An introduction to Benders decomposition

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Computational Aspects of Optimization

Content

Benders decomposition

Benders decomposition can be used to solve:

- linear programming
- mixed-integer (non)linear programming
- two-stage stochastic programming (L-shaped algorithm)
- multistage stochastic programming (Nested Benders decomposition)

Benders decomposition for two-stage linear programming problems

$$\begin{aligned} \min\ & c^T x + q^T y \\ \text{s.t. } & Ax = b, \\ & Tx + Wy = h, \\ & x \geq 0, \\ & y \geq 0. \end{aligned} \tag{1}$$

ASS. $\mathcal{B}_1 := \{x : \ Ax = b, x \geq 0\}$ is bounded and the problem has an optimal solution.

Benders decomposition

We define the **recourse function** (second-stage value function, slave problem)

$$f(x) \quad = \quad \min\{q^T y : \ Wy = h - Tx, \ y \geq 0\} \tag{2}$$

If for some x is $\{y : \ Wy = h - Tx, \ y \geq 0\} = \emptyset$, then we set $f(x) = \infty$. The recourse function is piecewise linear, convex, and bounded below ...

Benders decomposition

Proof (outline):

- **bounded below and piecewise linear (affine)**: There are finitely many optimal basis B chosen from W such that

$$f(x) = q_B^T B^{-1}(h - Tx),$$

where feasibility $B^{-1}(h - Tx) \geq 0$ is fulfilled for $x \in \mathcal{B}_1$. Optimality condition $q_B^T B^{-1} W - q \leq 0$ does not depend on x.

Benders decomposition

Proof (outline):

- **convex**: let $x_1, x_2 \in \mathcal{B}_1$ and y_1, y_2 be such that $f(x_1) = q^T y_1$ and $f(x_2) = q^T y_2$. For arbitrary $\lambda \in (0,1)$ and $x = \lambda x_1 + (1-\lambda)x_2$ we have
$$\lambda y_1 + (1-\lambda)y_2 \in \{y : \ Wy = h - Tx, \ y \geq 0\},$$
i.e. the convex combination of y's is feasible. Thus we have
$$\begin{aligned} f(x) &= \min\{q^T y : \ Wy = h - Tx, \ y \geq 0\} && (3)\\ &\leq q^T(\lambda y_1 + (1-\lambda)y_2) = \lambda f(x_1) + (1-\lambda) f(x_2). && (4) \end{aligned}$$

Benders decomposition

We have an equivalent NLP problem

$$\begin{aligned} \min\ & c^T x + f(x) \\ \text{s.t. } & Ax = b, \\ & x \geq 0. \end{aligned} \tag{5}$$

We solve the master problem (first-stage problem)

$$\begin{aligned} \min\ & c^T x + \theta \\ \text{s.t. } & Ax = b, \\ & f(x) \leq \theta, \\ & x \geq 0. \end{aligned} \tag{6}$$

We would like to approximate $f(x)$ (from below) ...

Algorithm – the feasibility cut

Solve

$$f(\hat{x}) = \min\{q^T y : \; Wy = h - T\hat{x}, \; y \geq 0\} \tag{7}$$
$$= \max\{(h - T\hat{x})^T u : \; W^T u \leq q\}. \tag{8}$$

If the dual problem is unbounded (primal is infeasible), then there exists a growth direction $\tilde{u}$ such that $W^T \tilde{u} \leq 0$ and $(h - T\hat{x})^T \tilde{u} > 0$. For any feasible x there exists some $y \geq 0$ such that $Wy = h - Tx$. If we multiply it by $\tilde{u}$

$$\tilde{u}^T (h - T\hat{x}) = \tilde{u}^T Wy \leq 0,$$

which has to hold for any feasible x, but is violated by $\hat{x}$. Thus by

$$\tilde{u}^T (h - Tx) \leq 0$$

the infeasible $\hat{x}$ is cut off.

