Life Insurance 2 – exercises

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5 Expense-Loaded Premium and Reserve

Please read the theory in Chapter 10 of Gerber book. You can[1] also solve the examples in Appendix C.10.

Expenses can be classified into following groups[2]:

Expenses	Abbreviation	Charged	Proportional to
acquisition	α	at the beginning	SI
collection	β	when premium is collected	premium
administration	γ	during entire contract period	SI
annuity	δ	when annuity is paid	SI

Expense-loaded annual premium is estimated using the (generalized) equivalence principle: the expected present value of the premium payments must be equal to the expected present value of the benefits and the incurred costs (expenses).

Expense-loaded premium reserve is defined as the difference between the expected present value of future benefits plus expenses minus expense-loaded premium related to the end of year k.

Example 5.1 *Consider the whole life insurance with the annual premium paid during the whole contract period. Derive the expense-loaded premium and reserve.*

i) Decompose the reserve.

ii) Do not decompose the reserve.

Solution: The expense-loaded annual premium P^B must satisfy the following (generalized) equivalence principle

$$P^B\,\ddot{a}_x = A_x + \alpha + \beta\,P^B\,\ddot{a}_x + \gamma\ddot{a}_x.$$

If we divide the equation by $\ddot{a}_x$, we obtain the decomposition of the premium in the form

$$P^B = P + P^\alpha + P^\beta + P^\gamma,$$

[1] It is not mandatory, but it can help you.

[2] We follow the notation by Gerber. Prof. Cipra is using different notation for collection expenses γ (instead of β) and administration expenses β (instead of γ).

where in our case it holds

$$P = \frac{A_x}{\ddot{a}_x},\ P^{\alpha} = \frac{\alpha}{\ddot{a}_x},\ P^{\beta} = \frac{\beta\, P^B\, \ddot{a}_x}{\ddot{a}_x} = \beta\, P^B,\ P^{\gamma} = \frac{\gamma\, \ddot{a}_x}{\ddot{a}_x} = \gamma.$$

where P is the net annual premium and the remaining components correspond to the expense groups. We can derive the expense-loaded annual premium in an explicit form

$$P^B = \frac{A_x + \alpha + \gamma\, \ddot{a}_x}{(1-\beta)\ddot{a}_x}.$$

i) Now, we will focus on the expense-loaded premium reserve, which can be also decomposed as

$${}_kV_x^B = {}_kV_x + {}_kV_x^{\alpha} + {}_kV_x^{\beta} + {}_kV_x^{\gamma},$$

where the net premium reserve is equal to

$${}_kV_x = A_{x+k} - P\, \ddot{a}_{x+k},\ k = 0, \ldots,$$

the reserve for the acquisition expenses is equal to

$${}_kV_x^{\alpha} = I(k=0)\, \alpha - P^{\alpha}\, \ddot{a}_{x+k},\ k = 0, \ldots,$$

the reserve for the collection expenses is

$${}_kV_x^{\beta} = \beta\, P^B\, \ddot{a}_{x+k} - P^{\beta}\, \ddot{a}_{x+k} = 0,\ k = 0, \ldots,$$

which is always equal to zero, and the reserve for the administration expenses is

$${}_kV_x^{\gamma} = \gamma\, \ddot{a}_{x+k} - P^{\gamma}\, \ddot{a}_{x+k} = 0,\ k = 0, \ldots,$$

which is equal to zero only if the premium collection period is the same as the whole contract period. Realize that the last two components are equal to zero and at the same time the α component is negative (nonpositive). Therefore, in this case, the expense-loaded premium reserve is lower or equal to the net premium reserve.

ii) When we are not interested into the decomposition and particular components, we can derive the expense-loaded premium reserve directly

$${}_kV_x^B = A_{x+k} + I(k=0)\, \alpha + \beta\, P^B\, \ddot{a}_{x+k} + \gamma \ddot{a}_{x+k} - P^B\, \ddot{a}_{x+k},\ k = 0, \ldots.$$

□

Example 5.2 *Consider the m year deferred life annuity due for n years with premium paid during the deferment period. Derive the expense-loaded premium and reserve*

1. *without premium refund,*
2. *with premium refund (= paid premium is returned to a beneficiary at the end of the year of death of the insured person during the deferment period).*

