An introduction to Benders decomposition

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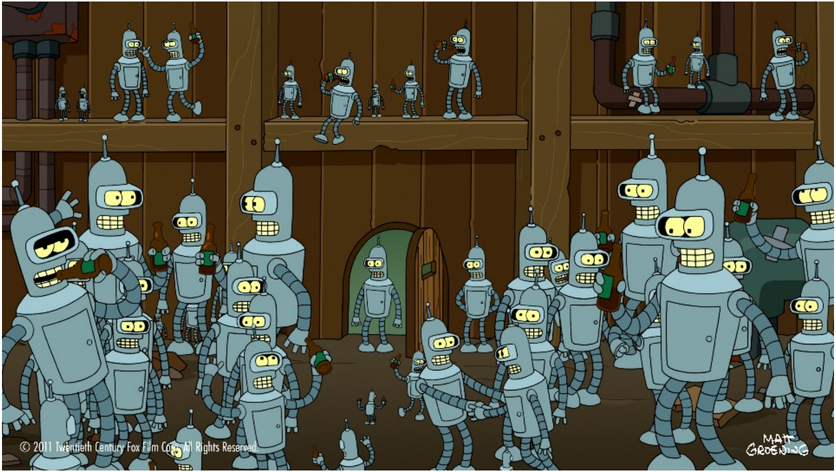
Computational Aspects of Optimization

Benders decomposition



Benders decomposition

Benders decomposition can be used to solve:

- linear programming
- mixed-integer (non)linear programming
- two-stage stochastic programming (L-shaped algorithm)
- multistage stochastic programming (Nested Benders decomposition)

Benders decomposition for two-stage linear programming problems

$$\begin{aligned} \min c^T x &+ q^T y && (1)\\ \text{s.t. } Ax &= b, && (2)\\ Tx + Wy &= h, && (3)\\ x &\geq 0, && (4)\\ y &\geq 0. && (5) \end{aligned}$$

ASS. $\mathcal{B}_1 := \{x : Ax = b, x \geq 0\}$ is bounded and the problem has an optimal solution.

We define the **recourse function** (second-stage function)

$$f(x) \quad = \quad \min\{q^T y : \; Wy = h - Tx, \; y \geq 0\} \tag{6}$$

If for some x is $\{y : \; Wy = h - Tx, \; y \geq 0\} = \emptyset$, then we set $f(x) = \infty$.
The recourse function is piecewise linear, convex, and bounded below ...

Proof (outline):

- **bounded below and piecewise linear**: There are finitely many optimal basis B chosen from W such that

$$f(x) = q_B^T B^{-1}(h - Tx),$$

where feasibility $B^{-1}(h - Tx) \geq 0$ is fulfilled for $x \in \mathcal{B}_1$. Optimality condition $q_B^T B^{-1} W - q \leq 0$ does not depend on x.

Proof (outline):

- **convex**: let $x_1, x_2 \in \mathcal{B}_1$ and y_1, y_2 be such that $f(x_1) = q^T y_1$ and $f(x_2) = q^T y_2$. For arbitrary $\lambda \in (0,1)$ and $x = \lambda x_1 + (1-\lambda)x_2$ we have
$$\lambda y_1 + (1-\lambda)y_2 \in \{y : \ Wy = h - Tx, \ y \geq 0\},$$
i.e. the convex combination of y's is feasible. Thus we have
$$\begin{aligned} f(x) &= \min\{q^T y : \ Wy = h - Tx, \ y \geq 0\} && (7)\\ &\leq q^T(\lambda y_1 + (1-\lambda)y_2) = \lambda f(x_1) + (1-\lambda) f(x_2). && (8) \end{aligned}$$

A simple example

$$
\begin{aligned}
\min\ & 2x + 2y_1 + 3y_2 \\
\text{s.t. } & x + y_1 + 2y_2 = 3, \\
& 3x + 2y_1 - y_2 = 4, \\
& x, y_1, y_2 \geq 0.
\end{aligned} \tag{9}
$$

We have an equivalent NLP problem

$$\begin{aligned} \min c^T x &+ f(x) && (10)\\ \text{s.t. } Ax &= b, && (11)\\ x &\geq 0. && (12) \end{aligned}$$

We solve the master problem (first-stage problem)

$$\begin{aligned} \min c^T x &+ \theta && (13)\\ \text{s.t. } Ax &= b, && (14)\\ f(x) &\leq \theta, && (15)\\ x &\geq 0. && (16) \end{aligned}$$

We would like to approximate $f(x)$ (from below) ...

