Optimization with application in finance – exercises

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2 Parametric nonlinear optimization

We start with a simple example where we apply the Robinson–Fiacco Theorem.

Example 2.1 *Let p be a real parameter and b be a fixed positive constant. Consider*

$$\begin{aligned} &\min\ p\,x \\ &\text{s.t. } 0 \le x \le b. \end{aligned}$$

Elaborate the stability of the problem with respect to p.

Solution: Formulate the Lagrange function

$$L(x,u,v) = p\,x - u\,x + v(x-b),\ \ u,v \ge 0,$$

and the Karush–Kuhn–Tucker conditions:

i. feasibility: $0 \le x \le b$,

ii. complementarity: $u\,x = 0$, $v(x-b) = 0$, $u, v \ge 0$,

iii. optimality:
$$\frac{d\,L(x,u,v)}{dx} = p - u + v = 0.$$

Now, we will verify the assumptions of the Robinson–Fiacco theorem for KKT point, which we obtain for particular value of parameter p, namely Strong Complementarity (SC), Linear Independence Constraint Qualification (LI) and Second Order Sufficient Condition (SOSC).

1. $p > 0$: The optimal solution is $x = 0$. From complementarity ii) we immediately get $v = 0$ and from optimality iii) we have $u = p > 0$. Thus we have KKT point $(0, p, 0)$. Now we can discuss the assumption of R–F theorem:

RF1. SC is fulfilled because the first constraint is active and the corresponding multiplier is positive, and the second one is inactive and corresponding multiplier is equal to zero.

RF2. LI is fulfilled, the gradient of the active constraint is equal to -1.

RF3. SOSC: We have that $I_g(0) = I_g^+(0) = \{1\}$ and
$$Z(0) = \{z \ne 0 :\ -z = 0\} = \emptyset.$$
Since the set of (adjusted) feasible directions is empty, SOSC is fulfilled.

We have verified the assumption of R–F theorem, therefore the local optimal solution is "'stable"', i.e it is continuous and differentiable on some neighborhood of each $p > 0$. Using the theorem, we can obtain the derivative of the reduced KKT point with respect to the parameter

$$w(p) := \big(x(p), u_i(p), i \in I_g(x(p))\big).$$

Set

$$D(p) = \begin{pmatrix} \nabla^2_{xx} L & \nabla_x g_1 \\ \nabla_x g_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \; B(p) = \begin{pmatrix} \nabla^2_{px} L \\ \nabla_p g_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1}$$

then the derivative is

$$\frac{d\, w(p)}{dp} = -D^{-1}(p) B(p) = (0,1)^T.$$

2. $p = 0$: Since all feasible solutions are optimal, we must split this case to three sub-cases. First, $x = 0$ leads again to $u = p = 0$ and $v = 0$, i.e. KKT point $(0,0,0)$. Then

RF1. SC is NOT fulfilled because the first constraint is active and the corresponding multiplier is equal to zero. (The second one is inactive and corresponding multiplier is equal to zero.)

RF2. LI is fulfilled, the gradient of the active constraint is equal to -1.

RF3. SOSC: We have that $I_g(0) = I_g^0(0) = \{1\}$,

$$Z(0) = \{z \neq 0 : \; -z \leq 0\} = \{z > 0\} \neq \emptyset,$$

and

$$\nabla^2_{xx} L(x,u,v) = 0,$$

thus it cannot hold that

$$z\, \nabla^2_{xx} L(0,0,0)\, z > 0.$$

Therefore also SOSC is not fulfilled.

Now, $x = b$ leads to KKT point (b,0,0), where $v = -p = 0$. Then

RF1. SC is NOT fulfilled because the second constraint is active and the corresponding multiplier is equal to zero. (The first one is inactive and corresponding multiplier is equal to zero.)

RF2. LI is fulfilled, the gradient of the active constraint is equal to 1.