Solution: 1. The (generalized) equivalence principle is

$$P^B\,\ddot{a}_{x:\overline{m}|} = {}_{m|}\ddot{a}_{x:\overline{n}|} + \alpha + \beta\,P^B\,\ddot{a}_{x:\overline{m}|} + \gamma\,\ddot{a}_{x:\overline{m+n}|} + \delta\,{}_{m|}\ddot{a}_{x:\overline{n}|},$$

i.e. we get the components

$$P = \frac{{}_{m|}\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}},\ P^\alpha = \frac{\alpha}{\ddot{a}_{x:\overline{m}|}},\ P^\beta = \frac{\beta\,P^B\,\ddot{a}_{x:\overline{m}|}}{\ddot{a}_{x:\overline{m}|}} = \beta\,P^B,\ P^\gamma = \frac{\gamma\,\ddot{a}_{x:\overline{m+n}|}}{\ddot{a}_{x:\overline{m}|}},\ P^\delta = \frac{\delta\,{}_{m|}\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}},$$

and the expense-loaded annual premium

$$P^B = P + P^\alpha + P^\beta + P^\gamma + P^\delta,$$

or

$$P^B = \frac{(1+\delta)\,{}_{m|}\ddot{a}_{x\overline{n}|} + \alpha + \gamma\,\ddot{a}_{x:\overline{m+n}|}}{(1-\beta)\,\ddot{a}_{x:\overline{m}|}}.$$

The components of the expense-loaded premium reserve are:
the net premium reserve

$$\begin{aligned} {}_kV_x &= {}_{m-k|}\ddot{a}_{x+k:\overline{n}|} - P\,\ddot{a}_{x+k:\overline{m-k}|}, && k = 0,\dots,m-1,\\ &= \ddot{a}_{x+k:\overline{n+m-k}|}, && k = m,\dots,m+n-1, \end{aligned}$$

the reserve for the acquisition expenses

$$\begin{aligned} {}_kV_x^\alpha &= I(k=0)\,\alpha - P^\alpha\,\ddot{a}_{x+k:\overline{m-k}|}, && k = 0,\dots,m-1,\\ &= 0, && k = m,\dots,m+n-1, \end{aligned}$$

the reserve for the collection expenses

$$\begin{aligned} {}_kV_x^\beta &= \beta\,P^B\,\ddot{a}_{x+k:\overline{m-k}|} - P^\beta\,\ddot{a}_{x+k:\overline{m-k}|} = 0, && k = 0,\dots,m-1,\\ &= 0, && k = m,\dots,m+n-1, \end{aligned}$$

the reserve for the administration expenses

$$\begin{aligned} {}_kV_x^\gamma &= \gamma\,\ddot{a}_{x+k:\overline{m+n-k}|} - P^\gamma\,\ddot{a}_{x+k:\overline{m-k}|}, && k = 0,\dots,m-1,\\ &= \gamma\,\ddot{a}_{x+k:\overline{m+n-k}|}, && k = m,\dots,m+n-1, \end{aligned}$$

and the reserve for the annuity expenses

$$\begin{aligned} {}_kV_x^\delta &= \delta\,{}_{m-k|}\ddot{a}_{x+k:\overline{n}|} - P^\delta\,\ddot{a}_{x+k:\overline{m-k}|}, && k = 0,\dots,m-1,\\ &= \delta\,\ddot{a}_{x+k:\overline{n+m-k}|}, && k = m,\dots,m+n-1. \end{aligned}$$

Then we have

$${}_kV_x^B = {}_kV_x + {}_kV_x^\alpha + {}_kV_x^\beta + {}_kV_x^\gamma + {}_kV_x^\delta,\ k = 0,\dots,m+n-1.$$

We provide also the non-decomposed formula for the expense-loaded premium reserve:

$$\begin{aligned} {}_kV_x^B &= (1+\delta)\, {}_{m-k|}\ddot{a}_{x+k:\overline{n}|} + I(k=0)\,\alpha + \beta\, P^B\, \ddot{a}_{x+k:\overline{m-k}|} + \gamma\, \ddot{a}_{x+k:\overline{m+n-k}|} - P^B\, \ddot{a}_{x+k:\overline{m-k}|}, \\ & \qquad\qquad k=0,\ldots,m-1, \\ &= (1+\delta)\, \ddot{a}_{x+k:\overline{n+m-k}|} + \gamma\, \ddot{a}_{x+k:\overline{m+n-k}|}, \qquad k=m,\ldots,m+n-1. \end{aligned}$$

