# MOMENTY DETERMINANTŮ NÁHODNÝCH MATIC

Plán přednášky:

- $\boxed{A}$ PŘIPOMENUTÍ ZÁKLADŮ
- $\boxed{B}$ NÁHODNÁ PROMĚNNÁ
- $\boxed{C}$ PERMUTACE
- $\boxed{D}$ DETERMINANT
- $\boxed{E}$ MOM. DET. NÁH. MATIC

# A PŘIPOMENUTÍ ZÁKLADŮ

- **Binomická věta** :
$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}}$$

binomické číslo $:= \dfrac{\overbrace{n\cdot(n-1)\cdots(n-k+1)}^{k}}{\underbrace{k(k-1)\cdots 3\cdot 2\cdot 1}_{k!}}$

* Dk :

$$(x+y)^n = (x+y)(x+y)(x+y)(x+y)\cdots(x+y)$$

$$= \cdots + c_{n,k}\, x^k y^{n-k} + \cdots \quad \left.\begin{matrix}\text{vybírám}\\ k \times \odot x\\ n-k \times \ominus y\end{matrix}\right\}$$

$$\therefore\ c_{n,k} = \frac{\#\text{výběrů } k \text{ indexů } x\text{-ů uspoř.}}{\#\text{prohození } k \text{ indexů}}$$

- **Exponenciální funkce** :
$$\boxed{e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$$

$e := \lim_{y\to\infty} \left(1+\frac{1}{y}\right)^y$ Eulerovo číslo

* Dk : $e^x$ také $\exp x$

$$e^x = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{xy} = \lim_{y\to\infty}\left(1+\frac{x}{\underbrace{xy}_{n}}\right)^{\overbrace{xy}^{n}} =$$

$$= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{n}\right)^k\cdot 1^{n-k} =$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^k}{n^k}\,\frac{n(n-1)\cdots(n-k+1)}{k!} =$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^k}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) = \sum_{k=0}^{\infty}\frac{x^k}{k!} \quad \blacksquare$$

- **Geometrická řada** :
$$\boxed{\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k} \quad |x|<1$$

* Dk : $S = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \quad /\cdot x$

$$xS = \phantom{1+{}} x + x^2 + x^3 + \cdots$$

$\ominus$ : $S - xS = 1 \implies S = \dfrac{1}{1-x}$

$\blacksquare$

# • Počítání s maticemi

* Definice matice: $A = (a_{ij})_{m \times n} :=$

$$m \text{ řádků} \left\{ \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{n \text{ sloupců}} \right.$$

* Transponovaná matice $(A^T)_{ij} = a_{ji}$

* Sčítání matic $A + B$

Př $\underset{A}{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}} + \underset{B}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 6 \end{pmatrix}} = \underset{A+B}{\begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 7 \end{pmatrix}}$

(obecně platí $A + B = B + A$)

* Násobení skalárem $\lambda A$

$$4 \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 12 \\ 0 & 4 & 4 \end{pmatrix}$$

* Maticové násobení $C = AB \overset{def}{\iff} (C)_{ij} = \sum_k a_{ik} b_{kj}$

Př $\overset{\xrightarrow{k}}{\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 1 \end{pmatrix} \Big\downarrow k = \begin{pmatrix} 11 & 5 \\ \underbrace{3}_{-1\cdot 1 + 2\cdot 2 + 0\cdot 2} & 2 \end{pmatrix}$

(obecně $AB \neq BA$, ale $A(BC) = (AB)C$ asociativita
$A(B+C) = AB + AC$ distributivita)

* Inverzní matice:

$A^{-1}$ je inv. mat k $A$ čtvercová $\iff AA^{-1} = A^{-1}A = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ jednotková matice

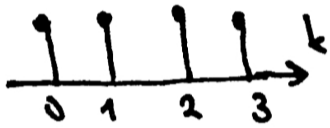
* Stopa čtvercové mat.:

$\mathrm{Tr}\, A := \sum a_{ii}$

$$\boxed{\begin{aligned} (AB)^T &= B^T A^T \\ (AB)^{-1} &= B^{-1} A^{-1} \end{aligned}}$$

# B NÁHODNÁ PROMĚNNÁ

- <u>Definice a vlastnosti</u>

| | | | |
|---|---|---|---|
| n.v. | $X$ | diskrétní vs. | spojitá |
| suport | $\operatorname{supp} X$ | konečná množ. | interval |
| pst | $P$ | $P(X=k)=p_k$ (pstní fce) | $P(X\in(a,b))=\int_a^b f(x)\,dx$ (hustota psti) |
| rozdělení | $P_X$ |  |  |



Př.

alternativní : $X\sim \mathrm{Alt}(p) \Leftrightarrow X\in\{0,1\},\ p_1=p,\ p_0=1-p$

rovnoměrné $X\sim \mathrm{Unif}(0,1)\Leftrightarrow X\in(0,1)\ ;\ f(x)=1$

exponenciální $X\sim \mathrm{Exp}(1) \Leftrightarrow X=-\ln Y\ ;\ Y\sim \mathrm{Unif}(0,1)$

normální $X\sim N(0,1) \Leftrightarrow f(x)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$

- <u>Střední hodnota</u> $\mathbb{E}X$

* Definice $\mathbb{E}X=\sum_k k p_k$, resp. $\mathbb{E}X=\int_{-\infty}^{\infty} x f(x)\,dx$

* Vlastnosti : $\mathbb{E}\varphi(X)=\sum_k \varphi(k)p_k$, resp $\mathbb{E}\varphi(X)=\int_{-\infty}^{\infty}\varphi(x)f(x)\,dx$

$\mathbb{E}(X\pm Y)=\mathbb{E}X\pm\mathbb{E}Y\ ;\ \mathbb{E}\lambda X=\lambda\mathbb{E}X$

$\mathbb{E}(XY)=(\mathbb{E}X)\cdot(\mathbb{E}Y)$ (pro nezávislé $X, Y$ !)

- <u>Momenty</u> : $\mu_k := \mathbb{E}X^k$

* Př $X\sim \mathrm{Unif}(0,1) \Rightarrow \mu_k=\mathbb{E}X^k=\int_0^1 x^k\,dx = \frac{1}{1+k}$

- <u>Moment-generující fce</u> : $\boxed{M(t):=\sum_{k=0}^{\infty}\frac{t^k}{k!}\mu_k=\mathbb{E}e^{Xt}}$

* Dk $\mathbb{E}e^{Xt}=\mathbb{E}\sum_{k=0}^{\infty}\frac{t^kX^k}{k!}$

* Př : $X\sim \mathrm{Unif}(0,1)\Rightarrow M(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\,\frac{1}{1+k}=\frac{e^t-1}{t}$

$Y\sim \mathrm{Exp}(1)\Rightarrow M(t)=\mathbb{E}e^{Yt}=\mathbb{E}e^{-t\ln X}=\mathbb{E}X^{-t}=\frac{1}{1-t}$

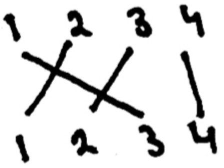
jelikož $\frac{1}{1-t}=\sum_{k=0}^{\infty}\frac{t^k}{k!}k! \Rightarrow \underline{\mathbb{E}Y^k=k!}$

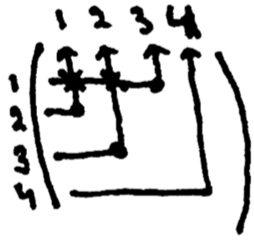
# C PERMUTACE

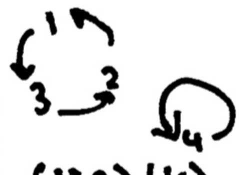
počet prvků, přeuspořádání

- **Definice**: Permutace $\pi$ (řádu $n$) je bijekce na $\{1,2,\dots,n\}$

* Př.: $\pi = \begin{cases} \pi(1)=3 \\ \pi(2)=1 \\ \pi(3)=2 \\ \pi(4)=4 \end{cases}$ $\qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$







(132)(4)

* Notace: Funkční | Cauchyho dvojřádková | Žebrová | Maticová | Grafická/cyklická

* Cyklická struktura: $C(\pi) = 2$ (počet cyklů)
  $C_S(\pi) = 0$ (počet cyklů sudé délky)
  $C_k(\pi)$ (počet cyklů délky $k$)

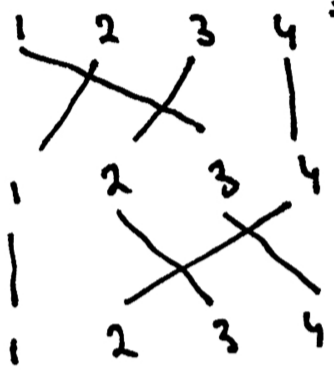
* $S_n :=$ množina všech permutací řádu $n$; $|S_n| = n!$

- **Speciální permutace**
  * identita $\quad id\ (= id_n) := \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$
  * transpozice $\quad \tau_{ij} := \begin{pmatrix} 1 & 2 & \dots & i & \dots & j & \dots & n-1 & n \\ 1 & 2 & \dots & j & \dots & i & \dots & n-1 & n \end{pmatrix}$
  * inverzní permutace $\quad \pi^{-1} \Leftrightarrow \pi^{-1}(\pi(i)) = i \ \forall i$

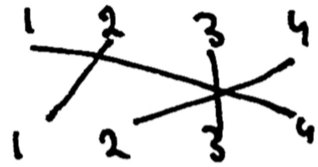
- **Skládání permutací** $\quad \sigma = \rho\pi \Leftrightarrow \sigma(i) = \rho(\pi(i)) \ \forall i$
  * Př.:

$\pi =$

$\rho =$



$\rho\pi =$ $(\leftarrow)$

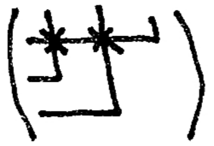


* Př. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \tau_{34} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$

* Vlastnosti: $\pi(\rho\sigma) = (\pi\rho)\sigma \qquad \pi\, id = id\, \pi = \pi$
  $(\pi\rho)^{-1} = \rho^{-1}\pi^{-1}$
  $\tau_{ij}^2 = id$

- Znaménko permutace $\operatorname{sgn}\pi \in \{-1, 1\}$

* Definice: $\operatorname{sgn}\pi := (-1)^{\#\text{překřížení}}$ (v maticové notaci