Algorithm – the feasibility cut

Solve

$$f(\hat{x}) = \min\{q^T y : \; Wy = h - T\hat{x}, \; y \geq 0\} \quad (17)$$
$$= \max\{(h - T\hat{x})^T u : \; W^T u \leq q\}. \quad (18)$$

If the dual problem is unbounded (primal is infeasible), then there exists a growth direction $\tilde{u}$ such that $W^T\tilde{u} \leq 0$ and $(h - T\hat{x})^T\tilde{u} > 0$. For any feasible x there exists some $y \geq 0$ such that $Wy = h - Tx$. If we multiply it by $\tilde{u}$

$$\tilde{u}^T(h - T\hat{x}) = \tilde{u}^T Wy \leq 0,$$

which has to hold for any feasible x, but is violated by $\hat{x}$. Thus by

$$\tilde{u}^T(h - Tx) \leq 0$$

the infeasible $\hat{x}$ is cut off.

Algorithm – the optimality cut

There is an optimal solution $\hat{u}$ of the dual problem such that

$$f(\hat{x}) = (h - T\hat{x})^T \hat{u}.$$

For arbitrary x we have

$$\begin{aligned} f(x) &= \sup_u \{(h - Tx)^T u : \; W^T u \leq q\}, && (19) \\ &\geq (h - Tx)^T \hat{u}. && (20) \end{aligned}$$

From inequality $f(x) \leq \theta$ we have the optimality cut

$$\hat{u}^T (h - Tx) \leq \theta.$$

If this cut is fulfilled for actual $(\hat{x}, \hat{\theta})$, then STOP, $\hat{x}$ is an optimal solution.

Algorithm – master problem

We solve the master problem with cuts

$$
\begin{aligned}
\min c^T x \quad &+ \quad \theta && (21)\\
\text{s.t. } Ax \quad &= \quad b, && (22)\\
\tilde{u}_l^T(h - Tx) \quad &\leq \quad 0, \; l = 1, \ldots, L && (23)\\
\tilde{u}_k^T(h - Tx) \quad &\leq \quad \theta, \; k = 1, \ldots, K, && (24)\\
x \quad &\geq \quad 0. && (25)
\end{aligned}
$$

Algorithm

0. INIC: Set $\theta = -\infty$.
1. Solve the master problem to obtain $\hat{x}$, $\hat{\theta}$.
2. For $\hat{x}$, solve the dual of the second-stage (recourse) problem to obtain a direction (feasibility cut) or an optimal solution (optimality cut).
3. STOP, if the current solution $\hat{x}$ fulfills the optimality cut. Otherwise GO TO Step 1.

(Generalization with lower and upper bounds.)

Convergence of the algorithm

Convergence of the algorithm: see Kall and Mayer (2005), Proposition 2.19.

Example

$$
\begin{aligned}
\min\ & 2x + 2y_1 + 3y_2 \\
\text{s.t. } & x + y_1 + 2y_2 = 3, \\
& 3x + 2y_1 - y_2 = 4, \\
& x,\ y_1,\ y_2 \geq 0.
\end{aligned}
\tag{26}
$$

Example

Recourse function

$$
\begin{aligned}
f(x) = \min\ & 2y_1 + 3y_2 \\
\text{s.t. } & y_1 + 2y_2 = 3 - x, \\
& 2y_1 - y_2 = 4 - 3x, \\
& y_1,\ y_2 \geq 0.
\end{aligned}
\tag{27}
$$

Iteration 1

Set $\theta = -\infty$ and solve master problem

$$\min_x 2x \text{ s.t. } x \geq 0. \tag{28}$$

Optimal solution $\hat{x} = 0$.

