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Charles University

ON RANDOM SIMPLEX PICKING

GPSD 2025

Dominik Beck

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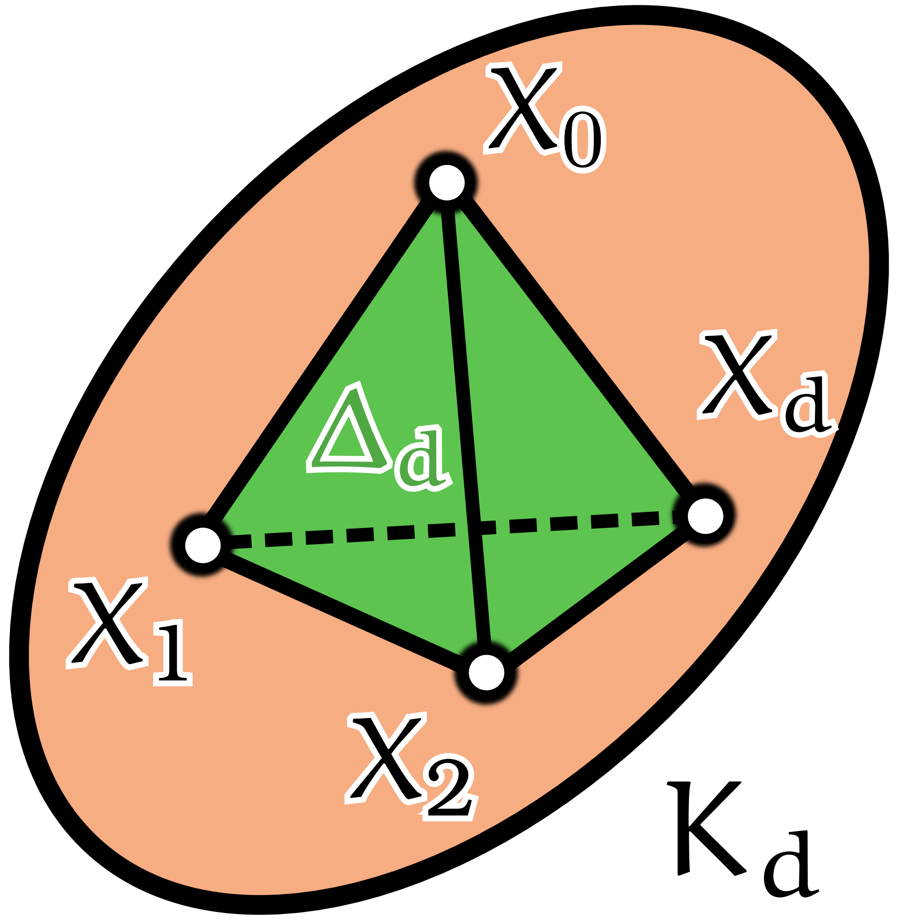
DEFINITION

Q: Let $K_d \subset \mathbb{R}^d$ be a convex d-body and $\mathbb{X} = (X_0, \ldots, X_d)$ be a collection of random points picked uniformly and independently from its interior. The convex hull of this collection is a d-simplex with volume $\Delta_d = \operatorname{vol}_d \operatorname{conv} \mathbb{X}$.



What are its k-th moments? More specifically, the affine invariant constants:

$$v_d^{(k)}(K_d) = \mathbb{E}\,\Delta_d^k / (\operatorname{vol}_d K_d)^k$$

KNOWN RESULTS

K_d		$v_d^{(1)}(K_d)$
T_2	triangle[1] 0.08333333	$\frac{1}{12}$
B_3	ball[2][3] 0.012587413	$\frac{9}{715}$
O_3	octahedron[6] 0.013637411	$\frac{19297\pi^2}{3843840} - \frac{6619}{184320}$
C_3	cube[5] 0.013842776	$\frac{3977}{216000} - \frac{\pi^2}{2160}$
T_3	tetrahedron[4] 0.017398239	$\frac{13}{720} - \frac{\pi^2}{15015}$

+ All 2D polygons
+ Other selected 3D polytopes
+ Ball in any dimension

REFERENCES [1] Sylvester (1865), Reed (1974) [2] Hostinský (1925) [3] Miles (1971) [4] Buchta & Reitzner (1992) [5] Zinani (2003), Philip (2007) [6] B. (2020)

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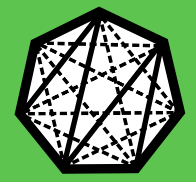
NEW RESULTS

K_d			$v_d^{(1)}(K_d)$
T_4		4-simplex 0.0031803708	$\frac{97}{27000} - \frac{2173\pi^2}{52026975}$
T_5		5-simplex 0.000523083	$\frac{2207}{3265920} - \frac{244129\pi^2}{14522729760} + \frac{73522\pi^4}{541513323351}$
T_6		6-simplex 0.000078805	$\frac{26609}{217818720} - \frac{3396146609\pi^2}{621871356506400} + \frac{1318349152898\pi^4}{1218020640129839045}$

Conjecture

$$v_r^{(k)}(T_r) = \sum_{s=1}^{\lceil r/2 \rceil} p_{rs}^{(k)} \pi^{2s-2}, \qquad p_{rs}^{(k)} \in \mathbb{Q}$$

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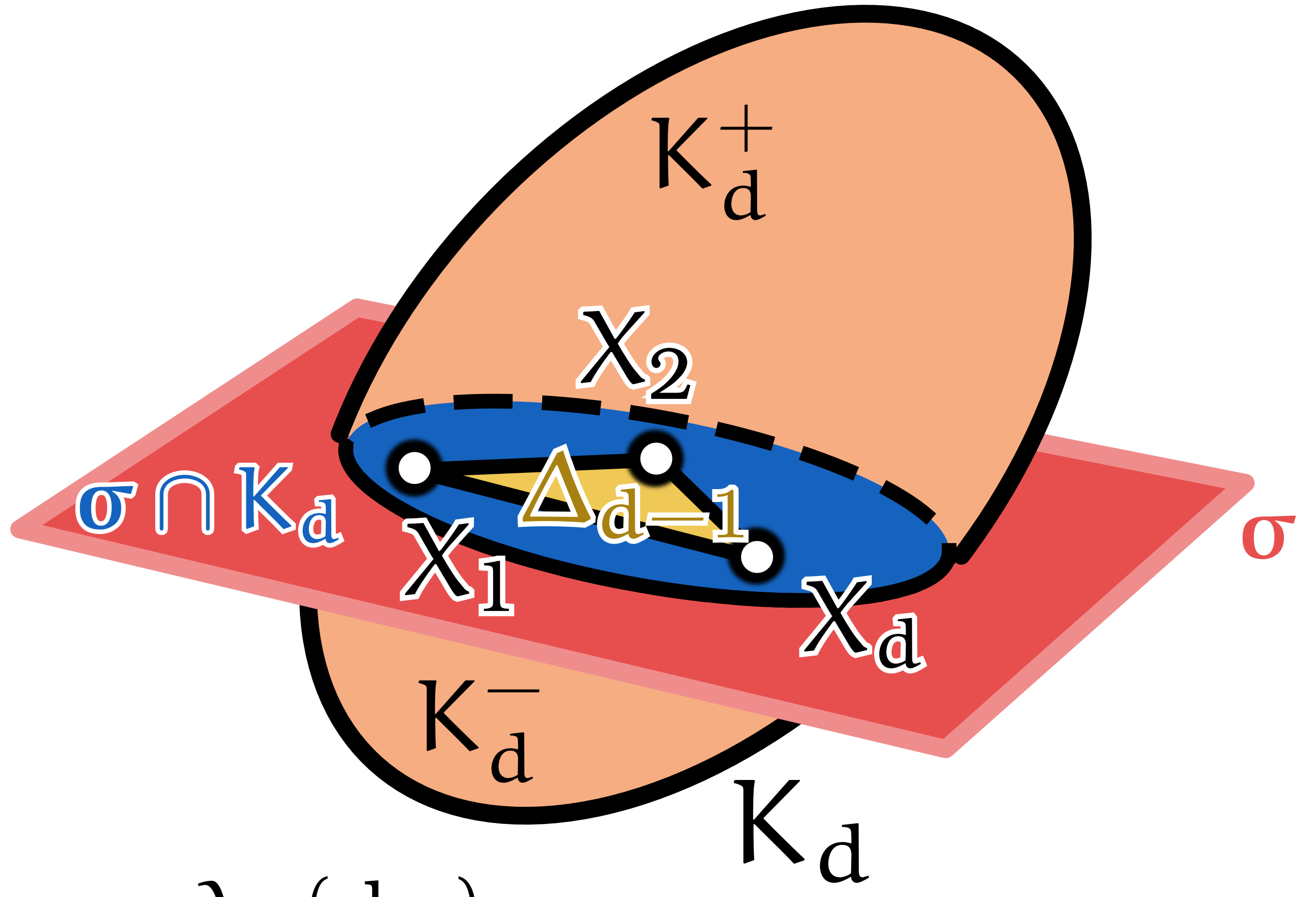
LEMMA (BLASHKE-PETKANCHIN)

$$\mathbb{E}\left[g(\sigma)\Delta_{d-1}^k\right] = \frac{(d-1)!\omega_d}{2(\mathrm{vol}_d K_d)^d}\int_{\mathbb{A}(d,d-1)} v_{d-1}^{(k+1)}(\sigma\cap K_d)(\mathrm{vol}_{d-1}(\sigma\cap K_d))^{k+d+1} g(\sigma)\mu_{d-1}(d\sigma)$$

