Linear programming – simplex algorithm, duality and dual simplex algorithm

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Linear programming

Standard form LP

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b, \\ & x \geq 0. \end{aligned}$$

$A \in \mathbb{R}^{m \times n}$, $h(A) = h(A|b) = m$.

$$M = \{x \in \mathbb{R}^n : \ Ax = b,\ x \geq 0\}.$$

Linear programming

Decomposition of M:

- **Convex polyhedron** P – uniquely determined by its vertices (convex hull)
- **Convex polyhedral cone** K – generated by extreme directions (positive hull)

Direct method (evaluate all vertices and extreme directions, compute the values of the objective function ...)

Linear programming trichotomy

One of these cases is valid:

1. $M = \emptyset$
2. $M \neq \emptyset$: the problem is unbounded
3. $M \neq \emptyset$: the problem has an optimal solution (at least one of the solutions is vertex)

1914–2005

Simplex algorithm – basis

Basis B = regular square submatrix of A, i.e.

$$A = (B|N).$$

We also consider $B = \{i_1, \dots, i_m\}$.
We split the objective coefficients and the decision vector accordingly:

$$c^T = (c_B^T, c_N^T),$$
$$x^T(B) = (x_B^T(B), x_N^T(B)),$$

where

$$B \cdot x_B(B) = b, \ x_N(B) \equiv 0.$$

- Feasible basis, optimal basis.
- Basic solution(s).

Simplex algorithm – simplex table

			c^T
			x^T
c_B	$x_B(B)$	$B^{-1}b$	$B^{-1}A$
		$c_B^T B^{-1}b$	$c_B^T B^{-1}A - c^T$

Simplex algorithm – simplex table

- **Feasibility condition:**
$$B^{-1}b \geq 0.$$
- **Optimality condition:**
$$c_B^T B^{-1}A - c^T \leq 0.$$

Simplex algorithm – a step

If the optimality condition is not fulfilled:

- Denote the criterion row by

$$\delta^T = c_B^T B^{-1} A - c^T.$$

- Find $\delta_i > 0$ and denote the corresponding column by

$$\rho = B^{-1} A_{\bullet,i},$$

  where $A_{\bullet,i}$ is the i–th column of A.
- Minimize the ratios

$$\hat{u} = \arg\min \left\{ \frac{x_u(B)}{\rho_u} : \ \rho_u > 0, \ u \in B \right\}.$$

- Substitute $x_{\hat{u}}$ by x_i in the basic variables, i.e. $\hat{B} = B \setminus \{\hat{u}\} \cup \{i\}$.

Simplex algorithm – a step

Denote by $\hat{B}$ the new basis. Define a **direction**

$$\begin{aligned} \Delta_u &= -\rho_u, \ u \in B, \\ \Delta_i &= 1, \\ \Delta_j &= 0, \ j \notin B \cup \{i\}. \end{aligned}$$

If $\rho \le 0$ ($\hat{u} = \emptyset$), then the problem is unbounded ($c^T x \to -\infty$). Otherwise, we can **move from the current basic solution to another one**

$$x(\hat{B}) = x(B) + t\Delta,$$

where $0 \le t \le \frac{x_{\hat{u}}(B)}{\rho_{\hat{u}}}$. We should prove that the new solution is a feasible basic solution and that the objective value decreases ...

Simplex algorithm – a step

New solution is feasible:

$$\begin{aligned} x(\hat{B}) &\ge 0, \\ Ax(\hat{B}) &= Ax(B) + tA\Delta \\ &= Ax(B) - tB\rho + tA_{\bullet,i} \\ &= b - tBB^{-1}A_{\bullet,i} + tA_{\bullet,i} = b. \end{aligned}$$

Objective value decreases

$$\begin{aligned} c^T x(\hat{B}) &= c^T x(B) + tc^T \Delta \\ &= c^T x(B) - tc_B^T \rho + tc_i \\ &= c^T x(B) - t(c_B^T B^{-1} A_{\bullet,i} - c_i) \\ &= c^T x(B) - t\delta_i, \end{aligned}$$

where $\delta_i > 0$ is the element of the criterion row.

