Optimization with application in finance – exercises

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2 Nonlinear programming problems: Karush–Kuhn–Tucker optimality conditions

2.1 A few pieces of the theory

We emphasize that this section contains just a basic summary and we refer the readers to the lecture notes for formal definitions and propositions.

Consider a **nonlinear programming problem** with inequality and equality constraints:

$$\begin{aligned} \min\ & f(x) \\ \text{s.t. } & g_i(x) \le 0,\ i = 1, \dots, m, \\ & h_j(x) = 0,\ j = 1, \dots, l, \end{aligned} \tag{1}$$

where $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$ are differentiable functions. We denote by M the set of feasible solutions.

We say that the **problem is convex** if functions f, $g_i, \forall_i$ are convex and $h_j, \forall_j$ are affine.

Define the **Lagrange function** by

$$L(x, u, v) = f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{j=1}^{l} v_j h_j(x),\ u_i \ge 0. \tag{2}$$

The **Karush–Kuhn–Tucker optimality conditions** are then (feasibility, complementarity and optimality):

$$\begin{aligned} \text{i) } & g_i(x) \le 0,\ i = 1, \dots, m,\ h_j(x) = 0,\ j = 1, \dots, l, \\ \text{ii) } & u_i g_i(x) = 0,\ u_i \ge 0,\ i = 1, \dots, m, \\ \text{iii) } & \nabla_x L(x, u, v) = 0, \end{aligned} \tag{3}$$

Any point (x, u, v) which fulfills the above conditions is called a KKT point.

If a Constraint Qualification (CQ) condition is fulfilled, then the KKT conditions are necessary for local optimality of a point. Basic CQ conditions are:

- **Slater CQ:** $\exists \tilde{x} \in M$ such that $g_i(\tilde{x}) < 0$ for all i and the gradients $\nabla_x h_j(\tilde{x})$, $j = 1, \dots, l$ are linearly independent.

- **Linear independence CQ** at $\hat{x} \in M$: all gradients

$$\nabla_x g_i(\hat{x}),\ i \in I_g(\hat{x}),\ \nabla_x h_j(\hat{x}),\ j = 1, \dots, l$$

are linearly independent.

These conditions are quite strong and are sufficient for weaker CQ conditions, e.g. the Kuhn–Tucker condition (Mangasarian–Fromovitz CQ, Abadie CQ, ...).

To summarize, we are going to practice the following relations:

1. KKT point and convex problem $\rightarrow$ global optimality at x.
2. Local optimality at x and a constraint qualification (CQ) condition $\rightarrow \exists (u, v)$ such that (x, u, v) is a KKT point.

2.2 Karush–Kuhn–Tucker optimality conditions

Example 2.1 *Verify that the point* $(x_1, x_2) = (\frac{4}{5}, \frac{8}{5})$ *is a local/global solution of the problem*

$$\begin{aligned} \min\ & x_1^2 + x_2^2, \\ s.t.\ & x_1^2 + x_2^2 \leq 5, \\ & x_1 + 2x_2 = 4, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Solution: Write the Lagrange function

$$L(x_1, x_2, u_1, u_2) = x_1^2 + x_2^2 + u_1(x_1^2 + x_2^2 - 5) - u_2 x_1 - u_3 x_2 + v(x_1 + 2x_2 - 4),\ u_1, u_2, u_3 \geq 0.$$

Derive the KKT conditions

$$\begin{aligned} \text{i)}\ & \text{feasibility}, \\ \text{ii)}\ & u_1(x_1^2 + x_2^2 - 5) = 0,\ u_1 \geq 0, \\ & u_2 x_1 = 0,\ u_2 \geq 0, \\ & u_3 x_2 = 0,\ u_3 \geq 0, \\ \text{iii)}\ & \frac{\partial L}{\partial x_1} = 2x_1 + 2u_1 x_2 - u_2 + v = 0, \\ & \frac{\partial L}{\partial x_2} = 2x_2 + 2u_1 x_2 - u_3 + 2v = 0. \end{aligned} \tag{4}$$

For point $(x_1, x_2) = (\frac{4}{5}, \frac{8}{5})$, we have that $u_{1,2,3} = 0$ (from complementarity conditions, i.e. none of the inequality constraints is active) and $v = -\frac{8}{5}$ which is feasible value for Lagrange multiplier corresponding to equality constraint. So we have obtained KKT point $(\frac{4}{5}, \frac{8}{5}, 0, 0, 0, -\frac{8}{5})$.

Since the objective function and inequality constraints are convex, and the equality constraint is linear (affine), $(x_1, x_2) = (\frac{4}{5}, \frac{8}{5})$ is a global solution.

Example 2.2 *Using the KKT conditions find the closest point to (0,0) in the set defined by*

$$M = \{x \in \mathbb{R}^2 : \ x_1 + x_2 \geq 4, \ 2x_1 + x_2 \geq 5\}.$$

Can several points (solutions) exist?

