ON RANDOM SIMPLEX PICKING

Dominik Beck
Department of Mathematics Education, Charles University, 186 75 Prague, Czech Republic

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KNOWN RESULTS / NEW RESULTS

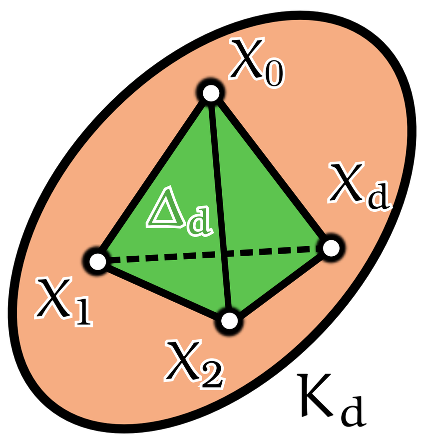
K_d		$v_d^{(1)}(K_d)$	$v_d^{(2)}(K_d)$	$v_d^{(3)}(K_d)$
T_2	triangle[1] 0.08333333	$\frac{1}{12}$	$\frac{1}{72}$	$\frac{31}{9000}$
C_2	square[4] 0.07638888	$\frac{11}{144}$	$\frac{1}{96}$	$\frac{137}{72000}$
B_3	3-ball[2] 0.012587413	$\frac{9}{715}$	$\frac{3}{1000\pi^2}$	$\frac{3}{29393\pi^2}$
O_3	octahedron[7] 0.013637411	$\frac{19297\pi^2}{3843840}-\frac{6619}{184320}$		$\frac{1628355709\pi^2}{19864965120000}-\frac{81932629}{103219200000}$
C_3	cube[6] 0.013842776	$\frac{3977}{216000}-\frac{\pi^2}{2160}$		$\frac{8411819}{450084600000}-\frac{\pi^2}{3402000}$
T_3	tetrahedron[5] 0.017398239	$\frac{13}{720}-\frac{\pi^2}{15015}$		$\frac{733}{12600000}+\frac{79\pi^2}{2424922500}$
T_4	4-simplex 0.0031803708	$\frac{97}{27000}-\frac{2173\pi^2}{52026975}$		$\frac{1955399}{3403417500000}+\frac{63065881\pi^2}{396699961407750000}$
T_5	5-simplex 0.000523083	$\frac{2207}{3265920}-\frac{244129\pi^2}{14522729760}+\frac{73522\pi^4}{541513323351}$		
T_6	6-simplex 0.000078805	$\frac{26609}{217818720}-\frac{3396146609\pi^2}{621871356506400}+\frac{1318349152898\pi^4}{12180206401298390455}$		

NEW RESULTS

DEFINITION

Q: Let $K_d \subset \mathbb{R}^d$ be a convex d-body and $\mathbb{X}=(X_0,\dots,X_d)$ be a collection of random points picked uniformly and independently from its interior. The convex hull of this collection is a d-simplex with volume $\Delta_d=\operatorname{vol}_d\operatorname{conv}\mathbb{X}$. What are its k-th moments? More specifically, the affine invariant constants:

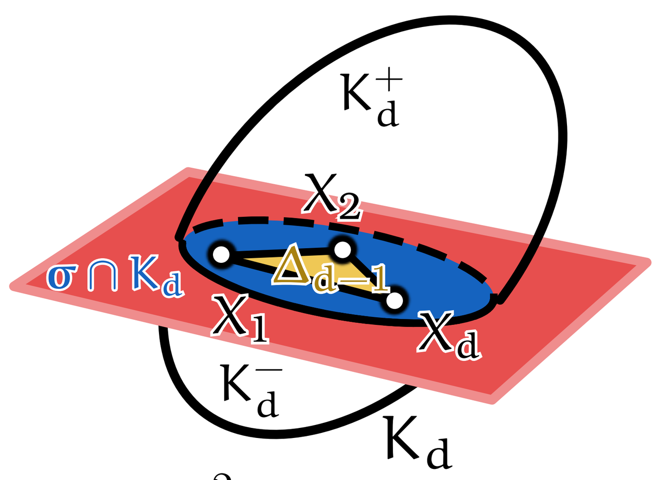
$$v_d^{(k)}(K_d)=\mathbb{E}\,\Delta_d^k/(\operatorname{vol}_d K_d)^k$$