RF3. SOSC: We have that $I_g(0) = I_g^0(0) = \{2\}$,

$$Z(b) = \{z \neq 0 : \; z \leq 0\} = \{z < 0\} \neq \emptyset,$$

and

$$\nabla^2_{xx} L(x,u,v) = 0,$$

thus it cannot hold that

$$z\, \nabla^2_{xx} L(b,0,0)\, z > 0.$$

Therefore also SOSC is not fulfilled.

Finally, $x \in (0, b)$ leads to KKT point $(x, 0, 0)$, realize that $I_g(x) = \emptyset$. Then

RF1. SC is fulfilled because both constraints are inactive and the corresponding multipliers are equal to zero.

RF2. LI is fulfilled, there is no active constraint.

RF3. SOSC: We have that $I_g(0) = \emptyset$,

$$Z(x) = \{z \neq 0\} \neq \emptyset,$$

and

$$\nabla^2_{xx} L(x, u, v) = 0,$$

thus it cannot hold that

$$z\, \nabla^2_{xx} L(b, 0, 0)\, z > 0.$$

Therefore SOSC is not fulfilled.

We can conclude that for $p = 0$ we cannot verify the stability of the problem.

3. $p < 0$: Left to the readers.
. $\square$

Example 2.2 *Let t be a real parameter. Consider*

$$\begin{aligned} &\min\ x\, y - x^2 \\ &\text{s.t. } x \geq t, \\ &\qquad y \leq 1. \end{aligned}$$

Elaborate the stability of the problem with respect to t.

Solution: First, realize that the problem is nonconvex and globally unbounded. So we will investigate stability of local optima. Formulate the Lagrange function

$$L(x, y, u, v) = x\, y - x^2 + u(t - x) + v(y - 1), \;\; u, v \geq 0,$$

and the Karush–Kuhn–Tucker conditions:

i. feasibility: $x \geq t$, $y \leq 1$,

ii. complementarity: $u(t - x) = 0$ $v(y - 1) = 0$, $u, v \geq 0$,

iii. optimality:

$$\frac{\partial\, L(x, y, u, v)}{\partial x} = y - 2x - u = 0,$$
$$\frac{\partial\, L(x, y, u, v)}{\partial y} = x + v = 0,$$

Now, we will verify the assumptions of the Robinson–Fiacco theorem for KKT point(s), which we obtain for particular value of parameter t, namely (SC), (LI) and (SOSC).

1. $t = 0$: First, let $u > 0$. Then from ii) $x = 0$ and from iii) $v = 0$ and $y = u$. From feasibility and assumptions we have $0 < y = u \leq 1$. So we have KKT points $(0, y, y, 0)$ for arbitrary $y \in (0, 1]$. Then

RF1. SC: If $y \in (0, 1)$, then (SC) is fulfilled. However, if $y = 1$ then the second constraint is active and the corresponding multiplier is equal to zero $v = 0$, i.e. (SC) does not hold.

RF2. LI: If $y \in (0, 1)$, then $I_g(0, y) = \{1\}$ and $\nabla g_1 = (-1, 0)^T$, i.e. (LI) is fulfilled. If $y = 1$, then $I_g(0, 1) = \{1, 2\}$, $\nabla g_2 = (0, 1)^T$ and (LI) is also fulfilled.

RF3. SOSC: If $y \in (0, 1)$, then $I_g(0, y) = I_g^+(0, y) = \{1\}$ and

$$Z(0, y) = \{z \neq 0 : \ -z_1 = 0\} = \{(0, z_2) : \ z_2 \neq 0\} \neq \emptyset,$$

and

$$\nabla^2_{xx} L(x, y, u, v) = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix},$$

i.e.

$$z^T \, \nabla^2_{xx} L(0, y, y, 0) \, z = 0, \ \ z \in Z(0, y).$$

Therefore (SOSC) is not fulfilled.
If $y = 1$, then $I_g^+(0, 1) = \{1\}$ and $I_g^0(0, 1) = \{2\}$ and

$$Z(0, 1) = \{z \neq 0 : \ -z_1 = 0, \ z_2 \leq 0\} = \{(0, z_2) : \ z_2 < 0\} \neq \emptyset,$$

and again it holds

$$z^T \, \nabla^2_{xx} L(0, 1, 1, 0) \, z = 0, \ \ z \in Z(0, 1),$$

i.e. (SOSC) is not fulfilled.

