Life Insurance 2 – exercises

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3 Multiple decrements

Please read the theory in Chapter 7 of Gerber book. You can[1] also solve the examples in Appendix C.7.

Notation

- J – cause of decrement (death, disability, cancelation, ...),
- m – number of decrements,
- $T = T_x$ – time of leaving the original status due to one of mutually exclusive decrements,
- ${}_tq_{j,x} = P(T < t,\ J = 1)$ – probability of leaving the original status due to decrement j,
- ${}_tp_x = 1 - \sum_{j=1}^{m} {}_tq_{j,x}$ – probability of surviving in the original status,
- $\mu_{j,x+t}$ – force of decrement j,
- $\mu_{x+t} = \sum_{j=1}^{m} \mu_{j,x+t}$,
- $g_j(t) = {}_tp_x \cdot \mu_{j,x+t}$ – pdf,
- $q_{j,x+k} = P(T < k+1,\ J = 1 \mid T > k)$,
- $q_{x+k} = \sum_{j=1}^{m} q_{j,x+k}$,
- $p_{x+k} = 1 - q_{x+k}$,

Example 3.1 *Consider term insurance for n years which provides SI to a beneficiary in the case of death of the insured person and $2SI$ in the case of death by accident. Derive the net single premium if the death benefit is paid*

a) at the end of the year of death,

b) immediately on death.

Derive the net annual premium which is paid during the whole contract life/until death.

Solution: We define two causes of decrement: $J = 1$ for death by accident, $J = 2$ for death from other causes. Then the net single premium is equal to

[1] It is not mandatory, but it can help you.

a)

$$\text{NSP} = \sum_{k=0}^{n-1} 2SI\ v^{k+1}\ {}_kp_x\ q_{1,x+k} + \sum_{k=0}^{n-1} SI\ v^{k+1}\ {}_kp_x\ q_{2,x+k},$$

b)

$$\text{NSP} = \int_0^n 2SI\ v^t\ {}_tp_x\ \mu_{1,x+t}\ dt + \int_0^n SI\ v^t\ {}_tp_x\ \mu_{2,x+t}\ dt.$$

The net annual premium can be derived in the standard form (using the equivalence principle)

$$\text{NAP} = \frac{\text{NSP}}{\ddot{a}_{x\overline{n}|}},$$

where $\ddot{a}_{x\overline{n}|}$ is the NSP for the standard life annuity due (=based on life tables without special decrements, i.e. with death only) for n years.

Example 3.2 *Consider special life tables with two causes of decrement for extreme sports and four races:*

year	*prob. of death*	*prob. of disability*	*survival prob.*
0	*0.15*	*0.25*	*0.60*
1	*0.10*	*0.20*	*0.70*
2	*0.05*	*0.15*	*0.80*
3	*0.00*	*0.10*	*0.90*

1000 extreme (iid) sportsmen start. Compute/estimate

a) *expected number and variance of survivors over four races,*

b) *expected number and variance of deaths during four races,*

c) *distribution of the causes of decrement J,*

d) *conditional distribution of ending the season during third race.*

Solution: We will consider three (!) causes of decrement: $J = 1$ death, $J = 2$ disability, $J = 3$ finishing the season. In this case, we must slightly modify our table[2]

k	$q_{1,k}$	$q_{2,k}$	$q_{3,k}$	p_k
0	0.15	0.25	0	0.60
1	0.10	0.20	0	0.70
2	0.05	0.15	0	0.80
3	0.00	0.10	0.90	0

a) Let X_1 denote the r.v. of the number of survivors over four races. We can solve it directly or we can realize that we work with binomial distribution with parameters

$$n = 1000,\ p^{(a)} = p_0\,p_1\,p_2\,q_{3,3} = 0.3024.$$

Therefore we obtain $E[X_1] = np^{(a)} = 302.4$ and $\text{var}(X_1) = np^{(a)}(1 - p^{(a)}) = 210.95$.

