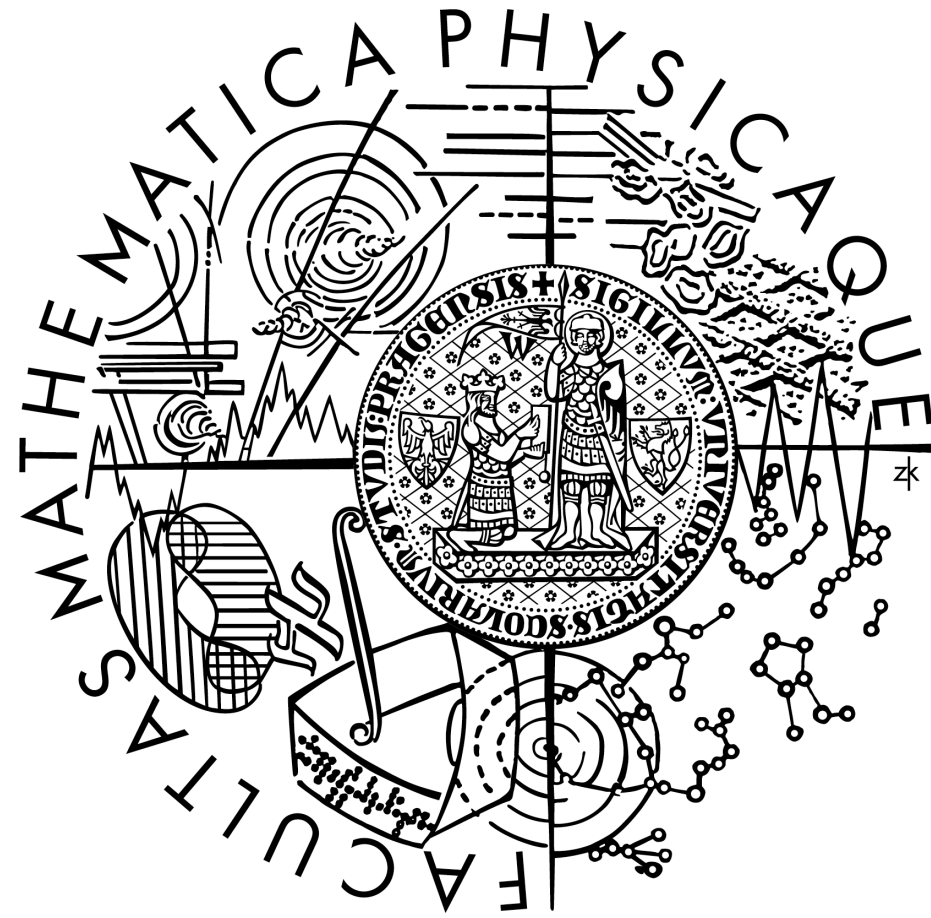




FACULTY OF MATHEMATICS AND PHYSICS

Charles University

ON RANDOM SIMPLEX PICKING

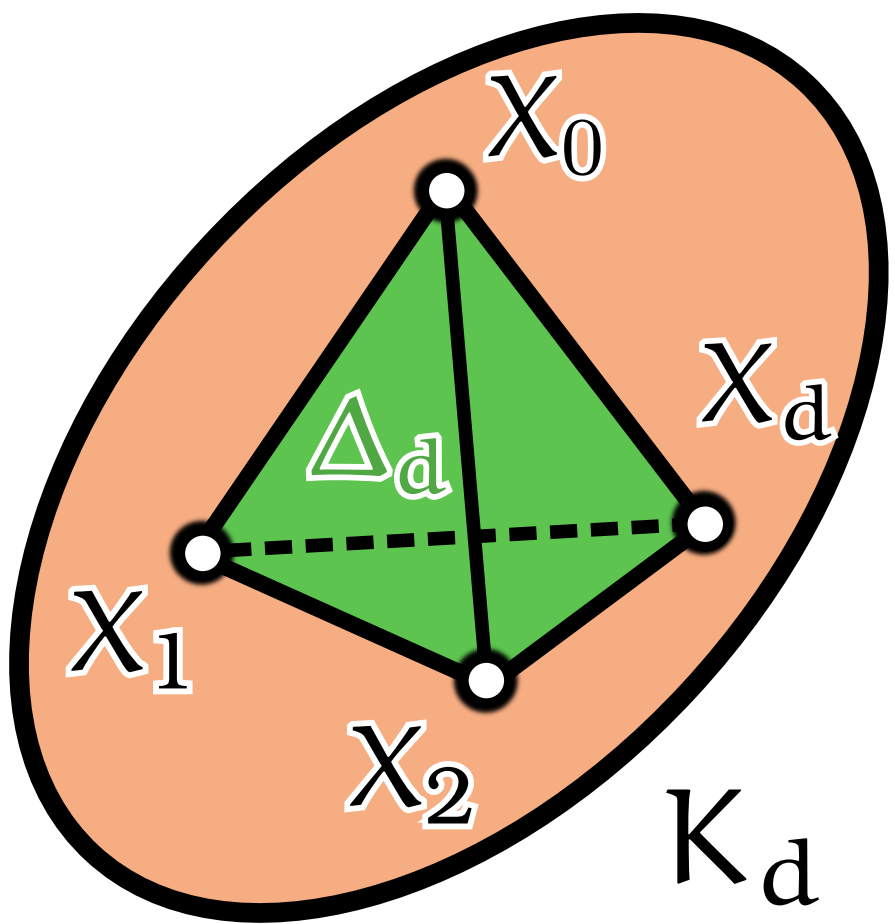
GPSD 2025

Dominik Beck



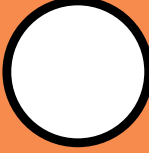
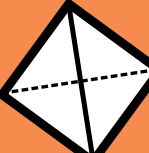
DEFINITION

Q: Let $K_d \subset \mathbb{R}^d$ be a convex d -body and $\mathbb{X} = (X_0, \dots, X_d)$ be a collection of random points picked uniformly and independently from its interior. The convex hull of this collection is a d -simplex with volume $\Delta_d = \text{vol}_d \text{conv } \mathbb{X}$.

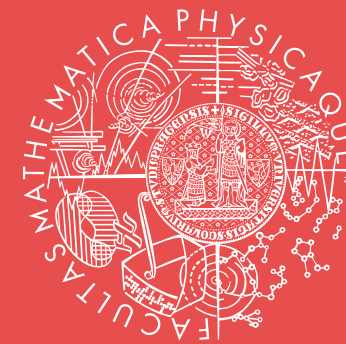


*What are its k -th moments?
More specifically, the affine
invariant constants:*

$$v_d^{(k)}(K_d) = \mathbb{E} \Delta_d^k / (\text{vol}_d K_d)^k$$

K_d	KNOWN RESULTS		$v_d^{(1)}(K_d)$
T_2		triangle ^[1] 0.08333333	$\frac{1}{12}$
B_3		ball ^{[2][3]} 0.012587413	$\frac{9}{715}$
O_3		octahedron ^[6] 0.013637411	$\frac{19297\pi^2}{3843840} - \frac{6619}{184320}$
C_3		cube ^[5] 0.013842776	$\frac{3977}{216000} - \frac{\pi^2}{2160}$
T_3		tetrahedron ^[4] 0.017398239	$\frac{13}{720} - \frac{\pi^2}{15015}$

- + All 2D polygons
- + Other selected 3D polytopes
- + Ball in any dimension



K_d

NEW RESULTS

$v_d^{(1)}(K_d)$

T_4



4-simplex
0.0031803708

$$\frac{97}{27000} - \frac{2173\pi^2}{52026975}$$

T_5



5-simplex
0.000523083

$$\frac{2207}{3265920} - \frac{244129\pi^2}{14522729760} + \frac{73522\pi^4}{541513323351}$$

T_6



6-simplex
0.000078805

$$\frac{26609}{217818720} - \frac{3396146609\pi^2}{621871356506400} + \frac{1318349152898\pi^4}{12180206401298390455}$$

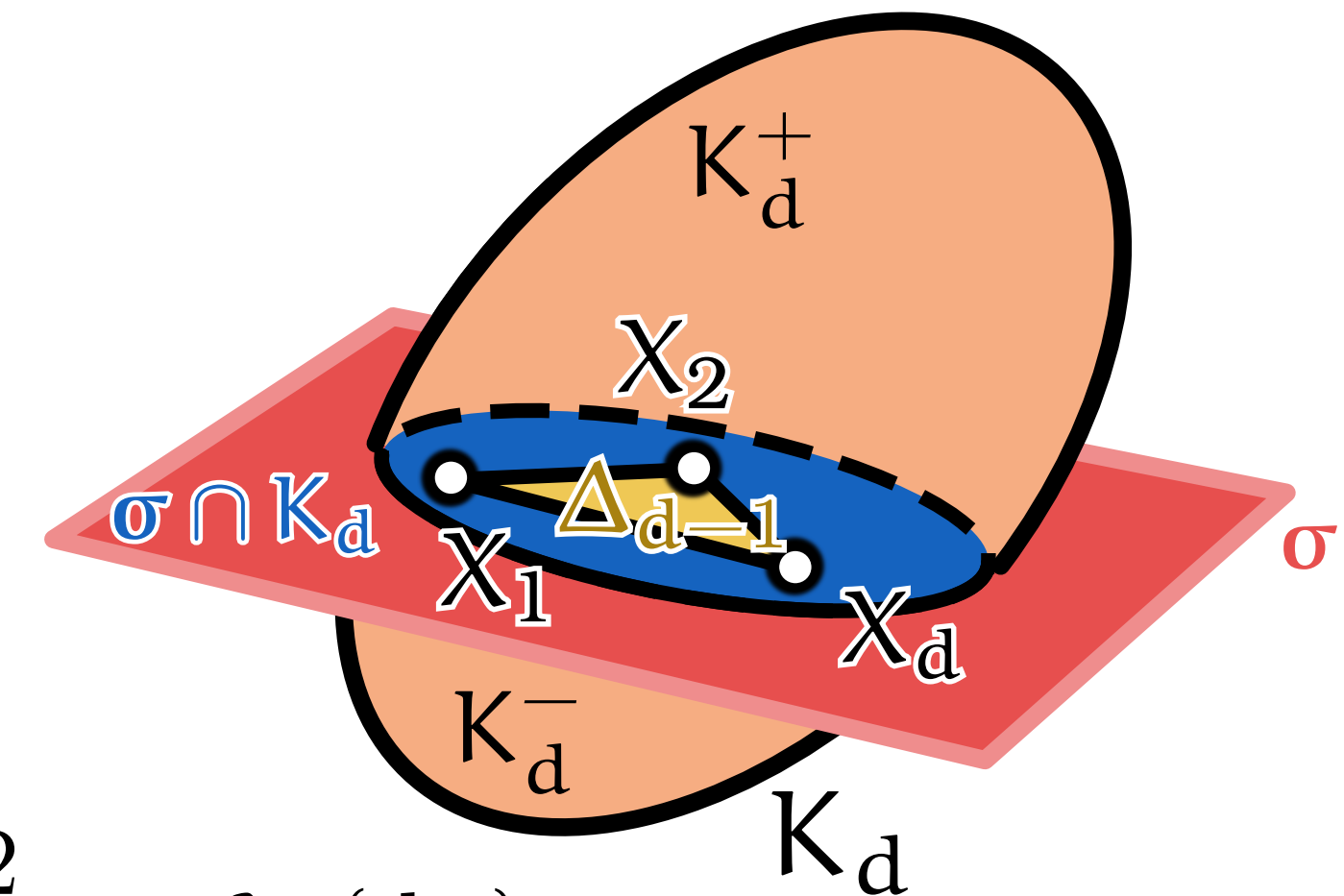
Conjecture

$$v_r^{(k)}(T_r) = \sum_{s=1}^{\lceil r/2 \rceil} p_{rs}^{(k)} \pi^{2s-2}, \quad p_{rs}^{(k)} \in \mathbb{Q}$$

