Linear programming – simplex algorithm, duality and dual simplex algorithm

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Computational Aspects of Optimization

Content

Linear programming

Standard form LP

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax = b, \\ & x \geq 0. \end{aligned}$$

$A \in \mathbb{R}^{m \times n}$, $h(A) = h(A|b) = m$.

$$M = \{x \in \mathbb{R}^n : \ Ax = b, \ x \geq 0\}.$$

Linear programming

Decomposition of M:

- **Convex polyhedron** P – uniquely determined by its vertices (convex hull)
- **Convex polyhedral cone** K – generated by extreme directions (positive hull)

Direct method (evaluate all vertices and extreme directions, compute the values of the objective function …)

Linear programming trichotomy

One of these cases is valid:

1. $M = \emptyset$
2. $M \neq \emptyset$: the problem is unbounded
3. $M \neq \emptyset$: the problem has an optimal solution (at least one of the solutions is vertex)

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Simplex algorithm – basis

Basis B = regular square submatrix of A, i.e.

$$A = (B|N).$$

We also consider $B = \{i_1, \dots, i_m\}$.
We split the objective coefficients and the decision vector accordingly:

$$c^T = (c_B^T, c_N^T),$$
$$x^T(B) = (x_B^T(B), x_N^T(B)),$$

where

$$B \cdot x_B(B) = b, \; x_N(B) \equiv 0.$$

- Feasible basis, optimal basis.
- Basic solution(s).
- **ASS.** non-degenerate problem (basic solutions have m positive elements) $\rightarrow$ finiteness of the simplex alg.

Simplex algorithm – simplex table

			x^T
			c^T
c_B	x_B	$B^{-1}b$	$B^{-1}A$
		$c_B^T B^{-1} b$	$c_B^T B^{-1} A - c^T$

Simplex algorithm – simplex table

- **Feasibility condition:**

$$B^{-1}b \geq 0.$$

- **Optimality condition:**

$$c_B^T B^{-1} A - c^T \leq 0.$$

Simplex algorithm – a step

If the optimality condition is not fulfilled:

- Denote the criterion row by

$$\delta^T = c_B^T B^{-1} A - c^T.$$

- Find $\delta_i > 0$ and denote the corresponding column by

$$\rho = B^{-1} A_{\cdot,i}.$$

- Minimize the ratios

$$\hat{u} = \arg\min \left\{ \frac{x_u(B)}{\rho_u} : \ \rho_u > 0, \ u \in B \right\}.$$

- Substitute $x_{\hat{u}}$ by x_i in the basic variables, i.e. $\hat{B} = B \setminus \{\hat{u}\} \cup \{i\}$.

Simplex algorithm – a step

Denote by $\hat{B}$ the new basis. Define a **direction**

$$\begin{aligned} \Delta_u &= -\rho_u,\ u \in B, \\ \Delta_i &= 1, \\ \Delta_j &= 0,\ j \notin B \cup \{i\}. \end{aligned}$$

If $\Delta \leq 0$, then the problem is unbounded ($c^T x(\hat{B}) \to -\infty$). Otherwise, we can **move from the current basic solution to another one**

$$x(\hat{B}) = x(B) + t\Delta,$$

where $0 \leq t \leq \frac{x_{\hat{u}}(B)}{\rho_{\hat{u}}}$. We should prove that the new solution is a feasible basic solution ($\hat{B}$ is regular, $x(\hat{B}) \geq 0$, $\hat{B}x(\hat{B}) = b$) and that the objective value decreases ...

Simplex algorithm – pivot rules

... rules for selecting the entering variable if there are several possibilities:

- **Largest coefficient** in the objective function
- **Largest decrease** of the objective function
- **Steepest edge** – choose an improving variable whose entering into the basis moves the current basic feasible solution in a direction closest to the direction of the vector c

$$\max \frac{c^T(x_{new} - x_{old})}{\|x_{new} - x_{old}\|}.$$

  Computationally the most successful.
- **Blands's rule** – choose the improving variable with the smallest index, and if there are several possibilities for the leaving variable, also take the one with the smallest index (prevents cycling)

Matoušek and Gärtner (2007).

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Transportation problem

- $i = 1, \dots, n$ – suppliers
- $j = 1, \dots, m$ – customers
- x_{ij} – decision variable: amount transported from i to j
- c_{ij} – costs for transported unit
- a_i – capacity
- b_j – demand

ASS. $\sum_{i=1}^{n} a_i \geq \sum_{j=1}^{m} b_j$.
(Sometimes $a_i, b_j \in \mathbb{N}$.)

Transportation problem

Primal problem

$$
\begin{aligned}
\min\ & \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij} \\
\text{s.t. } & \sum_{j=1}^{m} x_{ij} \le a_i,\ \ i = 1,\dots,n, \\
& \sum_{i=1}^{n} x_{ij} \ge b_j,\ \ j = 1,\dots,m, \\
& x_{ij} \ge 0.
\end{aligned}
$$

Transportation problem

Dual problem

$$\begin{aligned} \max \ & \sum_{i=1}^{n} a_i u_i + \sum_{j=1}^{m} b_j v_j \\ \text{s.t. } & u_i + v_j \le c_{ij}, \\ & u_i \le 0, \\ & v_j \ge 0. \end{aligned}$$

Interpretation: $-u_i$ (shadow) price for buying a unit of goods at i, v_j (shadow) price for selling at j.

