Life Insurance 2 – exercises

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4 Multiple Life Insurance

Please read the theory in Chapter 8 of Gerber book. You can[1] also solve the examples in Appendix C.8.

We consider m *independent*[2] lives with (random) future lifetimes

$$T_1 := T_{x_1},\ \dots,\ T_m := T_{x_m}.$$

Notation

- Joint-life status
  - status $u = x_1 : x_2 : \dots : x_m$ = all m participating lives survive,
  - failure time
    $$T(u) = \min\{T_1, \dots, T_m\},$$
  - survival probability
    $${}_tp_{x_1:x_2:\dots:x_m} = P(T(u) > t) = \prod_{k=1}^{m} P(T_k > t) = \prod_{k=1}^{m} {}_tp_{x_k},$$
    and
    $${}_tq_{x_1:x_2:\dots:x_m} = 1 - {}_tp_{x_1:x_2:\dots:x_m}.$$
    Under independence, we set
    $$l_{x_1:x_2:\dots:x_m} = \prod_{k=1}^{m} l_{x_k},\ \ d_{x_1:x_2:\dots:x_m} = l_{x_1:x_2:\dots:x_m} - l_{x_1+1:x_2+1:\dots:x_m+1}.$$
    Then
    $${}_tp_{x_1:x_2:\dots:x_m} = \frac{l_{x_1+t:x_2+t:\dots:x_m+t}}{l_{x_1:x_2:\dots:x_m}}.$$
- Last-survivor status
  - status $u = \overline{x_1 : x_2 : \dots : x_m}$ = at least one of the m lives survives,
  - failure time
    $$T(u) = \max\{T_1, \dots, T_m\},$$

[1] It is not mandatory, but it can help you.

[2] The independence is quite questionable assumption, especially when we consider a family insurance. There are several approaches how to elaborate the dependence, e.g., copula functions or conditional forces of mortality.

– survival probability

$$ {}_tp_{\overline{x_1:x_2:\cdots:x_m}} = P(T(u) > t) = S_1^t - S_2^t + \cdots (-1)^{m-1} S_m^t, $$

where

$$ S_k^t = \sum_{(j_1,\ldots,j_k)\subset\{1,\ldots,m\}} {}_tp_{x_{j_1}:x_{j_2}:\cdots:x_{j_k}}, $$

and

$$ {}_tq_{\overline{x_1:x_2:\cdots:x_m}} = 1 - {}_tp_{\overline{x_1:x_2:\cdots:x_m}}. $$

Example 4.1 *Consider the following insurances for a pair of independent lifes at ages x and y:*

a) joint-life whole life insurance payable on the first death,

b) joint-life life annuity-due.

c) joint-life life annuity-due for n years.

Derive a reasonable generalization of the commutation functions which enable you to simplify the computation of the net single premiums.

Solution: a) Under the independence of lifes

$$ \begin{aligned} A_{x:y} &= \sum_{k=0}^{\infty} v^{k+1} {}_kp_{x:y} q_{x:y} \\ &= \sum_{k=0}^{\infty} v^k \frac{l_{x+k:y+k}}{l_{x:y}} \frac{d_{x+k:y+k}}{l_{x+k:y+k}} \\ &= \sum_{k=0}^{\infty} \frac{v^{f(x+k,y+k)}}{v^{f(x,y)}} \frac{d_{x+k:y+k}}{l_{x:y}}. \end{aligned} $$

There are several possible choices of f:

$$ f(x,y) = \frac{x+y}{2},\ f(x,y) = \max\{x,y\},\ f(x,y) = \min\{x,y\}. $$

On the other hand, it is not possible to use simple sum of the ages as f, because we need a transformation which preserves v^k. So, if we define the commutation functions as

$$ \begin{aligned} C_{x:y} &= v^{f(x+1,y+1)} d_{x:y}, & D_{x:y} &= v^{f(x,y)} l_{x:y}, \\ M_{x:y} &= \sum_{k=0}^{\infty} C_{x+k:y+k}, & N_{x:y} &= \sum_{k=0}^{\infty} D_{x+k:y+k}, \\ R_{x:y} &= \sum_{k=0}^{\infty} M_{x+k:y+k}, & S_{x:y} &= \sum_{k=0}^{\infty} N_{x+k:y+k}, \end{aligned} $$

we can get the standard expression for NSP

$$A_{x:y} = \sum_{k=0}^{\infty} \frac{C_{x+k:y+k}}{D_{x:y}} = \frac{M_{x:y}}{D_{x:y}}.$$

b) For the life annuity-due, we obtain

$$\begin{aligned}
\ddot{a}_{x:y} &= \sum_{k=0}^{\infty} v^k {}_k p_{x:y} \\
&= \sum_{k=0}^{\infty} v^{k+1} \frac{l_{x+k:y+k}}{l_{x:y}} \\
&= \sum_{k=0}^{\infty} \frac{v^{f(x+k+1,y+k+1)}}{v^{f(x,y)}} \frac{l_{x+k:y+k}}{l_{x:y}} \\
&= \sum_{k=0}^{\infty} \frac{D_{x+k:y+k}}{D_{x:y}} = \frac{N_{x:y}}{D_{x:y}}.
\end{aligned}$$

c) For the life annuity-due for n years, we have

$$\begin{aligned}
\ddot{a}_{x:y\overline{n}|} &= \sum_{k=0}^{n-1} v^k {}_k p_{x:y} \\
&= \sum_{k=0}^{n-1} \frac{D_{x+k:y+k}}{D_{x:y}} = \frac{N_{x:y} - N_{x+n:y+n}}{D_{x:y}}.
\end{aligned}$$

