Linear programming – duality and shadow prices
Software tools for LP
Quadratic programming – Wolfe algorithm

Martin Branda

Charles University in Prague
Faculty of Mathematics and Physics
Department of Probability and Mathematical Statistics

COMPUTATIONAL ASPECTS OF OPTIMIZATION

Content

Linear programming duality

Primal problem

$$\begin{aligned} &\min\ c^T x \\ \text{(P)}\ &\text{s.t. } Ax \geq b, \\ &\quad\ x \geq 0. \end{aligned}$$

and corresponding **dual problem**

$$\begin{aligned} &\max\ b^T y \\ \text{(D)}\ &\text{s.t. } A^T y \leq c, \\ &\quad\ y \geq 0. \end{aligned}$$

Linear programming duality

		x_1	x_2	x_3		
		≥ 0	≤ 0	$\in \mathbb{R}$		
		1	2	3	$\leq$	b_1
		4	5	6	$\geq$	b_2
		7	8	9	$=$	b_3
		c_1	c_2	c_3	min	

Linear programming duality

		x_1	x_2	x_3		
		≥ 0	≤ 0	$\in \mathbb{R}$		
y_1	≥ 0	1	2	3	$\geq$	b_1
y_2	≤ 0	4	5	6	$\leq$	b_2
y_3	$\in \mathbb{R}$	7	8	9	$=$	b_3
		$\leq$	$\geq$	$=$		max
		c_1	c_2	c_3	min	

Linear programming duality

Denote

$$M = \{x \in \mathbb{R}^n : \; Ax \geq b, \; x \geq 0\},$$
$$N = \{y \in \mathbb{R}^m : \; A^T y \leq c, \; y \geq 0\}.$$

Weak duality theorem:

$$b^T y \leq c^T x, \; \forall x \in M, \forall y \in N.$$

Equality holds if and only if (iff) complementarity slackness conditions are fulfilled:

$$\begin{aligned} y^T(Ax - b) &= 0, \\ x^T(A^T y - c) &= 0. \end{aligned}$$

Linear programming duality

Apply KKT optimality conditions to primal LP ...

Linear programming duality

- **Duality theorem:** If $M \neq \emptyset$ and $N \neq \emptyset$, than the problems (P), (D) have optimal solutions.
- **Strong duality theorem:** The problem (P) has an optimal solution if and only if the dual problem (D) has an optimal solution. If one problem has an optimal solution, than the optimal values are equal.

Duality – production planning

Optimize the production of the following products V_1, V_2, V_3 made from materials M_1, M_2.

	V_1	V_2	V_3	Constraints
M_1	1	0	2	54 kg
M_2	2	3	1	30 kg
Gain (\$/kg)	10	15	10	

Duality

Primal problem

$$
(P)\quad
\begin{array}{lrcrcrcl}
\max & 10x_1 & + & 15x_2 & + & 10x_3 & & \\
\text{s.t.} & x_1 & & & + & 2x_3 & \leq & 54, \\
 & 2x_1 & + & 3x_2 & + & x_3 & \leq & 30, \\
 & x_1 & & & & & \geq & 0, \\
 & & & x_2 & & & \geq & 0, \\
 & & & & & x_3 & \geq & 0.
\end{array}
$$

Dual problem

$$
(D)\quad
\begin{array}{lrcrcl}
\min & 54y_1 & + & 30y_2 & & \\
\text{s.t.} & y_1 & + & 2y_2 & \geq & 10, \\
 & & & 3y_2 & \geq & 15, \\
 & 2y_1 & + & y_2 & \geq & 10, \\
 & y_1 & & & \geq & 0, \\
 & & & y_2 & \geq & 0.
\end{array}
$$

Duality

Optimal solution of (D) $\hat{y} = \left(\frac{5}{2}, 5\right)$.
Using the complementarity slackness conditions $\hat{x} = (0, 1, 27)$.
The optimal values (gains) of (P) and (D) are 285.

- Both (P) constraints are fulfilled with equality, thus there in no material left.
- Dual variables are called **shadow prices** and represent the prices of sources (materials).
- **Sensitivity**: If we increase (P) r.h.s. by one, then the objective value increases by the shadow price.
- The first constraint of (D) is fulfilled with strict inequality with the difference 2.5 $, called **reduced prices**, and the first product is not produced. The producer should increase the gain from V_1 by this amount to become profitable.

Transportation problem

- x_{ij} – decision variable: amount transported from i to j
- c_{ij} – costs for transported unit
- a_i – capacity
- b_j – demand

ASS. $\sum_{i=1}^{n} a_i \geq \sum_{j=1}^{m} b_j$.
(Sometimes $a_i, b_j \in \mathbb{N}$.)

Transportation problem

Primal problem

$$\begin{aligned}
\min \ & \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij} \\
\text{s.t.} \ & \sum_{j=1}^{m} x_{ij} \leq a_i, \ \ i = 1, \ldots, n, \\
& \sum_{i=1}^{n} x_{ij} \geq b_j, \ j = 1, \ldots, m, \\
& x_{ij} \geq 0.
\end{aligned}$$

Transportation problem

Dual problem

$$\begin{aligned} \max \ & \sum_{i=1}^{n} a_i u_i + \sum_{j=1}^{m} b_j v_j \\ \text{s.t. } & u_i + v_j \leq c_{ij}, \\ & u_i \leq 0, \\ & v_j \geq 0. \end{aligned}$$

Interpretation: $-u_i$ price for buying a unit of goods at i, v_j price for selling at j.

