2. písemka

Příklad [6b]:
Pomocí reziduové věty spočtěte integrály

$$\int_{-\infty}^{\infty} \frac{\sin(\omega x)}{(1+x^2)^2}\mathrm{d}x \quad \text{a} \quad \int_{-\infty}^{\infty} \frac{\cos(\omega x)}{(1+x^2)^2}\mathrm{d}x,$$

pro všechny $\omega \in \mathbb{R}$.

Všechny kroky pečlivě vysvětlete a okomentujte.

2. písemka

Příklad [6b]:
Pomocí reziduové věty spočtěte integrály

$$\int_{-\infty}^{\infty} \frac{\sin(\omega x)}{x(1+x^2)}\mathrm{d}x \quad \text{a} \quad \int_{-\infty}^{\infty} \frac{\cos(\omega x)}{x(1+x^2)}\mathrm{d}x.$$

pro všechny $\omega \in \mathbb{R}$.

Všechny kroky pečlivě vysvětlete a okomentujte.

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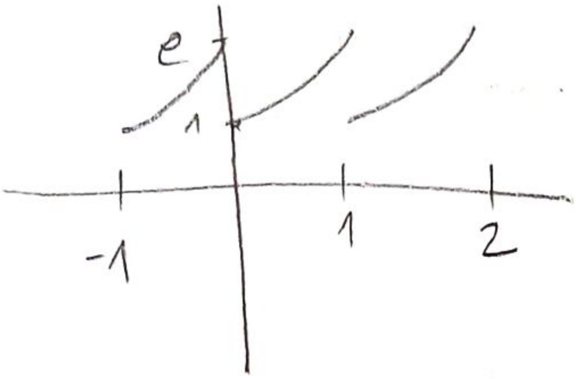
$f(x) = e^x$ na $[0,1]$

(1) Four. rada.



$$F(e^x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2\pi kx) + b_k \sin(2\pi kx)$$

kde $$a_0 = \frac{2}{1}\int_0^1 f(x)\,dx = 2\int_0^1 e^x\,dx = 2(e-1)$$

$$a_k = \frac{2}{1}\int_0^1 f(x)\cos(2\pi kx) = 2\int_0^1 e^x \cos(2\pi kx) \overset{PP}{=}$$

$$2\left[e^x\cos(2\pi kx)\right]_0^1 + 2\int_0^1 e^x (2\pi k)\sin(2\pi kx) \overset{PP}{=}$$

$$2(e-1) + 2\underbrace{\left[2\pi k\, e^x \sin(2\pi kx)\right]_0^1}_{0} - 2\int_0^1 (4\pi^2k^2)\, e^x \cos(2\pi kx)$$

$$\to a_k(1+4\pi^2k^2) = 2(e-1) \Rightarrow a_k = \frac{2(e-1)}{1+4\pi^2k^2}$$

$$b_k = \frac{2}{1}\int_0^1 f(x)\sin(2\pi kx) = 2\int_0^1 e^x \sin(2\pi kx) \overset{PP}{=}$$

$$= 2\overbrace{\left[e^x \sin(2\pi kx)\right]_0^1}^{0} - 2\int_0^1 (2\pi k)\, e^x \cos(2\pi kx) \overset{PP}{=}$$

2

$$= -2\left[e^x \cos(2\pi k x)(2\pi k)\right]_0^1 - 2\int_0^1 (4\pi^2 k^2) e^x \sin(2\pi k x)\,dx$$

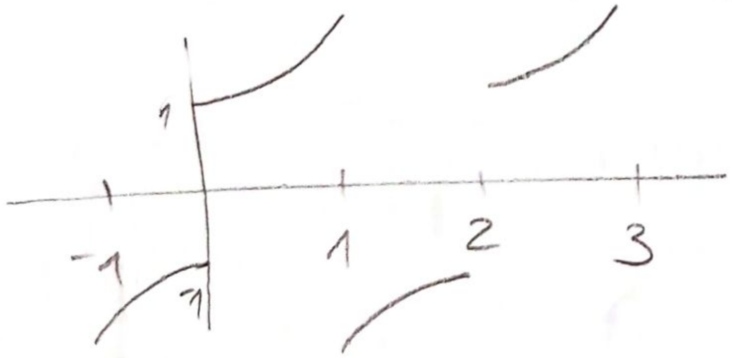
$$\Rightarrow \quad b_k = -\frac{2(e-1)(2\pi k)}{1+4\pi^2 k^2}$$

$$\mathcal{F}_f = (e-1)\left[1 + \sum_{k=1}^{\infty} \frac{2}{1+4\pi^2k^2}\cos(2\pi k x) - \frac{4\pi k}{1+4\pi^2 k^2}\sin(2\pi k x)\right] \quad (\#)$$