LEMMA (BLASHKE-PETKANCHIN)

$$\mathbb{E}\left[g(\sigma)\Delta_{d-1}^k\right]=\frac{(d-1)!\,\omega_d}{2(\operatorname{vol}_d K_d)^d}\int_{\mathbb{A}(d,d-1)} v_{d-1}^{(k+1)}(\sigma\cap K_d)(\operatorname{vol}_{d-1}(\sigma\cap K_d))^{k+d+1}g(\sigma)\mu_{d-1}(d\sigma)$$

- $\mathbb{X}'=(X_1,\dots,X_d)\implies\Delta_{d-1}=\operatorname{vol}_{d-1}\operatorname{conv}\mathbb{X}'$
- $\omega_d=\sigma_d(S_{d-1})=2\pi^{d/2}/\Gamma(d/2)$ surface area of the unit ball
- $\sigma=\mathcal{A}(\mathbb{X}')\in\mathbb{A}(d,d-1)$ affine (cutting) hyperplane
- $g:\mathbb{A}(d,d-1)\to\mathbb{R}$ any integrable function
- μ_{d-1} invariant measure on affine Grassmannian $\mathbb{A}(d,d-1)$



Cartesion reparametrisation

Let $x\in\sigma\Leftrightarrow\eta^\top x=1$, where $\eta=(\eta_1,\eta_2,\dots,\eta_d)^\top$ then $\mu_{d-1}(d\sigma)=\dfrac{2}{\omega_d\|\eta\|^{d+1}}\lambda_d(d\eta)$

Consequence

$$\mathbb{E}\left[g(\sigma)\Delta_{d-1}^k\right]=(d-1)!(\operatorname{vol}_d K)^{k+1}\int_{\mathbb{R}^d\setminus K_d^\circ} v_{d-1}^{(k+1)}(\sigma\cap K_d)\zeta_d^{d+k+1}(\sigma)g(\sigma)\|\eta\|^k\lambda_d(d\eta)$$

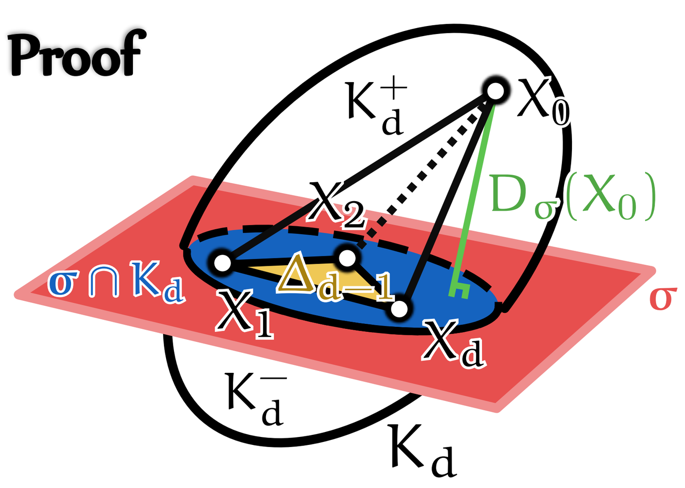
THEOREM (CANONICAL SECTION INTEGRAL)

Let $\sigma=\{x\in\mathbb{R}^d\mid\eta^\top x=1\}\in\mathbb{A}(d,d-1)$ be a section plane splitting convex d-body K_d into $K_d^+\sqcup K_d^-$, where $K_d^+=\{x\in K_d\mid\eta^\top x<1\}$, then

$$v_d^{(k)}(K_d)=\frac{(d-1)!}{d^k}\int_{\mathbb{R}^d\setminus K_d^\circ} v_{d-1}^{(k+1)}(\sigma\cap K_d)\,\zeta_d^{d+k+1}(\sigma)\iota_d^{(k)}(\sigma)\lambda_d(d\eta)$$

where $\zeta_d(\sigma)=\dfrac{\operatorname{vol}_{d-1}(\sigma\cap K_d)}{\|\eta\|\operatorname{vol}_d K_d}=-\dfrac{1}{\operatorname{vol}_d K_d}\sum_{j=1}^d\eta_j\dfrac{\partial\operatorname{vol}_d K_d^+}{\partial\eta_j}$ and $\iota_d^{(k)}(\sigma)=\int_{K_d}|\eta^\top x-1|^k\lambda_d(dx)$.