- $\mathbb{X}' = (X_1,\dots,X_d) \implies \Delta_{d-1} = \mathrm{vol}_{d-1}\,\mathrm{conv}\,\mathbb{X}'$
- $\omega_d = \sigma_d(S_{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ surface area of the unit ball
- $\sigma = \mathcal{A}(\mathbb{X}') \in \mathbb{A}(d,d-1)$ affine (cutting) hyperplane
- $g : \mathbb{A}(d,d-1) \to \mathbb{R}$ any integrable function
- μ_{d-1} invariant measure on affine Grassmannian $\mathbb{A}(d,d-1)$



Cartesion reparametrisation

Let $x\in\sigma \Leftrightarrow \eta^\top x = 1$, where $\eta = (\eta_1,\eta_2,\dots,\eta_d)^\top$ then $\mu_{d-1}(d\sigma) = \dfrac{2}{\omega_d\|\eta\|^{d+1}}\lambda_d(d\eta)$

$$\mathbb{E}\left[g(\sigma)\Delta_{d-1}^k\right] = (d-1)!(\mathrm{vol}_d K)^{k+1}\int_{\mathbb{R}^d\setminus K_d^\circ} v_{d-1}^{(k+1)}(\sigma\cap K_d)\zeta_d^{d+k+1}(\sigma)g(\sigma)\|\eta\|^k\lambda_d(d\eta)$$

where $\zeta_d(\sigma) = \dfrac{\mathrm{vol}_{d-1}(\sigma\cap K_d)}{\|\eta\|\,\mathrm{vol}_d K_d} = -\dfrac{1}{\mathrm{vol}_d K_d}\displaystyle\sum_{j=1}^d \eta_j \frac{\partial\,\mathrm{vol}_d K_d^+}{\partial\eta_j}$ $\qquad \iota_d^{(k)}(\sigma) = \displaystyle\int_{K_d} |\eta^\top x - 1|^k \lambda_d(dx)$

REFERENCES [1] Sylvester (1865), Reed (1974) [2] Hostinský (1925) [3] Miles (1971) [4] Buchta & Reitzner (1992) [5] Zinani (2003), Philip (2007) [6] B. (2020)

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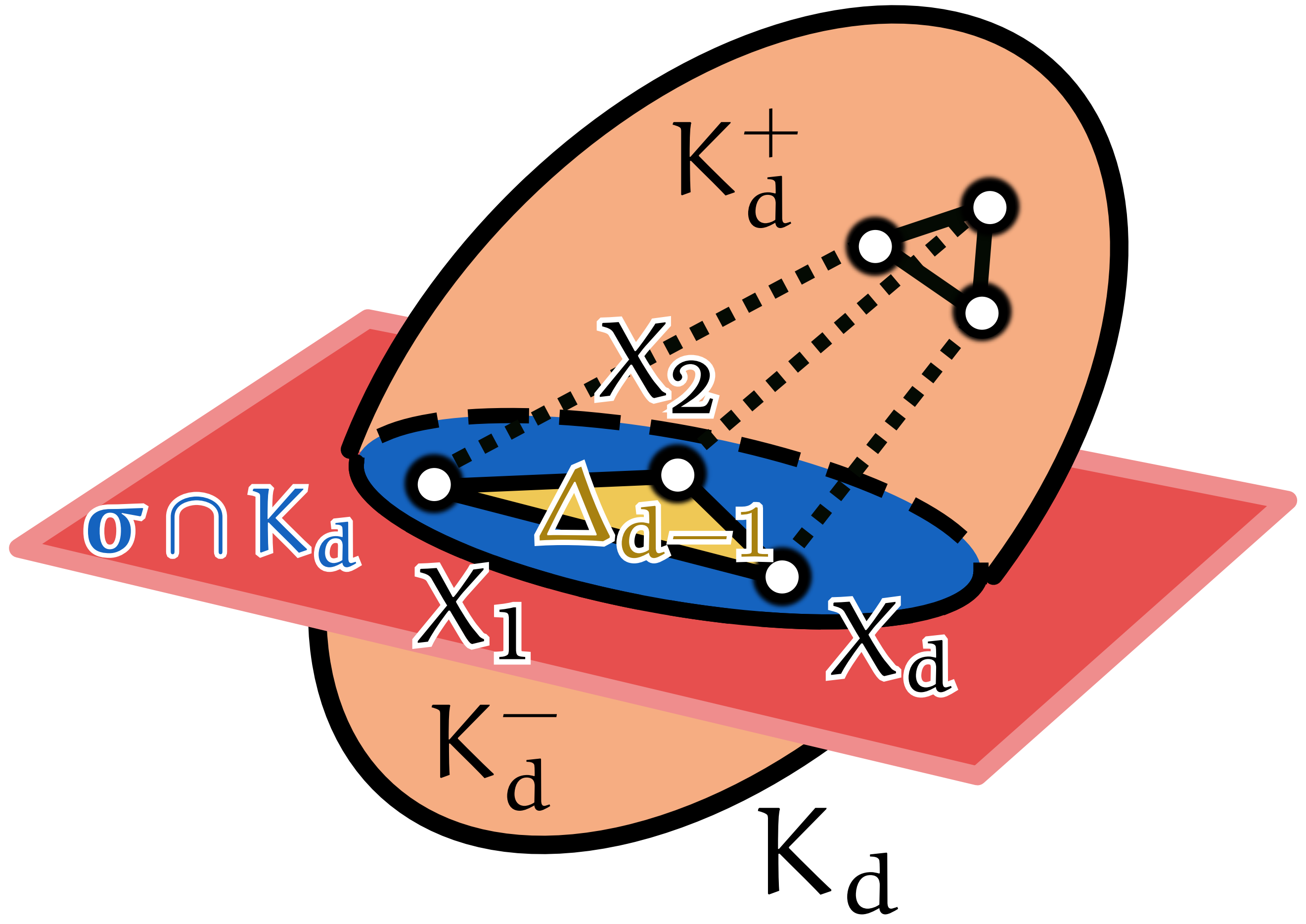
ORIGINAL APPROACH[4,5,6]

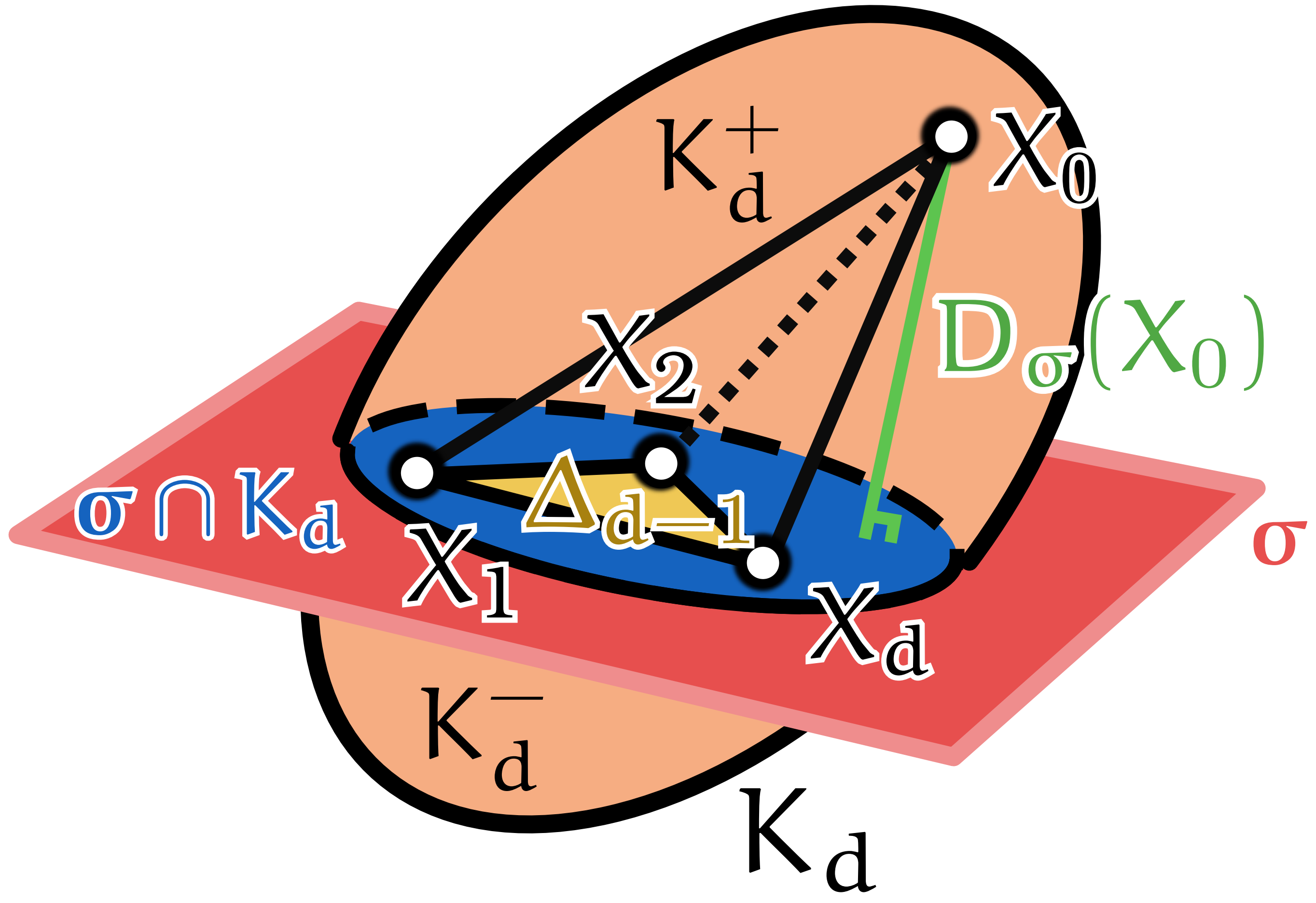


$$f_0(H_n) - f_1(H_n) + f_2(H_n) = 2$$

Efron's formula

$$\therefore v_n^{(1)}(K_3) = \frac{n-1}{n+1} - \frac{1}{6}\binom{n}{2}\mathbb{E}\left[\left(\frac{\mathrm{vol}_3 K_3^+}{\mathrm{vol}_3 K_3}\right)^{n-2} + \left(\frac{\mathrm{vol}_3 K_3^-}{\mathrm{vol}_3 K_3}\right)^{n-2}\right]$$