□

Remark 4.2 *The generalization of the commutation functions to m lifes is straightforward, e.g.*

$$C_{x_1:x_2:\dots:x_m} = v^{f(x_1+1,\dots,x_m+1)} d_{x_1:x_2:\dots:x_m}, \quad D_{x_1:x_2:\dots:x_m} = v^{f(x_1,\dots,x_m)} l_{x_1:x_2:\dots:x_m},$$

where

$$f(x_1,\dots,x_m) = \frac{\sum_{k=1}^m x_k}{m}, \text{or } f(x_1,\dots,x_m) = \max\{x_1,\dots,x_m\}, \text{or } f(x_1,\dots,x_m) = \min\{x_1,\dots,x_m\}.$$

Note that also the relations between CF which we know from the univariate case are valid, e.g.

$$M_{x:y} = D_{x:y} - d\,N_{x:y}.$$

Example 4.3 *Verify that*

$$\begin{aligned}
A_{x:y} &= 1 - d \cdot \ddot{a}_{x:y}, \\
A_{\overline{x:y}} &= 1 - d \cdot \ddot{a}_{\overline{x:y}}.
\end{aligned}$$

Solution: One possibility is to use the following identity

$$1 + v + \cdots + v^K = \frac{1 - v^{K+1}}{1 - v},$$

and compute the expected value with respect to one of the following distribution of curtate future lifetime K:

$$\begin{aligned} P(K = k) &= {}_k p_{x:y}\, q_{x:y}, \text{ or} \\ P(K = k) &= {}_k p_{\overline{x:y}}\, q_{\overline{x:y}}. \end{aligned}$$

Other possibilities are a direct derivation using the formula for the net single premiums, or using the relations between the commutation functions. □

Example 4.4 *Consider the following insurances for a pair of independent lifes at ages x and y:*

a) last-survival life annuity-due.

b) last-survival whole life insurance payable on the last death,

Using the above introduced commutation functions derive the net single premiums.

Solution: a)

$$\begin{aligned} \ddot{a}_{\overline{x:y}} &= \sum_{k=0}^{\infty} v^k\, {}_k p_{\overline{x:y}} \\ &= \sum_{k=0}^{\infty} v^k ({}_k p_x + {}_k p_y - {}_k p_{x:y}) \\ &= \sum_{k=0}^{\infty} v^k\, {}_k p_x + \sum_{k=0}^{\infty} v^k\, {}_k p_y - \sum_{k=0}^{\infty} v^k\, {}_k p_{x:y} \\ &= \frac{N_x}{D_x} + \frac{N_y}{D_y} - \frac{N_{x:y}}{D_{x:y}}. \end{aligned}$$

b) We can use the previous example to get

$$\begin{aligned} \ddot{A}_{\overline{x:y}} &= 1 - d\,\ddot{a}_{\overline{x:y}} \\ &= 1 + 1 - 1 - d\left(\frac{N_x}{D_x} + \frac{N_y}{D_y} - \frac{N_{x:y}}{D_{x:y}}\right) \\ &= \frac{D_x - d\,N_x}{D_x} + \frac{D_y - d\,N_y}{D_y} - \frac{D_{x:y} - d\,N_{x:y}}{D_{x:y}} \\ &= \frac{M_x}{D_x} + \frac{M_y}{D_y} - \frac{M_{x:y}}{D_{x:y}}. \end{aligned}$$

□

Example 4.5 *Consider*

a) widow's annuity-due (asymmetric) – payment stream of rate 1 starts at the death of husband x and terminates at the death of wife y.

b) widow's and widower's annuity-due (symmetric) – payment stream starts at the death of husband x or wife y and terminates at the death of wife y or husband x.

c) orphan's annuity-due – payment stream starts at the death of parents x, y and terminates at the death of child z or by reaching the age of 18.

Solution: a) Denote by u the status when wife is living and husband died

$$ {}_kp_u^{(a)} = {}_kp_y(1 - {}_kp_x). $$

Then

$$ \begin{aligned} \ddot{a}_u^{(a)} &= \sum_{k=0}^{\infty} v^k {}_kp_u^{(a)} \\ &= \sum_{k=0}^{\infty} v^k {}_kp_y(1 - {}_kp_x) \\ &= \ddot{a}_y - \ddot{a}_{x:y}. \end{aligned} $$

b) Denote by u the status when the wife is living and the husband died or vice versa

$$ {}_kp_u^{(b)} = {}_kp_y(1 - {}_kp_x) + {}_kp_x(1 - {}_kp_y). $$

Then

$$ \begin{aligned} \ddot{a}_u^{(b)} &= \sum_{k=0}^{\infty} v^k {}_kp_u^{(b)} \\ &= \sum_{k=0}^{\infty} v^k {}_kp_y(1 - {}_kp_x) + {}_kp_x(1 - {}_kp_y) \\ &= \ddot{a}_x + \ddot{a}_y - 2\ddot{a}_{x:y}. \end{aligned} $$

c) Denote by u the status when the child is living and the parents died and set $n = 18 - z$. Then

$$ {}_kp_u^{(c)} = {}_kp_z(1 - {}_kp_x)(1 - {}_kp_y), $$

and

$$ \begin{aligned} \ddot{a}_{u\overline{n}|}^{(c)} &= \sum_{k=0}^{n-1} v^k {}_kp_u^{(c)} \\ &= \sum_{k=0}^{n-1} v^k {}_kp_z(1 - {}_kp_x)(1 - {}_kp_y) \\ &= \ddot{a}_{z\overline{n}|} - \ddot{a}_{x:z\overline{n}|} - \ddot{a}_{y:z\overline{n}|} + \ddot{a}_{x:y:z\overline{n}|}. \end{aligned} $$

□

Example 4.6 *Consider orphan's annuity-due where payment stream starts at the death of parents x, y and terminates when both children z, w reach the age of 18 or at the death of last child.*