Optimization with application in finance – exercises

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HW: Examples 4.3, 4.6.

4 Chance constrained programming

Chance constrained problem

$$\min_{x \in X} f(x)$$
$$\text{s.t. } P\left(g_1(x,\xi) \le 0, \ldots, g_r(x,\xi) \le 0\right) \ge p,$$

where $p \in [0,1]$ is given level, $X \subseteq \mathbb{R}^n$ is a set of (deterministic) constraints, ξ is a real random vector on probability space $(\Omega, \mathcal{F}, P)$. If $r > 1$, then we speak about joint chance constraint, whereas $r = 1$ corresponds to individual chance constraint.

For a given $x \in X$, the probability of ξ for which the random constraint is fulfilled must be at least p:

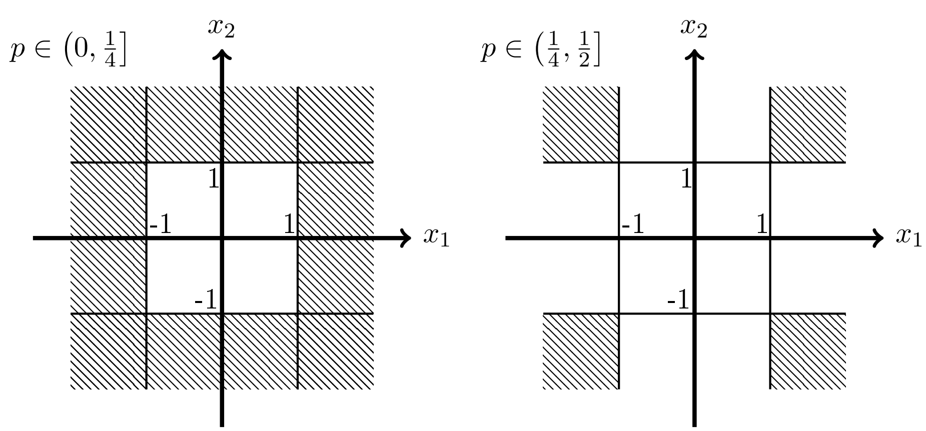
$$P\left(g_1(x,\xi) \le 0, \ldots, g_r(x,\xi) \le 0\right) = P\left(\{\xi : \ g_1(x,\xi) \le 0, \ldots, g_r(x,\xi) \le 0\}\right) \ge p.$$

Example 4.1 *Consider $\xi = (\xi_1, \xi_2, \xi_3)$ with equiprobable realizations $p_i = 1/4$: $(-1,0,-1)$, $(0,-1,-1)$, $(1,0,-1)$, $(0,1,-1)$. For $p \in [0,1]$, derive the set of feasible given by chance constraint:*

$$X(p) = \left\{(x_1,x_2) \in \mathbb{R}^2 : \ P(\xi_1 x_1 + \xi_2 x_2 \le \xi_3) \ge p\right\}.$$

Discuss the convexity of the set.

Solution: Realize that $p \in \left(0, \frac{1}{4}\right]$ means that the random constraint if fulfilled for at least one realization of random vector ξ, whereas $p \in \left(\frac{1}{4}, \frac{1}{2}\right]$ requires that the constraint is fulfilled for at least two realizations at the same time. This leads to these sets:



For $p \in \left(0, \frac{1}{4}\right]$, we have

$$X(p) = \{x_1 \geq 1\} \text{ or } \{x_2 \geq 1\} \text{ or } \{x_1 \leq -1\} \text{ or } \{x_2 \leq -1\},$$

whereas for $p \in \left(\frac{1}{4}, \frac{1}{2}\right]$

$$\begin{aligned} X(p) = &\{x_1 \geq 1, x_2 \geq 1\} \text{ or } \{x_1 \geq 1, x_2 \leq -1\} \\ &\text{or } \{x_1 \leq -1, x_2 \geq 1\} \text{ or } \{x_1 \leq -1, x_2 \leq -1\}. \end{aligned}$$

Moreover $X(0) = \mathbb{R}^2$ and $X(p) = \emptyset$ for $p \in (\frac{1}{2}, 1]$. Obviously the set is convex only in the trivial case for $p = 0$. Please realize that even though we used very simple chance constraint, the set is already nonconvex. □

Example 4.2 *Let ξ have a uniform distribution on interval $[-1, 1]$. Derive the reformulation of chance constraint*

$$P\left(\xi(x_1 - x_2) \geq \frac{1}{2}\right) \geq \frac{1}{4}.$$

Solution: We can use the explicit formula for the distribution function of ξ, i.e.