Simplex algorithm – a step

- If $\rho \le 0$, then $x(\hat{B})$ is feasible for all $t \ge 0$ and the objective value decreases in the direction Δ.
- Otherwise the step length t is bounded by $\frac{x_{\hat{u}}(B)}{\rho_{\hat{u}}}$. In this case, the new basis $\hat{B}$ is regular, because we interchange one unit vector by another one using the column i with $\rho_{\hat{u}} > 0$ element (on the right position).

Simplex algorithm – pivot rules

Rules for selecting the entering variable if there are several possibilities:

- **Largest coefficient** in the objective function
- **Largest decrease** of the objective function
- **Steepest edge** – choose an improving variable whose entering into the basis moves the current basic feasible solution in a direction closest to the direction of the vector c

$$\max \frac{c^T(x_{new} - x_{old})}{\|x_{new} - x_{old}\|}.$$

  Computationally the most successful.
- **Blands's rule** – choose the improving variable with the smallest index, and if there are several possibilities for the leaving variable, also take the one with the smallest index (prevents cycling)

Matoušek and Gärtner (2007).

Simplex algorithm – example

			3	-1	0	0
			x_1	x_2	x_3	x_4
0	x_3	2	-2	1	1	0
0	x_4	1	1	-2	0	1
		0	-3	1	0	0
-1	x_2	2	-2	1	1	0
0	x_4	5	-3	0	2	1
		-2	-1	0	-1	0

Moving in direction $\Delta^T = (0, 1, -1, 2)$, i.e.

$$(0, 2, 0, 5) = (0, 0, 2, 1) + t \cdot (0, 1, -1, 2),$$

where $t = 2$.

Simplex algorithm – unbounded problem

			-2	-1	0	0
			x_1	x_2	x_3	x_4
0	x_3	2	-2	1	1	0
0	x_4	1	1	-2	0	1
		0	2	1	0	0
-1	x_2	2	-2	1	1	0
0	x_4	5	-3	0	2	1
		-2	4	0	-1	0

Unbounded in direction $\Delta^T = (1, 2, 0, 3)$.

Linear programming duality

Primal problem

$$\begin{aligned} (\mathrm{P})\ \ & \min\ c^T x \\ & \text{s.t. } Ax \geq b, \\ & \quad\ \ x \geq 0. \end{aligned}$$

and corresponding **dual problem**

$$\begin{aligned} (\mathrm{D})\ \ & \max\ b^T y \\ & \text{s.t. } A^T y \leq c, \\ & \quad\ \ y \geq 0. \end{aligned}$$

Linear programming duality

Denote

$$M = \{x \in \mathbb{R}^n : \ Ax \geq b, \ x \geq 0\},$$
$$N = \{y \in \mathbb{R}^m : \ A^T y \leq c, \ y \geq 0\},$$

Weak duality theorem:

$$b^T y \leq c^T x, \ \forall x \in M, \forall y \in N.$$

Equality holds if and only if (iff) complementarity slackness conditions are fulfilled:

$$\begin{aligned} y^T(Ax - b) &= 0, \\ x^T(A^T y - c) &= 0. \end{aligned}$$

Linear programming duality

- **Duality theorem:** If $M \neq \emptyset$ and $N \neq \emptyset$, than the problems (P), (D) have optimal solutions.
- **Strong duality theorem:** The problem (P) has an optimal solution if and only if the dual problem (D) has an optimal solution. If one problem has an optimal solution, than the optimal values are equal.

Duality – production planning

Optimize the production of the following products V_1, V_2, V_3 made from materials M_1, M_2.

	V_1	V_2	V_3	Constraints
M_1	1	0	2	54 kg
M_2	2	3	1	30 kg
Gain ($/kg)	10	15	10	

Duality

Primal problem

$$(P) \quad \begin{array}{lccccccc} \max & 10x_1 & + & 15x_2 & + & 10x_3 & & \\ \text{s.t.} & x_1 & & & + & 2x_3 & \leq & 54, \\ & 2x_1 & + & 3x_2 & + & x_3 & \leq & 30, \\ & x_1 & & & & & \geq & 0, \\ & & & x_2 & & & \geq & 0, \\ & & & & & x_3 & \geq & 0. \end{array}$$