NEW APPROACH



$$\Delta_d = \frac{1}{d} D_\sigma(X_0)\Delta_{d-1}$$

Canonical section integral

$$\therefore v_d^{(k)}(K_d) = \frac{\mathbb{E}[D_\sigma^k(X_0)\Delta_{d-1}^k]}{d^k(\mathrm{vol}_d K_d)^k}$$

REFERENCES

[1] Sylvester (1865), Reed (1974)
[2] Hostinský (1925)
[3] Miles (1971)
[4] Buchta & Reitzner (1992)
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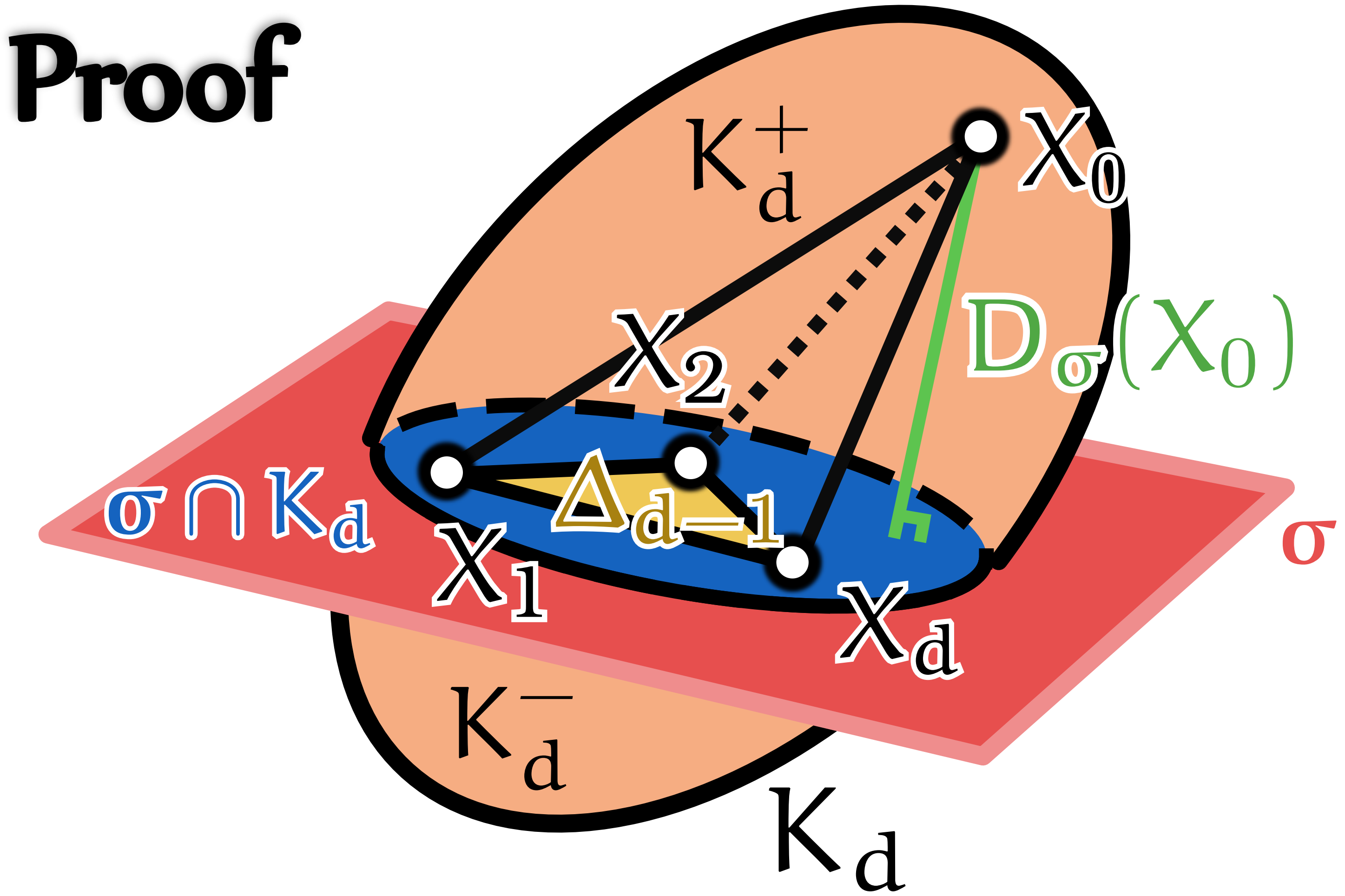
THEOREM (CANONICAL SECTION INTEGRAL) Let $\sigma = \{\mathbf{x} \in \mathbb{R}^d \mid \boldsymbol{\eta}^\top \mathbf{x} = 1\} \in \mathbb{A}(d, d-1)$ be a section plane splitting convex d-body K_d into $K_d^+ \sqcup K_d^-$, where $K_d^+ = \{\mathbf{x} \in K_d \mid \eta^\top \mathbf{x} < 1\}$, then

$$\nu_d^{(k)}(K_d) = \frac{(d-1)!}{d^k} \int_{\mathbb{R}^d \setminus K_d^\circ} \nu_{d-1}^{(k+1)}(\sigma \cap K_d)\, \zeta_d^{d+k+1}(\sigma)\, \iota_d^{(k)}(\sigma) \lambda_d(d\eta)$$

where $\zeta_d(\sigma) = \frac{\mathrm{vol}_{d-1}(\sigma \cap K_d)}{\|\boldsymbol{\eta}\| \,\mathrm{vol}_d K_d} = -\frac{1}{\mathrm{vol}_d K_d} \sum_{j=1}^{d} \eta_j \frac{\partial \,\mathrm{vol}_d K_d^+}{\partial \eta_j}$ and $\iota_d^{(k)}(\sigma) = \int_{K_d} |\boldsymbol{\eta}^\top \mathbf{x} - 1|^k \lambda_d(d\mathbf{x})$.

Proof



1. Base-height splitting: $\Delta_d = \frac{1}{d} D_\sigma(X_0) \Delta_{d-1}$ (distance from σ to X_0)

$$\therefore \nu_d^{(k)}(K_d) = \frac{\mathbb{E}[D_\sigma^k(X_0)\Delta_{d-1}^k]}{d^k (\mathrm{vol}_d K_d)^k} = \frac{\mathbb{E}[\mathbb{E}[D_\sigma^k(X_0) \mid \mathbb{X}'] \,\Delta_{d-1}^k]}{d^k (\mathrm{vol}_d K_d)^k}$$

2. Distance formula: $D_\sigma(X_0) = \frac{1}{\|\boldsymbol{\eta}\|} |\boldsymbol{\eta}^\top X_0 - 1|$

$$\therefore \mathbb{E}[D_\sigma^k(X_0) \mid \mathbb{X}'] = \frac{1}{\mathrm{vol}_d K_d} \int_{K_d} D_\sigma^k(\mathbf{x}_0) \lambda_d(d\mathbf{x}_0) = \frac{\iota_d^{(k)}(\sigma)}{\|\boldsymbol{\eta}\|^k \,\mathrm{vol}_d K_d}$$

3. The proof is conluded by taking $g(\sigma) = \mathbb{E}[D_\sigma^k(X_0) \mid \mathbb{X}']$.

REFERENCES [1] Sylvester (1865), Reed (1974) [2] Hostinský (1925) [3] Miles (1971) [4] Buchta & Reitzner (1992) [5] Zinani (2003), Philip (2007) [6] B. (2020)