2. We will modify only the net premium component where the premium refund is incorporated. The (generalized) equivalence principle is

$$\hat{P}^B\, \ddot{a}_{x:\overline{m}|} = {}_{m|}\ddot{a}_{x:\overline{n}|} + \hat{P}^B\, (IA)^1_{x:\overline{m}|} + \alpha + \beta\, \hat{P}^B\, \ddot{a}_{x:\overline{m}|} + \gamma\, \ddot{a}_{x:\overline{m+n}|} + \delta\, {}_{m|}\ddot{a}_{x:\overline{n}|},$$

i.e. the premium refund is modeled using the standard increasing term insurance component, so the net annual component is

$$\hat{P} = \frac{{}_{m|}\ddot{a}_{x:\overline{n}|} + \hat{P}^B\, (IA)^1_{x:\overline{m}|}}{\ddot{a}_{x:\overline{m}|}},$$

and the expense-loaded annual premium equals to

$$\hat{P}^B = \frac{(1+\delta)\, {}_{m|}\ddot{a}_{x:\overline{n}|} + \alpha + \gamma\, \ddot{a}_{x:\overline{m+n}|}}{(1-\beta)\ddot{a}_{x:\overline{m}|} - (IA)^1_{x:\overline{m}|}}.$$

The premium refund influences the net premium reserve component as follows

$$\begin{aligned} {}_k\hat{V}_x &= {}_{m-k|}\ddot{a}_{x+k:\overline{n}|} + \hat{P}^B\, (IA)^{\,1}_{x+k:\overline{m-k}|} + k\, \hat{P}^B\, A^{\,1}_{x+k:\overline{m-k}|} - \hat{P}\, \ddot{a}_{x+k:\overline{m-k}|}, \; k=0,\ldots,m-1, \\ &= \ddot{a}_{x+k:\overline{n+m-k}|}, \; k=m,\ldots,m+n-1, \end{aligned}$$

where we must use the correction term $k\, \hat{P}^B\, A^{\,1}_{x+k:\overline{m-k}|}$ to get the right premium refund level after k years. In principle, the β component is also influenced (it contains new $\hat{P}^B$), but it is again equal to zero. Thus, we have obtained

$${}_k\hat{V}_x^B = {}_k\hat{V}_x + {}_kV_x^\alpha (+ {}_kV_x^\beta) + {}_kV_x^\gamma + {}_kV_x^\delta.$$

□

Example 5.3 *Consider the m year deferred standard increasing term insurance for n years with premium paid during the deferment period. Derive the expense-loaded premium and reserve*

1. *using the standard definition of expenses,*
2. *using the modified definition of expenses as follows[3]:*
   - *acquisition expenses are divided into three consecutive payments,*

[3]This can really happen in practice, so always check the definition of expenses.

– *collection expenses are standard decreasing,*

– *administration expenses differ between deferment period and subsequent period.*

Assume $m \geq 3$.

Solution: 1. The (generalized) equivalence principle is

$$P^B \, \ddot{a}_{x:\overline{m}|} = {}_{m|}(IA)^{1}_{x:\overline{n}|} + \alpha + \beta \, P^B \, \ddot{a}_{x:\overline{m}|} + \gamma \, \ddot{a}_{x:\overline{m+n}|},$$

i.e. we get components of the expense-loaded annual premium

$$P = \frac{{}_{m|}(IA)^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{m}|}}, \; P^\alpha = \frac{\alpha}{\ddot{a}_{x:\overline{m}|}}, \; P^\beta = \frac{\beta \, P^B \, \ddot{a}_{x:\overline{m}|}}{\ddot{a}_{x:\overline{m}|}} = \beta \, P^B, \; P^\gamma = \frac{\gamma \, \ddot{a}_{x:\overline{m+n}|}}{\ddot{a}_{x:\overline{m}|}},$$

and the expense-loaded annual premium

$$P^B = P + P^\alpha + P^\beta + P^\gamma,$$

or

$$P^B \; = \frac{{}_{m|}(IA)^{1}_{x:\overline{n}|} + \alpha + \gamma \, \ddot{a}_{x:\overline{m+n}|}}{(1-\beta)\ddot{a}_{x:\overline{m}|}}.$$