Dual problem

$$(D) \quad \begin{array}{lccccc} \min & 54y_1 & + & 30y_2 & & \\ \text{s.t.} & y_1 & + & 2y_2 & \geq & 10, \\ & & & 3y_2 & \geq & 15, \\ & 2y_1 & + & y_2 & \geq & 10, \\ & y_1 & & & \geq & 0, \\ & & & y_2 & \geq & 0. \end{array}$$

Duality

Optimal solution of (D) $\hat{y} = \left(\frac{5}{2}, 5\right)^T$.
Using the complementarity slackness conditions $\hat{x} = (0, 1, 27)^T$.
The optimal values (gains) of (P) and (D) are 285.

- Both (P) constraints are fulfilled with equality, thus there in no material left.
- Dual variables are called **shadow prices** and represent the prices of sources (materials).
- **Sensitivity**: If we increase (P) r.h.s. by one, then the objective value increases by the shadow price.
- The first constraint of (D) is fulfilled with strict inequality with the difference 2.5 $, called **reduced prices**, and the first product is not produced. The producer should increase the gain from V_1 by this amount to become profitable.

Transportation problem

- x_{ij} – decision variable: amount transported from i to j
- c_{ij} – costs for transported unit
- a_i – capacity
- b_j – demand

ASS. $\sum_{i=1}^{n} a_i \geq \sum_{j=1}^{m} b_j$.
(Sometimes $a_i, b_j \in \mathbb{N}$.)

Transportation problem

Primal problem

$$\begin{aligned} \min \ & \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} \\ \text{s.t.} \ & \sum_{j=1}^{m} x_{ij} \leq a_i, \ i = 1, \ldots, n, \\ & \sum_{i=1}^{n} x_{ij} \geq b_j, \ j = 1, \ldots, m, \\ & x_{ij} \geq 0. \end{aligned}$$

Transportation problem

Dual problem

$$\begin{aligned} \max \ & \sum_{i=1}^{n} a_i u_i + \sum_{j=1}^{m} b_j v_j \\ \text{s.t.} \ & u_i + v_j \leq c_{ij}, \\ & u_i \leq 0, \\ & v_j \geq 0. \end{aligned}$$

Interpretation: $-u_i$ price for buying a unit of goods at i, v_j price for selling at j.

Transportation problem

Competition between the transportation company (which minimizes the transportation costs) and an "agent" (who maximizes the earnings):

$$\sum_{i=1}^{n} a_i u_i + \sum_{j=1}^{m} b_j v_j \leq \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}$$

Linear programming duality

Apply KKT optimality conditions to primal LP ... we will see relations with NLP duality.

Linear programming duality

Primal problem (standard form)

$$\begin{aligned} &\min\ c^T x \\ &\text{s.t. } Ax = b, \\ &\quad\ x \geq 0. \end{aligned}$$

and corresponding **dual problem**

$$\begin{aligned} &\max\ b^T y \\ &\text{s.t. } A^T y \leq c, \\ &\quad\ y \in \mathbb{R}^m. \end{aligned}$$

Dual simplex algorithm

Dual simplex algorithm works with

- **dual feasible basis** B and
- **basic dual solution** $y(B)$,

where

$$\begin{aligned} B^T y(B) &= c_B, \\ N^T y(B) &\leq c_N. \end{aligned}$$

Dual simplex algorithm

Primal feasibility $B^{-1}b \geq 0$ is violated until reaching the optimal solution.
Primal optimality condition is always fulfilled:

$$c_B^T B^{-1} A - c^T \leq 0.$$

Using $A = (B|N)$, $c^T = (c_B^T, c_N^T)$, we have

$$\begin{aligned} c_B^T B^{-1} B - c_B^T &= 0, \\ c_B^T B^{-1} N - c_N^T &\leq 0, \end{aligned}$$

Setting $\hat{y} = (B^{-1})^T c_B$

$$\begin{aligned} B^T \hat{y} &= c_B, \\ N^T \hat{y} &\leq c_N. \end{aligned}$$

Thus, $\hat{y}$ is a basic dual solution.

Dual simplex algorithm – a step

... uses the same simplex table.