Solution: We formulate a nonlinear programming problem using the Euclidean distance in the objective[1]:

$$\begin{aligned} &\min \ x_1^2 + x_2^2 \\ &\text{s.t.} \ -x_1 - x_2 + 4 \leq 0, \\ &\qquad -2x_1 - x_2 + 5 \leq 0. \end{aligned}$$

The problem is obviously convex (sum of one-dimensional quadratic functions in the objective, linear constraints). We can write the Lagrange function

$$L(x_1, x_2, u_1, u_2) = x_1^2 + x_2^2 + u_1(-x_1 - x_2 + 4) + u_2(-2x_1 - x_2 + 5), \ u_1, u_2 \geq 0.$$

Derive the KKT conditions

$$\begin{aligned} &\text{i) feasibility,} \\ &\text{ii) } u_1(-x_1 - x_2 + 4) = 0, \ u_1 \geq 0, \\ &\quad u_2(-2x_1 - x_2 + 5) = 0, \ u_2 \geq 0, \\ &\text{iii) } \frac{\partial L}{\partial x_1} = 2x_1 - u_1 - 2u_2 = 0, \\ &\quad \frac{\partial L}{\partial x_2} = 2x_2 - u_1 - u_2 = 0. \end{aligned} \tag{5}$$

Now, we will try to find the KKT point by analyzing the optimality conditions, where we proceed according to the complementarity conditions:

1. Set $u_1 = 0$, $u_2 = 0$: We have from iii) that $x_1 = 0$, $x_2 = 0$ which is not feasible point.

2. Set $x_1 + x_2 = 4$, $u_2 = 0$: Together with iii) we solve

$$\begin{aligned} &2x_1 - u_1 = 0, \\ &2x_2 - u_1 = 0, \\ &x_1 + x_2 = 4. \end{aligned} \tag{6}$$

We obtain $x_1 = 2$, $x_2 = 2$, $u_1 = 4 > 0$, i.e. we have KKT point $(2, 2, 4, 0)$.

3. Set $u_1 = 0$, $2x_1 + x_2 = 5$: Solve

$$\begin{aligned} &2x_1 - 2u_2 = 0, \\ &2x_2 - u_2 = 0, \\ &2x_1 + x_2 = 5. \end{aligned} \tag{7}$$

We get $x_1 = 2$, $x_2 = 1$, $u_2 = 2$, which is not feasible point.

4. Set $x_1 + x_2 = 4$, $2x_1 + x_2 = 5$: We get $x_1 = 1$, $x_2 = 3$ and compute the Lagrange multipliers by solving

$$\begin{aligned} &u_1 + 2u_2 = 2, \\ &u_1 + u_2 = 6. \end{aligned} \tag{8}$$

[1] The square root can be omitted.

We obtain $u_1 = 10$, $u_2 = -4 < 0$, i.e. the Lagrange multipliers are not nonnegative and (1,3,10,-4) is not KKT point.

Since the set M is convex, the closest point corresponding to the projection $(2,2)$ must be unique.

Example 2.3 *Let $n \geq 2$. Consider the problem*

$$\begin{aligned} &\min\ x_1 \\ &\text{s.t. } \sum_{i=1}^{n} \left(x_i - \frac{1}{n}\right)^2 \leq \frac{1}{n(n-1)}, \\ &\sum_{i=1}^{n} x_i = 1. \end{aligned}$$

Show that

$$\left(0, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)$$

is an optimal solution.

Solution: First, realize that the considered point is feasible. Write the Lagrange function

$$L(x_1, \ldots, x_n, u, v) = x_1 + u\left(\sum_{i=1}^{n} \left(x_i - \frac{1}{n}\right)^2 - \frac{1}{n(n-1)}\right) + v\left(\sum_{i=1}^{n} x_i - 1\right),$$

where $u \geq 0$ and $v \in \mathbb{R}$. The KKT conditions (feasibility, complementarity and optimality) are

$$\begin{aligned} &\text{i) } \sum_{i=1}^{n} \left(x_i - \frac{1}{n}\right)^2 \leq \frac{1}{n(n-1)}, \ \sum_{i=1}^{n} x_i = 1, \\ &\text{ii) } u\left(\sum_{i=1}^{n} \left(x_i - \frac{1}{n}\right)^2 - \frac{1}{n(n-1)}\right) = 0, \ u \geq 0, \\ &\text{iii) } \frac{\partial L}{\partial x_1} = 1 + 2u\left(x_1 - \frac{1}{n}\right) + v = 0, \\ &\frac{\partial L}{\partial x_i} = 2u\left(x_1 - \frac{1}{n}\right) + v = 0, \ i \neq 1. \end{aligned} \tag{9}$$

Realize that the inequality constraint is active at the considered point, i.e.