Transportation problem

Competition between the transportation company (which minimizes the transportation costs) and an "agent" (who maximizes the earnings):

$$\sum_{i=1}^{n} a_i u_i + \sum_{j=1}^{m} b_j v_j \leq \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij}$$

Content

Software tools for LP

- **Matlab** – perfect (maybe the best) for problems which can be formulated using vectors and matrices (LP, QP), difficult, but possible, to use for general nonlinear programming problems
- **Mathematica**
- **GAMS** – many supported problems and solvers, free version strongly limited
- MS Excel – problem size is limited (something like 200 decision variables, 100 constraints)
- SAS, R ...

Content

Quadratic programming problems (QPP)

Basic form suitable for Wolfe algorithm

$$\begin{aligned} \min\ & \frac{1}{2}x^T Cx + p^T x \\ \text{s.t. } & Ax \leq b, \\ & x \geq 0. \end{aligned}$$

i.e. (vector valued constraints)

$$\begin{aligned} f(x) &= \frac{1}{2}x^T Cx + p^T x \\ g_1(x) &= Ax - b, \\ g_2(x) &= -x. \end{aligned}$$

Quadratic programming problems – examples

- Least-squares problems with restrictions (constraints) on coefficients
- Markowitz portfolio problem (variance minimization)
- Second-order approximation of more difficult nonlinear programming problems (Sequential Quadratic Programming)

Quadratic programming problems

Ass. C is positive semidefinite (symmetric)

$$\begin{aligned} \nabla_x f(x) &= Cx + p, \\ \nabla^2_{x,x} f(x) &= C. \end{aligned}$$

KKT optimality conditions

Lagrange function

$$L(x, y, v) = \frac{1}{2}x^T Cx + p^T x + y^T (Ax - b) - v^T x,$$

where $y \geq 0,\ v \geq 0$.

KKT optimality conditions for a feasible point

$$\begin{aligned} \nabla_x L(x, y, v) = \ & Cx + p + A^T y - v = 0, \\ & y^T (Ax - b) = 0,\ v^T x = 0, \\ & y \geq 0, v \geq 0. \end{aligned}$$

KKT optimality conditions

Using additional slack variables $w \geq 0$ in $Ax + Iw = b$ we obtain the **linear system**

$$\begin{aligned} Cx + p + A^T y - v &= 0, \\ Ax + Iw &= b \end{aligned}$$

together with **complementarity slackness conditions**

$$v^T x = 0, \;\; w^T y = 0,$$

and **nonnegativity**

$$x, y, v, w \geq 0.$$

We can solve the linear system using the simplex algorithm and take into account the complementarity slackness conditions = **Wolfe algorithm**.

QPP – Example

$$\begin{aligned} \min\ & x_1^2 + x_2^2 + p_1 x_1 + p_2 x_2, \\ \text{s.t. } & x_1 \le 1, x_2 \le 1, \\ & x_1 \ge 0, x_2 \ge 0. \end{aligned}$$

$p = (0, 0)$ or $p = (-2, -2)$

Wolfe algorithm

1. Find a **primal basic solution** B of $Ax + Iw = b$.
   - If $b \geq 0$, then $B = I$.
   - Else you can use Gauss–Jordan elimination.

   If such basis does not exist, then END: the problem has no feasible solution.
2. Build a **linear programming problem** using new decision variables z:

$$\min \sum_{k=1}^{n} z_k,$$
$$Cx + A^T y - v + Dz = -p,$$
$$Ax + Iw = b,$$
$$x, y, v, w, z \geq 0,$$

   where D is a diagonal matrix with
   - $d_{kk} = -1$ if $\sum_j c_{kj} x_j(B) > -p_k$,
   - $d_{kk} = +1$ if $\sum_j c_{kj} x_j(B) \leq -p_k$.

Wolfe algorithm

3. Use $L = B \cup \{z_1, \ldots, z_n\}$ as the feasible basis[1] to start the **simplex algorithm**. During the algorithm run ensure that the **complementarity slackness conditions** $v^T x = 0$, $w^T y = 0$ are fulfilled, e.g.
   - If x_k is in basis, then v_k cannot be included into basis.
   - ...
4. Denote by L_{fin} the final basis.
   - If $\sum_{j=1}^{m} y_j(L_{fin}) = 0$, then $x(L_{fin})$ is an optimal solution of QPP.
   - If $\sum_{j=1}^{m} y_j(L_{fin}) > 0$, then the algorithm has not found an optimal solution of QPP.

FINITNESS ensured under C positive semidefinite and $h(C) = h(C|p)$.

[1] We overload the symbol for the basis (sub-matrix or corresponding decision variables).

Literature

- Bazaraa, M.S., Sherali, H.D., and Shetty, C.M. (2006). Nonlinear programming: theory and algorithms, Wiley, Singapore, 3rd edition.
- Boyd, S., Vandenberghe, L. (2004). Convex Optimization, Cambridge University Press, Cambridge.
- Lachout, P. (2011). Matematické programování. Skripta k (zaniklé) přednášce Optimalizace I (IN CZECH).
- Matoušek and Gärtner (2007). Understanding and using linear programming, Springer.
- Wolfe, P. (1959). The Simplex Method for Quadratic Programming. Econometrica 27(3), 382–398.