LEMMA (BLASHKE-PETKANCHIN)

$$\mathbb{E} \left[g(\boldsymbol{\sigma}) \Delta_{d-1}^k \right] = \frac{(d-1)! \omega_d}{2(\text{vol}_d K_d)^d} \int_{\mathbb{A}(d, d-1)} v_{d-1}^{(k+1)}(\boldsymbol{\sigma} \cap K_d) (\text{vol}_{d-1}(\boldsymbol{\sigma} \cap K_d))^{k+d+1} g(\boldsymbol{\sigma}) \mu_{d-1}(d\boldsymbol{\sigma})$$

- ❖ $\mathbb{X}' = (X_1, \dots, X_d) \implies \Delta_{d-1} = \text{vol}_{d-1} \text{conv } \mathbb{X}'$
- ❖ $\omega_d = \sigma_d(S_{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ **surface area of the unit ball**
- ❖ $\sigma = \mathcal{A}(\mathbb{X}') \in \mathbb{A}(d, d-1)$ **affine (cutting) hyperplane**
- ❖ $g : \mathbb{A}(d, d-1) \rightarrow \mathbb{R}$ **any integrable function**
- ❖ μ_{d-1} **invariant measure on affine Grassmannian $\mathbb{A}(d, d-1)$**



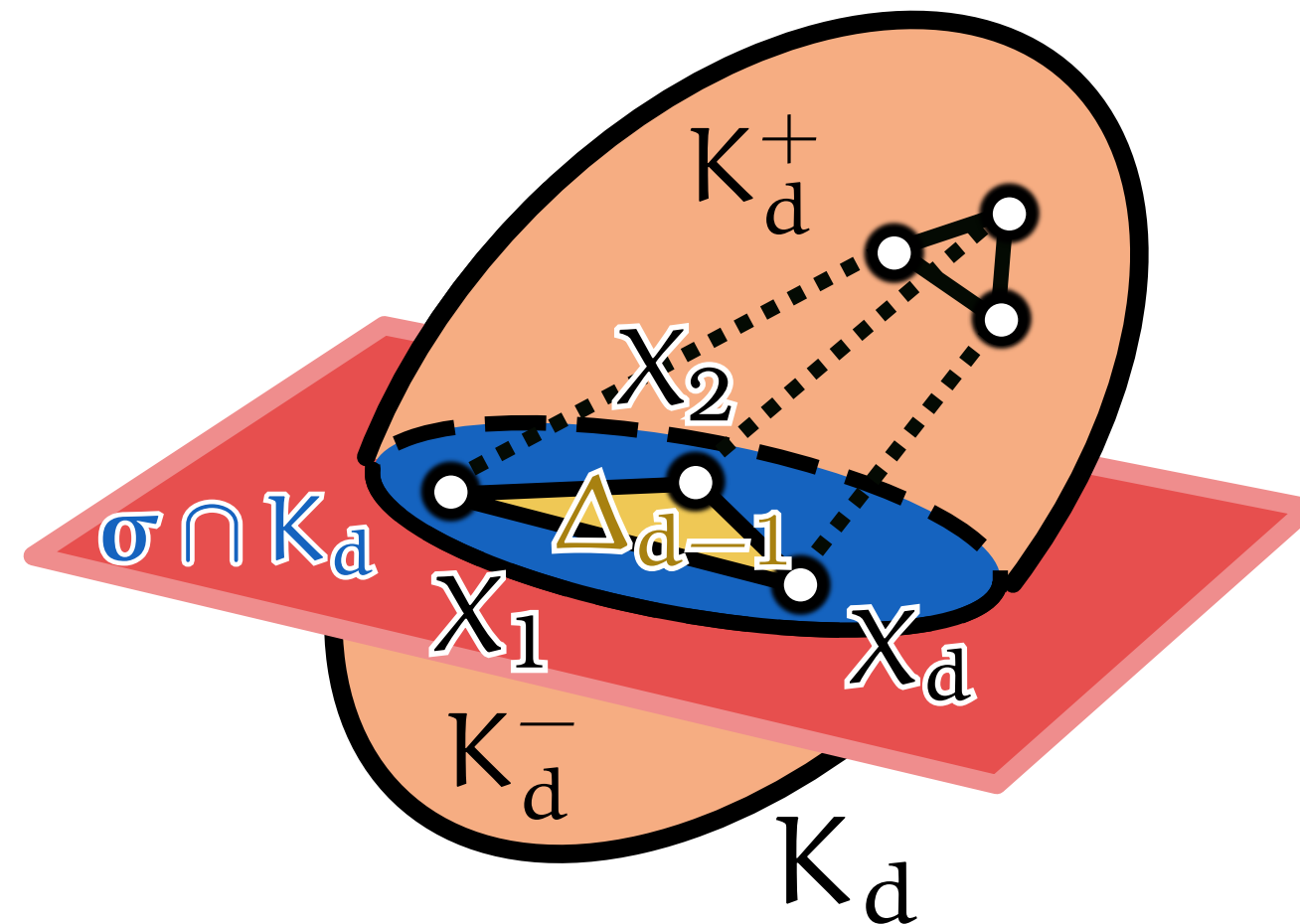
Cartesian reparametrisation

Let $\mathbf{x} \in \sigma \Leftrightarrow \boldsymbol{\eta}^\top \mathbf{x} = 1$, where $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_d)^\top$ then $\mu_{d-1}(d\boldsymbol{\sigma}) = \frac{2}{\omega_d \|\boldsymbol{\eta}\|^{d+1}} \lambda_d(d\boldsymbol{\eta})$

$$\mathbb{E} \left[g(\boldsymbol{\sigma}) \Delta_{\mathbf{d}-1}^k \right] = (\mathbf{d}-1)! (\text{vol}_{\mathbf{d}} \mathbf{K})^{k+1} \int_{\mathbb{R}^{\mathbf{d}} \setminus \mathbf{K}_{\mathbf{d}}^{\circ}} \mathbf{v}_{\mathbf{d}-1}^{(k+1)}(\boldsymbol{\sigma} \cap \mathbf{K}_{\mathbf{d}}) \zeta_{\mathbf{d}}^{\mathbf{d}+k+1}(\boldsymbol{\sigma}) g(\boldsymbol{\sigma}) \|\boldsymbol{\eta}\|^k \lambda_{\mathbf{d}}(\mathrm{d}\boldsymbol{\eta})$$

$$\text{where } \zeta_d(\boldsymbol{\sigma}) = \frac{\text{vol}_{d-1}(\boldsymbol{\sigma} \cap \mathbf{K}_d)}{\|\boldsymbol{\eta}\| \text{vol}_d \mathbf{K}_d} = -\frac{1}{\text{vol}_d \mathbf{K}_d} \sum_{j=1}^d \eta_j \frac{\partial \text{vol}_d \mathbf{K}_d^+}{\partial \eta_j} \quad \iota_d^{(k)}(\boldsymbol{\sigma}) = \int_{\mathbf{K}_d} |\boldsymbol{\eta}^\top \mathbf{x} - 1|^k \lambda_d(d\mathbf{x})$$

ORIGINAL APPROACH^[4,5,6]

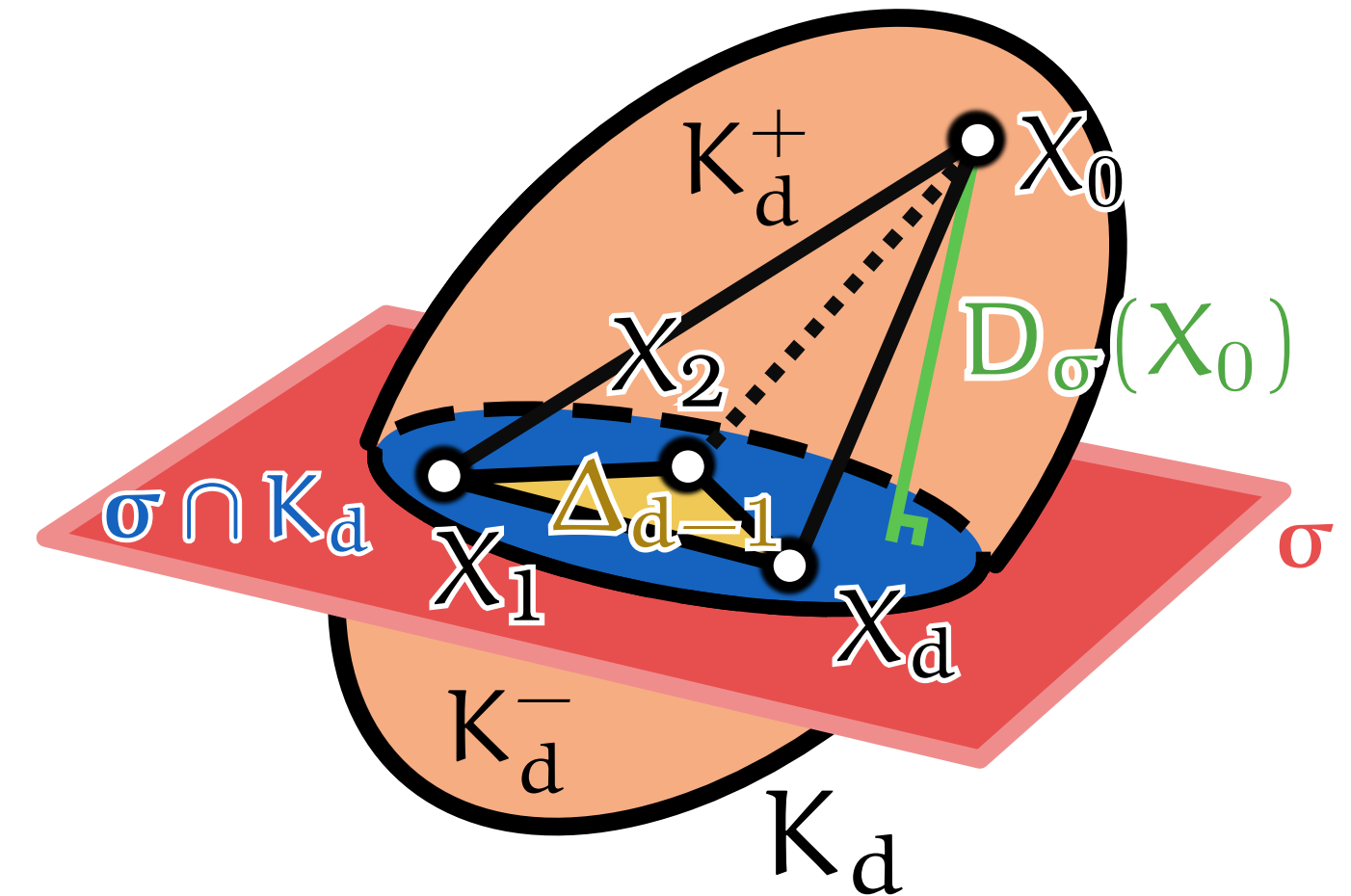


$$f_0(H_n) - f_1(H_n) + f_2(H_n) = 2$$

Efron's formula

$$\therefore v_n^{(1)}(K_3) = \frac{n-1}{n+1} - \frac{1}{6} \binom{n}{2} \mathbb{E} \left[\left(\frac{\text{vol}_3 K_3^+}{\text{vol}_3 K_3} \right)^{n-2} + \left(\frac{\text{vol}_3 K_3^-}{\text{vol}_3 K_3} \right)^{n-2} \right]$$