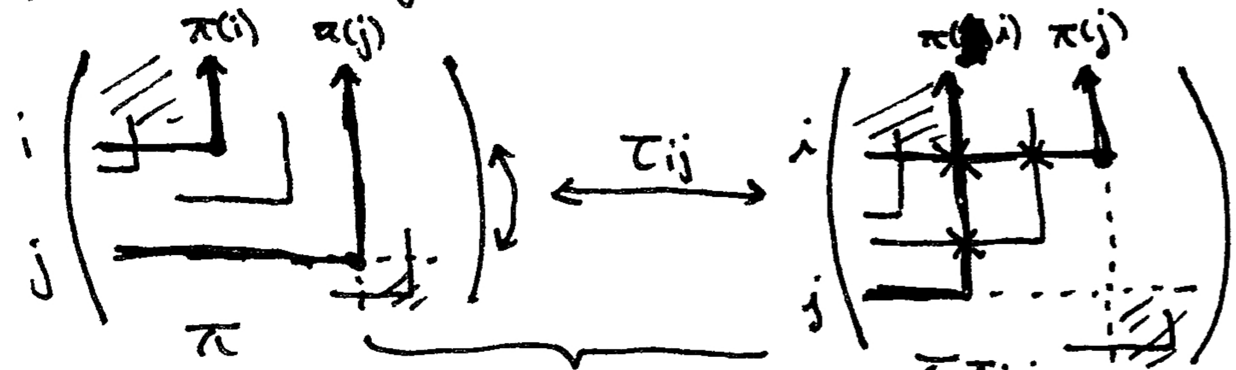
Př. $\operatorname{sgn}\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} = \operatorname{sgn}$  $= (-1)^2 = +1$

Př. $\operatorname{sgn}\mathrm{id} = +1$

* Charakterizace dle transpozic: $\boxed{\operatorname{sgn}\pi = (-1)^{\#\text{transpozic (z identity}}}$

Dk: 1) Libovolnou $\pi$ lze zapsat jako $\pi = \tau_1\tau_2\tau_3\cdots\tau_p\,\mathrm{id}$

2) Stačí dokázat jednoznačnost modulo 2: (Bubble sort



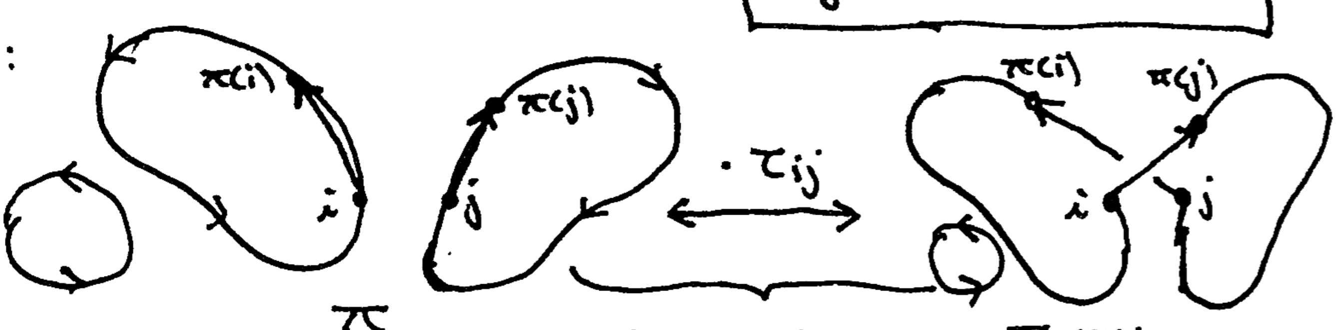

$\operatorname{sgn}(\pi\tau_{ij}) = -\operatorname{sgn}\pi$ ▨

* Znaménko složení: $\boxed{\operatorname{sgn}(\pi\rho) = \operatorname{sgn}\pi \cdot \operatorname{sgn}\rho}$

Dk: $\left.\begin{array}{l}\pi = \tau_1\cdots\tau_p\\ \rho = \sigma_1\cdots\sigma_q\end{array}\right\}$ $\operatorname{sgn}(\pi\rho) = \operatorname{sgn}\tau_1\cdots\tau_p\sigma_1\cdots\sigma_q = (-1)^{p+q} =$
$= (-1)^p\cdot(-1)^q = \operatorname{sgn}\pi\cdot\operatorname{sgn}\rho$ ▨

* Charakterizace dle počtu cyklů: $\boxed{\operatorname{sgn}\pi = (-1)^{n+C(\pi)}}$

Dk:



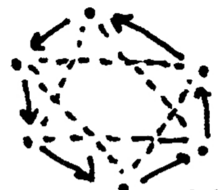

$C(\pi\tau_{ij}) = C(\pi)\pm 1$

$\therefore\ \operatorname{sgn}\pi = c_n(-1)^{C(\pi)}$; jest $\operatorname{sgn}\mathrm{id} = +1$ a $C(\mathrm{id}) = n$

čili $\operatorname{sgn}\pi = (-1)^{n+C(\pi)}$ $(\operatorname{sgn}\mathrm{id} = (-1)^{2n} = +1)$ ▨

* Charakterizace dle sudých cyklů: $\boxed{\operatorname{sgn} \pi = (-1)^{C_s(\pi)}}$

Dk

$\pi = \dots\dots$  $\dots\dots$ 

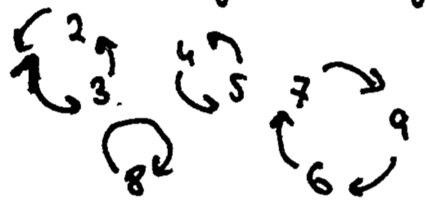
lichý cyklus      sudý cyklus

$\pi^2 = \dots\dots$  $\dots$ 

1 lichý cyklus      2 cykly

$$C(\pi^2) = C(\pi) + C_s(\pi)$$

čili $1 = (\operatorname{sgn}\pi)^2 = \operatorname{sgn}\pi \cdot \operatorname{sgn}\pi = \operatorname{sgn}(\pi^2) = (-1)^{n+C(\pi^2)} = (-1)^{n+C(\pi)+C_s(\pi)} = \underbrace{(-1)^{n+C(\pi)}}_{\operatorname{sgn}\pi} \cdot \underbrace{(-1)^{C_s(\pi)}}_{\operatorname{sgn}\pi}$ ■

* Př $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & 5 & 4 & 7 & 9 & 8 & 6 \end{pmatrix} =$ 



$\therefore\ C(\pi) = 4 \Rightarrow \operatorname{sgn}\pi = (-1)^{9+4} = -1$

$C_s(\pi) = 1 \Rightarrow \operatorname{sgn}\pi = (-1)^1 = -1$

- Generující fce počtu cyklů: $\boxed{g_n(x) := \sum_{\pi \in S_n} x^{C(\pi)} = \prod_{k=1}^{n} (x+k-1)}$

$$\underbrace{x(x+1)\dots(x+n-1)}_{n}$$

* Dk $\pi \in S_n$: $\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$

$$\pi' \in S_{n-1}: \underbrace{\begin{pmatrix} 1 & 2 & \dots n \\ \pi'(1) & \pi'(2) & \dots n \end{pmatrix}}_{C(\pi) = C(\pi')+1} \xleftrightarrow{\tau_{in}} \underbrace{\begin{pmatrix} 1 & 2 & \dots & i & \dots & n \\ \pi'(1) & \pi'(2) & \dots & n & \dots & \pi'(i) \end{pmatrix}}_{C(\pi) = C(\pi')}$$

$$\therefore\ g_n(x) = \sum_{\pi \in S_n} x^{C(\pi)} = \sum_{\substack{\pi \in S_n \\ \pi(n) = n}} x^{C(\pi)} + \sum_{i=1}^{n-1} \sum_{\substack{\pi \in S_n \\ \pi(i) = n}} x^{C(\pi)} =$$

$$= \sum_{\pi' \in S_{n-1}} x^{C(\pi')+1} + (n-1) \sum_{\pi' \in S_{n-1}} x^{C(\pi')} = g_{n-1}(x)(x+n-1)$$ ■

* Důsledky:

1) $|S_n| = \sum_{\pi \in S_n} 1 = g_n(1) = \prod_{k=1}^{n} k = n!$

2) $g_n'(x) = \sum_{\pi \in S_n} x^{C(\pi)-1} \cdot C(\pi) \Rightarrow g_n'(1) = \sum_{\pi \in S_n} C(\pi)$

na druhou stranu $\ln g_n(x) = \ln \prod_{k=1}^{n} (x+k-1) = \ln x + \ln(x+1) + \dots + \ln(x+n-1) \,/\, '$

$g_n'(x)/g_n(x) = \frac{1}{x} + \frac{1}{x+1} + \dots + \frac{1}{x+n-1} \,/\, x=1$ :

$\mathbb{E}\, C(\pi) := \frac{1}{|S_n|} \sum_{\pi \in S_n} C(\pi) = \frac{g_n'(1)}{g_n(1)} = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} \equiv H_n$

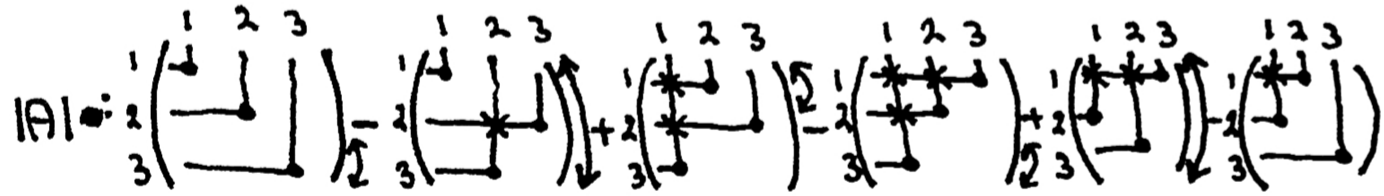
# D DETERMINANT

- **Definice**: Buď $A = (a_{ij})_{n\times n}$ čtvercová matice, pak

$$\boxed{\det A \, (= |A|) := \sum_{\pi \in S_n} \operatorname{sgn} \pi \; a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}}$$

* Př: Determinant matice $3\times 3$:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$ čili