Proof



1. Base-height splitting: $\Delta_d=\frac{1}{d}D_\sigma(X_0)\Delta_{d-1}$ (distance from σ to X_0)

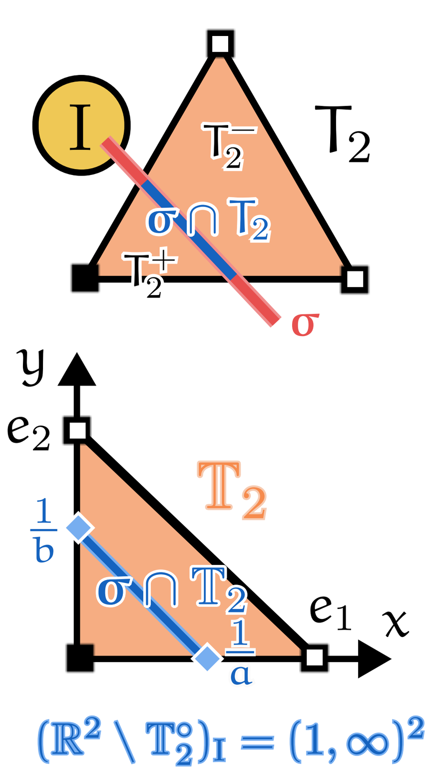
$$\therefore v_d^{(k)}(K_d)=\frac{\mathbb{E}[D_\sigma^k(X_0)\Delta_{d-1}^k]}{d^k(\operatorname{vol}_d K_d)^k}=\frac{\mathbb{E}[\mathbb{E}[D_\sigma^k(X_0)\mid\mathbb{X}']\,\Delta_{d-1}^k]}{d^k(\operatorname{vol}_d K_d)^k}$$

2. Distance formula: $D_\sigma(X_0)=\frac{1}{\|\eta\|}|\eta^\top X_0-1|$

$$\therefore\mathbb{E}[D_\sigma^k(X_0)\mid\mathbb{X}']=\frac{1}{\operatorname{vol}_d K_d}\int_{K_d}D_\sigma^k(x_0)\lambda_d(dx_0)=\frac{\iota_d^{(k)}(\sigma)}{\|\eta\|^k\operatorname{vol}_d K_d}$$

3. The proof is conluded by taking $g(\sigma)=\mathbb{E}[D_\sigma^k(X_0)\mid\mathbb{X}']$.

RE-DERIVATION OF REED'S RESULT IN A TRIANGLE (DEMONSTRATION 1)



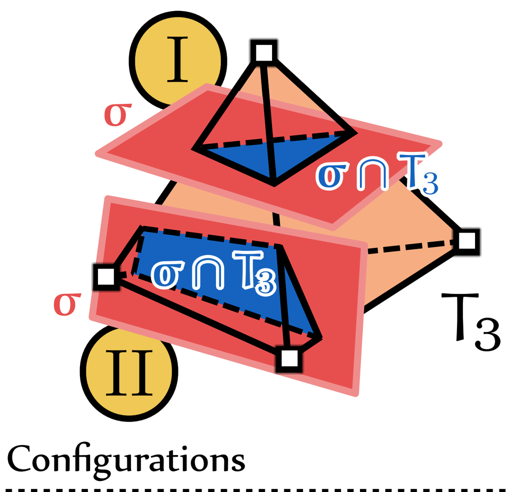
$(\mathbb{R}^2\setminus\mathbb{T}_2^\circ)_I=(1,\infty)^2$

$$v_2^{(k)}(T_2)=\frac{48\,k!}{(2+k)(3+k)}\sum_{j=0}^{k+1}\frac{j!}{(k-j+2)(k+j+3)!}$$