2

Pro sinusovou řadu rozšíříme eiše na $[-1,0]$ tedy

$$f^*(x) = \begin{cases} e^x & \text{na } [0,1] \\ -e^{-x} & \text{na } [-1,0] \end{cases}$$



a počítáme

$$\mathcal{F}_{f^*} = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(\pi k x) + b_k \sin(\pi k x)$$

a kvůli eichosti máme

$$a_0 = 0$$
$$a_k = 0 \quad \forall k \in \mathbb{N}$$

Počítáme tedy jen ③

$$b_k = \frac{2}{2}\int_{-1}^{-1} f_*(x)\sin(\pi k x) = 2\int_0^1 e^x \sin(\pi k x) =$$

$$\underbrace{2\left[e^x \sin(\pi k x)\right]_0^1}_{0} - 2\int_0^1 (\pi k)\, e^x \cos(\pi k x) =$$

$$= -2\pi k\left[e^x \cos(\pi k x)\right]_0^1 - 2\int_0^1 \pi^2 k^2\, e^x \sin(\pi k x)$$

$$\Rightarrow \quad b_k = -\frac{\left[(-1)^k e - 1\right]2\pi k}{1+\pi^2k^2}$$

$$\Rightarrow \quad F_{f^*} = \sum_{k=1}^{\infty} 2\pi k\, \frac{1-(-1)^k e}{1+\pi^2k^2}\sin(2\pi k x)$$

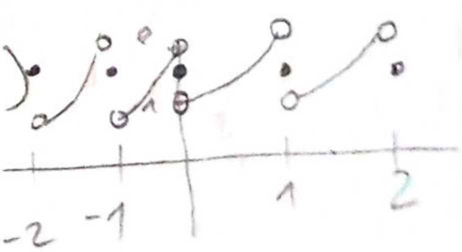
③ Obě funkce jsou po částech C^1 a tudíž podle přednášky víme, že

$$F_f = \frac{f(x^+) + f(x^-)}{2}$$

$$F_{f^*} = \frac{f^*(x^+) + f(x^-)}{2}$$

Tudíž

$$F_f = \begin{cases} \frac{e+1}{2} & \text{když } x=0 \\ e^x & \text{když } x\in(0,1) \\ \frac{e+1}{2} & \text{když } x=1 \end{cases}$$

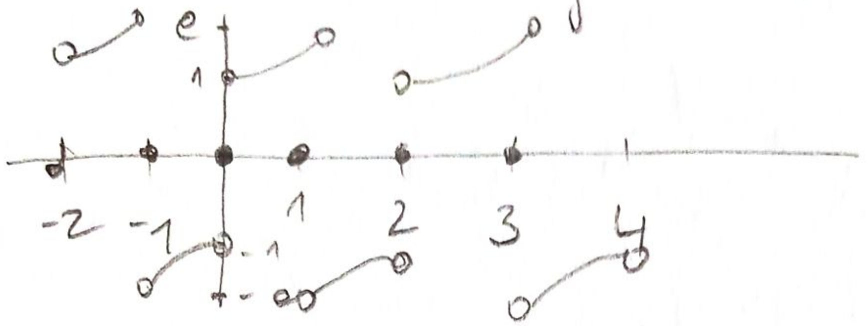


a dál periodick

a

(4)

$$F_{f^*} = \begin{cases} 0 & \text{když } x = -1 \\ -e^{-x} & \text{když } x \in (-1, 0) \\ 0 & \text{když } x = 0 \\ e^{x} & \text{když } x \in (0, 1) \\ 0 & \text{když } x = 1 \end{cases}$$



(4)