[2] You must be always sure that you are working with probability distribution.

b) Let X_2 denote the r.v. of the number of deaths over four races. Again we can employ the binomial distribution

$$n = 1000,\ p^{(b)} = q_{1,0} + p_0\, q_{1,1} + p_0\, p_1\, q_{1,2} + p_0\, p_1\, p_2\, q_{1,3} = 0.231.$$

Therefore we obtain $E[X_2] = np^{(b)} = 231$ and $\mathrm{var(X_2)} = np^{(b)}(1 - p^{(b)}) = 177,64$.

c) We have already some of the marginal probilities, namely $P(J = 1) = p^{(b)}$ and $P(J = 3) = p^{(a)}$. Therefore

$$P(J = 2) = 1 - p^{(a)} - p^{(b)} = q_{2,0} + p_0\, q_{2,1} + p_0\, p_1\, q_{2,2} + p_0\, p_1\, p_2\, q_{2,3} = 0.4666.$$

d) In general, we can derive the conditional probabilities as

$$P(J = j|K = k) = \frac{P(J = j, K = k)}{P(K = k)} = \frac{{}_k p_x\, q_{j,x+k}}{{}_k p_x\, q_{x+k}} = \frac{q_{j,x+k}}{q_{x+k}}.$$

Thus, in our case we have

$$\begin{aligned} P(J = 1|K = 2) &= \frac{q_{1,2}}{q_2} = \frac{0.05}{0.20} = 0.25, \\ P(J = 2|K = 2) &= \frac{q_{2,2}}{q_2} = \frac{0.15}{0.20} = 0.75, \\ P(J = 3|K = 2) &= \frac{q_{3,2}}{q_2} = \frac{0}{0.20} = 0. \end{aligned}$$

Example 3.3 *Pension plan for employees: Consider a person at age $x = 30$ and the following insurance:*

- *In the case of death in the original employment, there is a single payment of 5 mil. CZK to a beneficiary at the end of the year of death.*
- *If the employee stays with the same employer up to 70 years, he/she is entitled an annuity with annual payment* 300*times number of years in employment.*
- *if the employee leaves before reaching the age of 70, he/she is entitled an annuity with annual payment* 300*times number of finished years in employment with payments starting at the age of 70.*

Define a proper probabilistic model and derive a formula for the net single premium.

Solution: We can consider two causes of decrement: $J = 1$ death, $J = 2$ leaving the employer, i.e. $m = 2$. We can use slightly generalized formula for the net single premium (inspired by the general formula for net premium reserve at time 0, i.e. by ${}_0V_x$):

$$\mathrm{NSP} = \sum_{j=1}^{m} \sum_{k=0}^{\infty} c_{j,k+1}\ v^{k+1}\ {}_k p_x\ q_{j,x+k} - \sum_{k=0}^{\infty} \pi_k\ v^k\ {}_k p_x.$$

In our case, we set

- $c_{1,k+1} = 5 \cdot 10^6$ for $k = 0, \ldots, 39$, $c_{1,k+1} = 0$ for $k \geq 40$.
- $c_{2,k+1} = 300 \cdot k \cdot {}_{40-k-1|}\ddot{a}_{x+k+1}$ for $k = 0, \ldots, 39$, $c_{2,k+1} = 0$ for $k \geq 40$.
- $\pi_{40} = -300 \cdot 40 \cdot \ddot{a}_{x+40}$, $\pi_k = 0$ otherwise.

It is important to realize that the NSP for the annuities are based on the standard life tables (= with the cause of decrement death only).