- Decomposition into configurations: $v_2^{(k)}(T_2)=3v_2^{(k)}(T_2)_I=3v_2^{(k)}(\mathbb{T}_2)_I$
- Affine invariancy: $T_2\longrightarrow\mathbb{T}_2\overset{\text{def}}{=}\operatorname{conv}(\mathbf{0},e_1,e_2)$ canonical 2-simplex
- Section integral: $v_2^{(k)}(\mathbb{T}_2)_I=\dfrac{1}{2^k}\int_{(\mathbb{R}^2\setminus\mathbb{T}_2^\circ)_I}v_1^{(k+1)}(\sigma\cap\mathbb{T}_2)\,\zeta_2^{k+3}(\sigma)\iota_2^{(k)}(\sigma)\lambda_2(d\eta)$
- Section plane: $\eta=(a,b)^\top\Rightarrow\frac{1}{a}e_1,\frac{1}{b}e_2$ axes intersection points
- $\mathbb{T}_2^+=\operatorname{conv}(\mathbf{0},\frac{1}{a}e_1,\frac{1}{b}e_2)\therefore\operatorname{vol}_2\mathbb{T}_2^+=\frac{1}{2ab}=\frac{\operatorname{vol}_1(\sigma\cap\mathbb{T}_2)}{2\|\eta\|}\Rightarrow\zeta_2(\sigma)_I=\frac{\operatorname{vol}_1(\sigma\cap\mathbb{T}_2)}{\|\eta\|\operatorname{vol}_2\mathbb{T}_2}=\frac{2}{ab}$
- $\iota_2^{(k)}(\sigma)_I=\int_{\mathbb{T}_2^+}(1-\eta^\top x)^k\lambda_2(dx)+\int_{\mathbb{T}_2\setminus\mathbb{T}_2^+}(\eta^\top x-1)^k\lambda_2(dx)=\frac{b(a-1)^{k+2}-a(b-1)^{k+2}+a-b}{ab(a-b)(1+k)(2+k)}$
- Affine invariancy: $v_1^{(k+1)}(\sigma\cap\mathbb{T}_2)\overset{\text{aff.}}{=}v_1^{(k+1)}((0,1))=\frac{2}{(2+k)(3+k)}$

$$\therefore v_2^{(k)}(\mathbb{T}_2)_I=16\int_1^\infty\int_1^\infty\frac{b(a-1)^{2+k}-a(b-1)^{2+k}+a-b}{a^{k+4}b^{k+4}(a-b)(1+k)(2+k)^2(3+k)}\,dadb$$

RE-DERIVATION OF BUCHTA'S & REITZNER'S RESULT (DEMONSTRATION 2)



Configurations

$$v_3^{(1)}(T_3)=\frac{13}{720}-\frac{\pi^2}{15015}$$

- Decomposition into configurations: $v_3^{(k)}(T_3)=4v_3^{(k)}(T_3)_I+3v_3^{(k)}(T_3)_{II}$
- Affine invariancy: $T_3\longrightarrow\mathbb{T}_3\overset{\text{def}}{=}\operatorname{conv}(\mathbf{0},e_1,e_2,e_3)$ canonical 3-simplex
- Section integral: $v_3^{(1)}(\mathbb{T}_3)_C=\frac{2}{3}\int_{(\mathbb{R}^3\setminus\mathbb{T}^\circ)_C}v_2^{(2)}(\sigma\cap\mathbb{T}_3)\zeta_3^5(\sigma)_C\,\iota_3^{(1)}(\sigma)\lambda_3(d\eta)$
- Section plane: $\eta=(a,b,c)^\top\Rightarrow\frac{1}{a}e_1,\frac{1}{b}e_2,\frac{1}{c}e_3$ axes intersection points