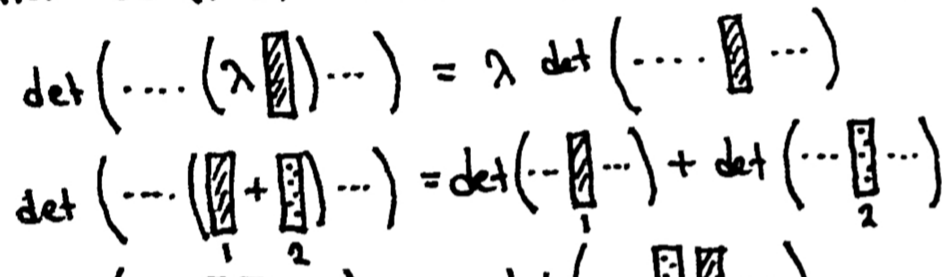
$$\pi: \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{\tau} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \xrightarrow{\tau} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \xrightarrow{\tau} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \xrightarrow{\tau} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow{\tau} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$|A| = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33}$$

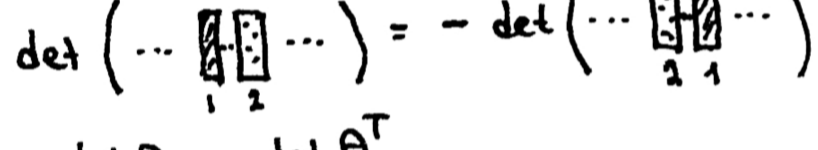
- **Vlastnosti** (bez Dk):

* Škálování matic: $\det(AB) = \det A \cdot \det B \implies \det A^{-1} = (\det A)^{-1}$

* Linearita:



* Antisymetrie:

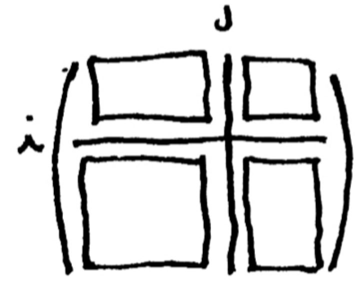


* Transpozice: $\det A = \det A^T$

* Gaussův součin: $\det \begin{pmatrix} a_{11} & & \\ & a_{22} & \\ 0 & & \ddots \; a_{nn} \end{pmatrix} = a_{11}a_{22}\cdots a_{nn}$

* Laplaceův rozvoj:

$$\boxed{|A| = \sum_{\substack{j \\ (i \text{ fixní})}} (-1)^{i+j} a_{ij} \, |A_{ij}|}$$ kde $A_{ij} =$



- **Explicitní vzorec pro inverzní matici:** $\boxed{(A^{-1})_{ij} = \frac{(-1)^{i+j}}{|A|}|A_{ji}|}$

* Dk: $(A^{-1}\cdot A)_{ij} = \sum_k (A^{-1})_{ik} a_{kj} = \frac{1}{|A|}\sum_k (-1)^{i+k}|A_{ki}|\, a_{kj}$

pro $i=j$ z Lapl. rozvoje $(A^{-1}\cdot A)_{ii} = \frac{1}{|A|}\cdot|A| = 1$

pro $i\neq j$ je $\sum_k (-1)^{i+k}|A_{ki}|\,a_{kj} = (-1)^{i-j}\det\left(\cdots \underset{i}{\|}\overset{=}{\cdots}\underset{j}{\|}\cdots\right) = 0$ ∎

- **Jacobiho formule:** Buď $A=(a_{ij}(\lambda))_{n\times n}$, pak $\boxed{\frac{d|A|}{d\lambda} = |A|\,\mathrm{Tr}\left(A^{-1}\frac{dA}{d\lambda}\right)}$

* Dk:

1) Buď $\varepsilon$ malé a $B=(b_{ij})_{n\times n}$, pak $|\overbrace{I+\varepsilon B}^{:=C}| = \begin{vmatrix}1+\varepsilon b_{11} & \varepsilon b_{12} & \cdots\\ \varepsilon b_{21} & 1+\varepsilon b_{22} & \cdots \\ \vdots & \vdots & \ddots\end{vmatrix}$

$= \sum_{\pi\in S_n}\mathrm{sgn}\,\pi\; c_{1\pi(1)}\cdots c_{n\pi(n)} = (1+\varepsilon b_{11})\cdots(1+\varepsilon b_{nn}) + O(\varepsilon^2) =$

$= 1+\varepsilon(b_{11}+b_{22}+\dots+b_{nn}) + O(\varepsilon^2) = 1+\varepsilon\,\mathrm{Tr}B + O(\varepsilon^2)$

2) $\frac{d|A|}{d\lambda} = \lim_{h\to0}\frac1h\left(|A(\lambda+h)|-|A(\lambda)|\right) \overset{\text{Tayl.}}{=} \lim_{h\to0}\frac1h\left(\left|A(\lambda)+h\frac{dA}{d\lambda}+O(h^2)\right|-|A|\right)$

$=\lim_{h\to0}\frac1h\left(\left|A\left(I+hA^{-1}\frac{dA}{d\lambda}+O(h^2)\right)\right|-|A|\right) = |A|\lim_{h\to0}\frac1h\left(\left|I+hA^{-1}\frac{dA}{d\lambda}+O(h^2)\right|-1\right)$

$\overset{1)}{=} |A|\lim_{h\to0}\frac1h\left(\cancel{1}+h\,\mathrm{Tr}\left(A^{-1}\frac{dA}{d\lambda}\right)-\cancel{1}+O(h^2)\right) = |A|\,\mathrm{Tr}\left(A^{-1}\frac{dA}{d\lambda}\right)$ ∎

- **Lemma 1:** Buď $A=\overset{\text{invertibilní}}{(a_{ij})_{n\times n}}$; $u=(u_i)_{n\times1}$, $v=(v_i)_{n\times1}$ (vektory $u,v$),

potom $\boxed{|A+\lambda uv^T| = |A|\,(1+\lambda v^TA^{-1}u)}$

* Dk

1) Označme $B(\lambda)=A+\lambda uv^T$, je tedy $\frac{dB}{d\lambda}=uv^T$.
Dále $f(\lambda):=|B(\lambda)|$ polynom v $\lambda$ stupně nejv. $n$; $f(0)=|A|$

2) Dle Jacobiho formule je $f'(\lambda)=|B|\,\mathrm{Tr}\left(B^{-1}\frac{dB}{d\lambda}\right) =$

$= f(\lambda)\,\mathrm{Tr}(B^{-1}uv^T) = f(\lambda)\,v^TB^{-1}u$; $f'(0)=|A|\,v^TA^{-1}u$

3) $\frac{d}{d\lambda}(B^{-1}) = \lim_{h\to0}\frac1h\left(B^{-1}(\lambda+h)-B^{-1}(\lambda)\right) = \lim_{h\to0}\frac1h\left(\left(B+h\frac{dB}{d\lambda}\right)^{-1}-B^{-1}\right) =$

$=\lim_{h\to0}\frac1h\left(\left(B\left(I+hB^{-1}\frac{dB}{d\lambda}\right)\right)^{-1}-B^{-1}\right) = \lim_{h\to0}\frac1h\left(\underbrace{\left(I+hB^{-1}\frac{dB}{d\lambda}\right)^{-1}}_{I-hB^{-1}\frac{dB}{d\lambda}+O(h^2)}B^{-1}-B^{-1}\right)$

$\therefore \frac{d}{d\lambda}(B^{-1}) = -B^{-1}\frac{dB}{d\lambda}B^{-1} = -B^{-1}uv^TB^{-1}$

4) $f''(\lambda) = (f(\lambda)\,v^TB^{-1}u)' = f'(\lambda)\,v^TB^{-1}u + f(\lambda)\,v^T\frac{d(B^{-1})}{d\lambda}u$

$= f(\lambda)\,v^TB^{-1}uv^TB^{-1}u + f(\lambda)\,v^T(-B^{-1}uv^TB^{-1})u = 0$ ∎

5) Jelikož $f''=0 \Rightarrow f(\lambda)=f(0)+\lambda f'(0)=|A|+\lambda|A|\,v^TA^{-1}u$

# E MOMENTY DETER. NÁH. MATIC

- <u>Znění problému</u>:

Buď $A = (X_{ij})_{n\times n}$; $X_{ij} \sim$ iid n.v. mající $\mu_p := \mathbb{E} X_{ij}^p$.

Vyjádřete $\xi_m^{(n)} := \mathbb{E}|A|^m$ jako fci $\mu = (\mu_1, \mu_2, \mu_3, \dots)$.

(Ekvivalentně lze pracovat s generujícími fcemi $G_m(\mu; t) := \sum_{n=0}^{\infty} \frac{t^n}{n!^2} \xi_m^{(n)}$)

* <u>Př $m=2, n=2$</u>: $A = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \Rightarrow |A| = X_{11}X_{22} - X_{12}X_{21}$,

čili $\xi_2^{(2)} = \mathbb{E}|A|^2 = \mathbb{E}\left(X_{11}^2 X_{22}^2 - 2X_{11}X_{22}X_{12}X_{21} + X_{12}^2 X_{21}^2\right) = 2(\mu_2^2 - \mu_1^4)$

* <u>Př $m=2, n=4$</u>: $\xi_4^{(2)} = \mathbb{E}\left(X_{11}X_{22} - X_{12}X_{21}\right)^4 =$

$= \mathbb{E}\left(X_{11}^4 X_{22}^4 - 4X_{11}^3 X_{22}^3 X_{12} X_{21} + 6X_{11}^2 X_{22}^2 X_{12}^2 X_{21}^2 - 4X_{11}X_{22}X_{12}^3 X_{21}^3 +\right.$

$\left. + X_{12}^4 X_{21}^4\right) = 2(\mu_4^2 - 4\mu_3^2\mu_1^2 + 3\mu_2^4)$

- <u>Případ $m=2$</u> (obecný)

$$\boxed{\xi_2^{(n)} = n!\left(\mu_2 + \mu_1^2(n-1)\right)\left(\mu_2 - \mu_1^2\right)^{n-1}, \quad \text{resp. } G_2 = (1 + \mu_1^2 t)\, e^{t(\mu_2 - \mu_1^2)}}$$

$\cdot \sum \frac{t^n}{n!^2}$

* <u>Dk</u> (Cauchy – Binet)