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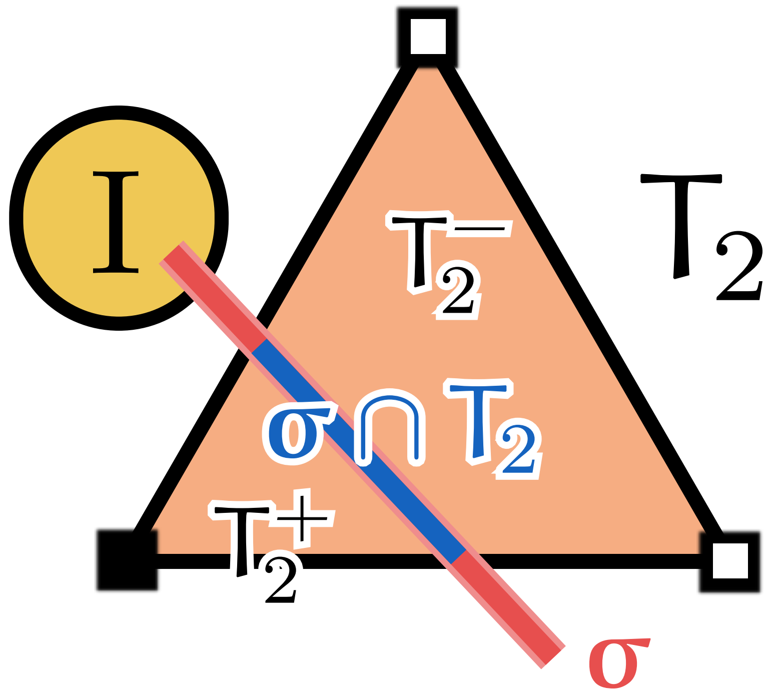
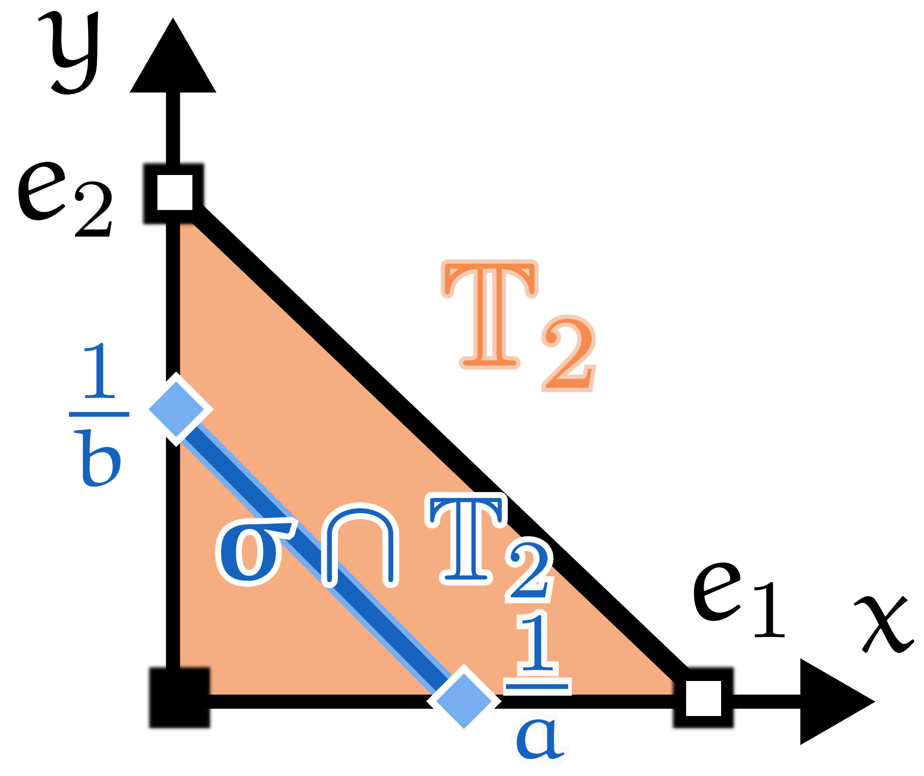
RE-DERIVATION OF REED'S RESULT IN A TRIANGLE (DEMONSTRATION 1)





$$(\mathbb{R}^2 \setminus \mathbb{T}_2^\circ)_I = (1,\infty)^2$$

$$v_2^{(k)}(T_2) = \frac{48\,k!}{(2+k)(3+k)} \sum_{j=0}^{k+1} \frac{j!}{(k-j+2)(k+j+3)!}$$

- Decomposition into configurations: $v_2^{(k)}(T_2) = 3v_2^{(k)}(T_2)_I = 3v_2^{(k)}(\mathbb{T}_2)_I$
- Affine invariancy: $T_2 \longrightarrow \mathbb{T}_2 \stackrel{\text{def}}{=} \operatorname{conv}(\mathbf{0}, e_1, e_2)$ canonical 2-simplex
- Section integral: $v_2^{(k)}(\mathbb{T}_2)_I = \frac{1}{2^k} \int_{(\mathbb{R}^2 \setminus \mathbb{T}_2^\circ)_I} v_1^{(k+1)}(\sigma \cap \mathbb{T}_2)\, \zeta_2^{k+3}(\sigma)\, \iota_2^{(k)}(\sigma) \lambda_2(\mathrm{d}\eta)$
- Section plane: $\eta = (a,b)^\top \Rightarrow \frac{1}{a}e_1, \frac{1}{b}e_2$ axes intersection points
- $\mathbb{T}_2^+ = \operatorname{conv}(\mathbf{0}, \frac{1}{a}e_1, \frac{1}{b}e_2) \therefore \operatorname{vol}_2 \mathbb{T}_2^+ = \frac{1}{2ab} = \frac{\operatorname{vol}_1(\sigma \cap \mathbb{T}_2)}{2\|\eta\|} \Rightarrow \zeta_2(\sigma)_I = \frac{\operatorname{vol}_1(\sigma \cap \mathbb{T}_2)}{\|\eta\| \operatorname{vol}_2 \mathbb{T}_2} = \frac{2}{ab}$
- $\iota_2^{(k)}(\sigma)_I = \int_{\mathbb{T}_2^+} (1 - \eta^\top x)^k \lambda_2(\mathrm{d}x) + \int_{\mathbb{T}_2 \setminus \mathbb{T}_2^+} (\eta^\top x - 1)^k \lambda_2(\mathrm{d}x) = \frac{b(a-1)^{k+2} - a(b-1)^{k+2} + a - b}{ab(a-b)(1+k)(2+k)}$
- Affine invariancy: $v_1^{(k+1)}(\sigma \cap \mathbb{T}_2) \stackrel{\text{aff.}}{=} v_1^{(k+1)}((0,1)) = \frac{2}{(2+k)(3+k)}$

$$\therefore v_2^{(k)}(\mathbb{T}_2)_I = 16 \int_1^\infty \int_1^\infty \frac{b(a-1)^{2+k} - a(b-1)^{2+k} + a - b}{a^{k+4} b^{k+4} (a-b)(1+k)(2+k)^2(3+k)} \,\mathrm{d}a\,\mathrm{d}b$$

REFERENCES [1] Sylvester (1865), Reed (1974) [2] Hostinský (1925) [3] Miles (1971) [4] Buchta & Reitzner (1992) [5] Zinani (2003), Philip (2007) [6] B. (2020)

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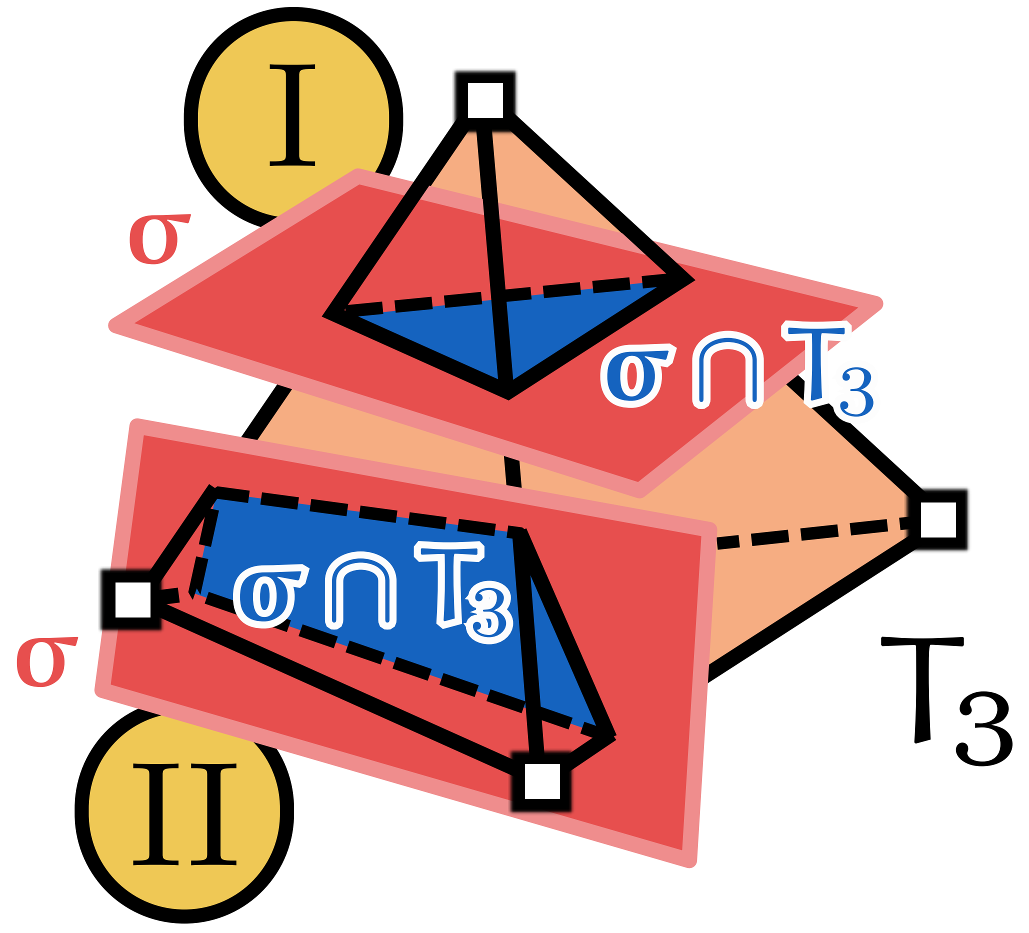
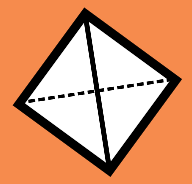
RE-DERIVATION OF BUCHTA'S & REITZNER'S RESULT (DEMONSTRATION 2)