Now, still we consider $t = 0$ and let $u = 0$. Then, $v > 0$ leads to a contradiction, because from ii) we have $x = -v < 0$ and at the same time from i) $x \geq 0$. So we can only consider $u = v = 0$. Then, from iii) we have $x = y = 0$, i.e. we have obtained the KKT point $(0, 0, 0, 0)$.

RF1. SC: The first constraint is active and the corresponding multiplier is equal to zero $u = 0$, i.e. (SC) does not hold.

RF2. LI: $I_g(0, 0) = \{1\}$ and $\nabla g_1 = (-1, 0)^T$, i.e. (LI) is fulfilled.

RF3. SOSC: $I_g(0, 0) = I_g^0(0, 0) = \{1\}$ and

$$Z(0, 0) = \{z \neq 0 : \ -z_1 \leq 0\} \neq \emptyset,$$

and if we take $z_1 > 0$ and $z_2 < 0$, then it holds

$$z^T \, \nabla^2_{xx} L(0, 0, 0, 0) \, z = -2z_1^2 + 2z_1 z_2 < 0,$$

i.e. (SOSC) is not fulfilled.

2. $t > 0$: Left to the readers (there is no KKT point).
3. $t < 0$: We elaborate the complementarity conditions:
3a. $u = 0$, $v = 0$: from iii) we get $x = y = 0$, which is feasible, thus we have KKT point $(0, 0, 0, 0)$. Then

RF1. SC: Both constraints are inactive and the corresponding multipliers are equal to zero, i.e. (SC) holds.

RF2. LI: $I_g(0, 0) = \emptyset$, therefore (LI) is fulfilled.

RF3. SOSC: $I_g(0, 0) = \emptyset$ implies

$$Z(0, 0) = \{z \neq 0\} \neq \emptyset,$$

and if we take $z_1 > 0$ and $z_2 < 0$, then it holds

$$z^T \nabla^2_{xx} L(0, 0, 0, 0)\, z = -2z_1^2 + 2z_1 z_2 < 0,$$

i.e. (SOSC) is not fulfilled.

3b. $u = 0$, $y = 1$: Left to the readers.
3c. $x = t$, $v = 0$: Left to the readers.
3d. $x = t$, $y = 1$: Using iii), we get $u = 1 - 2t$ and $v = -t$. Since $t < 0$, both multipliers are nonnegative, thus we have obtained KKT points

$$(t, 1, 1 - 2t, -t).$$

Then

RF1. SC: Both constraints are active and the corresponding multipliers are positive, i.e. (SC) holds.

RF2. LI: $I_g(t, 1) = \{1, 2\}$, and $\nabla g_1 = (-1, 0)^T$, $\nabla g_2 = (0, 1)^T$ therefore (LI) is fulfilled.

RF3. SOSC: $I_g^+(0, 0) = \{1, 2\}$ and

$$Z(0, 0) = \{z \neq 0 : \; -z_1 = 0, \; z_2 = 0\} = \emptyset,$$

i.e. (SOSC) is fulfilled.

We can conclude that the problem is stable on a neighborhood of the point $(t, 1)$ for arbitrary $t < 0$.

Example 2.3 *Let t be a real parameter. Consider*

$$\begin{aligned} \min\ & (x_1 - t)^2 + (x_2 + 1)^2 \\ \text{s.t.}\ & -x_1 + x_2 \geq 0, \\ & x_1 + x_2 \geq 0. \end{aligned}$$

Elaborate the stability of the problem on the neighborhood of $t = 1$.

HW

Example 2.4 *Consider problem from Example 1.1. Apply the postoptimization approach to add the constraint*

$$x_1 \geq \frac{1}{2}.$$