$$F(x) = \begin{cases} 0, & x \leq -1, \\ \frac{x+1}{2}, & x \in [-1, 1], \\ 1, & x \geq 1. \end{cases}$$

Assume that $x_1 \neq x_2$. If $x_1 > x_2$,

$$P\left(\xi(x_1 - x_2) \geq \frac{1}{2}\right) = 1 - P\left(\xi \leq \frac{1}{2(x_1 - x_2)}\right) = 1 - \frac{\frac{1}{2(x_1-x_2)} + 1}{2} = \frac{2(x_1 - x_2) - 1}{4(x_1 - x_2)} \geq \frac{1}{4},$$

which holds if

$$x_1 - x_2 \geq 1.$$

If $x_1 < x_2$,

$$P\left(\xi(x_1 - x_2) \geq \frac{1}{2}\right) = P\left(\xi \leq \frac{1}{2(x_1 - x_2)}\right) = \frac{\frac{1}{2(x_1-x_2)} + 1}{2} = \frac{2(x_1 - x_2) + 1}{4(x_1 - x_2)} \geq \frac{1}{4},$$

which holds if

$$x_1 - x_2 \leq -1.$$

Other cases lead to empty set, so the chance constrain is equivalent to

$$x_1 - x_2 \geq 1 \text{ or } \; x_1 - x_2 \leq -1.$$

Realize that it is a *union* of two disjunctive half-spaces. □

Example 4.3 *Let* ξ_1 *have a uniform distribution on interval* $[1,4]$, $\xi_2 \sim U[1/3,1]$ *and* ξ_1, ξ_2 *be independent. Derive the explicit form of the chance constrained problem*

$$\begin{aligned} \min_{x_{1,2}}\ & x_1 + x_2 \\ \text{s.t. } & P\left(\xi_1 x_1 + x_2 \geq 7,\ \xi_2 x_1 + x_2 \geq 4\right) \geq p, \\ & x_{1,2} \geq 0. \end{aligned}$$

Solution: Use the independence and explicit formula for cdf of the uniform distribution. Be careful with the bounds where some of the probabilities is equal to 0 or 1.

Example 4.4 *Consider chance constrained problem*

$$\begin{aligned} \max_{x_{1,2,3}}\ & x_3 \\ \text{s.t. } & P\left(\xi_1 x_1 + \xi_2 x_2 \leq x_3\right) = 0.37, \\ & P\left(3x_1 + 2x_2 \leq \xi_3\right) = 0.8, \\ & P\left(-x_1 + 4x_2 \leq \xi_4\right) = 0.9, \\ & x_{1,2} \geq 0. \end{aligned}$$

and assume that

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \sim \mathcal{N}_2\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 10 & 7 \\ 7 & 20 \end{pmatrix}\right), \quad \begin{pmatrix} \xi_3 \\ \xi_4 \end{pmatrix} \sim \mathcal{N}_2\left(\begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 & 0.4 \\ 0.4 & 1 \end{pmatrix}\right).$$

Find a nonlinear programming reformulation.

Solution: Realize that under our distributional assumption it holds

$$\xi_1 x_1 + \xi_2 x_2 \sim \mathcal{N}(-x_1 + 2x_2, 10x_1^2 + 14x_1x_2 + 20x_2^2) =: \mathcal{N}(\mu(x), \sigma^2(x)).$$

Then we have

$$\begin{aligned} P\left(\xi_1 x_1 + \xi_2 x_2 \leq x_3\right) &= P\left(\frac{\xi_1 x_1 + \xi_2 x_2 - \mu(x)}{\sigma(x)} \leq \frac{x_3 - \mu(x)}{\sigma(x)}\right) \\ &= \Phi\left(\frac{x_3 - \mu(x)}{\sigma(x)}\right), \end{aligned}$$

where Φ is cdf of $\mathcal{N}(0,1)$. Then

$$\begin{aligned} \Phi\left(\frac{x_3 - \mu(x)}{\sigma(x)}\right) = 0.38 \quad &\Leftrightarrow\ x_3 = \mu(x) + u_{0.38} \cdot \sigma(x) \\ &\Leftrightarrow\ x_3 = -x_1 + 2x_2 + u_{0.38} \cdot \sqrt{10x_1^2 + 14x_1x_2 + 20x_2^2}, \end{aligned}$$

where $u_{0.38} = \Phi^{-1}(0.38)$. We can use similar approach to the second chance constraint.