1) Lemma 2: $D_n(x) = \begin{pmatrix} x+1 & 1 & \cdots \\ 1 & x+1 & \\ \vdots & & \ddots \end{pmatrix}$, pak $|D_n(x)| = (x+n)x^{n-1}$

Dk: $D_n(x) = x\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & \ddots \end{pmatrix} = xI + uu^T$; $u = \left.\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}\right\} n$,

čili $|D_n(x)| = |xI|(1 + u^T(xI)^{-1}u)$ dle Lemmatu 1,

$\therefore |D_n(x)| = x^n\left(1 + \frac{1}{x}\underbrace{u^T u}_{n}\right) = (x+n)x^{n-1}$ ▨

2) $\xi_m^{(2)} = \mathbb{E}|A|^2 = \mathbb{E}\left(\sum_{\pi \in S_n} \operatorname{sgn}\pi \, X_{1\pi(1)} \cdots X_{n\pi(n)}\right)^2 =$

$= \mathbb{E}\sum_{\pi, \rho \in S_n} \operatorname{sgn}\pi \operatorname{sgn}\rho \,(X_{1\pi(1)} \cdots X_{n\pi(n)})(X_{1\rho(1)} \cdots X_{n\rho(n)})$

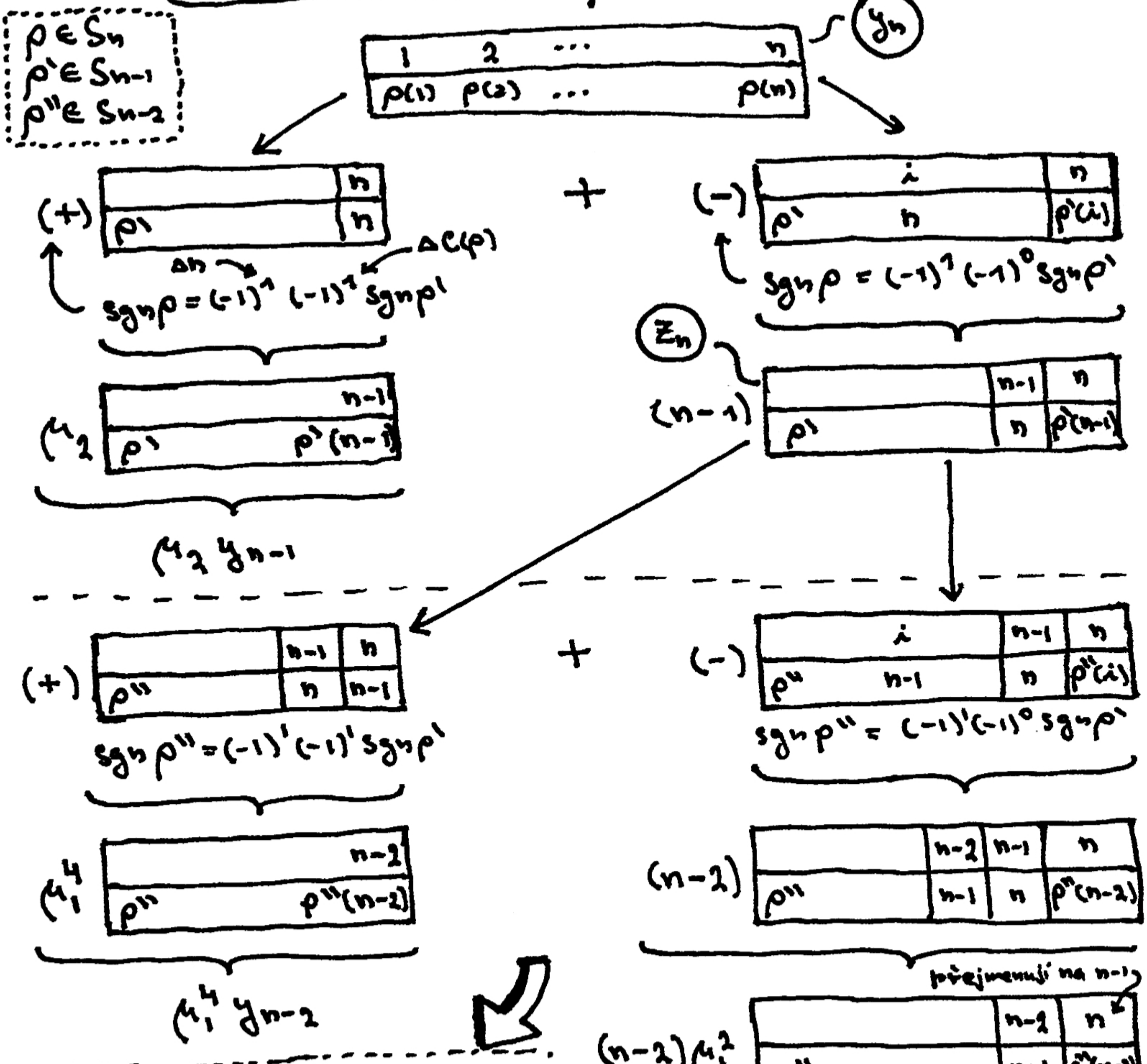
$\langle \pi = id \rangle$
$= n!\, \mathbb{E}\sum_{\rho \in S_n} \operatorname{sgn}\rho\, (X_{11}X_{22}\cdots X_{nn})(X_{1\rho(1)} \cdots X_{n\rho(n)}) =$

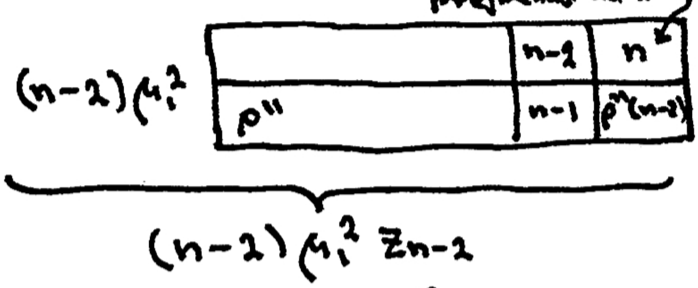
$= n! \sum_{\rho \in S_n} \operatorname{sgn}\rho\, \mu_2^{c_1(\rho)} (\mu_1^2)^{n - c_1(\rho)} = n!\, \mu_1^{2n} \sum_{\rho \in S_n} \operatorname{sgn}\rho\, \alpha^{c_1(\rho)} =$, $\alpha = \mu_2/\mu_1^2$

$= n!\, \mu_1^{2n} \begin{vmatrix} \alpha & 1 & \cdots \\ 1 & \alpha & \\ \vdots & & \ddots \end{vmatrix} = n!\, \mu_1^{2n} |D_n(\alpha - 1)| = n!\, \mu_1^{2n} (\alpha - 1 + n)\alpha^{n-1}$ ▨

* Dk (Nyquist 1954)

$$\xi_2^{(n)} = n! \underbrace{\sum_{\rho\in S_n} \operatorname{sgn}\rho\,(X_{11}X_{22}\cdots X_{nn})(X_{1\rho(1)}X_{2\rho(2)}\cdots X_{n\rho(n)})}$$




$$y_n = \mu_2 y_{n-1} - (n-1) z_n$$
$$z_n = \mu_1^4 y_{n-2} - (n-2)\mu_1^2 z_{n-1}$$

$\Sigma$( Jest $G_2 = \sum_{n=0}^{\infty} \frac{t^n}{n!\,2}\,\xi_2^{(n)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} y_n$ ; označme $Z_2 := \sum_{n=0}^{\infty} \frac{t^{n-1}}{n!}\, n(n-1)\, z_n$,

$$\left.\begin{aligned} G_2' &= \mu_2 G_2 - Z_2 \\ Z_2 &= \mu_1^4 t\, G_2 - \mu_1^2 t\, Z_2 \end{aligned}\right\} \quad Z_2 = \frac{\mu_1^4 t}{1+\mu_1^2 t} G_2, \qquad G_2' = \mu_2 G_2 - \frac{\mu_1^4 t}{1+\mu_1^2 t} G_2 \quad /\int$$

$$\therefore G_2 = \exp\int \mu_2 - \frac{\mu_1^4 t}{1+\mu_1^2 t}\,dt = \exp\int \mu_2 - \mu_1^2 + \frac{\mu_1^2}{1+\mu_1^2 t}\,dt =$$

$$= \exp\left(t(\mu_2-\mu_1^2) + \ln(1+\mu_1^2 t)\right) = (1+\mu_1^2 t)\, e^{t(\mu_2-\mu_1^2)}$$

▨

* Dk (dle Jacobiho formule)

1) Centrovaný případ: Buď $B=(Y_{ij})_{n\times n}$; $Y_{ij}\sim$ iid s momenty $\tilde{\mu}_p := \mathbb{E}\,Y_{ij}^p$ centrované n.v. (tj. $\tilde{\mu}_1=0$) a

$$\tilde{\xi}_m^{(n)} := \mathbb{E}|B|^m = \xi_m^{(n)}(\tilde{\mu}) = \xi_m^{(n)}(0,\tilde{\mu}_2,\tilde{\mu}_3,\dots\tilde{\mu}_m),\ \text{resp.}$$

$$\tilde{G}_m := \sum_{n=0}^{\infty}\frac{t^n}{n!^2}\tilde{\xi}_m^{(n)} = G_m(\tilde{\mu};t) = G_m(0,\tilde{\mu}_2,\dots,\tilde{\mu}_m;t).$$

Pro $m=2$:

$$\tilde{\xi}_2^{(n)} = \mathbb{E}|B|^2 = \mathbb{E}\left(\square\right)^2 \overset{\langle\square\rangle\ \text{rozvoj}}{=} n\tilde{\mu}_2\,\mathbb{E}\left(\square\right)^2$$

$$= n\tilde{\mu}_2\,\tilde{\xi}_2^{(n-1)} \overset{\text{rekur.}}{\Rightarrow} \tilde{\xi}_2^{(n)} = n!\,\tilde{\mu}_2^n$$

$$\therefore\ \tilde{G}_2 = \sum_{n=0}^{\infty}\frac{t^n}{n!^2}\,n!\,\tilde{\mu}_2^n = e^{t\tilde{\mu}_2}$$

2) Obecný případ: Pišme $Y_{ij} = X_{ij}-\mu_1$ ~~[illegible]~~

$\tilde{\mu}_1 = \mathbb{E}(X_{ij}-\mu_1) = \mu_1-\mu_2 = 0$ (splňuje centrovanost) $\quad u=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\Big\}n$