I

Zeta and Iota

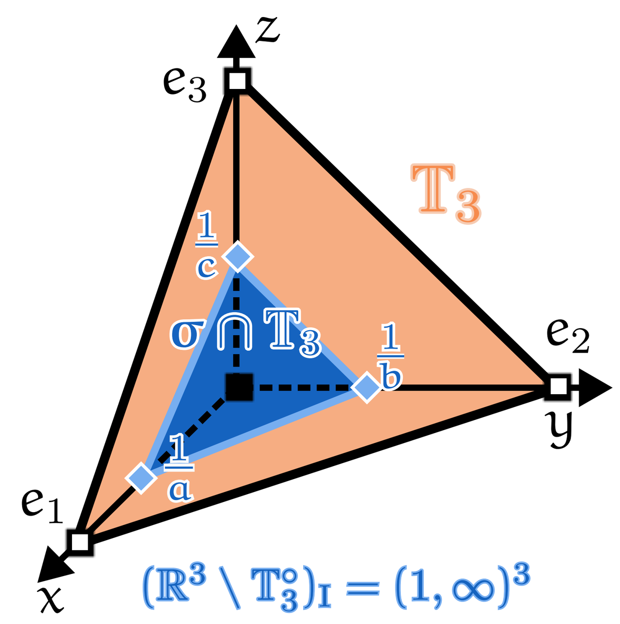
- $\mathbb{T}_3^+=\mathbb{T}_3^{abc}\overset{\text{def}}{=}\operatorname{conv}(\mathbf{0},\frac{1}{a}e_1,\frac{1}{b}e_2,\frac{1}{c}e_3)$, $D_\sigma(0)=1/\|\eta\|$

  $\therefore\operatorname{vol}_3\mathbb{T}_3^+=\frac{1}{6abc}=\frac{\operatorname{vol}_2(\sigma\cap\mathbb{T}_3)}{3\|\eta\|}\Rightarrow\zeta_3(\sigma)_I=\frac{\operatorname{vol}_2(\sigma\cap\mathbb{T}_3)}{\|\eta\|\operatorname{vol}_3\mathbb{T}_3}=\frac{3}{abc}$
- $M=[\frac{1}{4},\frac{1}{4},\frac{1}{4}]$, $M^+=[\frac{1}{4a},\frac{1}{4b},\frac{1}{4c}]$ centerpoints of $\mathbb{T}_3,\mathbb{T}_3^+$
- Iota integral: $\iota_3^{(1)}(\sigma)_I=\int_{\mathbb{T}_3}|\eta^\top x-1|\lambda_3(dx)$

  $=\int_{\mathbb{T}_3}(\eta^\top x-1)\lambda_3(dx)-2\int_{\mathbb{T}_3^+}(\eta^\top x-1)\lambda_3(dx)$

  $=(\eta^\top M-1)\operatorname{vol}_3\mathbb{T}_3-2(\eta^\top M^+-1)\operatorname{vol}_3\mathbb{T}_3^+$

  $=\left(\frac{a+b+c}{4}-1\right)\frac{1}{6}-2\left(\frac{3}{4}-1\right)\frac{1}{6abc}=\frac{1}{24}\left(a+b+c-4+\frac{2}{abc}\right)$

