Introduction to integer programming II

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Computational Aspects of Optimization

Introduction to complexity theory

Wolsey (1998): Consider **decision problems** having YES–NO answers.
Optimization problem

$$\max_{x \in M} c^T x$$

can be replaced by (for some k integral)

Is there an $x \in M$ with value $c^T x \geq k$?

For a problem instance X, the **length of the input** $L(X)$ is the length of the binary representation of a standard representation of the instance.
Instance $X = \{c, M\}$, $X = \{c, M, k\}$.

Example: Knapsack decision problem

For an instance

$$X = \left\{ \sum_{i=1}^{n} c_i x_i \geq k,\ \sum_{i=1}^{n} a_i x_i \leq b,\ x \in \{0,1\}^n \right\},$$

the length of the input is

$$L(X) = \sum_{i=1}^{n} \lceil \log c_i \rceil + \sum_{i=1}^{n} \lceil \log a_i \rceil + \lceil \log b \rceil + \lceil \log k \rceil$$

Running time

Definition

- $f_A(X)$ is the **number of elementary calculations** required to run the algorithm A on the instance $X \in P$.
- **Running time of the algorithm** A

$$f_A^*(l) = \sup_X \{ f_A(X) : \ L(X) = l \}.$$

- An algorithm A is **polynomial** for a problem P if $f_A^*(l) = O(l^p)$ for some $p \in \mathbb{N}$.

Classes $\mathcal{NP}$ and $\mathcal{P}$

Definition

- $\mathcal{NP}$ (Nondeterministic Polynomial) is the class of decision problems with the property that: for any instance for which the answer is YES, there is a polynomial proof of the YES.
- $\mathcal{P}$ is the class of decision problems in $\mathcal{NP}$ for which there exists a polynomial algorithm.

$\mathcal{NP}$ may be equivalently defined as the set of decision problems that can be solved in polynomial time on a non-deterministic Turing machine[1].

[1] NTM writes symbols one at a time on an endless tape by strictly following a set of rules. It determines what action it should perform next according to its internal state and what symbol it currently sees. It may have a set of rules that prescribes more than one action for a given situation. The machine "branches" into many copies, each of which follows one of the possible transitions leading to a "computation tree".

Alan Turing

The Imitation Game (2014)

Polynomial reduction and the class $\mathcal{NPC}$

Definition

- If problems $P, Q \in \mathcal{NP}$, and if an instance of P can be converted in polynomial time to an instance of Q, then P is **polynomially reducible** to Q.
- $\mathcal{NPC}$, the class of $\mathcal{NP}$-**complete** problems, is the subset of problems $P \in \mathcal{NP}$ such that for all $Q \in \mathcal{NP}$, Q is polynomially reducible to P.

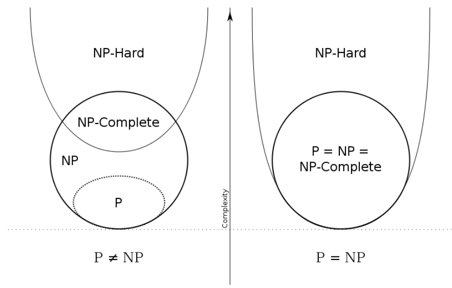
Proposition: Suppose that problems $P, Q \in \mathcal{NP}$.

- If $Q \in \mathcal{P}$ and P is polynomially reducible to Q, then $P \in \mathcal{P}$.
- If $P \in \mathcal{NPC}$ and P is polynomially reducible to Q, then $Q \in \mathcal{NPC}$.

Proposition: If $\mathcal{P} \cap \mathcal{NPC} \neq \emptyset$, then $\mathcal{P} = \mathcal{NPC}$.

Open question & Euler diagram

Is $\mathcal{P} = \mathcal{NP}$?



$\mathcal{NP}$-hard optimization problems

Definition

An optimization problem for which the decision problem lies in $\mathcal{NPC}$ is called $\mathcal{NP}$-hard.

Simplex algorithm

Klee–Minty (1972) example:

$$\begin{aligned} \max \ & \sum_{j=1}^{n} 10^{n-j} x_j \\ \text{s.t. } & 2 \sum_{j=1}^{i-1} 10^{i-j} x_j + x_i \le 100^{i-1}, \ i = 1, \dots, n, \\ & x_j \ge 0, \ j = 1, \dots, n. \end{aligned} \tag{1}$$

Can be easily reformulated in the standard form. The Simplex algorithm takes $2^n - 1$ **pivot steps**, i.e. it is not polynomial in the worst case.