Iteration 1

Solve the dual problem for $\hat{x} = 0$:

$$\begin{aligned} \max_u \ & (3 - x)u_1 + (4 - 3x)u_2 \\ \text{s.t. } & u_1 + 2u_2 \leq 2, \\ & 2u_1 - u_2 \leq 3. \end{aligned} \tag{29}$$

Optimal solution is $\hat{u} = (8/5, 1/5)$ with optimal value $28/5$, thus no feasibility cut is necessary. We can construct an optimality cut

$$(3 - x)8/5 + (4 - 3x)1/5 = 28/5 - 11/5x \leq \theta.$$

Add the optimality cut and solve

$$\begin{aligned} \min_{x,\theta}\ & 2x \\ \text{s.t. } & 28/5 - 11/5x \le \theta, \\ & x \ge 0. \end{aligned} \tag{30}$$

Optimal solution $(\hat{x}, \hat{\theta}) = (2.5455, 0)$ with optimal value 5.0909.

Iteration 2

Solve the dual problem for $\hat{x} = 2.5455$:

$$\begin{aligned} \max_u \; & (3 - x)u_1 + (4 - 3x)u_2 \\ \text{s.t. } & u_1 + 2u_2 \le 2, \\ & 2u_1 - u_2 \le 3. \end{aligned} \tag{31}$$

Optimal solution is $\hat{u} = (1.5, 0)$ with optimal value 0.6818, thus no feasibility cut is necessary. We can construct an optimality cut

$$(3 - x)1.5 + (4 - 3x)0 = 4.5 - 1.5x \le \theta.$$

Iteration 3

Add the optimality cut and solve

$$\begin{aligned} \min_{x,\theta}\ & 2x \\ \text{s.t. } & 28/5 - 11/5x \le \theta, \\ & 4.5 - 1.5x \le \theta, \\ & x \ge 0. \end{aligned} \tag{32}$$

...

Two-stage stochastic programming problem

Probabilities $0 < p_s < 1$, $\sum_s p_s = 1$,

$$
\begin{array}{rrrrrrl}
 & & & \min c^T x + \sum_{s=1}^{S} p_s q_s^T y_s & & & \\
 & & & & & & \text{s.t.} \\
 & & & & Ax & = & b, \\
Wy_1 & & & & +T_1 x & = & h_1, \\
 & Wy_2 & & & +T_2 x & = & h_2, \\
 & & \ddots & & \vdots & \vdots & \vdots \\
 & & & Wy_S & +T_S x & = & h_S, \\
 & & & & x \geq 0, & & y_s \geq 0.
\end{array}
$$

One master and S "second-stage" problems.

If the distribution of R_i is discrete with realizations r_{is} and probabilities $p_s = 1/S$, then we can use **linear programming** formulation

$$\min_{\xi, x_i} \xi + \frac{1}{(1-\alpha)S} \sum_{s=1}^{S} [-\sum_{i=1}^{n} x_i r_{is} - \xi]_+,$$

$$\text{s.t. } \sum_{i=1}^{n} x_i \overline{R}_i \geq r_0,$$

$$\sum_{i=1}^{n} x_i = 1, \; x_i \geq 0,$$

where $\overline{R}_i = 1/S \sum_{s=1}^{S} r_{is}$.

Conditional Value at Risk

Master problem

$$\min_{\xi, x_i} \xi + \frac{1}{(1-\alpha)S} \sum_{s=1}^{S} f_s(x,\xi),$$

$$\text{s.t. } \sum_{i=1}^{n} x_i \overline{R}_i \geq r_0, \ \sum_{i=1}^{n} x_i = 1, \ x_i \geq 0,$$

Second-stage problems

$$f_s(x,\xi) = \min_{y} \ y,$$

$$\text{s.t. } y \geq -\sum_{i=1}^{n} x_i r_{is} - \xi,$$

$$y \geq 0.$$

Solve the dual problem quickly ..

Literature

- L. Adam: Nelinearity v úlohách stochastického programování: aplikace na řízení portfolia. Diplomová práce MFF UK, 2011. (IN CZECH)
- P. Kall, J. Mayer: Stochastic Linear Programming: Models, Theory, and Computation. Springer, 2005.