NEW APPROACH



$$\Delta_d = \frac{1}{d} D_\sigma(X_0) \Delta_{d-1}$$

Canonical section integral

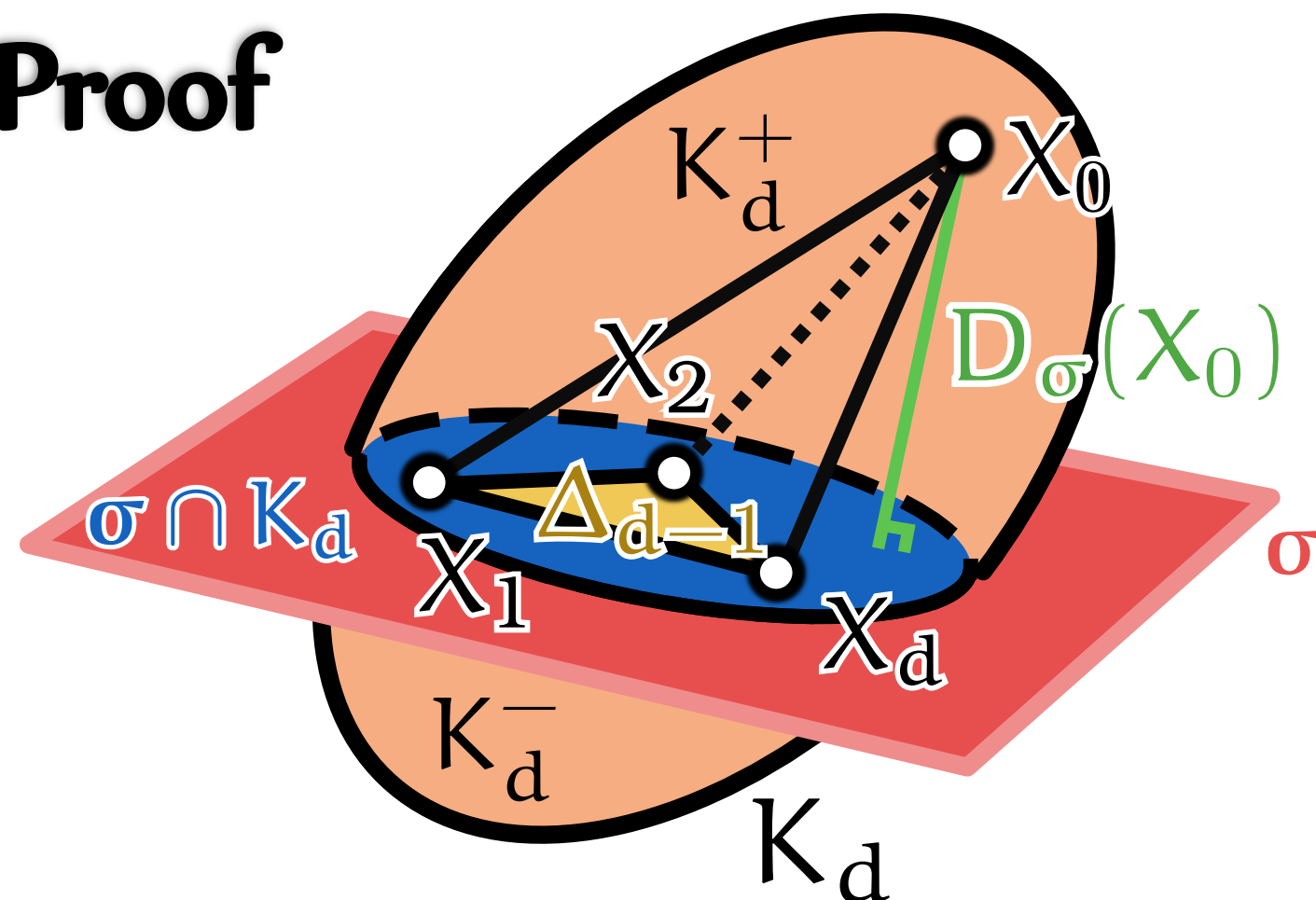
$$\therefore v_d^{(k)}(K_d) = \frac{\mathbb{E}[D_\sigma^k(X_0) \Delta_{d-1}^k]}{d^k (\text{vol}_d K_d)^k}$$

THEOREM (CANONICAL SECTION INTEGRAL) Let $\sigma = \{x \in \mathbb{R}^d \mid \eta^\top x = 1\} \in \mathbb{A}(d, d-1)$ be a section plane splitting convex d -body K_d into $K_d^+ \sqcup K_d^-$, where $K_d^+ = \{x \in K_d \mid \eta^\top x < 1\}$, then

$$v_d^{(k)}(K_d) = \frac{(d-1)!}{d^k} \int_{\mathbb{R}^d \setminus K_d^\circ} v_{d-1}^{(k+1)}(\sigma \cap K_d) \zeta_d^{d+k+1}(\sigma) \iota_d^{(k)}(\sigma) \lambda_d(d\eta)$$

where $\zeta_d(\sigma) = \frac{\text{vol}_{d-1}(\sigma \cap K_d)}{\|\eta\| \text{vol}_d K_d} = -\frac{1}{\text{vol}_d K_d} \sum_{j=1}^d \eta_j \frac{\partial \text{vol}_d K_d^+}{\partial \eta_j}$ and $\iota_d^{(k)}(\sigma) = \int_{K_d} |\eta^\top x - 1|^k \lambda_d(dx)$.

Proof



1. Base-height splitting: $\Delta_d = \frac{1}{d} D_\sigma(X_0) \Delta_{d-1}$ distance from σ to X_0

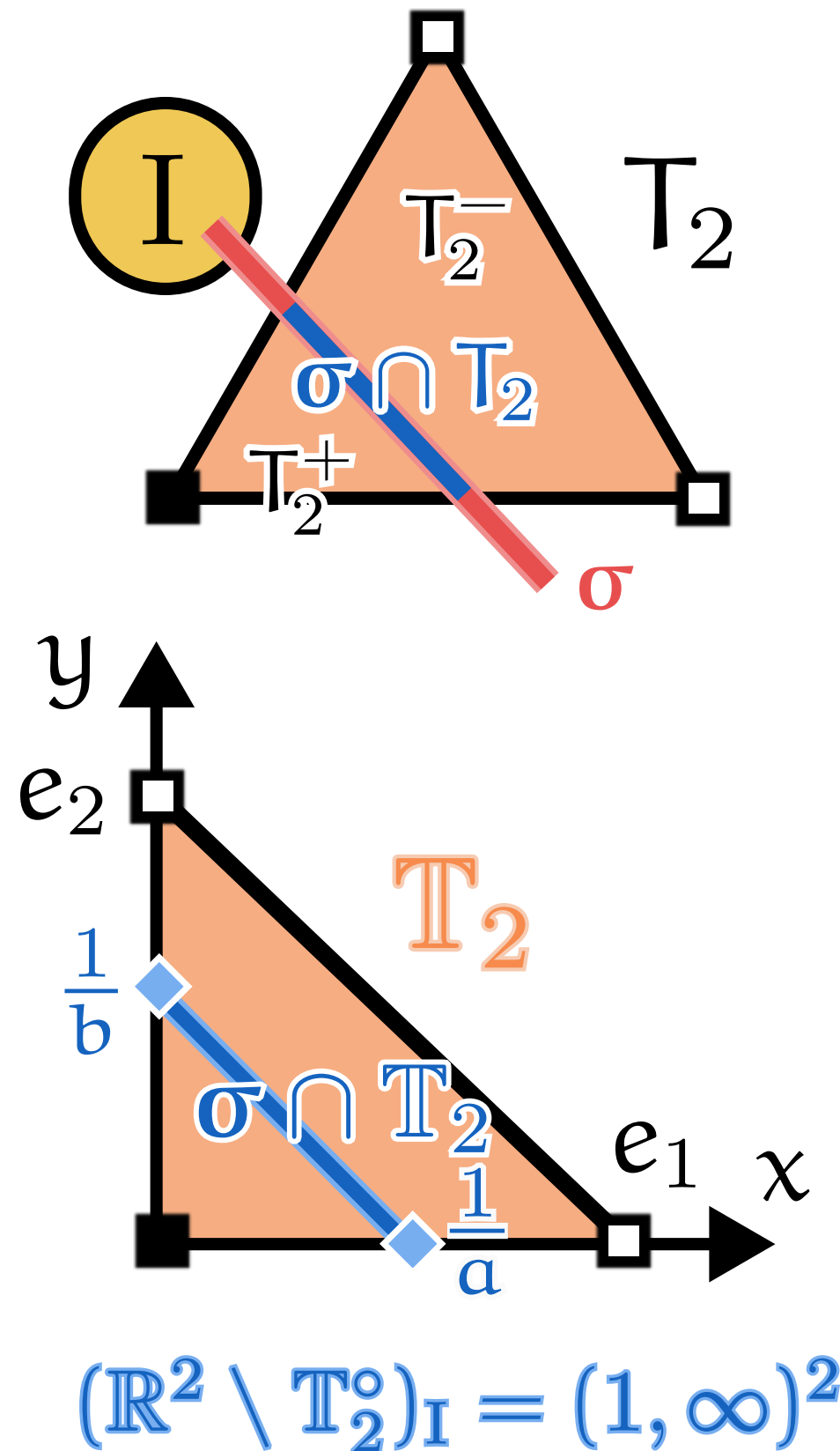
$$\therefore v_d^{(k)}(K_d) = \frac{\mathbb{E}[D_\sigma^k(X_0) \Delta_{d-1}^k]}{d^k (\text{vol}_d K_d)^k} = \frac{\mathbb{E}[\mathbb{E}[D_\sigma^k(X_0) \mid \mathbb{X}'] \Delta_{d-1}^k]}{d^k (\text{vol}_d K_d)^k}$$

2. Distance formula: $D_\sigma(X_0) = \frac{1}{\|\eta\|} |\eta^\top X_0 - 1|$

$$\therefore \mathbb{E}[D_\sigma^k(X_0) \mid \mathbb{X}'] = \frac{1}{\text{vol}_d K_d} \int_{K_d} D_\sigma^k(x_0) \lambda_d(dx_0) = \frac{\iota_d^{(k)}(\sigma)}{\|\eta\|^k \text{vol}_d K_d}$$

3. The proof is concluded by taking $g(\sigma) = \mathbb{E}[D_\sigma^k(X_0) \mid \mathbb{X}']$.

RE-DERIVATION OF REED'S RESULT IN A TRIANGLE (DEMONSTRATION 1)