Dosadíme 0 (#) a z minulého bodu máme

$$\frac{e+1}{2} = e-1\left[1 + 2\sum \frac{1}{1+4\pi^2k^2}\right]$$

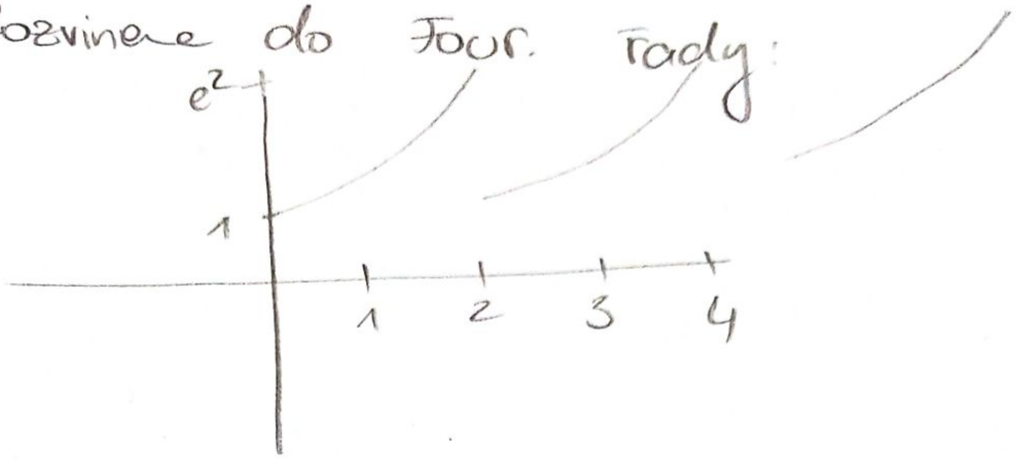
$$\Rightarrow \sum \frac{1}{1+4\pi^2k^2} = \frac{e+1}{4(e-1)} - \frac{1}{2} = \frac{1}{2}\,\frac{e+1-2e+2}{2e-2} =$$

$$= \frac{1}{4}\,\frac{3-e}{e-1}$$

Máme $f = e^x$ na $[0,2]$ (1)

(1)

Rozvinene do Four. řady:



$$F_f = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi}{2} x\right) + b_k \sin\left(\frac{2k\pi}{2} x\right)$$

perioda

kde $a_0 = \frac{2}{2}\int_0^2 e^x \, dx = e^2 - 1$

$$a_k = \frac{2}{2}\int_0^2 e^x \cos(k\pi x) dx \overset{P.P.}{=} \overbrace{\left[e^x \cos(k\pi x)\right]_0^2}^{e^2-1} +$$

$$+ \int_0^2 (k\pi) e^x \sin(k\pi x) \overset{P.P.}{=} (e^2-1) + \underbrace{\left[k\pi e^x \sin(k\pi x)\right]_0^2}_{0}$$

$$- \int_0^2 (k\pi)^2 e^x \cos(k\pi x) dx$$

$$\rightarrow a_k = \frac{e^2-1}{1+k^2\pi^2}$$

①

$$b_k = \frac{2}{2}\int_0^2 e^x \sin(k\pi x)\,dx \overset{P.P}{=} \overbrace{\left[e^x \sin(k\pi x)\right]_0^2}^{0} -$$

$$- \int_0^2 e^x (k\pi)\cos(k\pi x)\,dx = -\left[(k\pi)e^x \cos k\pi x\right]_0^2$$

$$- \int_0^2 (k\pi)^2 e^x \sin(k\pi x)\,dx$$

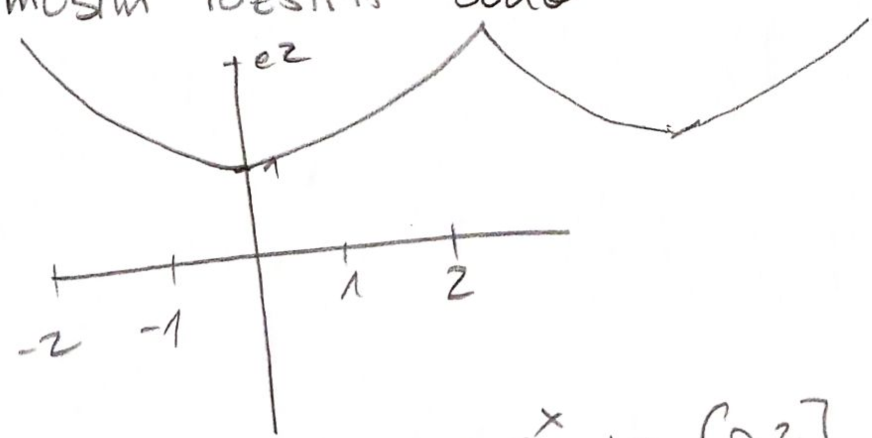
$$\Rightarrow \quad b_k = \frac{(1-e^2)(k\pi)}{1+k^2\pi^2}$$