$$\left(0 - \frac{1}{n}\right)^2 + \sum_{i=2}^{n} \left(\frac{1}{n-1} - \frac{1}{n}\right)^2 = \frac{1}{n(n-1)}.$$

To obtain the values of Lagrange multipliers, we solve the optimality conditions

$$\begin{aligned} &1 - \frac{2u}{n} + v = 0, \\ &2u\left(\frac{1}{n-1} - \frac{1}{n}\right) + v = 0, \ (\forall i \neq 1). \end{aligned} \tag{10}$$

By solving this linear system for u and v, we obtain the values

$$\begin{aligned} u &= \frac{n-1}{2} \geq 0, \\ v &= \frac{-1}{n} \in \mathbb{R}. \end{aligned} \tag{11}$$

Thus, we have obtained a KKT point

$$(x, u, v) = \left(0, \frac{1}{n-1}, \ldots, \frac{1}{n-1}, \frac{n-1}{2}, \frac{-1}{n}\right),$$

Since the objective function is convex (linear), the inequality constraint is convex and the equality constraint is linear, the considered point is a global solution (minimum) of the problem.

Example 2.4 *Consider the (water-filling[2]) problem*

$$\begin{aligned} \min \ & -\sum_{i=1}^{n} \log(\alpha_i + x_i) \\ \text{s.t.} \ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, \end{aligned}$$

where $\alpha_i > 0$ are parameters. Using the KKT conditions find the solutions.

Solution: First realize that the problem is convex, i.e. the objective is convex and the constraints are linear. Consider the Lagrange function

$$L(x, u, v) = -\sum_{i=1}^{n} \log(\alpha_i + x_i) - \sum_{i=1}^{n} u_i x_i + v\left(\sum_{i=1}^{n} x_i - 1\right), \ u_i \geq 0, v \in \mathbb{R}.$$

The KKT conditions are:

$$\begin{aligned} \text{i)} \ & \sum_{i=1}^{n} x_i = 1, \ x_i \geq 0, \ i = 1, \ldots, n \\ \text{ii)} \ & u_i x_i = 0, \ u_i \geq 0, \ i = 1, \ldots, n, \\ \text{iii)} \ & -\frac{1}{\alpha_i + x_i} - u_i + v = 0, \ i = 1, \ldots, n. \end{aligned}$$

We will proceed in several steps:

1. Since it holds

$$v = \frac{1}{\alpha_i + x_i} + u_i, \ \forall i,$$

and $\alpha_i > 0$ and $u_i \geq 0$, multiplier v must be positive.

[2] See Boyd and Vandenberghe (2004).

2. Now we can elaborate the complementarity conditions ii) for arbitrary $i \in \{1, \ldots, n\}$, i.e. $u_i = 0$ or $x_i = 0$:

   2.a. Let $u_i = 0$, then using iii) and 1. we obtain

   $$x_i = \frac{1}{v} - \alpha_i,$$

   which is nonnegative if and only if $v \leq 1/\alpha_i$.

   2.b. Let $x_i = 0$, then using iii) and 1. we obtain

   $$u_i = -1/\alpha_i + v,$$

   which is nonnegative if and only if $v \geq 1/\alpha_i$. Now realize that if $v \geq 1/\alpha_i$, then corresponding x_i cannot be positive because from iii) it would hold

   $$-\frac{1}{\alpha_i + x_i} + v = u_i > 0,$$

   which violates the complementarity condition (x_i and u_i cannot be both positive). In other words, x_i is positive if and only if $v \in (0, 1/\alpha_i)$.

   We have obtained two cases which are distinguished by relation between v and $1/\alpha_i$. Then we can write

   $$x_i = \max\left\{\frac{1}{v} - \alpha_i, 0\right\}.$$

3. It remains to determine the value of Lagrange multiplier v using the equality constraint

   $$\sum_{i=1}^{n} \max\left\{\frac{1}{v} - \alpha_i, 0\right\} = 1,$$

   which has a unique solution since the function of $\sum_{i=1}^{n} \max\{\cdot - \alpha_i, 0\}$ is piecewise-linear, continuous and increasing with breakpoints at points α_i. Note that there is no closed-form formula for v, we are satisfied with its existence.

2.3 Second Order Sufficient Condition (SOSC)

When the problem is not convex, then the solutions of the KKT conditions need not to correspond to global optima. The Second Order Sufficient Condition (SOSC) can be used to verify if the KKT point (its x part) is at least a local minimum.