$$\begin{aligned} P\left(3x_1 + 2x_2 \leq \xi_3\right) &= P\left(\frac{3x_1 + 2x_2 - 3}{\sqrt{2}} \leq \frac{\xi_3 - 3}{\sqrt{2}}\right) \\ &= 1 - \Phi\left(\frac{3x_1 + 2x_2 - 3}{\sqrt{2}}\right) = 0.8 \\ &\Leftrightarrow 3x_1 + 2x_2 = 3 + u_{0.2} \cdot \sqrt{2}. \end{aligned}$$

The reformulation of the last constraint is analogous. Note that the dependence between ξ_3 and ξ_4 will not effect anything. □

Example 4.5 *Let $f, g(\cdot, \xi) : \mathbb{R}^n \to \mathbb{R}$ be real functions, $X \subseteq \mathbb{R}^n$, ξ be a real random vector, $p \in (0,1)$:*

$$\begin{aligned} &\min_{x \in X} f(x) \\ &\text{s.t. } P\left(g(x,\xi) \le 0\right) \ge p. \end{aligned}$$

Let ξ has a finite discrete distribution with realizations $\xi^1, \ldots, \xi^S$ and probabilities $p_s > 0$, $\sum_{s=1}^S p_s = 1$. Find a mixed-integer programming reformulation.

Solution: We can use S binary variables y_s.

$$\begin{array}{rcl} \min_{x,y} f(x) & & \\ \text{s.t.} & & \\ \sum_{s=1}^S p_s y_s & \ge & p, \\ g(x,\xi_s) & \le & M(1-y_s),\ s = 1,\ldots,S \\ y_s & \in & \{0,1\},\ s = 1,\ldots,S, \\ x & \in & X, \end{array} \tag{1}$$

where $M \ge \max_{s=1,\ldots,S} \sup_{x\in X} g(x,\xi_s)$ is so called big-M constant. Realize that if $y_s = 1$, then the corresponding constraint $g(x,\xi_s) \le 0$ must be fulfilled and probability p_s contributes to fulfill the chance constraint, i.e. to reach the probability p. On the other hand, if the constraint is not fulfilled, i.e. $g(x,\xi_s) > 0$, then corresponding $y_s = 0$. □

Example 4.6 *Find reformulations of the Value at Risk portfolio optimization problem*

$$\begin{aligned} &\min_{z,x} z \\ &\text{s.t. } P\left(-\sum_{i=1}^n R_i x_i \le z\right) \ge p, \\ &\quad \sum_{i=1}^n \mathbb{E}[R_i] \cdot x_i \ge r_0, \\ &\quad \sum_{i=1}^n x_i = 1,\ x_i \ge 0, \end{aligned}$$

where R_i is random rate of return of i-th asset and minimal expected return r_0 is selected in such way that the problem is feasible. Assume that the distribution of returns is

1. *discrete with realizations r_{is}, $s = 1,\ldots,S$, and probabilities $p_s = 1/S$,*
2. *multivariate normal with mean μ and variance matrix Σ.*

4.1 Separated right-hand side

Let $g : \mathbb{R}^n \to \mathbb{R}$ and ξ be a random variable with distribution function F, $p \in [0,1]$. Then the individual chance constraint with separated right-hand side is

$$P\left(g(x) \geq \xi\right) \geq p,$$

and it can be reformulated using the quantile function as

$$g(x) \geq F^{-1}(p).$$

Using so called p-level efficient points (pLEPs), this approach can be generalized to the joint chance constrained case, i.e. to reformulate

$$P\big(g_1(x) \geq \xi_1, \ldots, g_r(x) \geq \xi_r\big) \geq p.$$

Definition 4.7 *Let $\mathbf{X}$ be a r-dimensional random vector with distribution function $F : \mathbb{R}^r \to [0,1]$ and $p \in [0,1]$. We say that $z \in \mathbb{R}^r$ is a p-level efficient point (pLEP) if $F(z) \geq p$ and there is no other $y \in \mathbb{R}^r$ for which $F(y) \geq p$ and $y \leq z$, $y \neq z$.*

Let $\{z_1, \ldots, z_n\} \subset \mathbb{R}^r$ be the set of pLEPs and $\mathbf{g}(x) = (g_1(x), \ldots, g_r(x))^T$. Then the joint chance constraint is equivalent to

$$\mathbf{g}(x) \geq z \text{ for at least one } z \in \{z_1, \ldots, z_n\}.$$

Sometimes it is written in the *disjunctive programming* form

$$\mathbf{g}(x) \in \bigcup_{i=1}^{n} \{z_i + \mathbb{R}^r_+\}.$$

Example 4.8 *Consider 6 equiprobable realizations of random vector:*

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Find pLEPs for arbitrary $p \in (0,1]$. Derive a deterministic reformulation for the chance constrained problem

$$\begin{aligned} &\min_{x \in X} \ f(x) \\ &\text{s.t. } P\big(g_1(x) \geq \xi_1,\ g_2(x) \geq \xi_2\big) \geq p. \end{aligned}$$