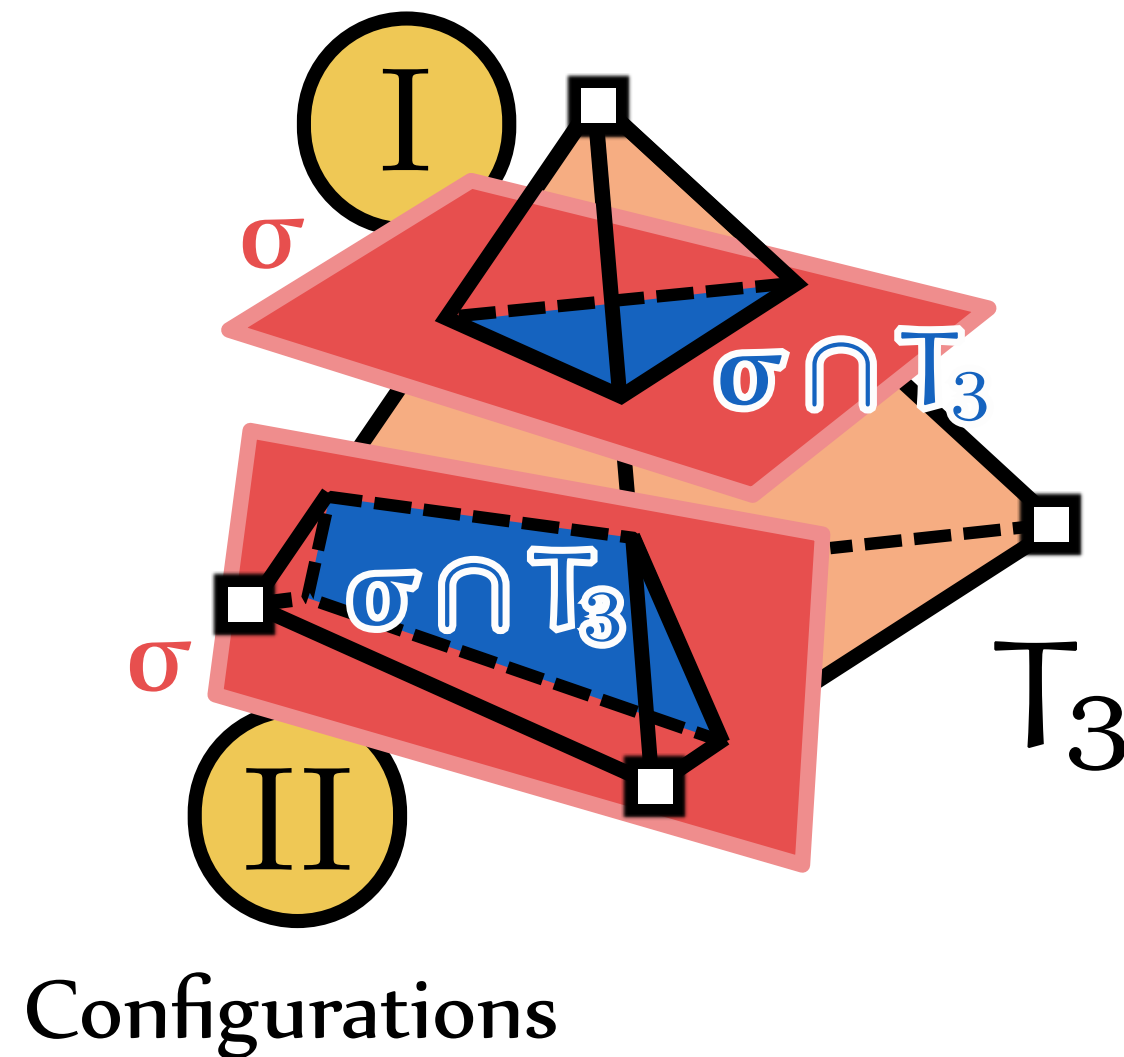


$$v_2^{(k)}(T_2) = \frac{48 k!}{(2+k)(3+k)} \sum_{j=0}^{k+1} \frac{j!}{(k-j+2)(k+j+3)!}$$

- ❖ Decomposition into configurations: $v_2^{(k)}(T_2) = 3v_2^{(k)}(T_2)_I = 3v_2^{(k)}(T_2)_{II}$
- ❖ Affine invariancy: $T_2 \longrightarrow \mathbb{T}_2 \stackrel{\text{def}}{=} \text{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2)$ canonical 2-simplex
- ❖ Section integral: $v_2^{(k)}(\mathbb{T}_2)_I = \frac{1}{2^k} \int_{(\mathbb{R}^2 \setminus \mathbb{T}_2^o)_I} v_1^{(k+1)}(\sigma \cap \mathbb{T}_2) \zeta_2^{k+3}(\sigma) \iota_2^{(k)}(\sigma) \lambda_2(d\eta)$
- ❖ Section plane: $\eta = (a, b)^\top \Rightarrow \frac{1}{a} \mathbf{e}_1, \frac{1}{b} \mathbf{e}_2$ axes intersection points
- ❖ $\mathbb{T}_2^+ = \text{conv}(\mathbf{0}, \frac{1}{a} \mathbf{e}_1, \frac{1}{b} \mathbf{e}_2) \therefore \text{vol}_2 \mathbb{T}_2^+ = \frac{1}{2ab} = \frac{\text{vol}_1(\sigma \cap \mathbb{T}_2)}{2\|\eta\|} \Rightarrow \zeta_2(\sigma)_I = \frac{\text{vol}_1(\sigma \cap \mathbb{T}_2)}{\|\eta\| \text{vol}_2 \mathbb{T}_2} = \frac{2}{ab}$
- ❖ $\iota_2^{(k)}(\sigma)_I = \int_{\mathbb{T}_2^+} (1 - \eta^\top \mathbf{x})^k \lambda_2(d\mathbf{x}) + \int_{\mathbb{T}_2 \setminus \mathbb{T}_2^+} (\eta^\top \mathbf{x} - 1)^k \lambda_2(d\mathbf{x}) = \frac{b(a-1)^{k+2} - a(b-1)^{k+2} + a - b}{ab(a-b)(1+k)(2+k)}$
- ❖ Affine invariancy: $v_1^{(k+1)}(\sigma \cap \mathbb{T}_2) \stackrel{\text{aff.}}{=} v_1^{(k+1)}((0, 1)) = \frac{2}{(2+k)(3+k)}$

$$\therefore v_2^{(k)}(\mathbb{T}_2)_I = 16 \int_1^\infty \int_1^\infty \frac{b(a-1)^{2+k} - a(b-1)^{2+k} + a - b}{a^{k+4} b^{k+4} (a-b)(1+k)(2+k)^2(3+k)} da db$$

RE-DERIVATION OF BUCHTA'S & REITZNER'S RESULT (DEMONSTRATION 2)



$$T_3 \quad \text{tetrahedron}^{[4]} \\ 0.017398239$$

$$v_3^{(1)}(T_3) = \frac{13}{720} - \frac{\pi^2}{15015}$$

- ❖ Decomposition into configurations: $v_3^{(k)}(T_3) = 4v_3^{(k)}(T_3)_I + 3v_3^{(k)}(T_3)_{II}$
- ❖ Affine invariancy: $T_3 \longrightarrow \mathbb{T}_3 \stackrel{\text{def}}{=} \text{conv}(\mathbf{0}, e_1, e_2, e_3)$ canonical 3-simplex
- ❖ Section integral: $v_3^{(1)}(\mathbb{T}_3)_C = \frac{2}{3} \int_{(\mathbb{R}^3 \setminus \mathbb{T}^\circ)_C} v_2^{(2)}(\sigma \cap \mathbb{T}_3) \zeta_3^5(\sigma)_C v_3^{(1)}(\sigma) \lambda_3(d\eta)$
- ❖ Section plane: $\eta = (a, b, c)^\top \Rightarrow \frac{1}{a}e_1, \frac{1}{b}e_2, \frac{1}{c}e_3$ axes intersection points

I



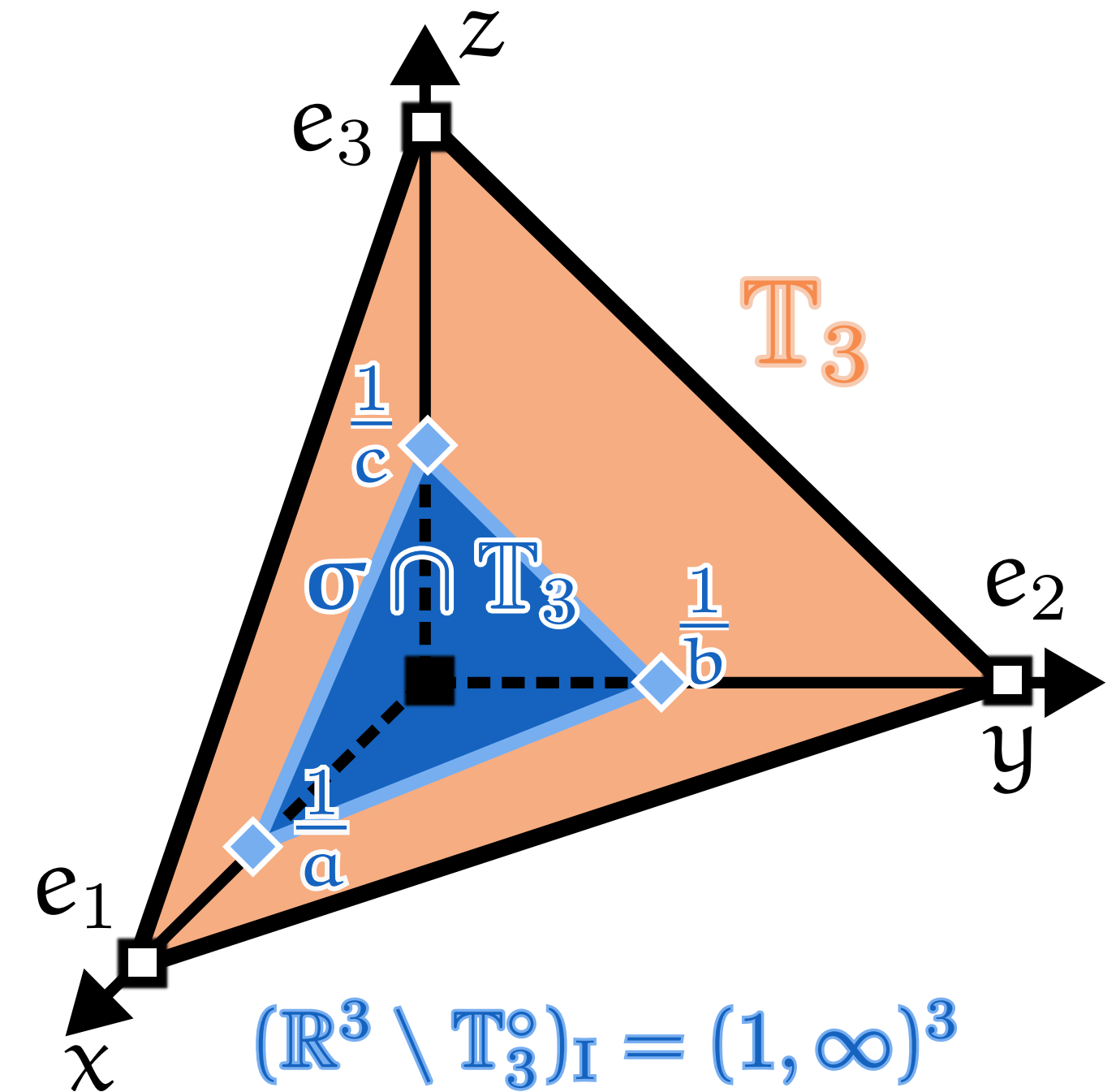
Zeta and Iota

$$\ast \mathbb{T}_3^+ = \mathbb{T}_3^{abc} \stackrel{\text{def}}{=} \text{conv}(\mathbf{0}, \frac{1}{a}\mathbf{e}_1, \frac{1}{b}\mathbf{e}_2, \frac{1}{c}\mathbf{e}_3), \quad \mathbf{D}_\sigma(\mathbf{0}) = 1/\|\boldsymbol{\eta}\|$$