Branch-and-Bound

Basic idea: **DIVIDE AND RULE**

Let $M = M_1 \cup M_2 \cup \dots \cup M_r$ be a partitioning of thefeasibility set and let

$$f_j = \min_{x \in M_j} f(x).$$

Then

$$\min_{x \in M} f(x) = \min_{j=1,\dots,r} f_j.$$

Branch-and-Bound

General principles:

- Solve LP problem without integrality only.
- Branch using additional constraints on integrality: $x_i \le \lfloor x_i^* \rfloor$, $x_i \ge \lfloor x_i^* \rfloor + 1$.
- Cut inperspective branches before solving (using bounds on the optimal value).

Branch-and-Bound

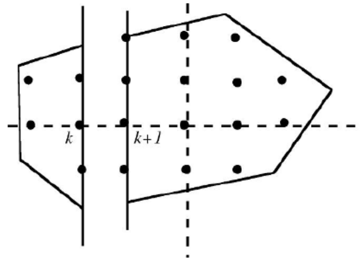
General principles:

- Solve only LP problems with relaxed integrality.
- **Branching**: if an optimal solution is not integral, e.g. $\hat{x}_i$, create and save two new problems with constraints $x_i \leq \lfloor \hat{x}_i \rfloor$, $x_i \geq \lceil \hat{x}_i \rceil$.
- **Bounding** ("different" cutting): save the objective value of the best integral solution and cut all problems in the queue created from the problems with higher optimal values[2].

Exact algorithm ..

[2] Branching cannot improve it.

Branch-and-Bound



P. Pedegral (2004). Introduction to optimization, Springer-Verlag, New York.

Branch-and-Bound

0. $f_{min} = \infty$, $x_{min} = \cdot$, list of problems $P = \emptyset$
   Solve LP-relaxed problem and obtain f^*, x^*. If the solution is integral, STOP. If the problem is infeasible or unbounded, STOP.
1. **BRANCHING**: There is x_i basic non-integral variable such that $k < x_i < k+1$ for some $k \in \mathbb{N}$:
   - Add constraint $x_i \leq k$ to previous problem and put it into list P.
   - Add constraint $x_i \geq k+1$ to previous problem and put it into list P.
2. Take problem from P and solve it: f^*, x^*.
3. - If $f^* < f_{min}$ and x^* is non-integral, GO TO 1.
   - **BOUNDING**: If $f^* < f_{min}$ a x^* is integral, set $f_{min} = f^*$ a $x_{min} = x^*$, GO TO 4.
   - **BOUNDING**: If $f^* \geq f_{min}$, GO TO 4.
   - Problem is infeasible, GO TO 4.
4. - If $P \neq \emptyset$, GO TO 2.
   - If $P = \emptyset$ a $f_{min} = \infty$, integral solution does not exist.
   - If $P = \emptyset$ a $f_{min} < \infty$, optimal value and solution are f_{min}, x_{min}.

Better ...

2./3. Take problem from list P and solve it: f^*, x^*. If for the optimal value of the current problem holds $f^* \geq f_{min}$, then the branching is not necessary, since by solving the problems with added branching constraints we can only increase the optimal value and obtain the same f_{min}.

Branch-and-Bound

Algorithmic issues:

- **Problem selection from the list** P: FIFO, LIFO (depth-first search), problem with the smallest f^*.
- **Selection of the branching variable** x_i^*: the highest/smallest violation of integrality OR the highest/smallest coefficient in the objective function.

B&B – Example I

$$\begin{aligned} \min 4x_1 + 5x_2 & \\ x_1 + 4x_2 &\geq 5, \\ 3x_1 + 2x_2 &\geq 7, \\ x_1, x_2 &\in \mathbb{Z}_+. \end{aligned}$$

After two iterations of the dual SIMPLEX algorithm ...