Solution: Compute the value of distribution function F in each point (in order):

$$\frac{1}{6}, \ \frac{1}{3}, \ \frac{1}{2}, \ \frac{1}{3}, \ \frac{2}{3}, \ 1.$$

Then, we can derive from the definition

$$\begin{aligned}
\text{pLEPs} &= \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}, \; p \in (2/3, 1], \\
&= \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}, \; p \in (1/2, 2/3], \\
&= \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}, \; p \in (1/3, 1/2], \\
&= \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}, \; p \in (1/6, 1/3], \\
&= \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \; p \in (0, 1/6].
\end{aligned}$$

Note that finding the set of pLEPs is in general highly demanding task and requires special algorithms. It is a serious research topic.

Then, the chance constrained problem is equivalent to

$$\begin{aligned}
&\min_{x \in X} f(x) \\
&\text{s.t. } g_1(x) \geq \xi_1, \; g_2(x) \geq \xi_2, \; \text{for at least one } (\xi_1, \xi_2) \in \text{ pLEPs}.
\end{aligned}$$

□

Example 4.9 *Consider compound Poisson distribution which characterizes the overall loss of a non-life insurance portfolio over 1 year period. Estimate the minimal level of portfolio premium such that the losses are covered with probability p.*

Solution: Let $\{X_i\}_{i=1}^{\infty}$ be iid positive random variables with finite second moment which model the claim severities and $N \sim Po(\lambda)$ be independent on $\{X_i\}$. Then, our goal is to solve

$$\min_{x \in \mathbb{R}_+} x \text{ s.t. } P\left(\sum_{i=1}^{N} X_i \leq x\right) \geq p.$$

In fact, we would like to compute the quantile, or Value at Risk. However, this is a difficult task for any compound distribution even using modern mathematical software tools. An easy approach is to use a conservative approximation, in our case based on the *one-sided Chebyshev's inequality*[1]: for $X \sim (\mu, \sigma^2)$, and $a > 0$,

$$P(X - \mu \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

[1]Its proof is not straightforward as for the traditional two-sided version. It is also known as Cantelli's inequality (under this name Wiki refers to several papers with a proof).

Now realize that for the moments of the compound Poisson distribution, it holds

$$\mu := \mathbb{E}\left(\sum_{i=1}^{N} X_i\right) = \mathbb{E}(N)\,\mathbb{E}(X_1) = \lambda\,\mathbb{E}(X_1),$$

$$\sigma^2 := \mathrm{var}\left(\sum_{i=1}^{N} X_i\right) = \mathbb{E}(N)\,\mathrm{var}(X_1) + (\mathbb{E}(X_1))^2\,\mathrm{var}(N) = \lambda\,\mathbb{E}(X_1^2).$$

By applying the one-sided Chebyshev's inequality for $x > \mu$, we obtain

$$P\left(\sum_{i=1}^{N} X_i \geq x\right) = P\left(\sum_{i=1}^{N} X_i - \mu \geq x - \mu\right) \leq \frac{\sigma^2}{\sigma^2 + (x-\mu)^2} \leq 1 - p,$$

which is equivalent to

$$\begin{aligned}(x-\mu)^2 &\geq \frac{\sigma^2}{1-p} - \sigma^2,\\ x &\geq \mu + \sqrt{\frac{p}{1-p}}\sigma,\\ x &\geq \lambda\,\mathbb{E}(X_1) + \sqrt{\frac{p}{1-p}}\sqrt{\lambda\,\mathbb{E}(X_1^2)}.\end{aligned}$$

To get the conservative estimate of safe premium, we can set x equal to the above expression. Realize that it is similar to the formula which we have obtained for the normal distribution, but instead of the normal quantile we are using $\sqrt{\frac{p}{1-p}}$. It can be shown that this is the most conservative (highest) value of quantile for the distribution with finite second moment.

Note that looking for new conservative tight approximations is still a serious research topic in stochastic optimization. □

Summary: You have seen several general approaches to deal with chance constraints:

- direct evaluation, Example 4.1
- reformulation using cumulative distribution function, Example 4.2, 4.3
- reformulation under Gaussian distribution using quantiles, Example 4.4
- mixed-integer reformulation using binary variables, Examples 4.5, 4.6
- generalized quantiles for separated random right-hand side, Example 4.8
- conservative approximation using probability inequalities, Example 4.9