$$\Rightarrow \quad \mathcal{F}_f = e^2-1\left[\frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{1+k^2\pi^2}\left[\cos(k\pi x) - (k\pi)\sin(k\pi x)\right]\right]$$

(#)

②

Pokud chci rozvinout e^x do kosinusové řady musím rozšířit sudě



tj $f^*(x) = \begin{cases} e^x & \text{na } [0,2] \\ e^{-x} & \text{na } [-2,0] \end{cases}$ a dál 4-per

(3)

a nyní $b_k = 0$

$$a_0 = \frac{2}{4}\int_{-2}^{2} f^*(x)\,dx = \int_0^2 e^x\,dx = e^2 - 1$$

$$a_k = \frac{2}{4}\int_{-2}^{2} f^*(x)\cos\left(\frac{\pi k}{2}x\right)dx \underset{\text{sudost}}{=} \int_0^2 e^x \cos\left(\frac{\pi k}{2}x\right)dx$$

$$\overset{PP}{=} \underbrace{\left[e^x \cos\left(\frac{\pi k}{2}x\right)dx\right]_0^2}_{e^2(-1)^k - 1} + \int_0^2 e^x\left(\frac{\pi k}{2}\right)\sin\left(\frac{\pi k}{2}x\right)$$

$$\overset{P.P}{=} e^2(-1)^k - 1 + \underbrace{\left[e^x\left(\frac{\pi k}{2}\right)\sin\left(\frac{\pi k}{2}x\right)\right]_0^2}_{0} -$$

$$- \int_0^2 e^x\left(\frac{\pi k}{2}\right)^2 \cos\left(\frac{\pi k}{2}x\right)dx$$

$$\Rightarrow a_k = \frac{(e^2(-1)^k - 1)4}{4 + \pi^2 k^2}$$

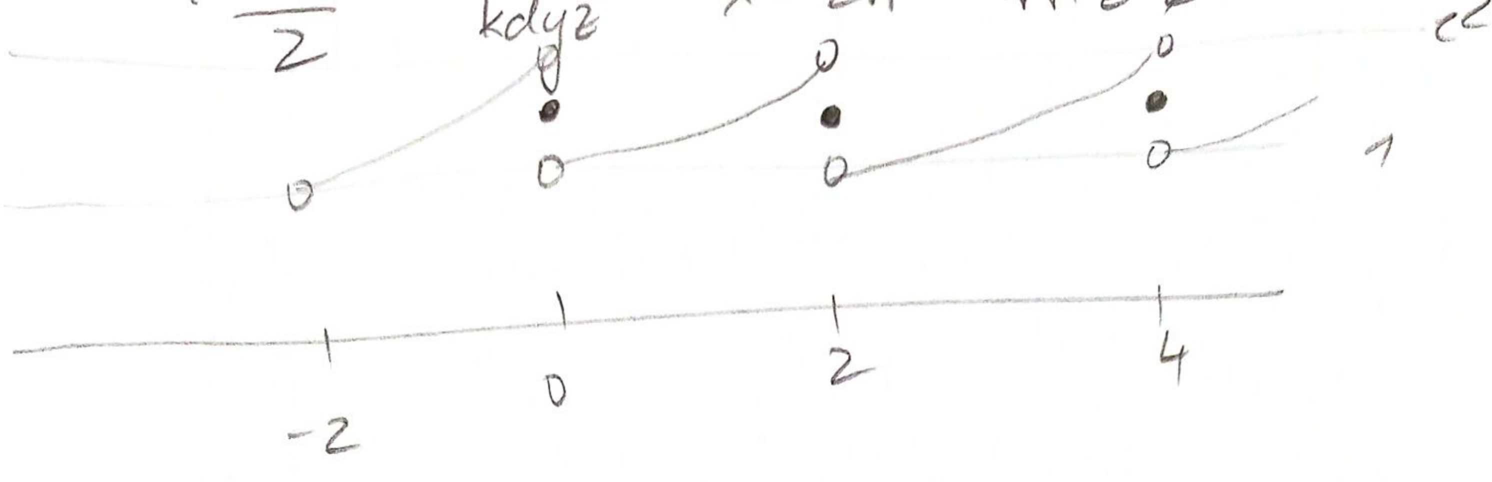
$$\Rightarrow \mathcal{F}_{f^*} = \frac{e^2 - 1}{2} + 4\sum_{k=1}^{\infty} \frac{e^2(-1)^k - 1}{4 + \pi^2 k^2}\cos\left(\frac{\pi k}{2}x\right)$$