Transportation problem

Competition between the transportation company (which minimizes the transportation costs) and an “agent” (who maximizes the earnings):

$$\sum_{i=1}^{n} a_i u_i + \sum_{j=1}^{m} b_j v_j \leq \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}$$

Linear programming duality

Primal problem

$$\begin{aligned}(\mathrm{P})\ \ \min\ & c^T x\\ \text{s.t. } & Ax \geq b,\\ & x \geq 0.\end{aligned}$$

and corresponding **dual problem**

$$\begin{aligned}(\mathrm{D})\ \ \max\ & b^T y\\ \text{s.t. } & A^T y \leq c,\\ & y \geq 0.\end{aligned}$$

Linear programming duality

Denote

$$M = \{x \in \mathbb{R}^n : \ Ax \geq b, \ x \geq 0\},$$
$$N = \{y \in \mathbb{R}^m : \ A^T y \leq c, \ y \geq 0\},$$

Weak duality theorem:

$$b^T y \leq c^T x, \ \forall x \in M, \forall y \in N.$$

Equality holds if and only if (iff) complementarity slackness conditions are fulfilled:

$$\begin{aligned} y^T(Ax - b) &= 0, \\ x^T(A^T y - c) &= 0. \end{aligned}$$

Linear programming duality

Apply KKT optimality conditions to primal LP ...

Linear programming duality

- **Duality theorem:** If $M \neq \emptyset$ and $N \neq \emptyset$, than the problems (P), (D) have optimal solutions.
- **Strong duality theorem:** The problem (P) has an optimal solution if and only if the dual problem (D) has an optimal solution. If one problem has an optimal solution, than the optimal values are equal.

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Linear programming duality

Primal problem (standard form)

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b, \\ & x \geq 0. \end{aligned}$$

and corresponding **dual problem**

$$\begin{aligned} \max\ & b^T y \\ \text{s.t. } & A^T y \leq c, \\ & y \in \mathbb{R}^m. \end{aligned}$$

Dual simplex algorithm

Dual simplex algorithm works with

- **dual feasible basis** B and
- **basic dual solution** $y(B)$,

where

$$\begin{aligned} B^T y(B) &= c_B^T, \\ N^T y(B) &\leq c_N^T. \end{aligned}$$

Dual simplex algorithm

Primal feasibility $B^{-1}b \geq 0$ **is violated** until reaching the optimal solution.

Primal optimality condition is always fulfilled:

$$c_B^T B^{-1} A - c^T \leq 0.$$

Using $A = (B|N)$, $c^T = (c_B^T, c_N^T)$, we have

$$\begin{aligned} c_B^T B^{-1} B - c_B^T &= 0, \\ c_B^T B^{-1} N - c_N^T &\leq 0, \end{aligned}$$

Setting $\hat{u} = (B^{-1})^T c_B$

$$\begin{aligned} B^T \hat{u} &= c_B^T, \\ N^T \hat{u} &\leq c_N^T. \end{aligned}$$

Thus, $\hat{u}$ is a basic dual solution.

Dual simplex algorithm – a step

… uses the same simplex table.

- Find index $u \in B$ such that $x_u(B) < 0$ and denote the corresponding row by
$$\tau^T = (B^{-1}A)_{u,\cdot}.$$
- Denote the criterion row by
$$\delta^T = c_B^T B^{-1} A - c^T \leq 0.$$
- Minimize the ratios
$$\hat{i} = \arg\min \left\{ \frac{\delta_i}{\tau_i} : \ \tau_i < 0 \right\}.$$
- Substitute x_u by $x_{\hat{i}}$ in the basic variables, i.e. $\hat{B} = B \setminus \{u\} \cup \{\hat{i}\}$. We move to another basic dual solution.

Example – dual simplex algorithm

$$
\begin{aligned}
\min 4x_1 + 5x_2 & & \\
x_1 + 4x_2 &\geq& 5, \\
3x_1 + 2x_2 &\geq& 7, \\
x_1, x_2 &\geq& 0.
\end{aligned}
$$

Example – dual simplex algorithm

			4	5	0	0
			x_1	x_2	x_3	x_4
0	x_3	-5	-1	-4	1	0
0	x_4	-7	-3	-2	0	1
		0	-4	-5	0	0
0	x_3	-8/3	0	-10/3	1	-1/3
4	x_1	7/3	1	2/3	0	-1/3
		28/3	0	-7/3	0	-4/3
5	x_2	8/10	0	1	-3/10	1/10
4	x_1	18/10	1	0	2/10	-4/10
		112/10	0	0	-7/10	-11/10

The last solution is primal and dual feasible, thus optimal.

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Software tools for LP

- Matlab
- Mathematica
- GAMS
- MS Excel
- ...

Literature

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Questions?

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