Configurations

T_3  tetrahedron[4] 0.017398239 $\quad v_3^{(1)}(T_3) = \frac{13}{720} - \frac{\pi^2}{15015}$

- Decomposition into configurations: $v_3^{(k)}(T_3) = 4v_3^{(k)}(T_3)_{\mathrm{I}} + 3v_3^{(k)}(T_3)_{\mathrm{II}}$
- Affine invariancy: $T_3 \longrightarrow \mathbb{T}_3 \overset{\text{def}}{=} \operatorname{conv}(\mathbf{0}, e_1, e_2, e_3)$ canonical 3-simplex
- Section integral: $v_3^{(1)}(\mathbb{T}_3)_C = \frac{2}{3}\int_{(\mathbb{R}^3 \setminus \mathbb{T}^\circ)_C} v_2^{(2)}(\sigma \cap \mathbb{T}_3)\zeta_3^5(\sigma)_C\, \iota_3^{(1)}(\sigma)\lambda_3(d\eta)$
- Section plane: $\eta = (a, b, c)^\top \Rightarrow \frac{1}{a}e_1, \frac{1}{b}e_2, \frac{1}{c}e_3$ axes intersection points

REFERENCES [1] Sylvester (1865), Reed (1974) [2] Hostinský (1925) [3] Miles (1971) [4] Buchta & Reitzner (1992) [5] Zinani (2003), Philip (2007) [6] B. (2020)

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I

Zeta and Iota

- $\mathbb{T}_3^+ = \mathbb{T}_3^{abc} \stackrel{\text{def}}{=} \operatorname{conv}(\mathbf{0}, \frac{1}{a}e_1, \frac{1}{b}e_2, \frac{1}{c}e_3)$, $D_\sigma(0) = 1/\|\eta\|$

  $\therefore \operatorname{vol}_3 \mathbb{T}_3^+ = \frac{1}{6abc} = \frac{\operatorname{vol}_2(\sigma \cap \mathbb{T}_3)}{3\|\eta\|} \Rightarrow \zeta_3(\sigma)_I = \frac{\operatorname{vol}_2(\sigma\cap\mathbb{T}_3)}{\|\eta\| \operatorname{vol}_3 \mathbb{T}_3} = \frac{3}{abc}$

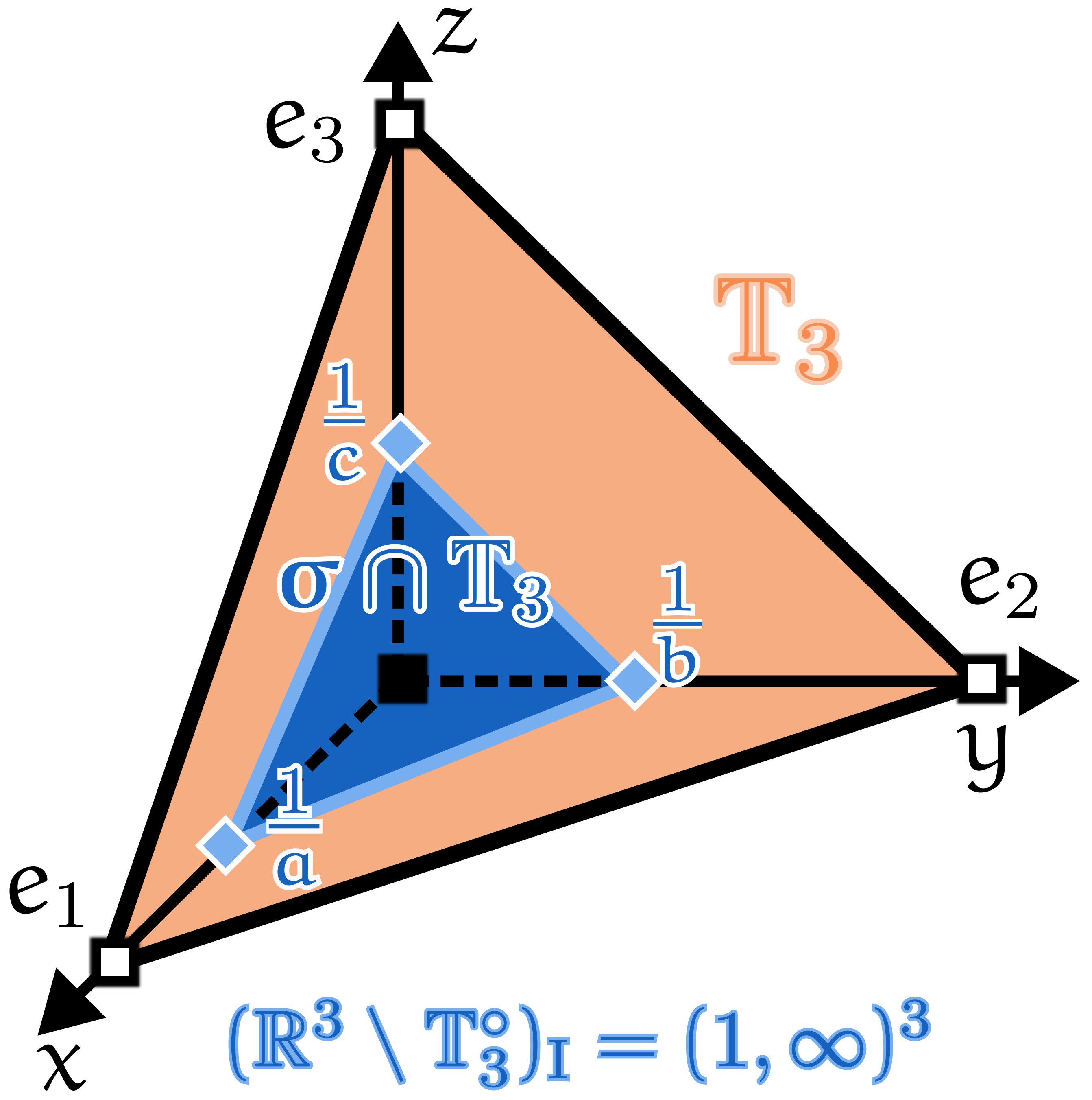
- $M = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$, $M^+ = [\frac{1}{4a}, \frac{1}{4b}, \frac{1}{4c}]$ centerpoints of $\mathbb{T}_3, \mathbb{T}_3^+$

- Iota integral: $\iota_3^{(1)}(\sigma)_I = \int_{\mathbb{T}_3} |\eta^\top \mathbf{x} - 1| \lambda_3(d\mathbf{x})$

$$= \int_{\mathbb{T}_3} (\eta^\top \mathbf{x} - 1)\lambda_3(d\mathbf{x}) - 2\int_{\mathbb{T}_3^+} (\eta^\top \mathbf{x} - 1)\lambda_3(d\mathbf{x})$$

$$= (\eta^\top M - 1) \operatorname{vol}_3 \mathbb{T}_3 - 2(\eta^\top M^+ - 1) \operatorname{vol}_3 \mathbb{T}_3^+$$

$$= \left(\frac{a+b+c}{4} - 1\right)\frac{1}{6} - 2\left(\frac{3}{4} - 1\right)\frac{1}{6abc} = \frac{1}{24}\left(a+b+c-4+\frac{2}{abc}\right)$$