Consider the set of active (inequality) constraints and its partitioning

$$\begin{aligned} I_g(x) &= \{i : \ g_i(x) = 0\}, \\ I_g^0(x) &= \{i : \ g_i(x) = 0, \ u_i = 0\}, \\ I_g^+(x) &= \{i : \ g_i(x) = 0, \ u_i > 0\}, \end{aligned} \tag{12}$$

i.e.

$$I_g(x) = I_g^0(x) \cup I_g^+(x).$$

Let all functions be twice differentiable. We say that the **second-order sufficient condition** (SOSC) is fulfilled at a KKT point (x, u, v) if for all $0 \neq z \in \mathbb{R}^n$ such that

$$\begin{aligned} z^T \nabla_x g_i(x) = 0,\ i \in I_g^+(x), \\ z^T \nabla_x g_i(x) \leq 0,\ i \in I_g^0(x), \\ z^T \nabla_x h_j(x) = 0,\ j = 1, \ldots, l, \end{aligned} \tag{13}$$

it holds

$$z^T \, \nabla_{xx}^2 L(x, u, v) \, z > 0. \tag{14}$$

Then x is a strict local minimum of the nonlinear programming problem (1).

Example 2.5 *Consider the problem*

$$\begin{aligned} \min\ & x^2 - y^2 \\ s.t.\ & x - y = 1 \\ & x, y \geq 0. \end{aligned}$$

Using the KKT optimality conditions find all stationary points. Using the SOSC verify if some of the points corresponds to a (strict) local minimum.

Solution: Write the Lagrange function

$$L(x, y, u_1, u_2, v) = x^2 - y^2 - u_1 x - u_2 y + v(x - y - 1),\ u_1, u_2 \geq 0.$$

Derive the KKT conditions

$$\begin{aligned} \text{i)}\ & \text{feasibility}, \\ \text{ii)}\ & -u_1 x = 0,\ u_1 \geq 0, \\ & -u_2 y = 0,\ u_2 \geq 0, \\ \text{iii)}\ & \frac{\partial L}{\partial x} = 2x - u_1 + v = 0, \\ & \frac{\partial L}{\partial y} = -2y - u_2 - v = 0. \end{aligned} \tag{15}$$

Solving this conditions together with feasibility leads to one feasible KKT point

$$(x, y, u_1, u_2, v) = (1, 0, 0, 2, -2).$$

Since the problem is non-convex, we can apply SOSC (13), (14). We have $I_g(1, 0) = I_g^+(1, 0) = \{2\}$ and $I_g^0(1, 0) = \emptyset$, so the conditions on $0 \neq z \in \mathbb{R}^2$ are:

$$\begin{aligned} z_1 - z_2 &= 0, \\ -z_2 &= 0. \end{aligned}$$

Since no $z \neq 0$ exists, the SOSC is fulfilled. (It is not necessary to compute $\nabla_{xx}^2 L$.)

Example 2.6 *Consider the problem*

$$\begin{aligned} &\min \; -x^2 - 4xy - y^2 \\ &s.t. \; x^2 + y^2 = 1. \end{aligned}$$

Using the SOSC verify that point $(\sqrt{2}/2, \sqrt{2}/2)$ *corresponds to a (strict) local minimum.*

Solution: Write the Lagrange function

$$L(x, y, v) = -x^2 - 4xy - y^2 + v\left(x^2 + y^2 - 1\right).$$

Derive the KKT conditions

$$\begin{aligned} &\text{i) feasibility,} \\ &\text{ii) } - \\ &\text{iii) } \frac{\partial L}{\partial x} = -2x - 4y + 2vx = 0, \\ &\quad\;\; \frac{\partial L}{\partial y} = -2y - 4x + 2vy = 0. \end{aligned} \tag{16}$$

We can compute the Lagrange multiplier and obtain the KKT point

$$(x, y, v) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 3\right).$$

Since the problem is non-convex, we can apply SOSC (13), (14). We have

$$\nabla h(\sqrt{2}/2, \sqrt{2}/2) = \begin{pmatrix} 2x \\ 2y \end{pmatrix} |_{(\sqrt{2}/2, \sqrt{2}/2)} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix},$$

so we have

$$Z(\sqrt{2}/2, \sqrt{2}/2) = \left\{z \in \mathbb{R}^2 : \; z_1 + z_2 = 0, \; z \neq 0\right\} = \{(z_1, -z_1) : \; z_1 \in \mathbb{R} \setminus \{0\}\}.$$

We must compute the Hessian matrix

$$\nabla^2_{xx} L\left(\sqrt{2}/2, \sqrt{2}/2, 3\right) = \begin{pmatrix} -2 + 2v & -4 \\ -4 & -2 + 2v \end{pmatrix} |_{(\sqrt{2}/2, \sqrt{2}/2, 3)} = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}.$$

Thus we have that $z^T \nabla^2_{xx} L(\sqrt{2}/2, \sqrt{2}/2, 3)\, z = 16 z_1^2 > 0$ for any $z_1 \in \mathbb{R} \setminus \{0\}$, which implies that $(\sqrt{2}/2, \sqrt{2}/2)$ is a strict local minimum of the problem.