$\tilde{\mu}_2 = \mathbb{E}(X_{ij}-\mu_1)^2 = \mathbb{E}(X_{ij}^2-2X_{ij}\mu_1+\mu_1^2) = \mu_2-\mu_1^2.$

Dále, položme $A=(X_{ij})_{n\times n} = (Y_{ij}+\mu_1)_{n\times n} = B+\mu_1\begin{pmatrix}1&\cdots&1\\ \vdots&\ddots&\\ 1&&1\end{pmatrix} = B+\mu_1 uu^T$,

čili (Lemma 1) $|A| = |B+\mu_1uu^T| = |B|(1+\mu_1u^TB^{-1}u) =$

$= |B|+\mu_1 S$, kde $S = u^T|B|B^{-1}u = \sum_{ij}(-1)^{i+j}|B_{ij}|$

$$\therefore\ \xi_2^{(n)} = \mathbb{E}|A|^2 = \mathbb{E}(|B|+\mu_1S)^2 = \underbrace{\mathbb{E}|B|^2}_{\tilde{\xi}_2^{(n)}} + 2\mu_1\underbrace{\mathbb{E}|B|S}_{q_n} + \mu_1^2\underbrace{\mathbb{E}S^2}_{r_n}$$

a) $q_n = \mathbb{E}|B|S \overset{\text{sym.}}{=} n^2\,\mathbb{E}|B||B_{11}| = n^2\,\mathbb{E}\left(\square\right)\left(\square\right) \overset{\langle\square\rangle}{=} 0$

b) $r_n = \mathbb{E}S^2 = \mathbb{E}\sum_{ijkl}(-1)^{i+j+k+l}|B_{ij}||B_{kl}| = n^2\,\mathbb{E}|B_{11}|^2 = n^2\tilde{\xi}_2^{(n-1)}$

| | | | |
|---|---|---|---|
| sgn: | $+$ | $-$ | $+$ |
| konfig: | | $\to 0$ | $\to 0$ |
| četnost: | $n^2$ | $2n^2(n-1)$ | $n^2(n-1)^2$ |

vzájemná pozice $B_{ij}, B_{kl}$

c) Celkově $\xi_2^{(n)} = \tilde{\xi}_2^{(n)} + \mu_1^2 n^2\tilde{\xi}_2^{(n-1)} \xrightarrow{} \Sigma$

$$\therefore\ G_2 = \tilde{G}_2 + \mu_1^2 t\,\tilde{G}_2 = (1+\mu_1^2t)\,\tilde{G}_2 = (1+\mu_1^2t)\,e^{t\tilde{\mu}_2},\quad \tilde{\mu}_2 = \mu_2-\mu_1^2$$

$\blacksquare$

- **Případ $m = 1, 3, 5, \dots$ (obecný)**

$$\boxed{\xi^{(n)}_{2k+1} = 0} \;\because\; \text{z antisymetrie determinantu} \quad (n \geq 2)$$

spec. $\xi^{(1)}_{2k+1} = \mathbb{E} X_{11}^{2k+1} = \mu_{2k+1} \Rightarrow G_{2k+1} = 1 + \mu_{2k+1} t$

- **Případ Gaussovský**

$$\boxed{\text{Buď } X_{ij} \overset{iid}{\sim} N(0,1), \text{ pak } \xi^{(n)}_{2k} = \mathbb{E}|A|^{2k} = \prod_{\ell=0}^{k-1} \frac{(n+2\ell)!}{(2\ell)!}}$$

* Dk (Prékopa 1967)

1) Pišme $A = (X_{ij})_{n \times n} = (\vec{X}_1, \vec{X}_2, \dots, \vec{X}_n)$ ; $\vec{X}_j = \left.\begin{pmatrix} X_{1j} \\ \vdots \\ X_{nj} \end{pmatrix}\right\} n$

2) Označme $\Delta^{(n)}_k := \sqrt{\mathrm{Gramm}(\vec{X}_1, \dots, \vec{X}_k)}$

k-objem rovnoběžnostěnu tvořeného vektory $\vec{X}_1, \dots \vec{X}_k \in \mathbb{R}^n$,

poté platí (sférická symetrie hust. fce) :

$$\Delta^{(n)}_k = \alpha^{(n)}_k \Delta^{(n)}_{k-1}, \text{ kde } \Delta^{(n)}_{k-1} \perp\!\!\!\perp \alpha^{(n)}_k \sim \chi_{n-k+1} \text{ (výška } \Delta^{(n)}_k).$$

čili $\Delta^{(n)}_n = \prod_{j=1}^{n} \alpha^{(n)}_j \equiv \text{abs} \det(A)$

3) Je známo $\mathbb{E}\chi_j^{2k} = \prod_{\ell=0}^{k-1} (j + 2\ell)$, proto

$$\xi^{(n)}_{2k} = \mathbb{E}|A|^{2k} = \mathbb{E}(\Delta^{(n)}_n)^{2k} = \mathbb{E} \prod_{j=1}^{n} (\alpha^{(n)}_j)^{2k} \overset{\perp\!\!\!\perp}{=} \prod_{j=1}^{n} \mathbb{E}(\alpha^{(n)}_j)^{2k} =$$

$$= \prod_{j=1}^{n} \prod_{\ell=0}^{k-1} (n - j + 1 + 2\ell) = \prod_{\ell=0}^{k-1} \underbrace{\prod_{j=1}^{n} (n - j + 1 + 2\ell)}_{(n+2\ell)!\,/\,(2\ell)!}$$

∎

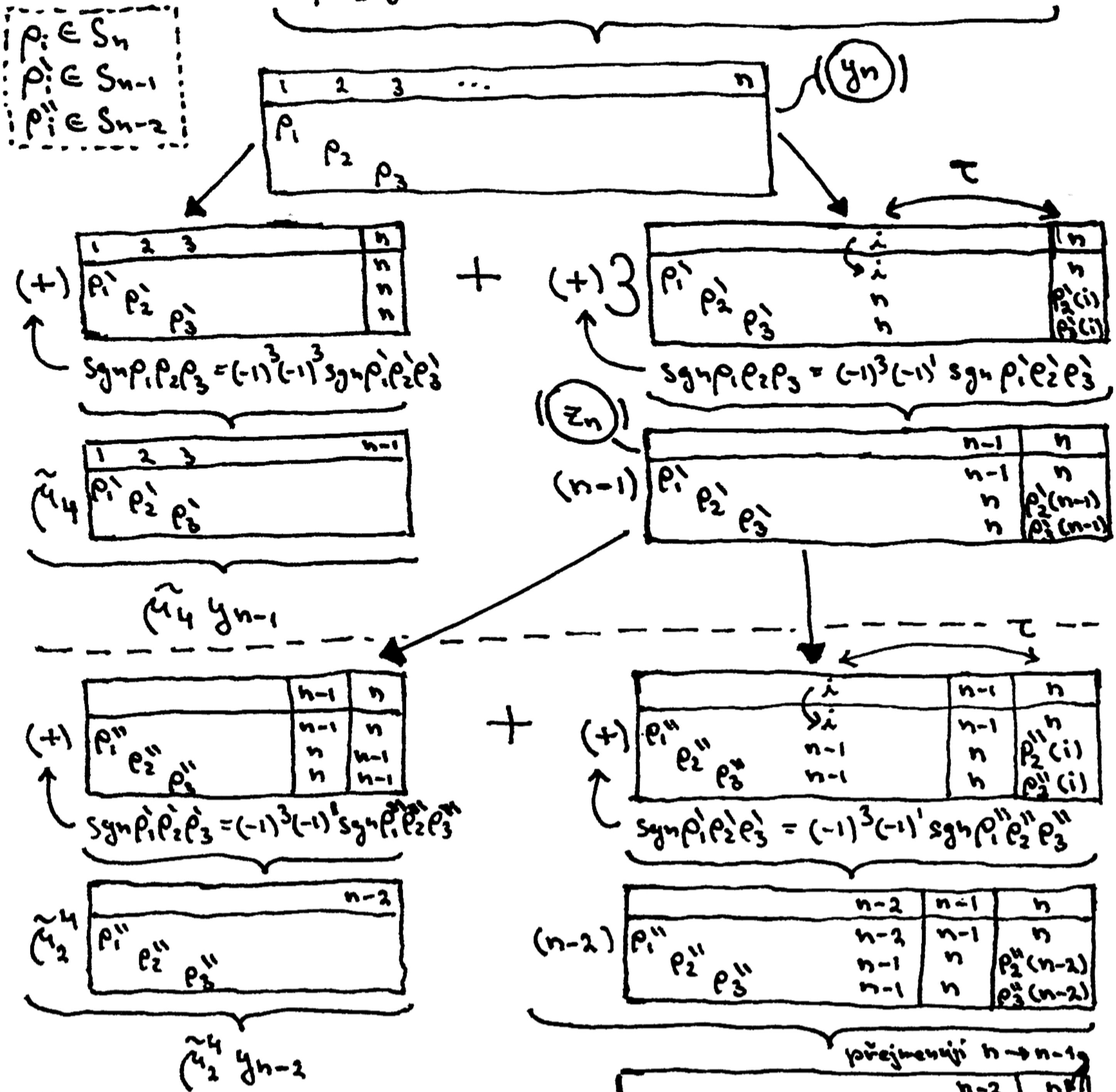
- **Případ $m=4$** (centrovaný, tj $\tilde{\mu}_1=0$)

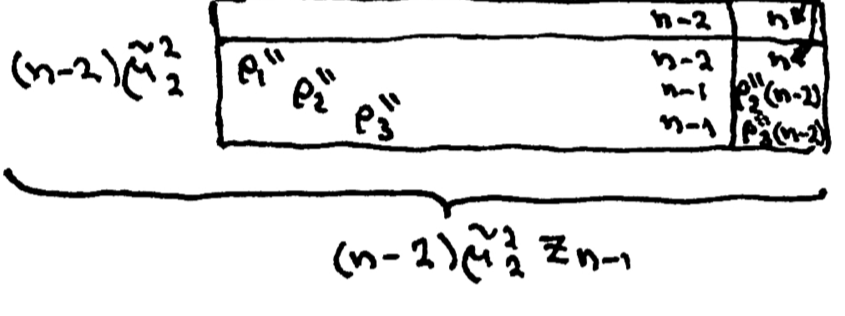
$$\boxed{\tilde{G}_4 = \frac{e^{t(\tilde{\mu}_4-3\tilde{\mu}_2^2)}}{(1-\tilde{\mu}_2^2 t)^3}}$$