$(\mathbb{R}^3 \setminus \mathbb{T}_3^\circ)_I = (1, \infty)^3$

REFERENCES [1] Sylvester (1865), Reed (1974) [2] Hostinský (1925) [3] Miles (1971) [4] Buchta & Reitzner (1992) [5] Zinani (2003), Philip (2007) [6] B. (2020)
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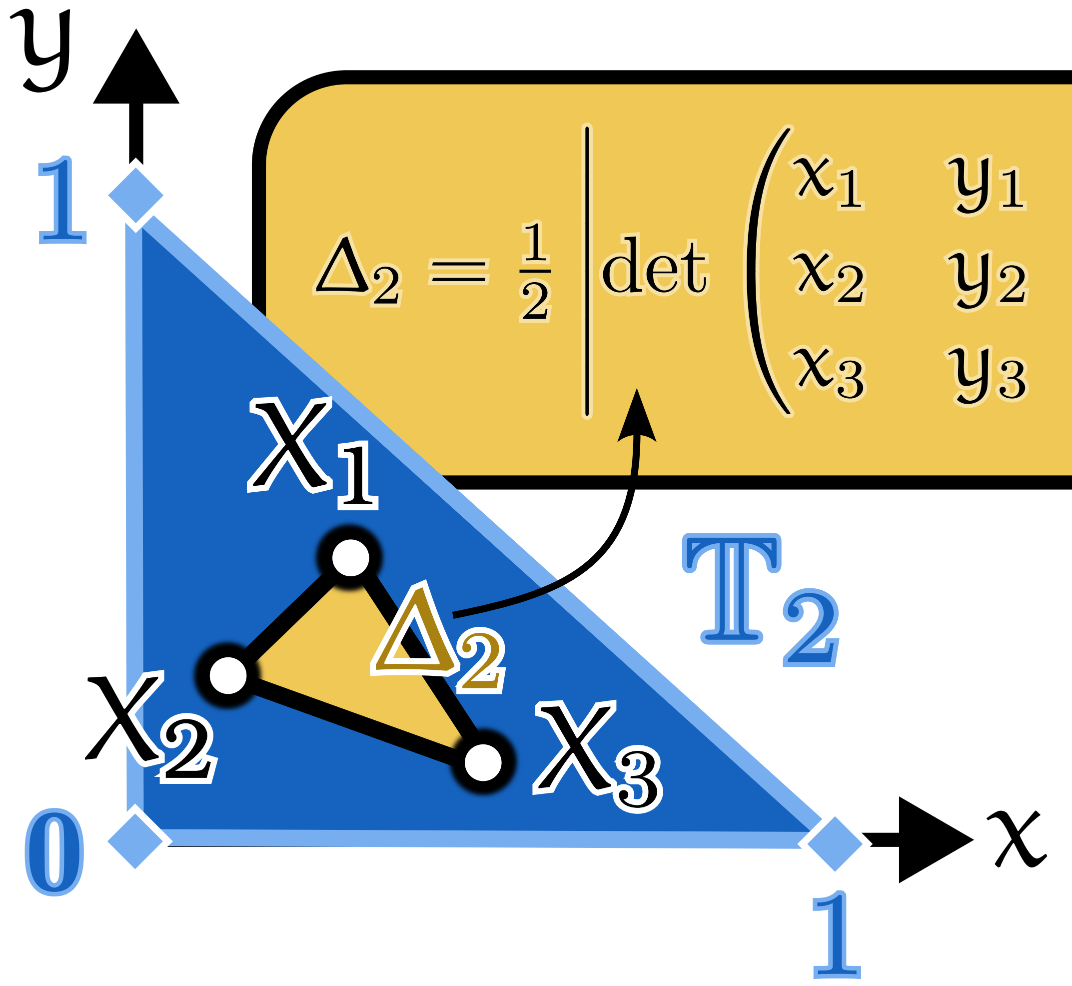
I ■

Mean cut function

- $\sigma \cap \mathbb{T}_3 = \mathrm{T}_2^{abc} \stackrel{\text{def}}{=} \mathrm{conv}(\tfrac{1}{a}e_1, \tfrac{1}{b}e_2, \tfrac{1}{c}e_3)$
- $X_1 = [x_1, y_1], X_2 = [x_2, y_2], X_3 = [x_3, y_3]$ by scale affinity
- Affine invariancy: $v_2^{(2)}(\sigma \cap \mathbb{T}_3) \stackrel{\text{aff.}}{=} v_2^{(2)}(\mathbb{T}_2)$

$$= \frac{\mathbb{E}\,\Delta_2^2}{\mathrm{vol}_2^2\,\mathbb{T}_2} = \frac{1}{\mathrm{vol}_2^5\,\mathbb{T}_2}\int_{\mathbb{T}_2^3} \Delta_2^2\,\mathrm{d}(x_1,y_1)\mathrm{d}(x_2,y_2)\mathrm{d}(x_3,y_3) = \frac{1}{72}$$



$$\Delta_2 = \frac{1}{2}\left|\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right|$$

Section integral

$$\therefore\ v_3^{(1)}(\mathbb{T}_3)_{\mathrm{I}} = \frac{2}{3}\int_1^\infty\int_1^\infty\int_1^\infty \frac{1}{72}\left(\frac{3}{abc}\right)^5 \frac{1}{24}\left(a+b+c-4+\frac{2}{abc}\right)\mathrm{d}a\mathrm{d}b\mathrm{d}c = \frac{3}{2000}$$

REFERENCES [1] Sylvester (1865), Reed (1974) [2] Hostinský (1925) [3] Miles (1971) [4] Buchta & Reitzner (1992) [5] Zinani (2003), Philip (2007) [6] B. (2020)
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II ■■

Zeta and Iota

- Intersection points $A = \frac{1}{c}e_3 + \alpha\left(\frac{1}{a}e_1 - \frac{1}{c}e_3\right) \in \overline{e_1 e_3}$

  i.e. $\frac{1}{c} + \alpha\left(\frac{1}{a} - \frac{1}{c}\right) = 1$, therefore $\alpha = a\frac{1-c}{a-c}$, similarly $\beta = b\frac{1-c}{b-c}$.

  $$\therefore \quad A = \left[\tfrac{1-c}{a-c}, 0, \tfrac{a-1}{a-c}\right], \quad B = \left[0, \tfrac{1-c}{b-c}, \tfrac{b-1}{b-c}\right]$$

- Jacobian of transformation $\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c = \frac{c^2(1-c)^2}{(\alpha+c-1)^2(\beta+c-1)^2}\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}c$

  and half-domain transformation $\frac{1}{2}(\mathbb{R}^3 \setminus \mathbb{T}_3^\circ)_{\mathrm{II}} \to (1-c, 1)^2 \times (0,1)$

- $\mathbb{T}_3^+ = \mathbb{T}_3^{abc} \setminus \mathbb{T}_3^*$, where $\mathbb{T}_3^* \overset{\text{def}}{=} \operatorname{conv}\left(A, B, e_3, \frac{1}{c}e_3\right) \Rightarrow \operatorname{vol}_3 \mathbb{T}_3^* = \frac{\alpha\beta(1-c)}{6abc}$

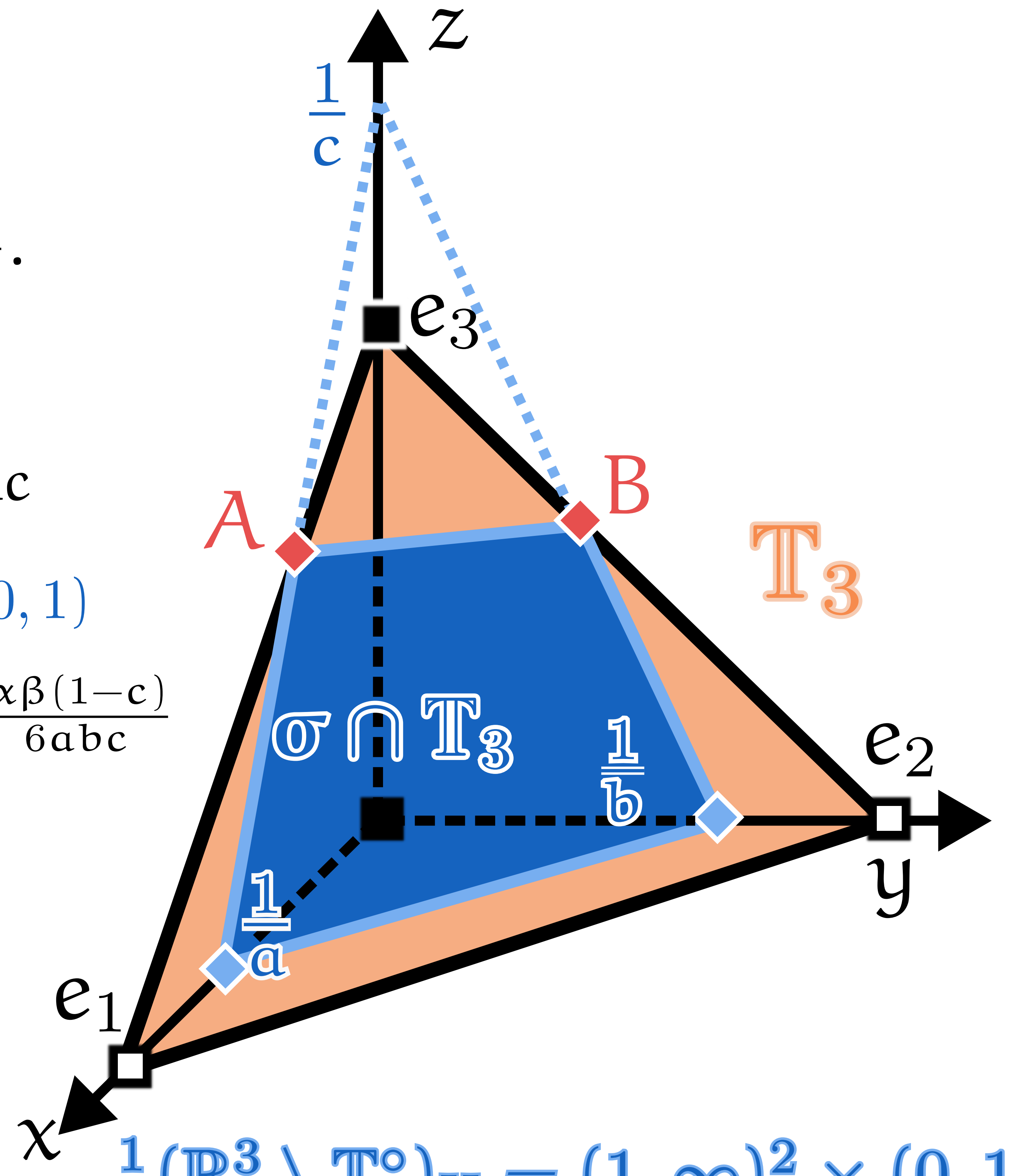
- Mass ballance: $M^+ \operatorname{vol}_3 \mathbb{T}_3^+ = M^{abc} \operatorname{vol}_3 \mathbb{T}_3^{abc} - M^* \operatorname{vol}_3 \mathbb{T}_3^*$