			4	5	0	0
			x_1	x_2	x_3	x_4
5	x_2	8/10	0	1	-3/10	1/10
4	x_1	18/10	1	0	2/10	-4/10
		112/10	0	0	-7/10	-11/10

B&B – Example I

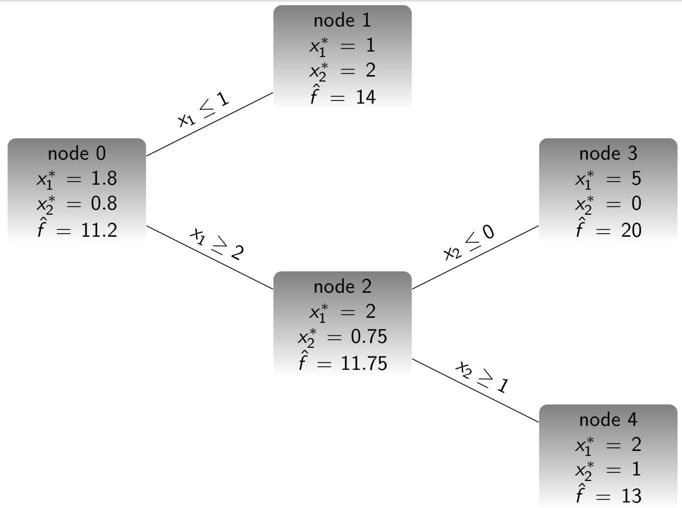
Branching means adding a cut of the form $x_1 \leq 1$, i.e.

$$x_1 + x_5 = 1, \ x_5 \geq 0.$$

$(\alpha = (1, 0, 0, 0, 1), \ \alpha_B = (1, 0))$

			4	5	0	0	0
			x_1	x_2	x_3	x_4	x_5
5	x_2	8/10	0	1	-3/10	1/10	0
4	x_1	18/10	1	0	2/10	-4/10	0
0	x_5	-8/10	0	0	- 2/10	4/10	1
		112/10	0	0	-7/10	-11/10	0

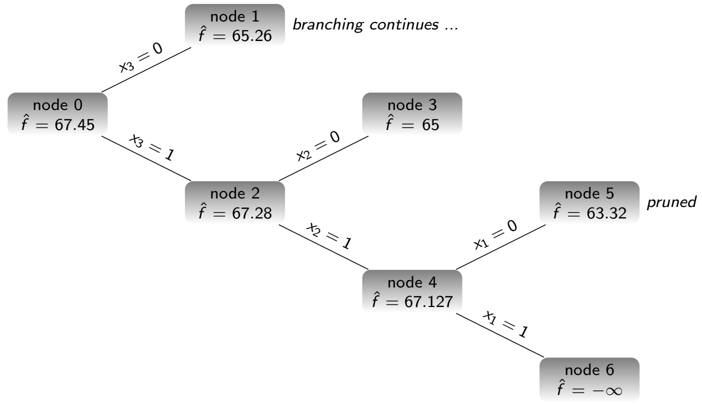
Dual feasible, primal infeasible – run the dual simplex ...



B&B – Example II

$$\begin{aligned}
\max\ & 23x_1 + 19x_2 + 28x_3 + 14x_4 + 44x_5 \\
\text{s.t. } & 8x_1 + 7x_2 + 11x_3 + 6x_4 + 19x_5 \le 25, \\
& x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}.
\end{aligned}$$

B&B – Example II



Branch-and-Bound – remarks

- If you are able to get a **feasible solution** quickly, deliver it to the software (solver).
- **Branch-and-Cut**: add cuts at the beginning/during B&B.
- **Algorithm termination**: (Relative) difference between a lower and an upper bound – construct the upper bound (for minimization) using a feasible solution, lower bound ...

Duality

Set $S(b) = \{x \in \mathbb{Z}^n_+ : \ Ax = b\}$ and define the **value function**

$$z(b) = \min_{x \in S(b)} c^T x. \tag{2}$$

A dual function $F : \mathbb{R}^m \to \overline{\mathbb{R}}$

$$F(b) \le z(b), \ \forall b \in \mathbb{R}^m. \tag{3}$$

A general form of **dual problem**

$$\max_F \{F(b) : \ \text{s.t. } F(b) \le z(b), \ b \in \mathbb{R}^m, \ F : \mathbb{R}^m \to \mathbb{R}\}. \tag{4}$$

We call F a **weak dual** function if it is feasible, and **strong dual** if moreover $F(b) = z(b)$.

Duality

A function F is **subadditive** over a domain Θ if

$$F(\theta_1 + \theta_2) \le F(\theta_1) + F(\theta_2)$$

for all $\theta_1 + \theta_2, \theta_1, \theta_2 \in \Theta$.

The value function z is subadditive over $\{b : S(b) \neq \emptyset\}$, since the sum of optimal x's is feasible for the problem with $b_1 + b_2$ r.h.s., i.e. $\hat{x}_1 + \hat{x}_2 \in S(b_1 + b_2)$.

Duality

If F is subadditive, then condition $F(Ax) \le c^T x$ for $x \in \mathbb{Z}^n_+$ is equivalent to $F(a_{\cdot j}) \le c_j$, $j = 1, \dots, m$.

