set of those tables (irrelevant column order), which in turn contribute to the sum $f_6(n)|_{m_1=0}$. These tables can be constructed out of the following columns (apart from permutation of rows):

Type:	6-column	4-column	2-column	3-column
$\mathcal{F}_6^{\text{cen}}$:	a,a,a,a,a,a	a,a,a,a,b,b	a,a,b,b,c,c	a,a,a,b,b,b
Weight:	m_6	m_4	1	m_3^2

In order to utilize inclusion/exclusion, we further divide the columns into two types *known* and *unknown*. An unknown column is a column where the numbers are, in addition, paired up (only the same ones). Let a, b, c be distinct integers different from integers a', b', c' (which themselves are **not necessarily distinct**, so we might have $a' = b'$). We construct our new structure of tables $\mathcal{F}_6^*$ built up from the following columns (apart from permutation of rows) with carefully designed weights:

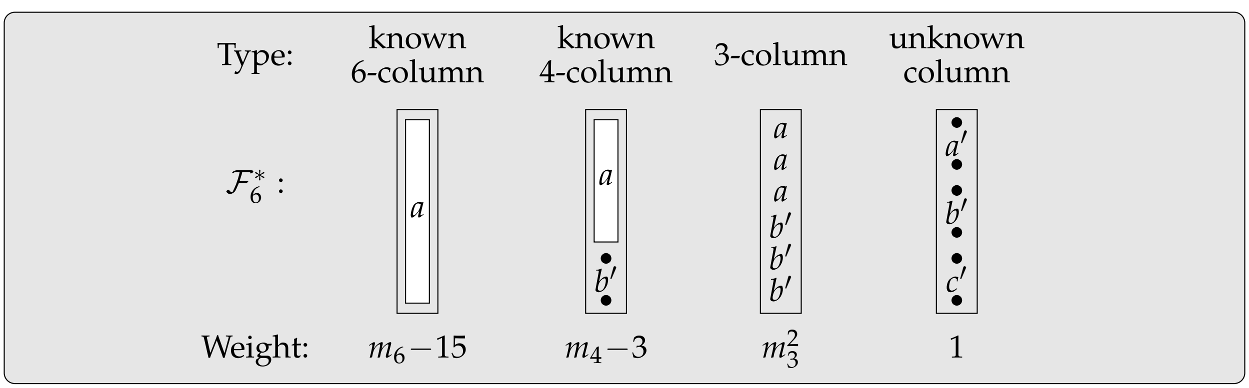


Figure 5 shows an example of a table $\tau \in \mathcal{F}^*_{6,9}$ with two known 4-columns (each with weight $m_4 - 3$) and one known 6-column. Note that since a 4-column can either be known (weight $m_4 - 3$) or unknown (there are 3 ways how we can pair up the four identical elements), the total contribution is $m_4 - 3 + 3 = m_4$, which matches the contribution of a 4-column to $\mathcal{F}^{\text{cen}}_6$. This decomposition is shown in Figure 6.

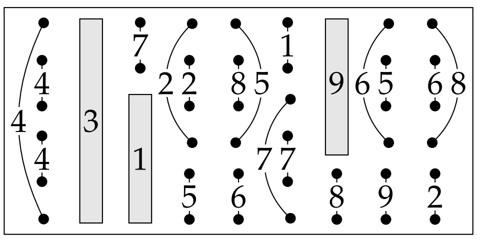


Figure 5: A table $\tau \in \mathcal{F}^*_{6,9}$ with weight $w(\tau) = (m_6 - 15)(m_4 - 3)^2$.