$$\therefore \text{vol}_3 \mathbb{T}_3^+ = \frac{1}{6abc} = \frac{\text{vol}_2(\sigma \cap \mathbb{T}_3)}{3\|\boldsymbol{\eta}\|} \Rightarrow \zeta_3(\sigma)_I = \frac{\text{vol}_2(\sigma \cap \mathbb{T}_3)}{\|\boldsymbol{\eta}\| \text{vol}_3 \mathbb{T}_3} = \frac{3}{abc}$$

$$\ast M = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}], M^+ = [\frac{1}{4a}, \frac{1}{4b}, \frac{1}{4c}] \text{ centerpoints of } \mathbb{T}_3, \mathbb{T}_3^+$$

$$\begin{aligned} \ast \text{Iota integral: } \iota_3^{(1)}(\sigma)_I &= \int_{\mathbb{T}_3} |\boldsymbol{\eta}^\top \mathbf{x} - 1| \lambda_3(d\mathbf{x}) \\ &= \int_{\mathbb{T}_3} (\boldsymbol{\eta}^\top \mathbf{x} - 1) \lambda_3(d\mathbf{x}) - 2 \int_{\mathbb{T}_3^+} (\boldsymbol{\eta}^\top \mathbf{x} - 1) \lambda_3(d\mathbf{x}) \\ &= (\boldsymbol{\eta}^\top M - 1) \text{vol}_3 \mathbb{T}_3 - 2(\boldsymbol{\eta}^\top M^+ - 1) \text{vol}_3 \mathbb{T}_3^+ \\ &= \left(\frac{a+b+c}{4} - 1\right) \frac{1}{6} - 2\left(\frac{3}{4} - 1\right) \frac{1}{6abc} = \frac{1}{24} \left(a + b + c - 4 + \frac{2}{abc}\right) \end{aligned}$$

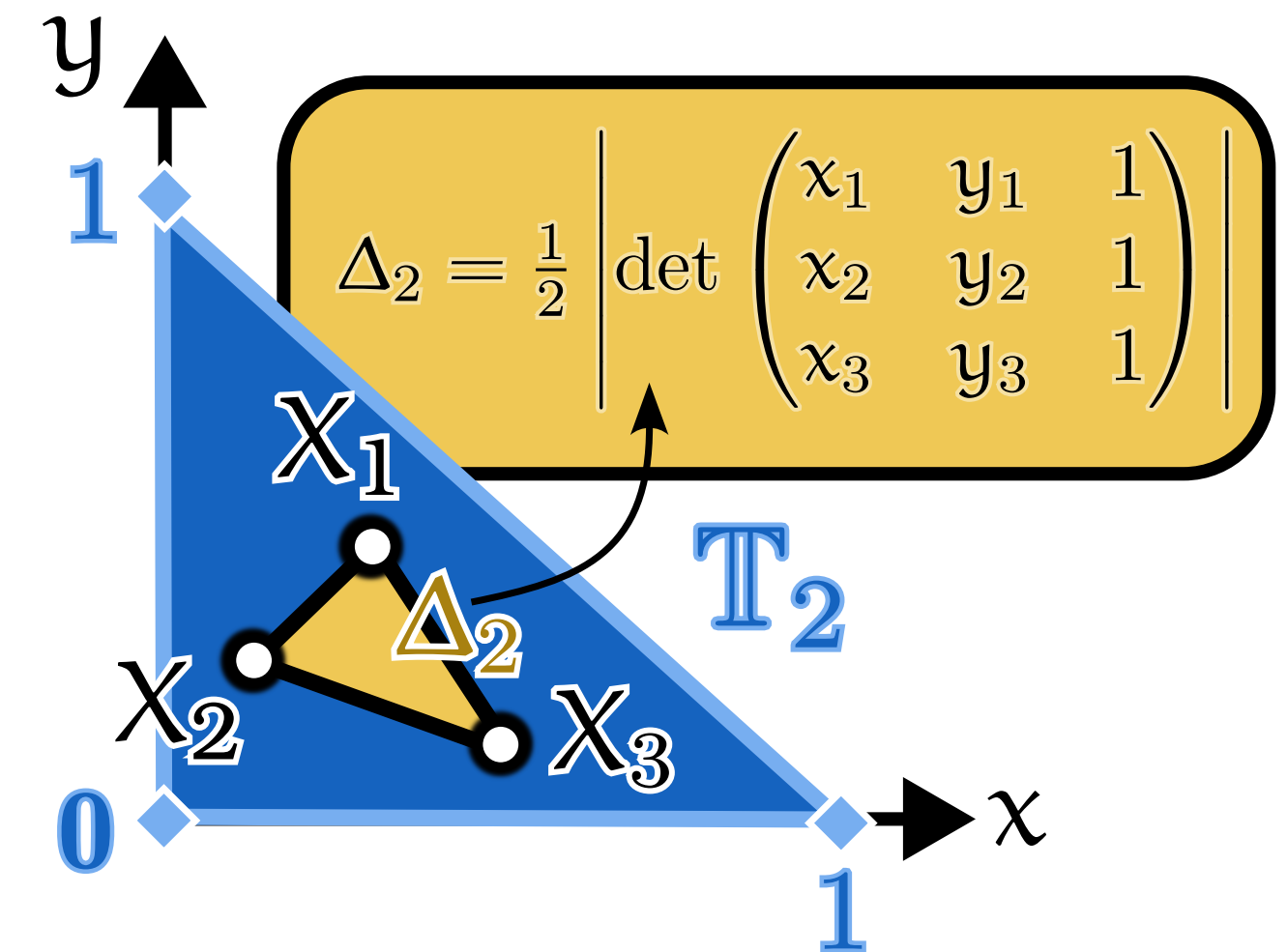


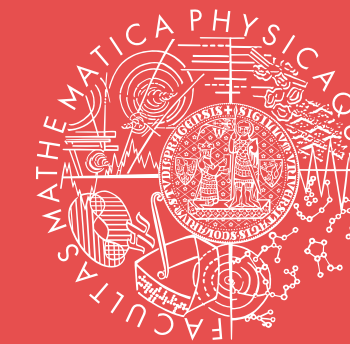
Mean cut function

- ❖ $\sigma \cap \mathbb{T}_3 = \mathbb{T}_2^{abc} \stackrel{\text{def}}{=} \text{conv}(\frac{1}{a}e_1, \frac{1}{b}e_2, \frac{1}{c}e_3)$
 - ❖ $X_1 = [x_1, y_1], X_2 = [x_2, y_2], X_3 = [x_3, y_3]$ by scale affinity
 - ❖ Affine invariancy: $v_2^{(2)}(\sigma \cap \mathbb{T}_3) \stackrel{\text{aff.}}{=} v_2^{(2)}(\mathbb{T}_2)$
- $$= \frac{\mathbb{E} \Delta_2^2}{\text{vol}_2^2 \mathbb{T}_2} = \frac{1}{\text{vol}_2^5 \mathbb{T}_2} \int_{\mathbb{T}_3^3} \Delta_2^2 d(x_1, y_1) d(x_2, y_2) d(x_3, y_3) = \frac{1}{72}$$

Section integral

$$\therefore v_3^{(1)}(\mathbb{T}_3)_I = \frac{2}{3} \int_1^\infty \int_1^\infty \int_1^\infty \frac{1}{72} \left(\frac{3}{abc} \right)^5 \frac{1}{24} \left(a + b + c - 4 + \frac{2}{abc} \right) da db dc = \frac{3}{2000}$$





Zeta and Iota

❖ Intersection points $A = \frac{1}{c}e_3 + \alpha\left(\frac{1}{a}e_1 - \frac{1}{c}e_3\right) \in \overline{e_1e_3}$

i.e. $\frac{1}{c} + \alpha\left(\frac{1}{a} - \frac{1}{c}\right) = 1$, therefore $\alpha = a\frac{1-c}{a-c}$, similarly $\beta = b\frac{1-c}{b-c}$.

$$\therefore A = \left[\frac{1-c}{a-c}, 0, \frac{a-1}{a-c}\right], \quad B = \left[0, \frac{1-c}{b-c}, \frac{b-1}{b-c}\right]$$

❖ Jacobian of transformation $dadbdc = \frac{c^2(1-c)^2}{(\alpha+c-1)^2(\beta+c-1)^2}d\alpha d\beta dc$

and half-domain transformation $\frac{1}{2}(\mathbb{R}^3 \setminus T_3^\circ)_{II} \rightarrow (1-c, 1)^2 \times (0, 1)$