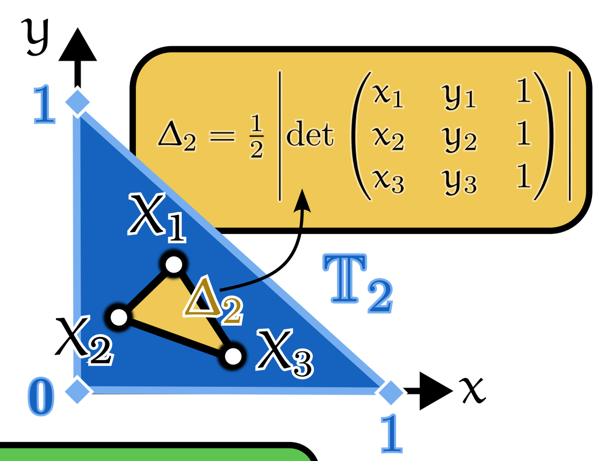


$(\mathbb{R}^3\setminus\mathbb{T}_3^\circ)_I=(1,\infty)^3$

Mean cut function

- $\sigma\cap\mathbb{T}_3=\mathbb{T}_2^{abc}\overset{\text{def}}{=}\operatorname{conv}(\frac{1}{a}e_1,\frac{1}{b}e_2,\frac{1}{c}e_3)$
- $X_1=[x_1,y_1],X_2=[x_2,y_2],X_3=[x_3,y_3]$ by scale affinity
- Affine invariancy: $v_2^{(2)}(\sigma\cap\mathbb{T}_3)\overset{\text{aff.}}{=}v_2^{(2)}(\mathbb{T}_2)$

$$=\frac{\mathbb{E}\,\Delta_2^2}{\operatorname{vol}_2^2\mathbb{T}_2}=\frac{1}{\operatorname{vol}_2^5\mathbb{T}_2}\int_{\mathbb{T}_2^3}\Delta_2^2\,d(x_1,y_1)d(x_2,y_2)d(x_3,y_3)=\frac{1}{72}$$



$$\therefore v_3^{(1)}(\mathbb{T}_3)_I=\frac{2}{3}\int_1^\infty\int_1^\infty\int_1^\infty\frac{1}{72}\left(\frac{3}{abc}\right)^5\frac{1}{24}\left(a+b+c-4+\frac{2}{abc}\right)dadbdc=\frac{3}{2000}$$

II

Zeta and Iota

- Intersection points $A=\frac{1}{c}e_3+\alpha\left(\frac{1}{a}e_1-\frac{1}{c}e_3\right)\in\overline{e_1e_3}$

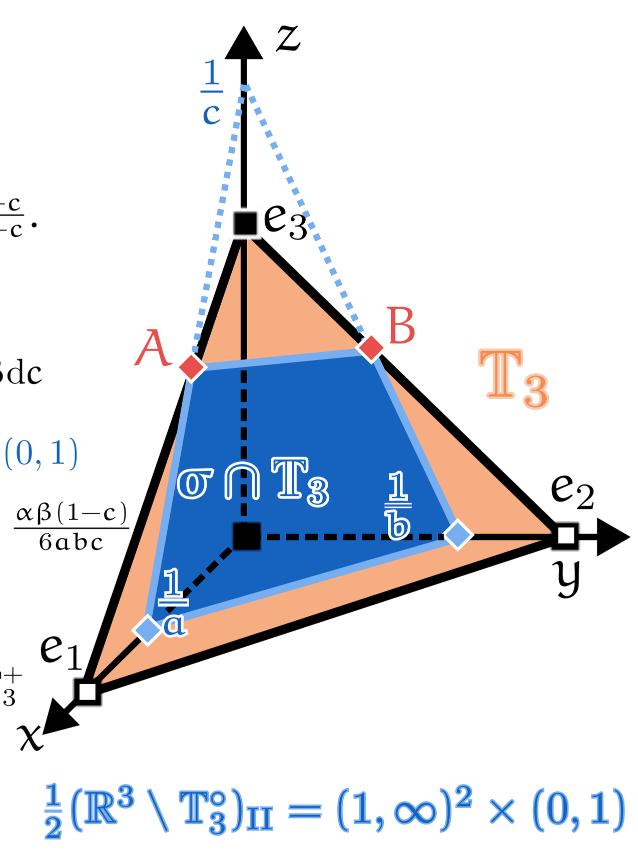
  i.e. $\frac{1}{c}+\alpha\left(\frac{1}{a}-\frac{1}{c}\right)=1$, therefore $\alpha=a\frac{1-c}{a-c}$, similarly $\beta=b\frac{1-c}{b-c}$.

