Introduction to computational complexity

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Computational Aspects of Optimization

Software

General mathematical and statistical software:

- **Matlab** and (free) OPTI toolbox: http://www.i2c2.aut.ac.nz/Wiki/OPTI/index.php/Main/HomePage
- **SAS**: OR package
- **R**: suitable packages
- **Mathematica**: findminimum, minimize, ...
- ...

Software

Modelling tools for optimization:

- **GAMS**: http://www.gams.com
- **AIMMS**: http://www.aimms.com
- **Gurobi**: http://www.gurobi.com
- **AMPL**: http://ampl.com
- **CPlex Studio**: http://www-03.ibm.com/software/products/cs/ibmilogcpleoptistud
- **MPL**: http://www.maximalsoftware.com/mpl
- ...

Software

Open Source libraries

- **COIN-OR**: http://www.coin-or.org/

Software

Solvers

- Baron (LP, NLP, MIP, MINLP, ...)
- Bonmin (NLP, MIP, MINLP, ...)
- Conopt (LP, NLP, ...)
- CPlex (LP, MIP, MIQCP, ...)
- Dicopt (MIQCP, MINLP, ...)
- Gurobi
- Knitro (LP, MINLP, MIQCP, NLP, ...)
- Lindo (LP, MINLP, MIP, MIQCP, NLP, ...)
- Minos (LP, NLP, ...)
- Mosek (LP, MIP, MIQCP, NLP, ...)
- Xpress (LP, MIP, MIQCP, ...)
- ...

Introduction to complexity theory

Wolsey (1998): Consider **decision problems** having YES–NO answers.
Optimization problem

$$\max_{x \in M} c^T x$$

can be replaced by (for some k integral)

$$\text{Is there an } x \in M \text{ with value } c^T x \geq k?$$

For a problem instance X, the **length of the input** $L(X)$ is the length of the binary representation of a standard representation of the instance.
Instance $X = \{c, M\}$, $X = \{c, M, k\}$

Example: Knapsack decision problem

For an instance

$$X = \left\{ \sum_{i=1}^{n} c_i x_i \geq k,\ \sum_{i=1}^{n} a_i x_i \leq b,\ x \in \{0,1\}^n \right\},$$

the length of the input is

$$L(X) = \sum_{i=1}^{n} \lceil \log c_i \rceil + \sum_{i=1}^{n} \lceil \log a_i \rceil + \lceil \log b \rceil + \lceil \log k \rceil$$

Running time

Definition

- $f_A(X)$ is the **number of elementary calculations** required to run the algorithm A on the instance $X \in P$.
- **Running time of the algorithm** A

$$f_A^*(l) = \sup_X \{f_A(X) : \; L(X) = l\}.$$

- An algorithm A is **polynomial** for a problem P if $f_A^*(l) = O(l^p)$ for some $p \in \mathbb{N}$.

Classes $\mathcal{NP}$ and $\mathcal{P}$

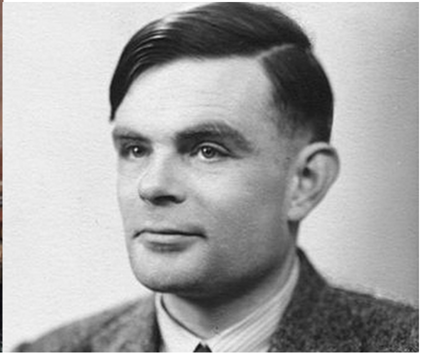
Definition

- $\mathcal{NP}$ (Nondeterministic Polynomial) is the class of decision problems with the property that: for any instance for which the answer is YES, there is a polynomial proof of the YES.
- $\mathcal{P}$ is the class of decision problems in $\mathcal{NP}$ for which there exists a polynomial algorithm.

$\mathcal{NP}$ may be equivalently defined as the set of decision problems that can be solved in polynomial time on a non-deterministic Turing machine[1].

[1] NTM writes symbols one at a time on an endless tape by strictly following a set of rules. It determines what action it should perform next according to its internal state and what symbol it currently sees. It may have a set of rules that prescribes more than one action for a given situation. The machine "branches" into many copies, each of which follows one of the possible transitions leading to a "computation tree".

Alan Turing

The Imitation Game (2014)

Polynomial reduction and the class $\mathcal{NPC}$

Definition

- If problems $P, Q \in \mathcal{NP}$, and if an instance of P can be converted in polynomial time to an instance of Q, then P is **polynomially reducible** to Q.
- $\mathcal{NPC}$, the class of $\mathcal{NP}$-**complete** problems, is the subset of problems $P \in \mathcal{NP}$ such that for all $Q \in \mathcal{NP}$, Q is polynomially reducible to P.

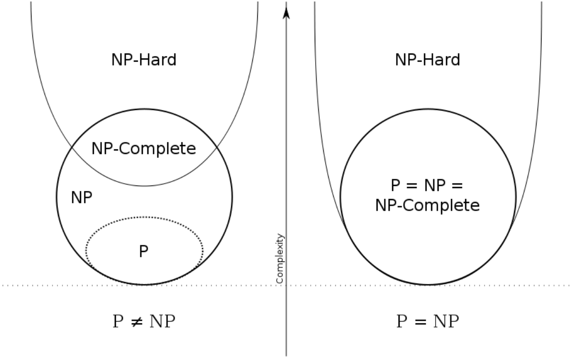
Proposition: Suppose that problems $P, Q \in \mathcal{NP}$.

- If $Q \in \mathcal{P}$ and P is polynomially reducible to Q, then $P \in \mathcal{P}$.
- If $P \in \mathcal{NPC}$ and P is polynomially reducible to Q, then $Q \in \mathcal{NPC}$.

Proposition: If $\mathcal{P} \cap \mathcal{NPC} \neq \emptyset$, then $\mathcal{P} = \mathcal{NPC}$.

Open question & Euler diagram

Is $\mathcal{P} = \mathcal{NP}$?



$\mathcal{NP}$-hard optimization problems

Definition

An optimization problem for which the decision problem lies in $\mathcal{NPC}$ is called $\mathcal{NP}$-hard.

Literature

- G.L. Nemhauser, L.A. Wolsey (1989). Integer Programming. Chapter VI in Handbooks in OR & MS, Vol. 1, G.L. Nemhauser et al. Eds.
- L.A. Wolsey (1998). Integer Programming. Wiley, New York.
- L.A. Wolsey, G.L. Nemhauser (1999). Integer and Combinatorial Optimization. Wiley, New York.