This is true since $F(Ae_j) \le c^T e_j$ is the same as $F(a_{\cdot j}) \le c_j$.

On the other hand, if F is subadditive and $F(a_{\cdot j}) \le c_j$, $j = 1, \dots, m$ imply

$$F(Ax) \le \sum_{j=1}^{m} F(a_{\cdot j}) x_j \le \sum_{j=1}^{m} c_j x_j = c^T x.$$

Duality

If we set

$$\Gamma^m = \{F : \mathbb{R}^m \to \mathbb{R},\ F(0) = 0,\ F \text{ subadditive}\},$$

then we can write a **subadditive dual** independent of x:

$$\max_F \{F(b) :\ \text{s.t. } F(a_{\cdot j}) \le c_j,\ F \in \Gamma^m\}. \tag{5}$$

Weak and strong duality holds.

An easy feasible solution based on LP duality (= weak dual)

$$F_{LP}(b) = \max_y b^T y \text{ s.t. } A^T y \le c. \tag{6}$$

Duality

Complementary slackness condition: if $\hat{x}$ is an optimal solution for IP, and $\hat{F}$ is an optimal subadditive dual solution, then

$$(\hat{F}(a_{\cdot j}) - c_j)\hat{x}_j = 0,\ j = 1, \dots, m.$$

Knapsack problem

$$\max \sum_{i=1}^{n} c_i x_i$$
$$\text{s.t.} \sum_{i=1}^{n} a_i x_i \leq b,$$
$$x_i \in \{0,1\}.$$

Dynamic programming

Let a_i, b be positive integers.

$$f_r(\lambda) = \max \sum_{i=1}^{r} c_i x_i$$
$$\text{s.t.} \sum_{i=1}^{r} a_i x_i \leq \lambda,$$
$$x_i \in \{0,1\}.$$

If

- $\hat{x}_r = 0$, then $f_r(\lambda) = f_{r-1}(\lambda)$,
- $\hat{x}_r = 1$ then $f_r(\lambda) = c_r + f_{r-1}(\lambda - a_r)$.

Thus we arrive at the recursion

$$f_r(\lambda) = \max \{f_{r-1}(\lambda), c_r + f_{r-1}(\lambda - a_r)\}.$$

Dynamic programming

0. Start with $f_1(\lambda) = 0$ for $0 \leq \lambda < a_1$ and $f_1(\lambda) = \max\{0, c_1\}$ for $\lambda \geq a_1$.
1. Use the **forward recursion**

   $$f_r(\lambda) = \max \{f_{r-1}(\lambda), c_r + f_{r-1}(\lambda - a_r)\}.$$

   to successively calculate $f_2, \dots, f_n$ for all $\lambda \in \{0, 1, \dots, b\}$; $p_n(b)$ is the optimal value.
2. Keep indicator $p_r(\lambda) = 0$ if $f_r(\lambda) = f_{r-1}(\lambda)$, and $p_r(\lambda) = 1$ otherwise.
3. Obtain the optimal solution by a **backward recursion**: if $p_n(b) = 0$ then set $\hat{x}_n = 0$ and continue with $f_{n-1}(b)$, else (if $p_n(b) = 1$) set $\hat{x}_n = 1$ and continue with $f_{n-1}(b - a_n)$ ...

Knapsack problem

Values $a_1 = 4$, $a_2 = 6$, $a_3 = 7$, costs $c_1 = 4$, $c_2 = 5$, $c_3 = 11$, budget $b = 10$:

$$\max \sum_{i=1}^{3} c_i x_i$$
$$\text{s.t.} \sum_{i=1}^{3} a_i x_i \leq 10,$$
$$x_i \in \{0,1\}.$$

Knapsack problem – Dynamic programming

$a_1 = 4,\ a_2 = 6,\ a_3 = 7,\ c_1 = 4,\ c_2 = 5,\ c_3 = 11$

	r/ λ	1	2	3	4	5	6	7	8	9	10
	1	0	0	0	4	4	4	4	4	4	4
f_r	2	0	0	0	4	4	5	5	5	5	9
	3	0	0	0	4	4	5	5	11	11	11
	1	0	0	0	1	1	1	1	1	1	1
p_r	2	0	0	0	0	0	1	1	1	1	1
	3	0	0	0	0	0	0	0	1	1	1

Dynamic programming

Other successful applications

- Uncapacitated lot-sizing problem
- Shortest path problem
- ...

Literature

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