  $\therefore\quad A=\left[\frac{1-c}{a-c},0,\frac{a-1}{a-c}\right],\quad B=\left[0,\frac{1-c}{b-c},\frac{b-1}{b-c}\right]$
- Jacobian of transformation $dadbdc=\frac{c^2(1-c)^2}{(\alpha+c-1)^2(\beta+c-1)^2}d\alpha d\beta dc$

  and half-domain transformation $\frac{1}{2}(\mathbb{R}^3\setminus\mathbb{T}_3^\circ)_{II}\to(1-c,1)^2\times(0,1)$
- $\mathbb{T}_3^+=\mathbb{T}_3^{abc}\setminus\mathbb{T}_3^*$, where $\mathbb{T}_3^*\overset{\text{def}}{=}\operatorname{conv}\left(A,B,e_3,\frac{1}{c}e_3\right)\Rightarrow\operatorname{vol}_3\mathbb{T}_3^*=\frac{\alpha\beta(1-c)}{6abc}$
- Mass ballance: $M^+\operatorname{vol}_3\mathbb{T}_3^+=M^{abc}\operatorname{vol}_3\mathbb{T}_3^{abc}-M^*\operatorname{vol}_3\mathbb{T}_3^*$

  $\therefore M^{abc}=\left[\frac{a}{4},\frac{b}{4},\frac{c}{4}\right],M^+=\left[\frac{\alpha}{a},\frac{\beta}{b},\frac{3+c-\alpha-\beta}{c}\right]$ centerpoints of $\mathbb{T}_3^{abc},\mathbb{T}_3^+$
- Iota: $\iota_3^{(1)}(\sigma)_{II}=(\eta^\top M-1)\operatorname{vol}_3\mathbb{T}_3-2(\eta^\top M^+-1)\operatorname{vol}_3\mathbb{T}_3^+$

  $=\iota_3^{(1)}(\sigma)_I+2(\eta^\top M^*-1)\operatorname{vol}_3\mathbb{T}^*=\iota_3^{(1)}(\sigma)_I-\frac{\alpha\beta(1-c)^2}{12abc}$
- Zeta: $\zeta_3(\sigma)_{II}=\frac{\operatorname{vol}_2(\sigma\cap\mathbb{T}_3)}{\|\eta\|\operatorname{vol}_3\mathbb{T}_3}=(1-\alpha\beta)\frac{\operatorname{vol}_2 T_2^{abc}}{\|\eta\|\operatorname{vol}_3\mathbb{T}_3}=(1-\alpha\beta)\zeta_3(\sigma)_I=\frac{3(1-\alpha\beta)}{abc}$

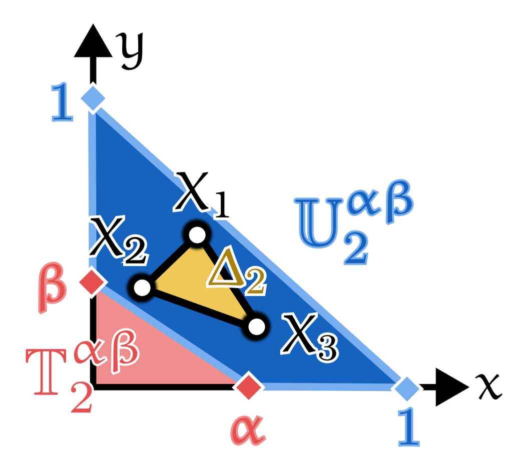


$\frac{1}{2}(\mathbb{R}^3\setminus\mathbb{T}_3^\circ)_{II}=(1,\infty)^2\times(0,1)$

Mean cut function

- $v_2^{(2)}(\sigma\cap\mathbb{T}_3)\overset{\text{aff.}}{=}v_2^{(2)}(\mathbb{U}_2^{\alpha\beta})=\dfrac{1}{\operatorname{vol}_2^5\mathbb{U}_2^{\alpha\beta}}\int_{(\mathbb{U}_2^{\alpha\beta})^3}\Delta_2^2\,dX_1dX_2dX_3$

$$=\frac{\left\{\begin{matrix}\alpha^4\beta^4-8\alpha^3\beta^3+8\alpha^3\beta^2-4\alpha^3\beta+8\alpha^2\beta^3\\-10\alpha^2\beta^2+8\alpha^2\beta-4\alpha\beta^3+8\alpha\beta^2-8\alpha\beta+1\end{matrix}\right\}}{72(1-\alpha\beta)^4}$$



polynomials in c

$$\therefore v_3^{(1)}(\mathbb{T}_3)_{II}=2\cdot\frac{2}{3}\int_0^1\int_{1-c}^1\int_{1-c}^1\cdots\,d\alpha d\beta dc=\int_0^1\sum_{r=0}^2\frac{q_r(c)\ln^r(1-c)}{c^{16}}dc=\frac{217}{54000}-\frac{\pi^2}{450450}$$