* Dk (Nyqist 1954)

$$\tilde{\xi}_4^{(n)} = \mathbb{E}|B|^4 = n!\,\mathbb{E}\sum_{\rho_1\rho_2\rho_3\in S_n} \operatorname{sgn}\rho_1\rho_2\rho_3\,(X_{11}\cdots X_{nn})(X_{1\rho_1(1)}\cdots)(X_{1\rho_2(1)}\cdots)(X_{1\rho_3(1)}\cdots)$$

$\rho_i \in S_n$
$\rho_i' \in S_{n-1}$
$\rho_i'' \in S_{n-2}$




$$y_n = \tilde{\mu}_4\, y_{n-1} + 3(n-1)\, z_n$$

$$z_n = \tilde{\mu}_2^4\, y_{n-2} + (n-2)\tilde{\mu}_2^2\, z_{n-1}$$

Připomenutí $\tilde{G}_4 = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} \tilde{\xi}_4^{(n)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} y_n$, dále označme

$Z_4 := \sum_{n=0}^{\infty} \frac{t^{n-1}}{n!} n(n-1) z_n$, potom z rekurentních relací:

$$\left| \begin{aligned} \tilde{G}_4' &= \tilde{\mu}_4 + 3 Z_4 \\ Z_4 &= \tilde{\mu}_2^4 t \tilde{G}_4 + \tilde{\mu}_2^2 t Z_4 \end{aligned} \right\} \begin{aligned} Z_4 &= \frac{\tilde{\mu}_2^4 t}{1-\tilde{\mu}_2^2 t} \tilde{G}_4 \text{, z čehož} \\ \tilde{G}_4' &= \tilde{\mu}_4 \tilde{G}_4 + \frac{3\tilde{\mu}_2^4 t}{1-\tilde{\mu}_2^2 t} \tilde{G}_4 \quad / \int \end{aligned}$$

$$\therefore \tilde{G}_4 = \exp \int \tilde{\mu}_4 + \frac{3\tilde{\mu}_2^4 t}{1-\tilde{\mu}_2^2 t} \, dt = \exp \int \tilde{\mu}_4 - 3\tilde{\mu}_2^2 + \frac{3\tilde{\mu}_2^2}{1-\tilde{\mu}_2^2 t} \, dt =$$

$$= \exp\left(t(\tilde{\mu}_4 - 3\tilde{\mu}_2^2) - 3\ln(1-\tilde{\mu}_2^2 t)\right) = \frac{\exp(t(\tilde{\mu}_4 - 3\tilde{\mu}_2^2))}{(1-\tilde{\mu}_2^2)^3} \quad \blacksquare$$

* Dk (dle Laplaceova rozvoje)

$$\tilde{\xi}_4^{(n)} = \mathbb{E}|B|^4 = \mathbb{E}\left(\square\right)^4 \overset{\langle \square \rangle}{=}$$

$$= \mathbb{E} \sum_{ijkl} (-1)^{i+j+k+l} Y_{1i} Y_{1j} Y_{1k} Y_{1l} |B_{1i}||B_{1j}||B_{1k}||B_{1l}|$$

$$= n\tilde{\mu}_4 \underbrace{\mathbb{E}|B_{11}|^4}_{\tilde{\xi}_4^{(n-1)}} + 3n(n-1)\tilde{\mu}_2^2 \underbrace{\mathbb{E}|B_{11}|^2|B_{12}|^2}_{e_n}$$

| četnosti: | konfig.: |
| --- | --- |
| $n$ | $+$ |
| $4n(n-1)$ | $-$ |
| $3n(n-1)$ | $+$ |
| $6n(n-1)(n-2)$ | $-$ |
| $n(n-1)(n-2)(n-3)$ | $+$ |

kde $e_n = \mathbb{E}|B_{11}|^2|B_{12}|^2 = \mathbb{E}\left(\square\right)^2\left(\square\right)^2 \overset{\langle \square \rangle}{=}$

$$= (n-1)\tilde{\mu}_2 \, \mathbb{E}\left(\square\right)^2\left(\square\right)^2 = (n-1)\tilde{\mu}_2 \left[\overbrace{\mathbb{E}|B|^2|B_{11}|^2}^{@}\right]_{n \to n-1}$$

dále $@ = \mathbb{E}|B|^2|B_{11}|^2 = \mathbb{E}\left(\square\right)^2\left(\square\right)^2 \overset{\langle \square \rangle}{=} \tilde{\mu}_2 \underbrace{\mathbb{E}\left(\square\right)^2}_{\tilde{\xi}_4^{(n-1)}} + (n-1)\tilde{\mu}_2 \underbrace{\mathbb{E}\left(\square\right)\left(\square\right)^2}_{e_n}$

$$\therefore e_n = (n-1)\tilde{\mu}_2^2 \, \tilde{\xi}_4^{(n-2)} + (n-1)(n-2)\tilde{\mu}_2^2 \, e_{n-1}$$

Označme $E(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2} n^2(n-1) e_n$, potom z rekurencí a $\Sigma$:

$$\left. \begin{aligned} \tilde{G}_4' &= \tilde{\mu}_4 \tilde{G}_4 + 3\tilde{\mu}_2^2 E \\ E &= \tilde{\mu}_2^2 t \tilde{G}_4 + t\tilde{\mu}_2^2 E \end{aligned} \right\} \begin{aligned} E &= \frac{\tilde{\mu}_2^2 t}{1-\tilde{\mu}_2^2 t} \tilde{G}_4 \text{, z čehož} \\ \tilde{G}_4' &= \tilde{\mu}_4 \tilde{G}_4 + \frac{3\tilde{\mu}_2^4 t}{1-\tilde{\mu}_2^2 t} \tilde{G}_4 \end{aligned}$$

(toto je stejná ODR jako v Nyqistově důkazu)

$\blacksquare$

- <u>**Případ $m=4$**</u> (obecný)

$$\boxed{\begin{aligned} G_4 &= \frac{e^{t(\tilde\mu_4 - 3\tilde\mu_2^2)}}{(1-\tilde\mu_2^2 t)^5}\sum_{k=0}^{6} p_k t^k; \quad \text{kde} \\ p_0 &= 1 \\ p_1 &= \mu_1^4 + 6\mu_1^2\tilde\mu_2 - 2\tilde\mu_2^2 + 4\mu_1\tilde\mu_3 \\ p_2 &= 7\mu_1^4\tilde\mu_2^2 - 6\mu_1^2\tilde\mu_2^3 + \tilde\mu_2^4 + 12\mu_1^3\tilde\mu_2\tilde\mu_3 - 8\mu_1\tilde\mu_2^2\tilde\mu_3 + 6\mu_1^2\tilde\mu_3^2 \\ p_3 &= 2\mu_1(2\mu_1^3\tilde\mu_2^4 - 6\mu_1^2\tilde\mu_2^3\tilde\mu_3 + 2\tilde\mu_2^4\tilde\mu_3 + 3\mu_1^3\tilde\mu_2\tilde\mu_3^2 - 6\mu_1\tilde\mu_2^2\tilde\mu_3^2 + 2\mu_1^2\tilde\mu_3^3) \\ p_4 &= \mu_1^2\tilde\mu_3^2(\mu_1^2\tilde\mu_3^2 - 6\mu_1^2\tilde\mu_2^3 + 6\tilde\mu_2^4 - 8\mu_1\tilde\mu_2^2\tilde\mu_3) \\ p_5 &= 2\mu_1^3\tilde\mu_2^2\tilde\mu_3^3(2\tilde\mu_2^2 - \mu_1\tilde\mu_3) \\ p_6 &= \mu_1^4\tilde\mu_2^4\tilde\mu_3^4 \end{aligned}}$$

* Dk (dle Jacobiho formule) (B. 2022):

Jako v případě $m=2$ pišme $Y_{ij} = X_{ij} - \mu_1$; $\tilde\mu_p := \mathbb{E}\,Y_{ij}^p$, čili $\tilde\mu_1 = 0$; $\tilde\mu_2 = \mu_2 - \mu_1^2$; $\tilde\mu_3 = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3$; $\tilde\mu_4 = \mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1^2 - 3\mu_1^4$

$A = (X_{ij})_{n\times n}$; $B = (Y_{ij})_{n\times n}$; $A = B + \mu_1 uu^T$; $u = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\Big\}n$; dle Lemma 1

$|A| = |B| + \mu_1 S$, kde $S = \sum_{ij}(-1)^{i+j}|B_{ij}|$.

$$\therefore\ \xi_4^{(n)} = \mathbb{E}|A|^4 = \mathbb{E}(|B| + \mu_1 S)^4 = \underbrace{\mathbb{E}|B|^4}_{\tilde\xi_4^{(n)}} + 4\mu_1\underbrace{\mathbb{E}|B|^3 S}_{q_n} + 6\mu_1^2\underbrace{\mathbb{E}|B|^2 S^2}_{r_n} + 4\mu_1^3\underbrace{\mathbb{E}|B|S^3}_{s_n} + \mu_1^4\underbrace{\mathbb{E}S^4}_{t_n}$$

$$\Big\updownarrow \Sigma$$

$$G_4 = \tilde G_4 + 4\mu_1 Q_4 + 6\mu_1^2 R_4 + 4\mu_1^3 S_4 + \mu_1^4 T_4$$

1) $q_n = \mathbb{E}|B|^3 S \overset{\text{sym.}}{=} n^2\,\mathbb{E}|B|^3|B_{11}| = n^2\,\mathbb{E}(\square)^3(\square) \overset{\langle\square\rangle}{=}$

$= n^2\tilde\mu_3\,\mathbb{E}(\square)^4 - n^2(n-1)\tilde\mu_3\,\cancel{\mathbb{E}(\square)(\square)}^{\,0} = n^2\tilde\mu_3\,\tilde\xi_4^{(n-1)}$