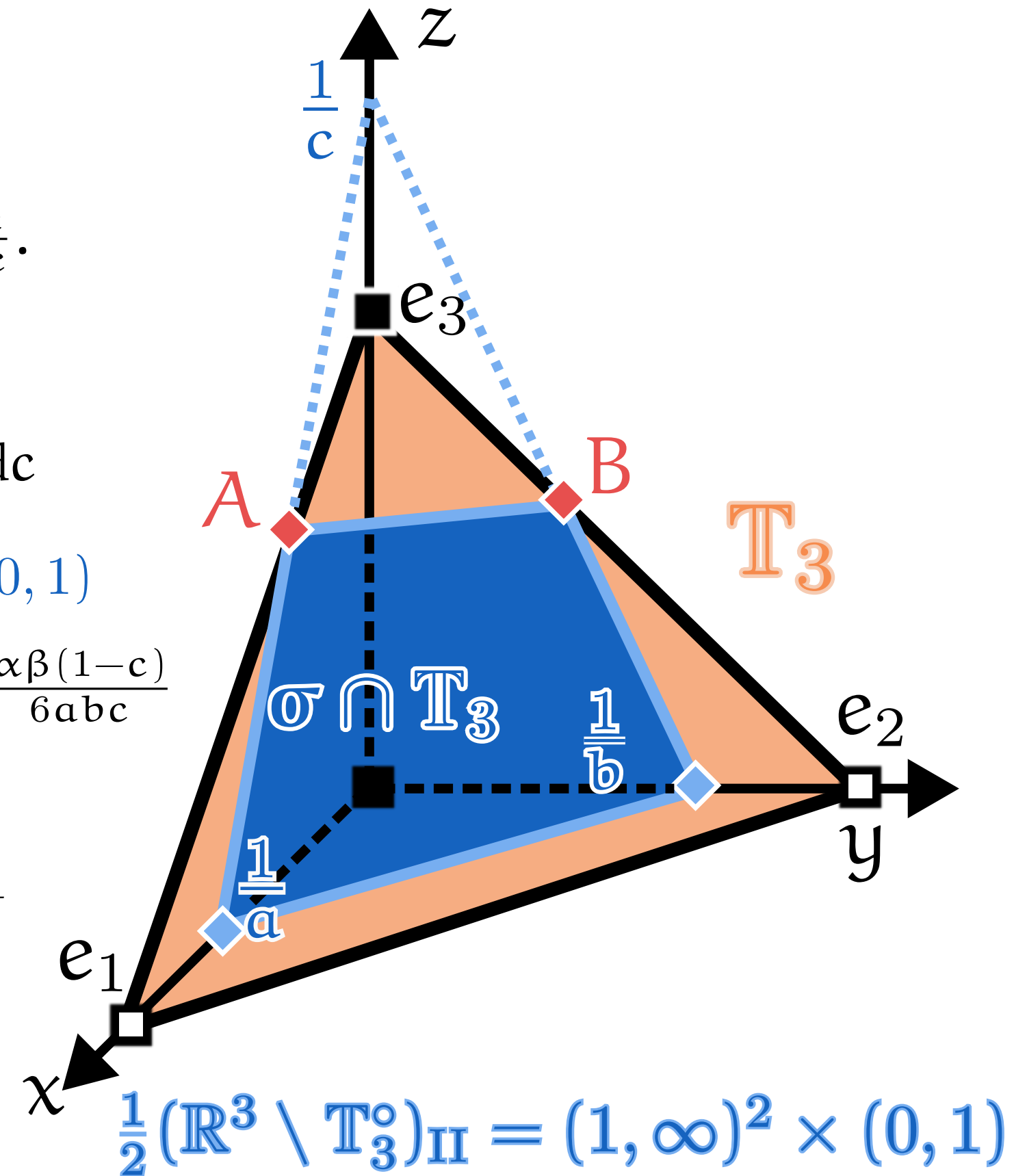
❖ $T_3^+ = T_3^{abc} \setminus T_3^*$, where $T_3^* \stackrel{\text{def}}{=} \text{conv}(A, B, e_3, \frac{1}{c}e_3) \Rightarrow \text{vol}_3 T_3^* = \frac{\alpha\beta(1-c)}{6abc}$

❖ Mass balance: $M^+ \text{vol}_3 T_3^+ = M^{abc} \text{vol}_3 T_3^{abc} - M^* \text{vol}_3 T_3^*$

$\therefore M^{abc} = \left[\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right], M^+ = \left[\frac{\alpha}{a}, \frac{\beta}{b}, \frac{3+c-\alpha-\beta}{c}\right]$ centerpoints of T_3^{abc}, T_3^+

❖ Iota: $\iota_3^{(1)}(\sigma)_{II} = (\eta^\top M - 1) \text{vol}_3 T_3 - 2(\eta^\top M^+ - 1) \text{vol}_3 T_3^+$
 $= \iota_3^{(1)}(\sigma)_I + 2(\eta^\top M^* - 1) \text{vol}_3 T_3^* = \iota_3^{(1)}(\sigma)_I - \frac{\alpha\beta(1-c)^2}{12abc}$

❖ Zeta: $\zeta_3(\sigma)_{II} = \frac{\text{vol}_2(\sigma \cap T_3)}{\|\eta\| \text{vol}_3 T_3} = (1 - \alpha\beta) \frac{\text{vol}_2 T_2^{abc}}{\|\eta\| \text{vol}_3 T_3} = (1 - \alpha\beta) \zeta_3(\sigma)_I = \frac{3(1-\alpha\beta)}{abc}$



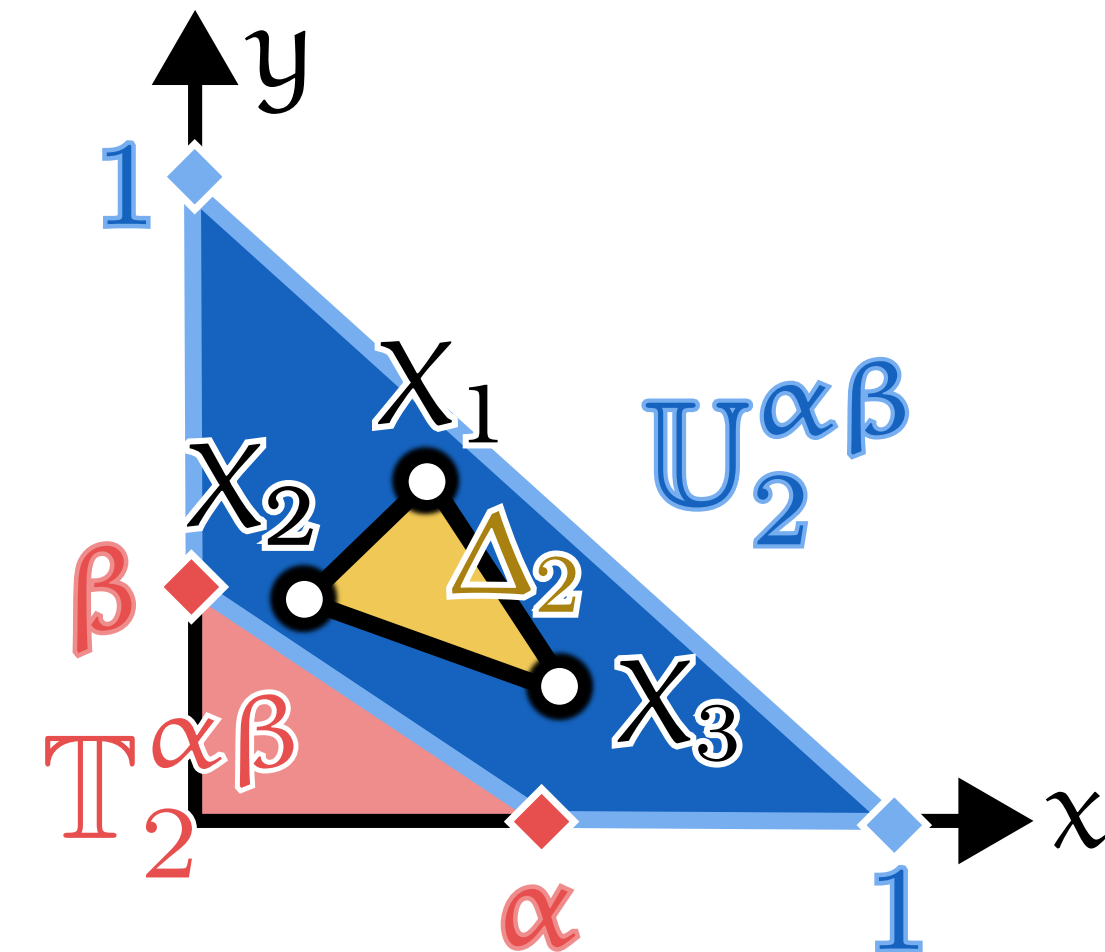
II



II

Mean cut function

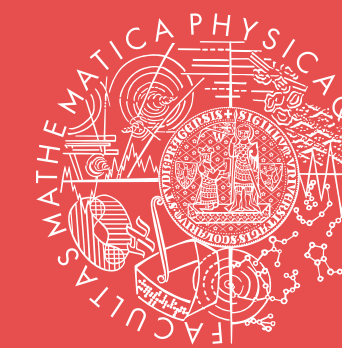
$$\begin{aligned} \ast \quad v_2^{(2)}(\sigma \cap \mathbb{T}_3) &\stackrel{\text{aff.}}{=} v_2^{(2)}(U_2^{\alpha\beta}) = \frac{1}{\text{vol}_2^5 U_2^{\alpha\beta}} \int_{(U_2^{\alpha\beta})^3} \Delta_2^2 dX_1 dX_2 dX_3 \\ &= \frac{\begin{Bmatrix} \alpha^4 \beta^4 - 8\alpha^3 \beta^3 + 8\alpha^3 \beta^2 - 4\alpha^3 \beta + 8\alpha^2 \beta^3 \\ -10\alpha^2 \beta^2 + 8\alpha^2 \beta - 4\alpha \beta^3 + 8\alpha \beta^2 - 8\alpha \beta + 1 \end{Bmatrix}}{72(1 - \alpha\beta)^4} \end{aligned}$$



Section integral

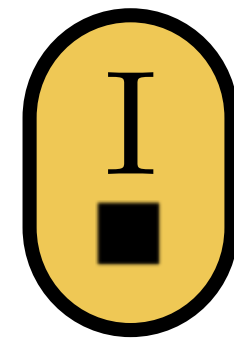
polynomials in c ←

$$\therefore v_3^{(1)}(\mathbb{T}_3)_{\text{II}} = 2 \cdot \frac{2}{3} \int_0^1 \int_{1-c}^1 \int_{1-c}^1 \cdots d\alpha d\beta dc = \int_0^1 \sum_{r=0}^2 \frac{q_r(c) \ln^r(1-c)}{c^{16}} dc = \frac{217}{54000} - \frac{\pi^2}{450450}$$

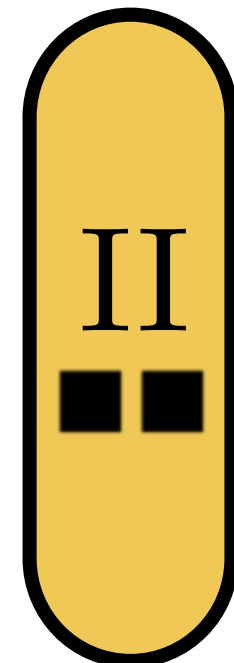


PENTACHORON (4-SIMPLEX) (DEMONSTRATION 3)