PENTACHORON (4-SIMPLEX) (DEMONSTRATION 3)

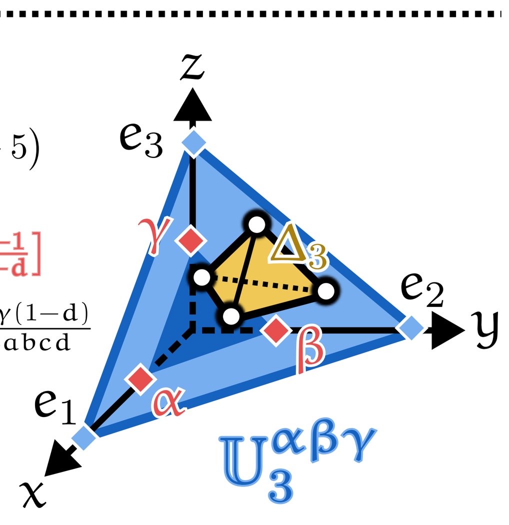
- Decomposition into configurations: $v_4^{(k)}(T_4)=5v_4^{(k)}(T_4)_I+10v_4^{(k)}(T_4)_{II}$, aff. $T_4\longrightarrow\mathbb{T}_4\overset{\text{def}}{=}\operatorname{conv}(\mathbf{0},e_1,e_2,e_3,e_4)$
- Section plane: $\eta=(a,b,c,d)^\top\Rightarrow\frac{1}{a}e_1,\frac{1}{b}e_2,\frac{1}{c}e_3,\frac{1}{d}e_4$ axes intersection points

I

- Intersection $\mathbb{T}_4^+=\mathbb{T}_4^{abcd}\overset{\text{def}}{=}\operatorname{conv}(\mathbf{0},\frac{1}{a}e_1,\frac{1}{b}e_2,\frac{1}{c}e_3,\frac{1}{d}e_4)$
- Zeta and iota: $\zeta_2(\sigma)_I=\frac{24}{abcd}$, $\iota_4^{(1)}(\sigma)_I=\frac{1}{120}\left(\frac{2}{abcd}+a+b+c+d-5\right)$

II

- Int. pts. $A=\left[\frac{1-d}{a-d},0,0,\frac{a-1}{a-d}\right],B=\left[0,\frac{1-d}{b-d},0,\frac{b-1}{b-d}\right],C=\left[0,0,\frac{1-d}{c-d},\frac{c-1}{c-d}\right]$
- $\mathbb{T}_4^+=\mathbb{T}_4^{abcd}\setminus\mathbb{T}_4^*$, where $\mathbb{T}_4^*\overset{\text{def}}{=}\operatorname{conv}\left(A,B,C,e_4,\frac{1}{d}e_4\right)\Rightarrow\operatorname{vol}_4\mathbb{T}_4^*=\frac{\alpha\beta\gamma(1-d)}{24abcd}$
- $\zeta_4(\sigma)_{II}=(1-\alpha\beta\gamma)\zeta_4(\sigma)_I$, $\iota_4^{(1)}(\sigma)_{II}=\iota_4^{(1)}(\sigma)_I-\frac{(1-d)^5}{60d(a-d)(b-d)(c-d)}$
- Affine invariancy: $v_3^{(k+1)}(\sigma\cap\mathbb{T}_4)\overset{\text{aff.}}{=}v_3^{(k+1)}(\mathbb{U}_3^{\alpha\beta\gamma})$



Conjecture $v_r^{(k)}(T_r)=\sum_{s=1}^{\lceil r/2\rceil}p_{rs}^{(k)}\pi^{2s-2}$, $p_{rs}^{(k)}\in\mathbb{Q}$

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