2) $r_n = \mathbb{E}|B|^2 S^2 = \mathbb{E}|B|^2 \sum_{ijkl}(-1)^{i+j+k+l}|B_{ij}||B_{kl}|$

| | | | |
|---|---|---|---|
| sgn.: | $+$ | $-$ | $+$ |
| kontig.: | $\boxed{\square}$ | $\square\square$ |  |
| četnost: | $\underbrace{n^2}_{f_n^{(0)}}$ | $\underbrace{2n^2(n-1)}_{f_n^{(1)}}$ | $\underbrace{n^2(n-1)^2}_{f_n^{(2)}}$ |

vzájemná poloha $B_{ij}, B_{kl}$

a) $f_n^{(0)} = \mathbb{E}|B|^2|B_{11}|^2 = \mathbb{E}(\;)^2(\;)^2 \overset{\langle\;\rangle}{=} \tilde{\mu}_2[\mathbb{E}|B|^4]_{n\to n-1}$

$+ (n-1)\tilde{\mu}_2\,\mathbb{E}(\;)^2(\;)^2 \overset{\langle\;\rangle}{=} \tilde{\mu}_2\tilde{\xi}_4^{(n-1)} + (n-1)^2\tilde{\mu}_2^2\,\mathbb{E}(\;)^2(\;)^2$

$= \tilde{\mu}_2\tilde{\xi}_4^{(n-1)} + (n-1)^2\tilde{\mu}_2^2 f_{n-1}^{(0)}$

b) $f_n^{(1)} = \mathbb{E}(\;)^2(\;)(\;) \overset{\langle\;\rangle}{=} \tilde{\mu}_2\,\cancelto{0}{\mathbb{E}(\;)^3(\;)} +$

$+ \tilde{\mu}_2\,\cancelto{0}{\mathbb{E}(\;)^3(\;)} + (n-2)\tilde{\mu}_2\,\mathbb{E}(\;)^2(\;)(\;)$

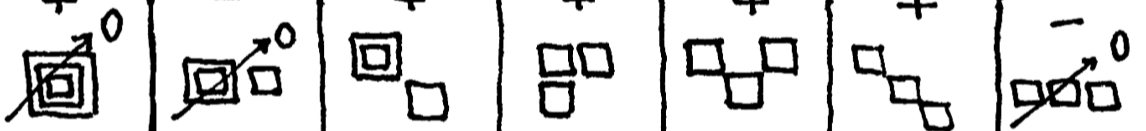
$\overset{\langle\;\rangle}{=} (n-1)(n-2)\tilde{\mu}_2\tilde{\mu}_3\,\mathbb{E}(\;)^2(\;)(\;) \overset{\langle\;\rangle}{=} 0$

c) $f_n^{(2)} = \mathbb{E}(\;)^2(\;)(\;) \overset{\langle\;\rangle}{=} \tilde{\mu}_3\,\mathbb{E}(\;)^3(\;) -$

$- (n-2)\tilde{\mu}_3\,\cancelto{0}{\mathbb{E}(\;)^2(\;)(\;)} \overset{\langle\;\rangle}{=} \tilde{\mu}_3\,\mathbb{E}[|B|^3|B_{11}|]_{n\to n-1} = \tilde{\mu}_3^2\tilde{\xi}_4^{(n-2)}$

$\therefore\; r_n = n^2 f_n^{(0)} + n^2(n-1)^2\tilde{\mu}_3^2\tilde{\xi}_4^{(n-2)}$

3) $s_n = \mathbb{E}|B|S^3 = \mathbb{E}|B|\sum_{ijkluv}(-1)^{i+j+k+l+u+v}|B_{ij}||B_{kl}||B_{uv}|$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| sgn: | + | − | + | + | + | + | − |
| konfig.: |  | | | | | | |
| četnost: | $n^2$ | $6n^2(n-1)$ | $\underbrace{3n^2(n-1)^2}_{f_n^{(3)}}$ | $\underbrace{6n^2(n-1)^2}_{f_n^{(4)}}$ | $\underbrace{6n^2(n-1)^2(n-2)}_{f_n^{(5)}}$ | $\underbrace{n^2(n-1)^2(n-2)^2}_{f_n^{(6)}}$ | $2n^2(n-1)(n-2)$ |

a) $f_n^{(3)} = \mathbb{E}|B||B_{11}|^2|B_{22}| = \mathbb{E}(\;)(\;)^2(\;) \overset{\langle\;\rangle}{=}$

$= \tilde{\mu}_3\,\mathbb{E}(\;)^2(\;)^2 - (n-2)\tilde{\mu}_3\,\cancelto{0}{\mathbb{E}(\;)(\;)^2(\;)} \overset{\langle\;\rangle}{=}$

$= \tilde{\mu}_3[\mathbb{E}(\;)^2(\;)^2]_{n\to n-1} = \tilde{\mu}_3[\mathbb{E}|B|^2|B_{11}|^2]_{n\to n-1} = \tilde{\mu}_3 f_{n-1}^{(0)}$

b) $f_n^{(4)} = \mathbb{E}|B||B_{12}||B_{21}||B_{22}|] = \mathbb{E}(\;)(\;)(\;)(\;) \overset{\langle\;\rangle}{=}$

$= (n-2)\tilde{\mu}_3\,\mathbb{E}(\;)(\;)(\;)(\;) \overset{\langle\;\rangle}{=} 0$

c) $f_n^{(5)} = \mathbb{E}|B||B_{11}||B_{13}||B_{22}| = \mathbb{E}(\;)(\;)(\;)(\;) \overset{\langle\;\rangle}{=}$

$= (n-2)\tilde{\mu}_3\,\mathbb{E}(\;)(\;)(\;)(\;) \overset{\langle\;\rangle}{=} 0$

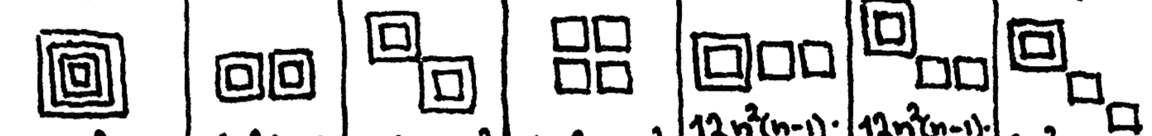
d) $f_n^{(6)} = \mathbb{E}|B||B_{11}||B_{22}||B_{33}| = \mathbb{E}(\;)(\;)(\;)(\;) =$

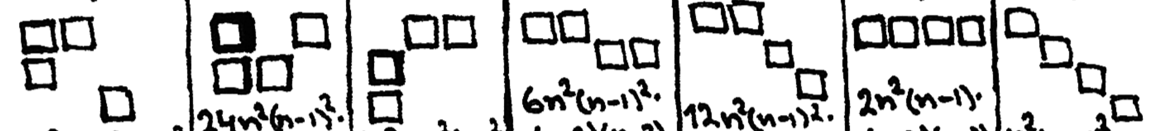
$\overset{\langle\;\rangle}{=} \tilde{\mu}_3\,\mathbb{E}(\;)^2(\;)(\;) + (n-3)\tilde{\mu}_3\,\cancelto{0}{\mathbb{E}(\;)(\;)(\;)(\;)}$

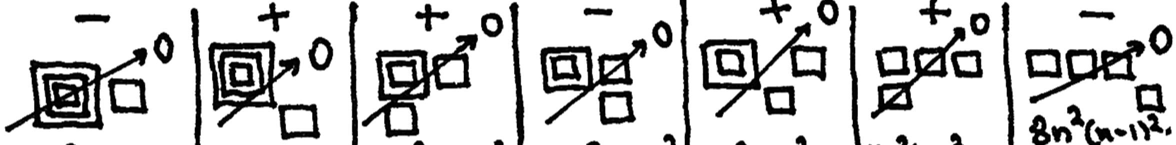
$\overset{\langle\;\rangle}{=} \tilde{\mu}_3[\mathbb{E}|B|^2|B_{11}||B_{22}|]_{n\to n-1} = \tilde{\mu}_3 f_{n-1}^{(2)} = \tilde{\mu}_3^3\tilde{\xi}_4^{(n-3)}$

$\therefore\; s_n = 3n^2(n-1)^2\tilde{\mu}_3 f_{n-1}^{(0)} + n^2(n-1)^2(n-2)^2\tilde{\mu}_3^3\tilde{\xi}_4^{(n-3)}$

4) $t_n = \mathbb{E} S^4 = \mathbb{E} \sum_{ijkluvpq} (-1)^{i+j+k+l+u+v+p+q} |B_{ij}||B_{kl}||B_{uv}||B_{pq}|$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| sgn: | + | + | + | + | − | − | + |
| konfig: |  | | | | | | |
| četnost: | $n^2$ | $6n^2(n-1)$ | $3n^2(n-1)^2$ | $6n^2(n-1)^2$ | $12n^2(n-1)\cdot(n-2)$ | $12n^2(n-1)\cdot(n-2)$ | $6n^2(n-1)^2\cdot(n-2)^2$ |
| | $f_n^{(7)}$ | $f_n^{(8)}$ | $f_n^{(9)}$ | $f_n^{(10)}$ | $f_n^{(11)}$ | $f_n^{(12)}$ | $f_n^{(13)}$ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| sgn: | + | − | + | + | − | + | + |
| konfig: |  | | | | | | |
| četnost: | $24n^2(n-1)^2(n-2)^2$ | $24n^2(n-1)^2\cdot(n-2)$ | $6n^2(n-1)^2(n-2)^2$ | $6n^2(n-1)^2\cdot(n-2)(n-3)$ | $12n^2(n-1)^2\cdot(n-2)^2(n-3)$ | $2n^2(n-1)\cdot(n-2)(n-3)$ | $n^2(n-1)^2\cdot(n-2)^2(n-3)^2$ |
| | $f_n^{(14)}$ | $f_n^{(15)}$ | $f_n^{(16)}$ | $f_n^{(17)}$ | $f_n^{(18)}$ | $f_n^{(19)}$ | $f_n^{(20)}$ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| sgn: | − | + | + | − | + | + | − |
| konfig: |  | | | | | | |
| četnost: | $8n^2(n-1)$ | $4n^2(n-1)^2$ | $12n^2(n-1)^2$ | $24n^2(n-1)^2$ | $24n^2(n-1)^2(n-2)$ | $24n^2(n-1)^2(n-2)$ | $8n^2(n-1)^2\cdot(n-2)(n-3)$ |

a) $f_n^{(7)} = \mathbb{E}|B_{11}|^4 = [\mathbb{E}|B|^4]_{n\to n-1} = \tilde{\xi}_4^{(n-1)}$

b) $f_n^{(8)} = \mathbb{E}|B_{11}|^2|B_{12}|^2 = \mathbb{E}(\square)^2(\square)^2 \overset{\langle\blacksquare\rangle}{=} (n-1)\tilde{\mu}_2 \mathbb{E}(\square)^2(\square)^2 =$