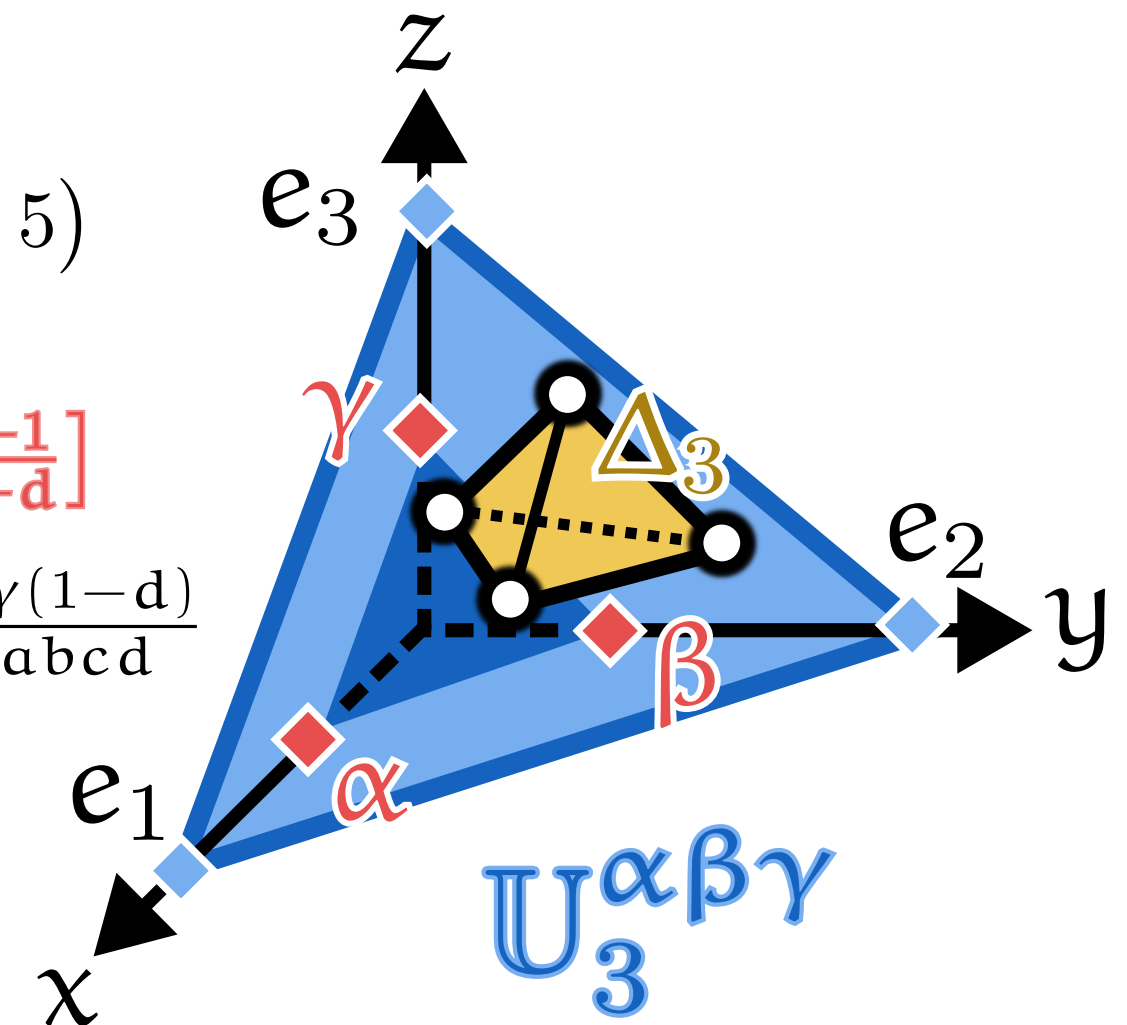
- ❖ Decomposition into configurations: $v_4^{(k)}(T_4) = 5v_4^{(k)}(T_4)_I + 10v_4^{(k)}(T_4)_{II}$, aff. $T_4 \longrightarrow T_4 \stackrel{\text{def}}{=} \text{conv}(\mathbf{0}, e_1, e_2, e_3, e_4)$
- ❖ Section plane: $\eta = (a, b, c, d)^T \Rightarrow \frac{1}{a}e_1, \frac{1}{b}e_2, \frac{1}{c}e_3, \frac{1}{d}e_4$ axes intersection points



- ❖ Intersection $T_4^+ = T_4^{abcd} \stackrel{\text{def}}{=} \text{conv}(\mathbf{0}, \frac{1}{a}e_1, \frac{1}{b}e_2, \frac{1}{c}e_3, \frac{1}{d}e_4)$
- ❖ Zeta and iota: $\zeta_2(\sigma)_I = \frac{24}{abcd}$, $\iota_4^{(1)}(\sigma)_I = \frac{1}{120} \left(\frac{2}{abcd} + a + b + c + d - 5 \right)$



- ❖ Int. pts. $A = [\frac{1-d}{a-d}, 0, 0, \frac{a-1}{a-d}]$, $B = [0, \frac{1-d}{b-d}, 0, \frac{b-1}{b-d}]$, $C = [0, 0, \frac{1-d}{c-d}, \frac{c-1}{c-d}]$
- ❖ $T_4^+ = T_4^{abcd} \setminus T_4^*$, where $T_4^* \stackrel{\text{def}}{=} \text{conv}(A, B, C, e_4, \frac{1}{d}e_4) \Rightarrow \text{vol}_4 T_4^* = \frac{\alpha\beta\gamma(1-d)}{24abcd}$
- ❖ $\zeta_4(\sigma)_{II} = (1 - \alpha\beta\gamma)\zeta_4(\sigma)_I$, $\iota_4^{(1)}(\sigma)_{II} = \iota_4^{(1)}(\sigma)_I - \frac{(1-d)^5}{60d(a-d)(b-d)(c-d)}$
- ❖ Affine invariancy: $v_3^{(k+1)}(\sigma \cap T_4) \stackrel{\text{aff.}}{=} v_3^{(k+1)}(U_3^{\alpha\beta\gamma})$



T_4



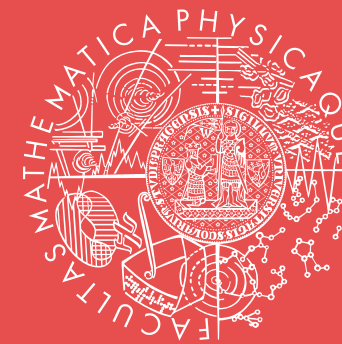
4-simplex
0.0031803708

$$v_4^{(1)}(T_4) = \frac{97}{27000} - \frac{2173\pi^2}{52026975}$$

ON RANDOM SIMPLEX PICKING

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Dominik Beck



FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University



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K_d	KNOWN RESULTS	$v_d^{(1)}(K_d)$	$v_d^{(2)}(K_d)$	$v_d^{(3)}(K_d)$
T_2	triangle ^[1] 0.083333333	$\frac{1}{12}$	$\frac{1}{72}$	$\frac{31}{9000}$
B_3	ball ^[2,3] 0.012587413	$\frac{9}{715}$	$\frac{3}{1000\pi^2}$	$\frac{3}{29393\pi^2}$
O_3	octahedron ^[6] 0.013637411	$\frac{19297\pi^2}{3843840} - \frac{6619}{184320}$	$\frac{1628355709\pi^2}{19864965120000} - \frac{81932629}{103219200000}$	
C_3	cube ^[5] 0.013842776	$\frac{3977}{216000} - \frac{\pi^2}{2160}$	$\frac{8411819}{450084600000} - \frac{\pi^2}{3402000}$	
T_3	tetrahedron ^[4] 0.017398239	$\frac{13}{720} - \frac{\pi^2}{15015}$	$\frac{733}{12600000} + \frac{79\pi^2}{2424922500}$	
T_4	4-simplex 0.0031803708	$\frac{97}{27000} - \frac{2173\pi^2}{52026975}$	$\frac{1955399}{3403417500000} + \frac{63065881\pi^2}{39669996140775000}$	
T_5	5-simplex 0.000523083	$\frac{2207}{3265920} - \frac{244129\pi^2}{14522729760} + \frac{73522\pi^4}{541513523351}$		
T_6	6-simplex 0.00078805	$\frac{26609}{217818720} - \frac{3396146609\pi^2}{621871356506400} + \frac{1318349152898\pi^4}{12180206401298390455}$		

DEFINITION

Q: Let $K_d \subset \mathbb{R}^d$ be a convex d -body and $X = (X_0, \dots, X_d)$ be a collection of random points picked uniformly and independently from its interior. The convex hull of this collection is a d -simplex with volume $\Delta_d = \text{vol}_d \text{conv } X$. What are its k -th moments? More specifically, the affine invariant constants:

$$v_d^{(k)}(K_d) = \mathbb{E} \Delta_d^k / (\text{vol}_d K_d)^k$$

LEMMA (BLASCHKE-PETKANCHIN)

$$\mathbb{E}[g(\sigma) \Delta_{d-1}^k] = \frac{(d-1)! \omega_d}{2(\text{vol}_d K_d)^d} \int_{\Delta(d, d-1)} v_{d-1}^{(k+1)}(\sigma \cap K_d) (\text{vol}_{d-1}(\sigma \cap K_d))^{k+d+1} g(\sigma) \mu_{d-1}(d\sigma)$$

- $X' = (X_1, \dots, X_d) \Rightarrow \Delta_{d-1} = \text{vol}_{d-1} \text{conv } X'$
- $\omega_d = \sigma_d(S_{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ surface area of the unit ball
- $\sigma = A(X') \in \Delta(d, d-1)$ affine (cutting) hyperplane
- $g: \Delta(d, d-1) \rightarrow \mathbb{R}$ any integrable function
- μ_{d-1} invariant measure on affine Grassmannian $\Delta(d, d-1)$

Cartesian reparametrisation

Let $x \in \sigma \Leftrightarrow \eta^T x = 1$, where $\eta = (\eta_1, \eta_2, \dots, \eta_d)^T$ then $\mu_{d-1}(d\sigma) = \frac{2}{\omega_d \|\eta\|^{d+1}} \lambda_d(d\eta)$

Consequence

$$\mathbb{E}[g(\sigma) \Delta_{d-1}^k] = (d-1)! (\text{vol}_d K)^{k+1} \int_{\mathbb{R}^d \setminus K_d^+} v_{d-1}^{(k+1)}(\sigma \cap K_d) \zeta_d^{d+k+1}(\sigma) g(\sigma) \|\eta\|^k \lambda_d(d\eta)$$

THEOREM (CANONICAL SECTION INTEGRAL) Let $\sigma = (x \in \mathbb{R}^d \mid \eta^T x = 1) \in \Delta(d, d-1)$ be a section plane splitting convex d -body K_d into $K_d^+ \sqcup K_d^-$, where $K_d^+ = \{x \in K_d \mid \eta^T x < 1\}$, then

$$v_d^{(k)}(K_d) = \frac{(d-1)!}{d!} \int_{\mathbb{R}^d \setminus K_d^+} v_{d-1}^{(k+1)}(\sigma \cap K_d) \tau_d^{d+k+1}(\sigma) v_d^{(k)}(\sigma) \lambda_d(d\eta)$$

where $\zeta_d(\sigma) = \frac{\text{vol}_{d-1}(\sigma \cap K_d)}{\|\eta\| \text{vol}_d K_d} = -\frac{1}{\text{vol}_d K_d} \sum_{i=1}^d \eta_i \frac{\partial \text{vol}_d K_d^+}{\partial \eta_i}$ and $v_d^{(k)}(\sigma) = \int_{K_d^+} \|\eta^T x - 1\|^k \lambda_d(dx)$.

Proof

- Base-height splitting: $\Delta_d = \frac{1}{d} D_\sigma(X_0) \Delta_{d-1}$
 $\therefore v_d^{(k)}(K_d) = \frac{\mathbb{E}[D_\sigma^k(X_0) \Delta_{d-1}^k]}{d^k (\text{vol}_d K_d)^k} = \frac{\mathbb{E}[\mathbb{E}[D_\sigma^k(X_0) \mid X'] \Delta_{d-1}^k]}{d^k (\text{vol}_d K_d)^k}$
- Distance formula: $D_\sigma(X_0) = \frac{1}{\|\eta\|} \|\eta^T X_0 - 1\|$
 $\therefore \mathbb{E}[D_\sigma^k(X_0) \mid X'] = \frac{1}{\text{vol}_d K_d} \int_{K_d^+} D_\sigma^k(x_0) \lambda_d(dx_0) = \frac{v_d^{(k)}(\sigma)}{\|\eta\|^k \text{vol}_d K_d}$
- The proof is concluded by taking $g(\sigma) = \mathbb{E}[D_\sigma^k(X_0) \mid X']$.