- Find index $u \in B$ such that $x_u(B) < 0$ and denote the corresponding row by
$$\tau^T = (B^{-1}A)_{u,\bullet}.$$
- Denote the criterion row by
$$\delta^T = c_B^T B^{-1} A - c^T \leq 0.$$
- Minimize the ratios
$$\hat{i} = \arg\min \left\{ \frac{\delta_i}{\tau_i} : \ \tau_i < 0 \right\}.$$
If there is no i such that $\tau_i < 0$, then STOP: the dual problem is unbounded and primal is infeasible.
- Substitute x_u by $x_{\hat{i}}$ in the basic variables, i.e. $\hat{B} = B \setminus \{u\} \cup \{\hat{i}\}$. We move to another **basic dual solution**.

Dual simplex algorithm – an assumption

The problem is **dual nondegenerate** if for all dual feasible basis B it holds

$$\begin{aligned} (A^T y(B) - c)_j = 0, \ j \in B, \\ (A^T y(B) - c)_j < 0, \ j \notin B. \end{aligned}$$

If the problem is dual nondegenerate, then the dual simplex algorithm ends after finitely many steps.

Dual simplex algorithm – a step

A general step in the dual simplex algorithm

$$y(\hat{B}) = y(B) - t(B^{-1})^T_{\bullet,u}$$

with

$$t := \frac{\delta_{\hat{i}}}{\tau_{\hat{i}}}.$$

Then it can be shown that the **dual feasibility** is preserved, i.e.

$$A^T y(\hat{B}) = A^T y(B) - tA^T (B^{-1})^T_{\bullet,u} \leq c,$$

e.g.

$$\left(A^T y(\hat{B})\right)_{\hat{i}} = \delta_{\hat{i}} + c_{\hat{i}} - \frac{\delta_{\hat{i}}}{\tau_{\hat{i}}} \tau_{\hat{i}} = c_{\hat{i}},$$

or

$$\left(A^T y(\hat{B})\right)_u = \delta_u + c_u - \frac{\delta_{\hat{i}}}{\tau_{\hat{i}}} \tau_u \leq c_u.$$

Dual simplex algorithm – a step

A general step in the dual simplex algorithm

$$y(\hat{B}) = y(B) - t(B^{-1})^T_{\bullet,u}$$

with

$$t := \frac{\delta_{\hat{i}}}{\tau_{\hat{i}}} > 0.$$

Then it can be shown that the **objective function increases** if the problem is dual nondegenerate, i.e.

$$\begin{aligned} b^T y(\hat{B}) &= b^T y(B) - t b^T (B^{-1})^T_{\bullet,u}, \\ &= b^T y(B) - \frac{\delta_{\hat{i}}}{\tau_{\hat{i}}} x_u(B) > b^T y(B), \end{aligned}$$

because $x_u(B) < 0$.

Example – dual simplex algorithm

$$\begin{aligned} \min 4x_1 + 5x_2 & & \\ x_1 + 4x_2 &\geq 5, \\ 3x_1 + 2x_2 &\geq 7, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Example – dual simplex algorithm

Dual problem

$$\begin{aligned} \max\ & -5y_1 - 7y_2 \\ \text{s.t.}\ & -y_1 - 3y_2 \leq 4 \\ & -4y_1 - 2y_2 \leq 5 \\ & y_1 \leq 0 \\ & y_2 \leq 0. \end{aligned}$$

Example – dual simplex algorithm

			4	5	0	0
			x_1	x_2	x_3	x_4
0	x_3	-5	-1	-4	1	0
0	x_4	-7	-3	-2	0	1
		0	-4	-5	0	0
0	x_3	-8/3	0	-10/3	1	-1/3
4	x_1	7/3	1	2/3	0	-1/3
		28/3	0	-7/3	0	-4/3
5	x_2	8/10	0	1	-3/10	1/10
4	x_1	18/10	1	0	2/10	-4/10
		112/10	0	0	-7/10	-11/10

The last solution is primal and dual feasible, thus optimal.

Example – dual simplex algorithm

A general step in the dual simplex algorithm

$$y(\hat{B}) = y(B) - t(B^{-1})^T_{\bullet,u}$$

i.e.

$$(0, -4/3) = (0, 0) - 4/3(0, 1),$$

which can be seen in the criterion row in the columns corresponding to the initial basis. Dual constraints 1 and 3 are then active.