  $\therefore M^{abc} = \left[\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right], M^+ = \left[\frac{\alpha}{a}, \frac{\beta}{b}, \frac{3+c-\alpha-\beta}{c}\right]$ centerpoints of $\mathbb{T}_3^{abc}, \mathbb{T}_3^+$

- Iota: $\iota_3^{(1)}(\boldsymbol{\sigma})_{\mathrm{II}} = (\boldsymbol{\eta}^\top M - 1)\operatorname{vol}_3 \mathbb{T}_3 - 2(\boldsymbol{\eta}^\top M^+ - 1)\operatorname{vol}_3 \mathbb{T}_3^+$

  $$= \iota_3^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}} + 2(\boldsymbol{\eta}^\top M^* - 1)\operatorname{vol}_3 \mathbb{T}^* = \iota_3^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}} - \frac{\alpha\beta(1-c)^2}{12abc}$$

- Zeta: $\zeta_3(\boldsymbol{\sigma})_{\mathrm{II}} = \frac{\operatorname{vol}_2(\boldsymbol{\sigma}\cap\mathbb{T}_3)}{\|\boldsymbol{\eta}\|\operatorname{vol}_3\mathbb{T}_3} = (1-\alpha\beta)\frac{\operatorname{vol}_2 \mathbb{T}_2^{abc}}{\|\boldsymbol{\eta}\|\operatorname{vol}_3\mathbb{T}_3} = (1-\alpha\beta)\zeta_3(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{3(1-\alpha\beta)}{abc}$



$\frac{1}{2}(\mathbb{R}^3 \setminus \mathbb{T}_3^\circ)_{\mathrm{II}} = (1,\infty)^2 \times (0,1)$

REFERENCES [1] Sylvester (1865), Reed (1974) [2] Hostinský (1925) [3] Miles (1971) [4] Buchta & Reitzner (1992) [5] Zinani (2003), Philip (2007) [6] B. (2020)

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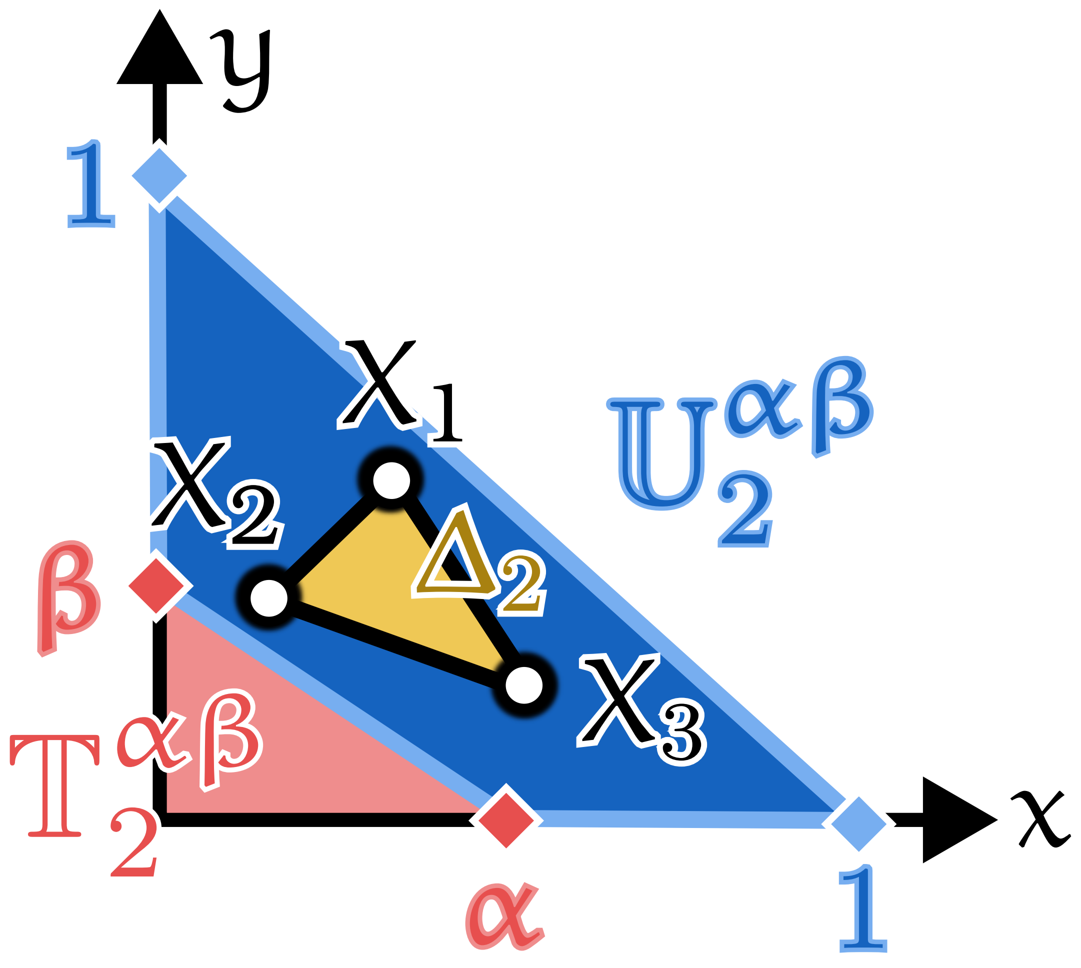
II

Mean cut function

- $\nu_2^{(2)}(\sigma \cap \mathbb{T}_3) \stackrel{\text{aff.}}{=} \nu_2^{(2)}(\mathbb{U}_2^{\alpha\beta}) = \frac{1}{\operatorname{vol}_2^5 \mathbb{U}_2^{\alpha\beta}} \int_{(\mathbb{U}_2^{\alpha\beta})^3} \Delta_2^2 \, dX_1 dX_2 dX_3$

$$= \frac{\left\{\begin{matrix} \alpha^4\beta^4 - 8\alpha^3\beta^3 + 8\alpha^3\beta^2 - 4\alpha^3\beta + 8\alpha^2\beta^3 \\ -10\alpha^2\beta^2 + 8\alpha^2\beta - 4\alpha\beta^3 + 8\alpha\beta^2 - 8\alpha\beta + 1 \end{matrix}\right\}}{72(1-\alpha\beta)^4}$$



Section integral

polynomials in c

$$\therefore \nu_3^{(1)}(\mathbb{T}_3)_{\text{II}} = 2\cdot\frac{2}{3}\int_0^1\int_{1-c}^1\int_{1-c}^1 \cdots \, d\alpha \, d\beta \, dc = \int_0^1 \sum_{r=0}^{2} \frac{q_r(c)\ln^r(1-c)}{c^{16}} dc = \frac{217}{54000} - \frac{\pi^2}{450450}$$

REFERENCES [1] Sylvester (1865), Reed (1974) [2] Hostinský (1925) [3] Miles (1971) [4] Buchta & Reitzner (1992) [5] Zinani (2003), Philip (2007) [6] B. (2020)
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PENTACHORON (4-SIMPLEX) (DEMONSTRATION 3)