RE-DERIVATION OF REED'S RESULT IN A TRIANGLE (DEMONSTRATION 1)

$$v_2^{(k)}(T_2) = \frac{48k!}{(2+k)!(3+k)!} \sum_{j=0}^{k-1} \frac{j!}{(k-j+2)!(k+j+3)!}$$

- Decomposition into configurations: $v_2^{(k)}(T_2) = 3v_2^{(k)}(T_2)_I = 3v_2^{(k)}(T_2)_{II}$
- Affine invariance: $T_2 \rightarrow T_2 \cong \text{conv}(0, e_1, e_2)$ canonical 2-simplex
- Section integral: $v_2^{(k)}(T_2)_I = \frac{1}{2!} \int_{\mathbb{R}^2 \setminus T_2} v_1^{(k+1)}(\sigma \cap T_2) \zeta_2^{k+3}(\sigma) v_2^{(k)}(\sigma) \lambda_2(d\eta)$
- Section plane: $\eta = (a, b)^T \Rightarrow \frac{1}{a} e_1, \frac{1}{b} e_2$ axes intersection points
- $T_2^+ = \text{conv}(0, \frac{1}{a} e_1, \frac{1}{b} e_2) \therefore \text{vol}_2 T_2^+ = \frac{\text{vol}_2(\sigma \cap T_2)}{ab} = \frac{\text{vol}_2(\sigma \cap T_2)}{2ab}$
- $v_2^{(k)}(\sigma)_I = \int_{T_2^+} (1 - \eta^T x)^k \lambda_2(dx) + \int_{T_2 \setminus T_2^+} (\eta^T x - 1)^k \lambda_2(dx) = \frac{b(a-1)^{k+2} - a(b-1)^{k+2} + a-b}{ab(a-b)(1+k)(2+k)}$
- Affine invariance: $v_2^{(k+1)}(\sigma \cap T_2) \cong v_2^{(k+1)}((0, 1)) = \frac{1}{(2+k)(3+k)}$

$$\therefore v_2^{(k)}(T_2)_I = 16 \int_0^{\infty} \int_0^{\infty} \frac{b(a-1)^{k+2} - a(b-1)^{k+2} + a-b}{a^{k+3} b^{k+3} (a-b)(1+k)(2+k)!} da db$$

RE-DERIVATION OF BUCHTA'S & REITZNER'S RESULT (DEMONSTRATION 2)

$$v_3^{(1)}(T_3) = \frac{13}{720} - \frac{\pi^2}{15015}$$

- Decomposition into configurations: $v_3^{(k)}(T_3) = 4v_3^{(k)}(T_3)_I + 3v_3^{(k)}(T_3)_{II}$
- Affine invariance: $T_3 \rightarrow T_3 \cong \text{conv}(0, e_1, e_2, e_3)$ canonical 3-simplex
- Section integral: $v_3^{(1)}(T_3)_I = \frac{1}{3!} \int_{\mathbb{R}^3 \setminus T_3} v_2^{(2)}(\sigma \cap T_3) \zeta_3^3(\sigma) v_3^{(1)}(\sigma) \lambda_3(d\eta)$
- Section plane: $\eta = (a, b, c)^T \Rightarrow \frac{1}{a} e_1, \frac{1}{b} e_2, \frac{1}{c} e_3$ axes intersection points

Zeta and Iota

- $T_3^+ = T_3^{abc} \cong \text{conv}(0, \frac{1}{a} e_1, \frac{1}{b} e_2, \frac{1}{c} e_3)$, $D_\sigma(0) = 1/\|\eta\|$
 $\therefore \text{vol}_3 T_3^+ = \frac{1}{6abc} = \frac{\text{vol}_3(\sigma \cap T_3)}{3\|\eta\|} \Rightarrow \zeta_3(\sigma)_I = \frac{\text{vol}_3(\sigma \cap T_3)}{3\|\eta\| \text{vol}_3 T_3} = \frac{3}{abc}$
- $M = [\frac{1}{a}, \frac{1}{b}, \frac{1}{c}]$, $M^+ = [\frac{1}{2a}, \frac{1}{2b}, \frac{1}{2c}]$ centerpoints of T_3, T_3^+
- Iota integral: $i_3^{(1)}(\sigma)_I = \int_{T_3} \eta^T x - 1 \lambda_3(dx) = \int_{T_3} (\eta^T x - 1) \lambda_3(dx) = \int_{T_3} (\eta^T M - 1) \text{vol}_3 T_3 - 2(\eta^T M^+ - 1) \text{vol}_3 T_3^+ = (\frac{a+b+c}{2} - 1) \frac{1}{6abc} - 2(\frac{a+b+c}{4} - 1) \frac{1}{6abc} = \frac{1}{24} (a+b+c-4 + \frac{2}{abc})$

Mean cut function

- $\sigma \cap T_3 = T_3^{abc} \cong \text{conv}(\frac{1}{a} e_1, \frac{1}{b} e_2, \frac{1}{c} e_3)$
- $X_1 = [x_1, y_1], X_2 = [x_2, y_2], X_3 = [x_3, y_3]$ by scale affinity
- Affine invariance: $v_3^{(2)}(\sigma \cap T_3) \cong v_3^{(2)}(T_2)$
- $\frac{\mathbb{E} \Delta_2^2}{\text{vol}_2^2 T_2} = \frac{1}{\text{vol}_2^2 T_2} \int_{T_2} \Delta_2^2 d(x_1, y_1) d(x_2, y_2) d(x_3, y_3) = \frac{1}{72}$
- $\therefore v_3^{(1)}(T_3)_I = \frac{2}{3} \int_1^\infty \int_1^\infty \int_1^\infty \frac{1}{72} \left(\frac{3}{abc}\right)^5 \frac{1}{24} (a+b+c-4 + \frac{2}{abc}) da db dc = \frac{3}{2000}$

Zeta and Iota

- Intersection points $A = \frac{1}{c} e_3 + \alpha(\frac{1}{a} e_1 - \frac{1}{b} e_2) \in \overline{e_1 e_3}$
i.e. $\frac{1}{c} + \alpha(\frac{1}{a} - \frac{1}{b}) = 1$, therefore $\alpha = \frac{a-b}{a-b-c}$, similarly $\beta = \frac{b-c}{a-b-c}$
 $\therefore A = [\frac{1-b}{a-b-c}, 0, \frac{a-b}{a-b-c}]$, $B = [0, \frac{b-c}{a-b-c}, \frac{c-a}{a-b-c}]$
- Jacobian of transformation $da db dc = \frac{c^2(1-c)^2}{(a-b-c-1)^2(b-c-1)^2} da db dc$ and half-domain transformation $\frac{1}{2}(\mathbb{R}^3 \setminus T_3)_I \rightarrow (1-c, 1)^2 \times (0, 1)$
- $T_3^+ = T_3^{abc} \setminus T_3$, where $T_3^+ \cong \text{conv}(A, B, e_3, \frac{1}{a} e_1) \Rightarrow \text{vol}_3 T_3^+ = \frac{a\beta(1-c)}{6abc}$
- Mass balance: $M^+ \text{vol}_3 T_3^+ = M^{abc} \text{vol}_3 T_3^{abc} - M^+ \text{vol}_3 T_3$
 $\therefore M^{abc} = [\frac{a}{6}, \frac{b}{6}, \frac{c}{6}]$, $M^+ = [\frac{a}{6}, \frac{b}{6}, \frac{3+c-a-\beta}{6}]$ centerpoints of T_3^{abc}, T_3^+
- Iota: $i_3^{(1)}(\sigma)_I = (\eta^T M - 1) \text{vol}_3 T_3 - 2(\eta^T M^+ - 1) \text{vol}_3 T_3^+ = i_3^{(1)}(\sigma)_I + 2(\eta^T M^+ - 1) \text{vol}_3 T^+ = i_3^{(1)}(\sigma)_I - \frac{a\beta(1-c)^2}{12abc}$
- Zeta: $\zeta_3(\sigma)_I = \frac{\text{vol}_3(\sigma \cap T_3)}{\|\eta\| \text{vol}_3 T_3} = (1 - \alpha\beta) \frac{\text{vol}_3 T_3^{abc}}{\|\eta\| \text{vol}_3 T_3} = (1 - \alpha\beta) \zeta_3(\sigma)_I = \frac{3(1-\alpha\beta)}{abc}$

Mean cut function

- $v_3^{(2)}(\sigma \cap T_3) \cong v_3^{(2)}(U_2^{\alpha\beta}) = \frac{1}{\text{vol}_2^2 U_2^{\alpha\beta}} \int_{(U_2^{\alpha\beta})^2} \Delta_2^2 dX_1 dX_2 dX_3 = \frac{\alpha^4 \beta^4 - 8\alpha^3 \beta^3 + 8\alpha^2 \beta^2 - 4\alpha \beta + 8\alpha^2 \beta^3}{72(1-\alpha\beta)^4}$
- $\therefore v_3^{(1)}(T_3)_I = 2 \frac{2}{3} \int_0^1 \int_{1-c}^1 \int_{1-c}^1 \dots da db dc = \int_0^1 \sum_{c=0}^1 \frac{q_c(c) \ln^2(1-c)}{c^{16}} dc = \frac{217}{54000} - \frac{\pi^2}{450450}$

PENTACHORON (4-SIMPLEX) (DEMONSTRATION 3)

- Decomposition into configurations: $v_4^{(k)}(T_4) = 5v_4^{(k)}(T_4)_I + 10v_4^{(k)}(T_4)_{II}$, aff. $T_4 \rightarrow T_4 \cong \text{conv}(0, e_1, e_2, e_3, e_4)$
- Section plane: $\eta = (a, b, c, d)^T \Rightarrow \frac{1}{a} e_1, \frac{1}{b} e_2, \frac{1}{c} e_3, \frac{1}{d} e_4$ axes intersection points
- Intersection $T_4^+ = T_4^{abcd} \cong \text{conv}(0, \frac{1}{a} e_1, \frac{1}{b} e_2, \frac{1}{c} e_3, \frac{1}{d} e_4)$
- Zeta and Iota: $\zeta_4(\sigma)_I = \frac{24}{\|\eta\|} i_4^{(1)}(\sigma)_I = \frac{24}{\|\eta\|} (\frac{2}{a+b+c+d} + a+b+c+d-5)$
- Int. pts. $A = [\frac{a-d}{a-b-c-d}, 0, 0, \frac{a-d}{a-b-c-d}]$, $B = [0, \frac{b-d}{a-b-c-d}, 0, \frac{b-d}{a-b-c-d}]$, $C = [0, 0, \frac{c-d}{a-b-c-d}, \frac{c-d}{a-b-c-d}]$
- $T_4^+ = T_4^{abcd} \setminus T_4$, where $T_4^+ \cong \text{conv}(A, B, C, e_4, \frac{1}{a} e_1) \Rightarrow \text{vol}_4 T_4^+ = \frac{a\beta\gamma(1-d)}{24abcd}$
- $\zeta_4(\sigma)_I = (1 - \alpha\beta\gamma) \zeta_4(\sigma)_I$, $i_4^{(1)}(\sigma)_I = i_4^{(1)}(\sigma)_I - \frac{(1-d)^2}{24d(a-b)(1+k)(2+k)}$
- Affine invariance: $v_4^{(k+1)}(\sigma \cap T_4) \cong v_4^{(k+1)}(U_3^{\alpha\beta\gamma})$

$$\text{Conjecture } v_r^{(k)}(T_r) = \sum_{s=1}^r \frac{\tau_r/2!}{p_{rs}} \frac{1}{r^{2s-2}}, \quad p_{rs} \in \mathbb{Q}$$