Adding new constraint

We would like to add a new constraint to the problem and use previous solution = post-optimization approach

$$\alpha^T x \leq \beta,$$

where $\alpha \neq 0$. We add new slack variable $x_{n+1} \geq 0$ to get

$$\alpha^T x + x_{n+1} = \beta.$$

The problem and the simplex table is extended by one row and one column

$$\tilde{A} = \begin{pmatrix} A & 0 \\ \alpha^T & 1 \end{pmatrix}, \; \tilde{b} = \begin{pmatrix} b \\ \beta \end{pmatrix}, \; \tilde{c} = \begin{pmatrix} c \\ 0 \end{pmatrix}$$

The basis can be extended by the new variable $B = B \cup \{n+1\}$.

Adding new constraint

We obtain the new basis, the matrix is obviously regular

$$\tilde{B} = \begin{pmatrix} B & 0 \\ \alpha_B^T & 1 \end{pmatrix}.$$

The inverse matrix can be derived

$$\tilde{B}^{-1} = \begin{pmatrix} B^{-1} & 0 \\ -\alpha_B^T B^{-1} & 1 \end{pmatrix},$$

so we try to verify feasibility

$$\tilde{B}^{-1}\tilde{b} = \begin{pmatrix} B^{-1} & 0 \\ -\alpha_B^T B^{-1} & 1 \end{pmatrix} \begin{pmatrix} b \\ \beta \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ -\alpha_B^T B^{-1} b + \beta \end{pmatrix},$$

where obviously $B^{-1}b \geq 0$, but the second row corresponds to $\alpha_B^T B^{-1} b \leq \beta$, which is fulfilled only if the current basic solution satisfies the new constraint.

Adding new constraint

New optimality condition (criterion row) is

$$\begin{aligned} \tilde{c}_{\tilde{B}} \tilde{B}^{-1} \tilde{A} - \tilde{c}^T &= (c_B^T, 0) \begin{pmatrix} B^{-1} & 0 \\ -\alpha_B^T B^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ \alpha^T & 1 \end{pmatrix} - (c^T, 0) \\ &= (c_B^T, 0) \begin{pmatrix} B^{-1}A & 0 \\ -\alpha_B^T B^{-1} A + \alpha^T & 1 \end{pmatrix} - (c^T, 0) \\ &= (c_B^T B^{-1} A - c^T, 0) \leq 0, \end{aligned}$$

which is obviously fulfilled no matter what constraints we have added.

Adding new constraint

To summarize, if

$$\alpha_B^T B^{-1} b \le \beta,$$

- is fulfilled, the previously obtained optimal solution remains optimal,
- is not fulfilled, then the primal feasibility (dual optimality) condition is violated and we continue by iteration(s) of the dual simplex algorithm with initial table

			c^T	0
			x^T	x_{n+1}
c_B	x_B	$B^{-1}b$	$B^{-1}A$	0
0	x_{n+1}	$-\alpha_B^T B^{-1} b + \beta$	$-\alpha_B^T B^{-1} A + \alpha^T$	1
		$c_B^T B^{-1} b$	$c_B^T B^{-1} A - c^T$	0

Adding new constraint

Consider final table after several iterations of the simplex algorithm:

			2	-1	0	0
			x_1	x_2	x_3	x_4
-1	x_2	1	-1	1	1	0
0	x_4	2	1	0	-1	1
		-1	-1	0	-1	0

We would like to add constraint $x_2 \le \frac{1}{2}$. Obviously the current optimal solution is not feasible, so we add the constraint to the simplex table. We have $\alpha_B^T = (1, 0)$

			2	-1	0	0	
			x_1	x_2	x_3	x_4	x_5
-1	x_2	1	-1	1	1	0	0
0	x_4	2	1	0	-1	1	0
0	x_5	$\frac{-1}{2}$	1	0	-1	0	1
		-1	-1	0	-1	0	0

Software tools for LP

- Matlab
- Mathematica
- GAMS
- Cplex studio
- AIMMS
- ...
- R
- MS Excel
- ...

Literature

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