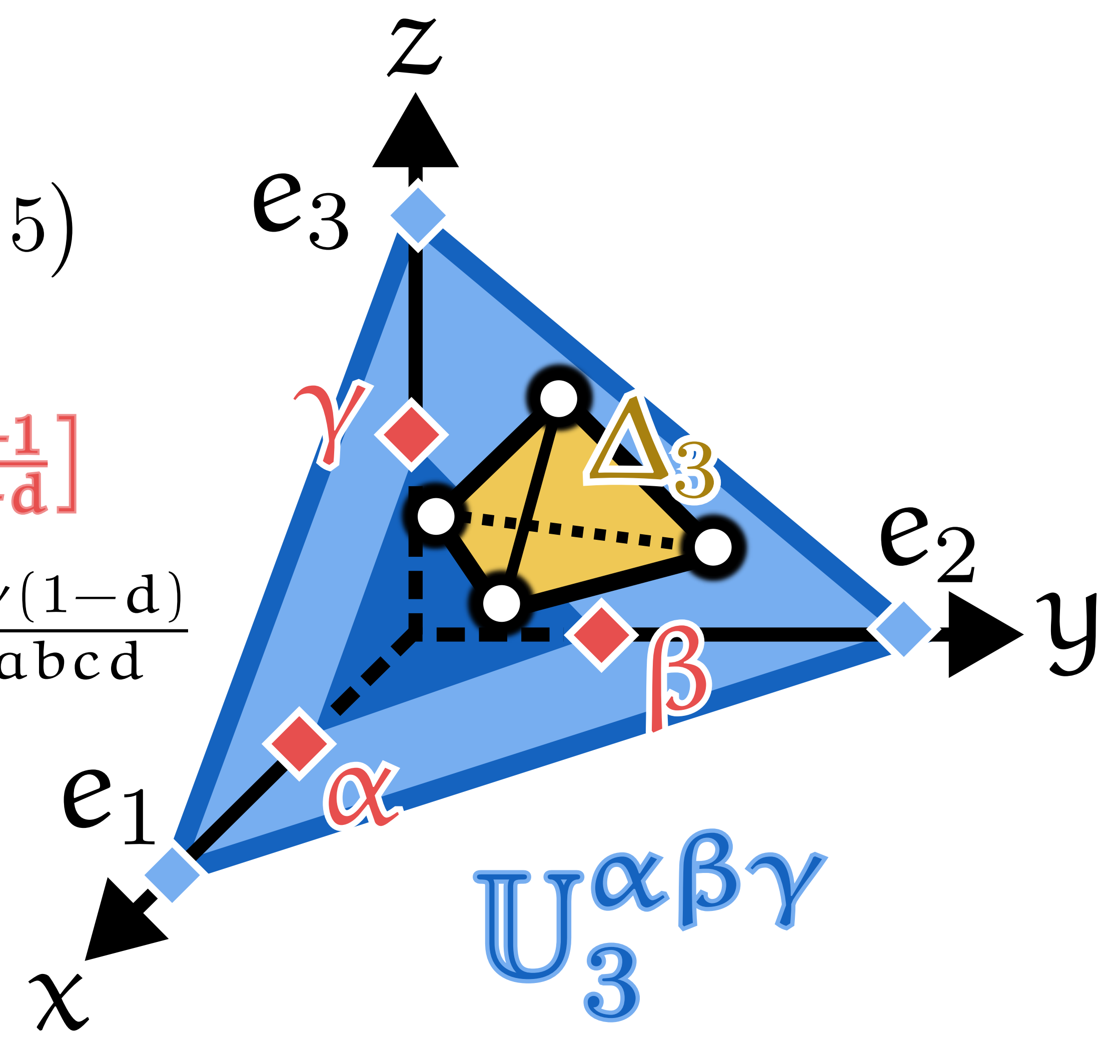
- Decomposition into configurations: $v_4^{(k)}(T_4) = 5v_4^{(k)}(T_4)_{\mathrm{I}} + 10v_4^{(k)}(T_4)_{\mathrm{II}}$, aff. $T_4 \longrightarrow \mathbb{T}_4 \stackrel{\text{def}}{=} \operatorname{conv}(\mathbf{0}, e_1, e_2, e_3, e_4)$
- Section plane: $\eta = (a, b, c, d)^\top \Rightarrow \frac{1}{a}e_1, \frac{1}{b}e_2, \frac{1}{c}e_3, \frac{1}{d}e_4$ axes intersection points

I

- Intersection $\mathbb{T}_4^+ = \mathbb{T}_4^{abcd} \stackrel{\text{def}}{=} \operatorname{conv}(\mathbf{0}, \frac{1}{a}e_1, \frac{1}{b}e_2, \frac{1}{c}e_3, \frac{1}{d}e_4)$
- Zeta and iota: $\zeta_2(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{24}{abcd}$, $\iota_4^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}} = \frac{1}{120}\left(\frac{2}{abcd} + a + b + c + d - 5\right)$

II

- Int. pts. $A = \left[\frac{1-d}{a-d}, 0, 0, \frac{a-1}{a-d}\right]$, $B = \left[0, \frac{1-d}{b-d}, 0, \frac{b-1}{b-d}\right]$, $C = \left[0, 0, \frac{1-d}{c-d}, \frac{c-1}{c-d}\right]$
- $\mathbb{T}_4^+ = \mathbb{T}_4^{abcd} \setminus \mathbb{T}_4^*$, where $\mathbb{T}_4^* \stackrel{\text{def}}{=} \operatorname{conv}\left(A, B, C, e_4, \frac{1}{d}e_4\right) \Rightarrow \operatorname{vol}_4 \mathbb{T}_4^* = \frac{\alpha\beta\gamma(1-d)}{24abcd}$
- $\zeta_4(\boldsymbol{\sigma})_{\mathrm{II}} = (1 - \alpha\beta\gamma)\zeta_4(\boldsymbol{\sigma})_{\mathrm{I}}$, $\iota_4^{(1)}(\boldsymbol{\sigma})_{\mathrm{II}} = \iota_4^{(1)}(\boldsymbol{\sigma})_{\mathrm{I}} - \frac{(1-d)^5}{60d(a-d)(b-d)(c-d)}$
- Affine invariancy: $v_3^{(k+1)}(\boldsymbol{\sigma} \cap \mathbb{T}_4) \stackrel{\text{aff.}}{=} v_3^{(k+1)}(\mathbb{U}_3^{\alpha\beta\gamma})$



T_4  4-simplex 0.0031803708 $\quad v_4^{(1)}(T_4) = \frac{97}{27000} - \frac{2173\pi^2}{52026975}$

REFERENCES [1] Sylvester (1865), Reed (1974) [2] Hostinský (1925) [3] Miles (1971) [4] Buchta & Reitzner (1992) [5] Zinani (2003), Philip (2007) [6] B. (2020)

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ON RANDOM SIMPLEX PICKING

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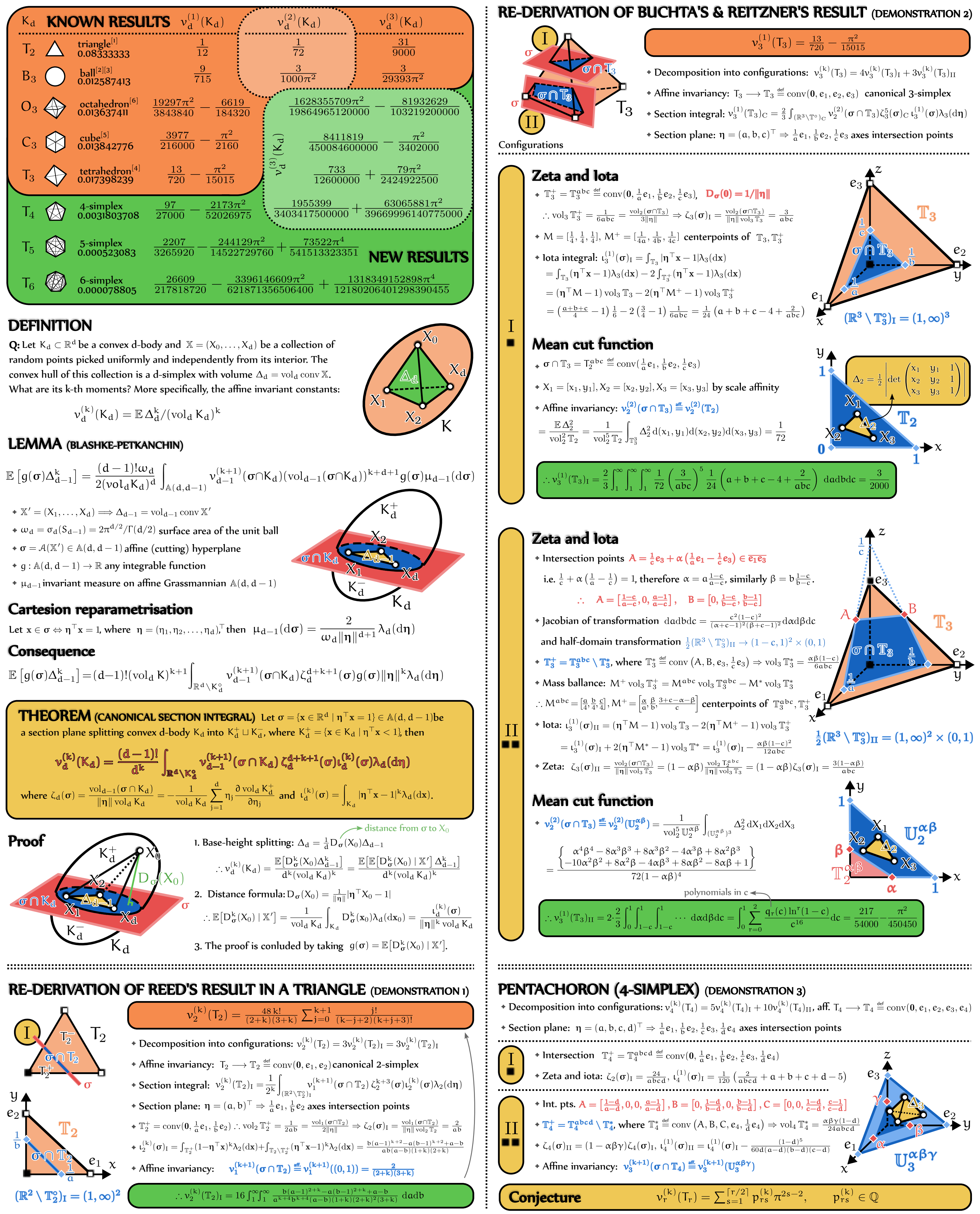

download the poster here!

REFERENCES

[1] Sylvester (1865), Reed (1974) [2] Hostinský (1925) [3] Miles (1971) [4] Buchta & Reitzner (1992) [5] Zinani (2003), Philip (2007) [6] B. (2020)

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