Algorithm – the optimality cut

There is an optimal solution $\hat{u}$ of the dual problem such that

$$f(\hat{x}) = (h - T\hat{x})^T \hat{u}.$$

For arbitrary x we have

$$\begin{aligned} f(x) &= \sup_u \{(h - Tx)^T u : \ W^T u \leq q\}, & (9) \\ &\geq (h - Tx)^T \hat{u}, & (10) \end{aligned}$$

because $\hat{u}$ is feasible for arbitrary x. From inequality $f(x) \leq \theta$ we have the optimality cut

$$\hat{u}^T (h - Tx) \leq \theta.$$

If this cut is fulfilled for actual $(\hat{x}, \hat{\theta})$, then STOP, $\hat{x}$ is an optimal solution.

Algorithm – master problem

We solve the **master problem with cuts**

$$\begin{aligned} \min\ & c^T x + \theta \\ \text{s.t. } & Ax = b, \\ & \tilde{u}_l^T (h - Tx) \le 0,\ \ l = 1, \dots, L, \\ & \tilde{u}_k^T (h - Tx) \le \theta,\ \ k = 1, \dots, K, \\ & x \ge 0. \end{aligned} \tag{11}$$

Algorithm

0. INIC: Set $\theta = -\infty$, $L = 0$, $K = 0$.
1. Solve the **master problem** to obtain $(\hat{x}, \hat{\theta})$.
2. For $\hat{x}$, solve the **dual of the second-stage** (recourse) problem to obtain
    - a direction of unbounded decrease (feasibility cut), $L = L + 1$,
    - or an optimal solution (optimality cut), $K = K + 1$.
3. STOP, if the current solution $(\hat{x}, \hat{\theta})$ fulfills the optimality cuts. Otherwise GO TO Step 1.

Convergence of the algorithm

There are finitely many extreme directions that can generate the feasibility cuts and finitely many (dual) feasible basis which can produce the optimality cuts.

Let (x^*, θ^*) be an optimal solution of the reformulated original problem.

1. The feasibility set of the master problem (6) is always contained in the feasibility set of the master problem with cuts (11) (no feasible solutions are cut).
2. The optimal solution $(\hat{x}, \hat{\theta})$ obtained by the algorithm is feasible for the master problem (6), because

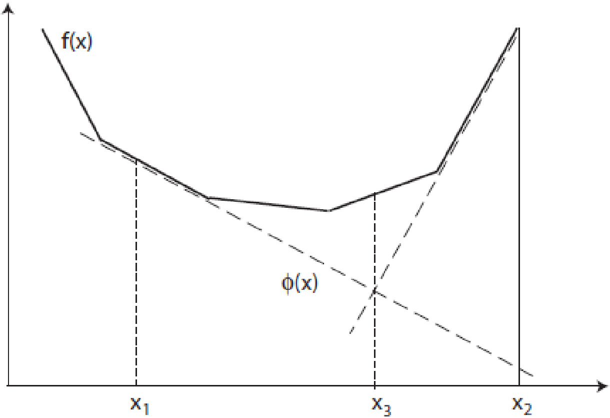
$$\hat{\theta} \geq (h - T\hat{x})^T \hat{u} = f(\hat{x}).$$

Thus, from 1. and 2. we obtain

$$c^T x^* + \theta^* \geq c^T \hat{x} + \hat{\theta} \geq c^T x^* + \theta^*.$$

Kall and Mayer (2005), Proposition 2.19

Benders optimality cuts



Kall and Mayer (2005)

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Example

$$\begin{aligned} \min\ & 2x + 2y_1 + 3y_2 \\ \text{s.t.}\ & x + y_1 + 2y_2 = 3, \\ & 3x + 2y_1 - y_2 = 4, \\ & x,\ y_1,\ y_2 \geq 0. \end{aligned} \tag{12}$$

Example

Recourse function

$$\begin{aligned} f(x) = \min\ & 2y_1 + 3y_2 \\ \text{s.t. } & y_1 + 2y_2 = 3 - x, \\ & 2y_1 - y_2 = 4 - 3x, \\ & y_1,\ y_2 \geq 0. \end{aligned} \tag{13}$$

Iteration 1

Set $\theta = -\infty$ and solve master problem

$$\min_x 2x \text{ s.t. } x \geq 0. \tag{14}$$

Optimal solution $\hat{x} = 0$.