We can consider also the third cause of decrement $J = 3$ for staying with the original employer up to 70 years, but the situation is then much more difficult. You must be very careful with the definition of the probability distribution, compare with Example 3.2. □

Example 3.4 *Consider a person at age $x = 30$ and the following insurance valid until reaching the age of 65: In the case of death, there is a single payment of 5 mil. CZK at the end of the year of death. In the case of disability, an annuity of 0.5 mil. CZK is paid until reaching the age 65 and then it is increased by 50 thousands CZK every year until the death. However, the payment in the case of death is not further valid. Define a proper probabilistic model and derive a formula for the net single premium.*

Solution: We can consider two causes of decrement: $J = 1$ death, and $J = 2$ disability, i.e. $m = 2$. Using the general formula, we can set

- $c_{1,k+1} = 5 \cdot 10^6$ for all $k = 0, \ldots, 34$, $c_{1,k+1} = 0$ otherwise,
- $c_{2,k+1} = 0.5 \cdot 10^6 \cdot \ddot{a}_{x+k+1} + 0.05 \cdot 10^6 \cdot {}_{35-k-1|}(I\ddot{a})_{x+k+1}$ for $k = 0, \ldots, 34$, $c_{2,k+1} = 0$ otherwise,
- $\pi_k = 0$ for all k.

The NSP for the annuities are based on the standard life tables (= with the cause of decrement death only). □

4 Construction of life tables with multiple decrements

When we want to prepare a new life insurance product based on multiple decrements, we must construct suitable life tables (with all considered causes of decrement). However, very often the causes of decrements are parametrized separately. Then, our goal is to mix them. You can find the proper way below.

Consider (continuous) compound model with one decrement where we consider only one cause of decrement. Then, the probability of surviving over time t without observing the cause of decrement j is

$${}_tp'_{j,x} = \exp\left\{-\int_0^t \mu_{j,x+s}\, ds\right\},$$

and we set

$${}_tq'_{j,x} = 1 - {}_tp'_{j,x}.$$

Example 4.1 *Show that*

a) ${}_tp_x \le {}_tp'_{j,x}$,

b) ${}_tq_x \ge {}_tq'_{j,x}$,

c) ${}_tq_{j,x} \le {}_tq'_{j,x}$.

Solution: a)

$$\begin{aligned} {}_tp_x &= \exp\left\{-\int_0^t \mu_{x+s}\,ds\right\} \\ &= \exp\left\{-\int_0^t \sum_{j=1}^m \mu_{j,x+s}\,ds\right\} \\ &= \prod_{j=1}^m \exp\left\{-\int_0^t \mu_{j,x+s}\,ds\right\} \\ &= \prod_{j=1}^m {}_tp'_{j,x}. \end{aligned}$$

Since ${}_tp'_{j,x} \in (0,1)$ for all j, we obtain the inequality.
b) It is a consequence of a) if we realize that ${}_tq_x = 1 - {}_tp_x$ and ${}_tq'_{j,x} = 1 - {}_tp'_{j,x}$.
c) It is again a consequence of a) if we realize

$$ {}_tq_{j,x} = \int_0^t {}_sp_x\,\mu_{j,x+s}\,ds \le \int_0^t {}_sp'_{j,x}\,\mu_{j,x+s}\,ds = {}_tq'_{j,x}. $$

□

Example 4.2 *Under the assumption of linearity for each cause of decrement, i.e.*

$$ {}_uq'_{j,x} = u \cdot q'_{j,x},\ \ u \in (0,1), $$

derive an exact relation between $q_{j,x}$ *and* $q'_{j,x}$.

Solution:

$$\begin{aligned} q'_{j,x} &= 1 - p'_{j,x} \\ &= 1 - \exp\left\{-\int_0^1 \mu_{j,x+u}\,du\right\} \\ &= \left[\text{lectures/Gerber formula (7.3.6) : ass. of linearity} \Rightarrow \mu_{j,x+u} = \frac{q_{j,x}}{1-u\,q_x}\right] \\ &= 1 - \exp\left\{-q_{j,x}\int_0^1 \frac{1}{1-u\,q_x}\,du\right\} \\ &= 1 - \exp\left\{-q_{j,x}\left[\frac{-1}{q_x}\ln(1-u\,q_x)\right]_0^1\right\} \\ &= 1 - \exp\left\{\frac{q_{j,x}}{q_x}\ln(1-\,q_x)\right\} \\ &= 1 - (1-\,q_x)^{\frac{q_{j,x}}{q_x}}. \end{aligned}$$