$= (n-1)\tilde{\mu}_2[\mathbb{E}|B|^2|B_{11}|^2]_{n\to n-1} = (n-1)\tilde{\mu}_2 f_{n-1}^{(0)}$

c) $f_n^{(9)} = \mathbb{E}|B_{11}|^2|B_{22}|^2 = \mathbb{E}(\square)^2(\square)^2 \overset{\langle\blacksquare\rangle}{=} \tilde{\mu}_2 \mathbb{E}(\square)^2(\square)^2 +$

$+ (n-2)\tilde{\mu}_2 \mathbb{E}(\square)^2(\square)^2 \overset{\langle\blacksquare\rangle}{=} \tilde{\mu}_2 f_{n-1}^{(0)} + (n-2)\tilde{\mu}_2^2 \mathbb{E}(\square)^2(\square)^2 +$

$+ (n-2)^2\tilde{\mu}_2^2 \mathbb{E}(\square)^2(\square)^2 = \tilde{\mu}_2 f_{n-1}^{(0)} + (n-2)\tilde{\mu}_2^2[\mathbb{E}|B_{11}|^2|B_{12}|^2]_{n\to n-1} +$

$+ (n-2)^2\tilde{\mu}_2^2[\mathbb{E}|B_{21}|^2|B_{12}|^2]_{n\to n-1} = \langle \text{symmetry } 11,22 \leftrightarrow 12,21\rangle =$

$= \tilde{\mu}_2 f_{n-1}^{(0)} + (n-2)\tilde{\mu}_2^2 f_{n-1}^{(8)} + (n-2)^2\tilde{\mu}_2^2[\mathbb{E}|B_{11}|^2|B_{22}|^2]_{n\to n-1} =$

$= \tilde{\mu}_2 f_{n-1}^{(0)} + (n-2)^2\tilde{\mu}_2^3 f_{n-2}^{(0)} + (n-2)^2\tilde{\mu}_2^2 f_{n-1}^{(9)}$

d) $f_n^{(10)} = \mathbb{E}|B_{11}||B_{12}||B_{21}||B_{22}| = \mathbb{E}(\square)(\square)(\square)(\square) \overset{\langle\blacksquare\rangle}{=}$

$= (n-2)\tilde{\mu}_2 \mathbb{E}(\square)(\square)(\square)(\square) \overset{\langle\blacksquare\rangle}{=}$

$= (n-2)\tilde{\mu}_2^2 \mathbb{E}(\square)^2(\square)^2 + (n-2)^2\tilde{\mu}_2^2 \mathbb{E}(\square)(\square)(\square)(\square)$

$= (n-2)\tilde{\mu}_2^2[\mathbb{E}|B_{11}|^2|B_{12}|^2]_{n\to n-1} + (n-2)^2\tilde{\mu}_2^2[\mathbb{E}|B_{21}||B_{22}||B_{11}||B_{12}|]_{n\to n-1} =$

$= (n-2)^2\tilde{\mu}_2^3 f_{n-2}^{(0)} + (n-2)^2\tilde{\mu}_2^2 f_{n-1}^{(10)}$

e) $f_n^{(11)} = \mathbb{E}|B_{11}|^2|B_{12}||B_{13}| = \mathbb{E}(\square)^2(\square)(\square) \overset{\langle\blacksquare\rangle}{=} (n-1)\tilde{\mu}_3 \xcancel{\mathbb{E}(\square)^2(\square)(\square)}^{\langle\blacksquare\rangle} = 0$

f) $f_n^{(12)} = \mathbb{E}|B_{11}|^2|B_{22}||B_{23}| = \mathbb{E}(\ )^2$ [redacted] $(\ )(\ ) =$

$= -(n-2)\tilde{\mu}_3\,\mathbb{E}(\ )^2(\ )(\ ) = 0$

g) $f_n^{(13)} = \mathbb{E}|B_{11}|^2|B_{22}||B_{33}| = \mathbb{E}(\ )^2(\ )(\ ) =$

$= \tilde{\mu}_3\,\mathbb{E}(\ )^2(\ )(\ ) + \cancelto{0}{(n-3)\tilde{\mu}_3\,\mathbb{E}(\ )(\ )(\ )} =$

$= \tilde{\mu}_3\left[\mathbb{E}|B_{11}|^2|B||B_{22}|\right]_{n\to n-1} = \tilde{\mu}_3 f_{n-1}^{(3)} = \tilde{\mu}_3^2 f_{n-2}^{(0)}$

h) $f_n^{(14)} = \mathbb{E}|B_{11}||B_{12}||B_{21}||B_{33}| = \mathbb{E}(\ )(\ )(\ )(\ ) =$

$= (n-3)\tilde{\mu}_3\,\mathbb{E}(\ )(\ )(\ )$ [redacted] $(\ ) = 0$

i) $f_n^{(15)} = \mathbb{E}|B_{11}||B_{21}||B_{22}||B_{13}| = \mathbb{E}(\ )(\ )(\ )(\ ) =$

$= (n-2)\tilde{\mu}_3\,\mathbb{E}(\ )(\ )(\ )(\ ) = 0$

j) $f_n^{(16)} = \mathbb{E}|B_{12}||B_{13}||B_{21}||B_{31}| = \mathbb{E}(\ )(\ )(\ )(\ ) =$

$= (n-3)\tilde{\mu}_3\,\mathbb{E}(\ )(\ )(\ )(\ ) = 0$

k) $f_n^{(17)} = \mathbb{E}|B_{11}||B_{12}||B_{23}||B_{24}| = \mathbb{E}(\ )(\ )(\ )(\ ) =$

$= -(n-2)\tilde{\mu}_3\,\mathbb{E}(\ )(\ )(\ )(\ ) = 0$

l) $f_n^{(18)} = \mathbb{E}|B_{11}||B_{12}||B_{23}||B_{34}| = \mathbb{E}(\ )(\ )(\ )(\ ) =$

$= (n-3)\tilde{\mu}_3\,\mathbb{E}(\ )(\ )(\ )(\ ) = 0$

m) $f_n^{(19)} = \mathbb{E}|B_{11}||B_{12}||B_{13}||B_{14}| = \mathbb{E}(\ )(\ )(\ )(\ ) =$

$= (n-1)\tilde{\mu}_3\,\mathbb{E}(\ )(\ )(\ )(\ ) = 0$

n) $f_n^{(20)} = \mathbb{E}|B_{11}||B_{22}||B_{33}||B_{44}| = \mathbb{E}(\ )(\ )(\ )(\ ) =$

$= \tilde{\mu}_3\,\mathbb{E}(\ )(\ )(\ )(\ ) - \cancelto{0}{(n-4)\tilde{\mu}_3\,\mathbb{E}(\ )(\ )(\ )(\ )}$

$= \tilde{\mu}_3\left[\mathbb{E}|B||B_{11}||B_{22}||B_{33}|\right]_{n\to n-1} = \tilde{\mu}_3 f_{n-1}^{(6)} = \tilde{\mu}_3^4\,\tilde{\xi}_4^{(n-4)}$

[redacted] $\therefore\ t_n = n^2\tilde{\xi}_4^{(n-1)} + 6n^2(n-1)^2\tilde{\mu}_2 f_{n-1}^{(0)} + 3n^2(n-1)^2 f_n^{(9)} + 6n^2(n-1)^2 f_n^{(10)} + 6n^2(n-1)^2(n-2)^2\tilde{\mu}_3^2 f_{n-2}^{(0)} + n^2(n-1)^2(n-2)^2(n-3)^2\tilde{\mu}_3^4\tilde{\xi}_4^{(n-4)}$

5) Označme $F_0 := \sum_{n=0}^{\infty}\frac{t^n}{n!^2}n^2 f_n^{(0)}$; $F_9 := \sum_{n=0}^{\infty}\frac{t^n}{n!^2}n^2(n-1)^2 f_n^{(9)}$; $F_{10} := \sum_{n=0}^{\infty}\frac{t^n}{n!^2}n^2(n-1)^2 f_n^{(10)}$

$\Sigma$

$$\left.\begin{aligned} F_0 &= \tilde{\mu}_2 t\,\tilde{G}_4 + \tilde{\mu}_2^2 t F_0\\ F_9 &= \tilde{\mu}_2 t F_0 + \tilde{\mu}_2^3 t^2 F_0 + \tilde{\mu}_2^2 t F_9\\ F_{10} &= \tilde{\mu}_2^3 t^2 F_0 + \tilde{\mu}_2^2 t F_{10}\end{aligned}\right\}\quad \begin{aligned} F_0 &= \frac{\tilde{\mu}_2 t}{1-\tilde{\mu}_2^2 t}\tilde{G}_4\\ F_9 &= \tilde{\mu}_2 t\,\frac{1+\tilde{\mu}_2^2 t}{1-\tilde{\mu}_2^2 t}F_0 = \tilde{\mu}_2^2 t^2\frac{1+\tilde{\mu}_2^2 t}{(1-\tilde{\mu}_2^2 t)^2}\tilde{G}_4\\ F_{10} &= \frac{\tilde{\mu}_2^3 t^2}{1-\tilde{\mu}_2^2 t}F_0 = \frac{\tilde{\mu}_2^4 t^3}{(1-\tilde{\mu}_2^2 t)^2}\tilde{G}_4\end{aligned}$$

$\therefore\ G_4 = \tilde{G}_4 + 4\mu_1\tilde{\mu}_3 t\tilde{G}_4 + 6\mu_1^2(F_0 + \tilde{\mu}_3^2 t^2\tilde{G}_4) + 4\mu_1^3(3\tilde{\mu}_3 t F_0 + \tilde{\mu}_3^3 t^3\tilde{G}_4) +$

$+ \mu_1^4(t\tilde{G}_4 + 6\tilde{\mu}_2 t F_0 + 3F_9 + 6F_{10} + 6\tilde{\mu}_3^2 t^2 F_0 + \tilde{\mu}_3^4 t^4\tilde{G}_4)$ ∎