Iteration 1

Solve the dual problem for $\hat{x} = 0$:

$$\begin{aligned} \max_{u}\ & (3-x)u_1 + (4-3x)u_2 \\ \text{s.t.}\ & u_1 + 2u_2 \le 2, \\ & 2u_1 - u_2 \le 3. \end{aligned} \tag{15}$$

Optimal solution is $\hat{u} = (8/5, 1/5)$ with optimal value $28/5$, thus no feasibility cut is necessary. We can construct an optimality cut

$$(3-x)8/5 + (4-3x)1/5 = 28/5 - 11/5x \le \theta.$$

Iteration 2

Add the optimality cut and solve

$$\begin{aligned} \min_{x,\theta}\ & 2x \\ \text{s.t. } & 28/5 - 11/5x \le \theta, \\ & x \ge 0. \end{aligned} \tag{16}$$

Optimal solution $(\hat{x}, \hat{\theta}) = (2.5455, 0)$ with optimal value 5.0909.

Iteration 2

Solve the dual problem for $\hat{x} = 2.5455$:

$$\begin{aligned} \max_u \ & (3-x)u_1 + (4-3x)u_2 \\ \text{s.t. } & u_1 + 2u_2 \le 2, \\ & 2u_1 - u_2 \le 3. \end{aligned} \tag{17}$$

Optimal solution is $\hat{u} = (1.5, 0)$ with optimal value 0.6818, thus no feasibility cut is necessary. We can construct an optimality cut

$$(3-x)1.5 + (4-3x)0 = 4.5 - 1.5x \le \theta.$$

Iteration 3

Add the optimality cut and solve

$$\begin{aligned} \min_{x,\theta}\ & 2x \\ \text{s.t.}\ & 28/5 - 11/5x \le \theta, \\ & 4.5 - 1.5x \le \theta, \\ & x \ge 0. \end{aligned} \tag{18}$$

...

Content

Two-stage stochastic programming problem

Probabilities $0 < p_s < 1$, $\sum_s p_s = 1$,

$$
\begin{aligned}
\min\ & c^T x + \sum_{s=1}^{S} p_s q_s^T y_s \\
\text{s.t. } & Ax = b, \\
& \begin{array}{ccccccc}
Wy_1 & & & +T_1 x & = & h_1, \\
& Wy_2 & & +T_2 x & = & h_2, \\
& & \ddots & \vdots & \vdots & \vdots \\
& & Wy_S & +T_S x & = & h_S,
\end{array} \\
& x \geq 0,\ y_s \geq 0,\ s = 1, \dots, S.
\end{aligned}
\tag{19}
$$

One master and S “second-stage” problems – apply the dual approach to each of them.

Minimization of Conditional Value at Risk

If the distribution of R_i is discrete with realizations r_{is} and probabilities $p_s = 1/S$, then we can use **linear programming** formulation

$$\min_{\xi, x_i} \xi + \frac{1}{(1-\alpha)S} \sum_{s=1}^{S} [- \sum_{i=1}^{n} x_i r_{is} - \xi]_+,$$

$$\text{s.t.} \sum_{i=1}^{n} x_i \overline{R}_i \geq r_0,$$

$$\sum_{i=1}^{n} x_i = 1, \; x_i \geq 0,$$

where $\overline{R}_i = 1/S \sum_{s=1}^{S} r_{is}$, $[\cdot]_+ = \max\{\cdot, 0\}$.

Conditional Value at Risk

Master problem

$$\min_{\xi, x_i} \xi + \frac{1}{(1-\alpha)S} \sum_{s=1}^{S} f_s(x,\xi),$$

$$\text{s.t. } \sum_{i=1}^{n} x_i \overline{R}_i \geq r_0, \ \sum_{i=1}^{n} x_i = 1, \ x_i \geq 0,$$

Second-stage problems

$$f_s(x,\xi) = \min_{y} \ y,$$

$$\text{s.t. } y \geq -\sum_{i=1}^{n} x_i r_{is} - \xi,$$

$$y \geq 0.$$

Solve the dual problems quickly ..

Literature

- L. Adam: Nelinearity v úlohách stochastického programování: aplikace na řízení portfolia. Diplomová práce MFF UK, 2011. (IN CZECH)
- J.F. Benders (1962): Partitioning procedures for solving mixed-variables programming problems, Numerische Mathematik 4(3), 238–252.
- P. Kall, J. Mayer: Stochastic Linear Programming: Models, Theory, and Computation. Springer, 2005.