Then, we can express

$$q_{j,x} = q_x \frac{\ln(1 - q'_{j,x})}{\ln(1 - q_x)} = q_x \frac{\ln\, p'_{j,x}}{\ln\, p_x}.$$

□

Example 4.3 *Under the assumption of constant force of decrement for each cause, i.e.* $\mu_{j,x+u} = \mu_{j,x+\frac{1}{2}}$, $u \in (0,1)$, *derive an exact relation between* $q_{j,x}$ *and* $q'_{j,x}$.

Solution: The assumption stays valid also for the aggregate force of decrement, i.e. $\mu_{x+u} = \mu_{x+\frac{1}{2}}$, $u \in (0,1)$. Then

$$\begin{aligned}
q_{j,x} &= \int_0^1 {}_u p_x \cdot \mu_{j,x+u}\, du \\
&= \int_0^1 {}_u p_x \cdot \mu_{j,x+\frac{1}{2}}\, du \\
&= \frac{\mu_{j,x+\frac{1}{2}}}{\mu_{x+\frac{1}{2}}} \int_0^1 {}_u p_x \cdot \mu_{x+\frac{1}{2}}\, du \\
&= \frac{\mu_{j,x+\frac{1}{2}}}{\mu_{x+\frac{1}{2}}} \int_0^1 {}_u p_x \cdot \mu_{x+u}\, du \\
&= \frac{\mu_{j,x+\frac{1}{2}}}{\mu_{x+\frac{1}{2}}} q_j.
\end{aligned}$$

Moreover, under our assumption

$$\begin{aligned}
p'_{j,x} &= \exp\left\{ - \int_0^1 \mu_{j,x+u}\, du \right\} = \exp\left\{ -\mu_{j,x+\frac{1}{2}} \right\}, \\
p_x &= \exp\left\{ - \int_0^1 \mu_{x+u}\, du \right\} = \exp\left\{ -\mu_{x+\frac{1}{2}} \right\},
\end{aligned}$$

i.e. we get the ratio

$$\frac{\mu_{j,x+\frac{1}{2}}}{\mu_{x+\frac{1}{2}}} = \frac{-\ln\, p'_{j,x}}{-\ln\, p_x}.$$

Thus, we get the same formula as in the previous example

$$q_{j,x} = q_x \frac{\ln(1 - q'_{j,x})}{\ln(1 - q_x)} = q_x \frac{\ln\, p'_{j,x}}{\ln\, p_x}.$$

□

We can summarize the construction of multiple decrement life tables from m compound models with one decrement:

1. Compute $q'_{j,x}$ and $p'_{j,x} = 1 - q'_{j,x}$ for all j.

2. Derive
$$p_x = \prod_{j=1}^{m} p'_{j,x} \text{ and } q_x = 1 - p_x.$$

3. Apply the formula
$$q_{j,x} = q_x \frac{\ln\, p'_{j,x}}{\ln\, p_x}.$$

Example 4.4 *Using the above introduced approach, construct the multiple decrement life table with* $m = 3$ *given the following compound models with one decrement:*

x	$q'_{1,x}$	$q'_{2,x}$	$q'_{3,x}$
25	*0.020*	*0.030*	*0.200*
26	*0.022*	*0.034*	*0.100*
27	*0.028*	*0.040*	*0.120*

i.e. compute $q_{1,x}$*,* $q_{2,x}$*,* $q_{3,x}$*.*

Solution:

x	$q'_{1,x}$	$q'_{2,x}$	$q'_{3,x}$	p_x	q_x	$q_{1,x}$	$q_{2,x}$	$q_{3,x}$
25	0.020	0.030	0.200	0.760	0.240	0.018	0.027	0.195
26	0.022	0.034	0.100					
27	0.028	0.040	0.120					

□