(3)

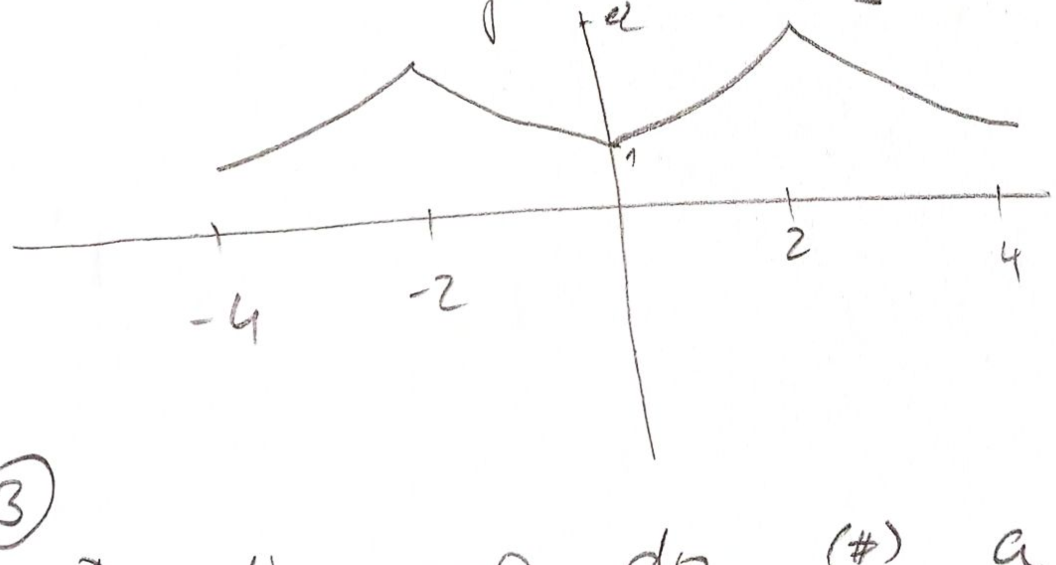
a v bodech nespojitosti mají vlastní limitě

obě funkce jsou po částech C^1 a tudíž využijeme větu že $\mathcal{F}_f = \frac{f(x^+) + f(x^-)}{2}$

$$\mathcal{F}_f = \begin{cases} e^{(x-2n)} & \text{když } x \in (2n, 2(n+1)) \ \forall n \in \mathbb{Z} \\ \frac{e^2+1}{2} & \text{když } x = 2n \ \forall n \in \mathbb{Z} \end{cases}$$ (4)



$$\mathcal{F}_f = \begin{cases} e^{x} & \text{když } x \in (0,2] \\ e^{-x} & \text{když } x \in [-2,0] \end{cases}$$ a dál 4-pe



(3)

Dosadím $x = 0$ do $(\#)$ a

$$\frac{e^2+1}{2} = \frac{e^2-1}{2} + e^2-1 \sum_{k=1}^{\infty} \frac{1}{1+k^2\pi^2}$$

$$\frac{e^2+1-e^2+1}{2} = e^2-1 \sum_{k=1}^{\infty} \frac{1}{1+k^2\pi^2}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{1+k^2\pi^2} = \frac{1}{(e^2-1)}$$