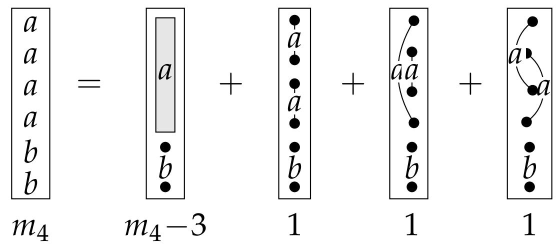


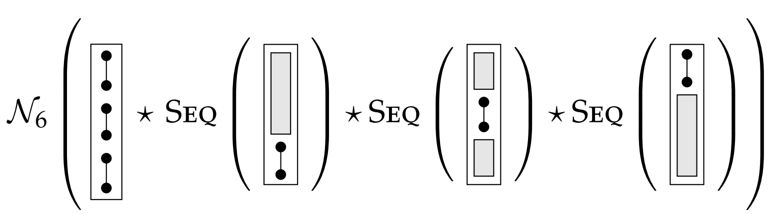
Figure 6: Inclusion/Exclusion of 4-columns

A similar analysis holds for the 6-columns. Overall, we obtain that

$$f_6(n)|_{m_1,m_3=0} = n! \sum_{\tau \in \mathcal{F}^{\text{cen}}_6} w(\tau)\,\text{sgn}(\tau) = n! \sum_{\tau \in \mathcal{F}^*_6} w(\tau)\,\text{sgn}(\tau). \tag{4.1}$$

The key idea is that permutation tables $\mathcal{F}^*_6$ can be decomposed into several components.

1. Known 6-columns, composed as $\textsc{Set}\big((m_6 - 15)\ \text{①}\big)$ with EGF equal to $e^{(m_6-15)t}$.
2. Cycles of known 4-columns, or $\textsc{Set}\big(15\,\textsc{Cyc}_{\geq 2}\big((m_4-3)\ \text{①}\big)\big)$ with EGF $\frac{e^{-15(m_4-3)t}}{(1-(m_4-3)t)^{15}}$.
3. Cycles of 3-columns, or $\textsc{Set}\big(-10\,\textsc{Cyc}_{\geq 2}\big(-m_3^2\ \text{①}\big)\big)$ with EGF $(1+m_3^2 t)^{10} e^{-10 m_3^2 t}$.
4. A "core" of unknown columns together with paths of known 4-columns leading to pairs in the core. We can analyze this part as follows. Letting $\mathcal{N}_6$ be the structure for the Gaussian case, $\mathcal{N}_6$ is also the structure for the core without the attached paths of known 4-columns with EGF being the known $N_6(t) = \frac{1}{48}\sum_{n=0}^{\infty}(n+1)(n+2)(n+4)!\,t^n$. We now observe that each column of the core has three paths (possibly of length 0) of known 4-columns leading to it. Structurally, this gives



(4.2)

Since the EGF for $\textsc{Seq}\big((m_4 - 3)\ \text{①}\big)$ is $\frac{1}{1-(m_4-3)t}$, the EGF for the core and the paths of known 4-columns leading to it is $N_6\big(\frac{t}{(1-(m_4-3)t)^3}\big)$.

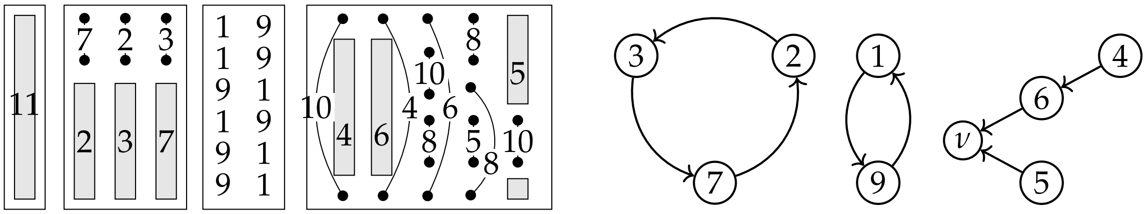


Figure 7: An $\mathcal{F}^*_{6,11}$ table with disjoint components (known 6-columns, cycles of known 4-columns, cycles of 6-columns and the core with attached paths of known 4-columns)

By joining all disjoint components (see example in Figure 7), we get, in terms of EGF's,

$$F_6(t)|_{m_1=0} = e^{(m_6-15)t} \frac{e^{-15(m_4-3)t}}{(1-(m_4-3)t)^{15}} (1+m_3^2 t)^{10} e^{-10m_3^2 t} N_6 \left(\frac{t}{(1-(m_4-3)t)^3} \right).$$

■

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