The components of the expense-loaded premium reserve are:

$$\begin{aligned} {}_kV_x \; &= {}_{m-k|}(IA)_{x+k:\overline{n}|}^{\;1} - P \, \ddot{a}_{x+k:\overline{m-k}|}, && k = 0, \ldots, m-1, \\ &= (IA)_{x+k:\overline{n+m-k}|}^{\;1} + (k-m) \, A_{x+k:\overline{n+m-k}|}^{\;1}, && k = m, \ldots, m+n-1, \end{aligned}$$

$$\begin{aligned} {}_kV_x^\alpha \; &= I(k=0) \, \alpha - P^\alpha \, \ddot{a}_{x+k:\overline{m-k}|}, && k = 0, \ldots, m-1, \\ &= 0, && k = m, \ldots, m+n-1, \end{aligned}$$

$$\begin{aligned} {}_kV_x^\beta \; &= \beta \, P^B \, \ddot{a}_{x+k:\overline{m-k}|} - P^\beta \, \ddot{a}_{x+k:\overline{m-k}|} = 0, && k = 0, \ldots, m-1, \\ &= 0, && k = m, \ldots, m+n-1, \end{aligned}$$

$$\begin{aligned} {}_kV_x^\gamma \; &= \gamma \, \ddot{a}_{x+k:\overline{m+n-k}|} - P^\gamma \, \ddot{a}_{x+k:\overline{m-k}|}, && k = 0, \ldots, m-1, \\ &= \gamma \, \ddot{a}_{x+k:\overline{m+n-k}|}, && k = m, \ldots, m+n-1, \end{aligned}$$

Then we have

$${}_kV_x^B = {}_kV_x + {}_kV_x^\alpha + {}_kV_x^\beta + {}_kV_x^\gamma, \; k = 0, \ldots, m+n-1.$$

2. The (generalized) equivalence principle with the modified expenses

$$P^B \, \ddot{a}_{x:\overline{m}|} = {}_{m|}(IA)^{1}_{x:\overline{n}|} + \alpha \, \ddot{a}_{x:\overline{3}|} + \beta \, P^B \, (D\ddot{a})_{x:\overline{m}|} + \gamma_1 \, \ddot{a}_{x:\overline{m}|} + \gamma_2 \, {}_{m|}\ddot{a}_{x:\overline{n}|}.$$

We can derive the components of the expense-loaded annual premium

$$P = \frac{{}_{m|}(IA)^{1}_{x:\overline{n|}}}{\ddot{a}_{x:\overline{m|}}},\ P^{\alpha} = \frac{\alpha\,\ddot{a}_{x:\overline{3|}}}{\ddot{a}_{x:\overline{m|}}},\ P^{\beta} = \frac{\beta\,P^{B}\,(D\ddot{a})_{x:\overline{m|}}}{\ddot{a}_{x:\overline{m|}}},\ P^{\gamma} = \frac{\gamma_1\,\ddot{a}_{x:\overline{m|}} + \gamma_2\,{}_{m|}\ddot{a}_{x:\overline{n|}}}{\ddot{a}_{x:\overline{m|}}},$$

The components of the expense-loaded premium reserve:

$$\begin{aligned} {}_kV_x &= {}_{m-k|}(IA)^{1}_{x+k:\overline{n|}} - P\,\ddot{a}_{x+k:\overline{m-k|}}, && k = 0,\ldots,m-1,\\ &= (IA)^{1}_{x+k:\overline{n+m-k|}} + (k-m)\,A^{1}_{x+k:\overline{n+m-k|}}, && k = m,\ldots,m+n-1, \end{aligned}$$

$$\begin{aligned} {}_kV_x^{\alpha} &= \alpha\,\ddot{a}_{x+k:\overline{3-k|}} - P^{\alpha}\,\ddot{a}_{x+k:\overline{m-k|}}, && k = 0,\ldots,2,\\ &= -P^{\alpha}\,\ddot{a}_{x+k:\overline{m-k|}}, && k = 3,\ldots,m-1,\\ &= 0, && k = m,\ldots,m+n-1, \end{aligned}$$

$$\begin{aligned} {}_kV_x^{\beta} &= \beta\,P^{B}\,(D\ddot{a})_{x+k:\overline{m-k|}} - P^{\beta}\,\ddot{a}_{x+k:\overline{m-k|}}, && k = 0,\ldots,m-1,\\ &= 0, && k = m,\ldots,m+n-1, \end{aligned}$$

which is not equal to zero in this case, and

$$\begin{aligned} {}_kV_x^{\gamma} &= \gamma_1\,\ddot{a}_{x+k:\overline{m-k|}} + \gamma_2\,{}_{m-k|}\ddot{a}_{x+k:\overline{n|}} - P^{\gamma}\,\ddot{a}_{x+k:\overline{m-k|}}, && k = 0,\ldots,m-1,\\ &= \gamma_2\,\ddot{a}_{x+k:\overline{m+n-k|}}, && k = m,\ldots,m+n-1, \end{aligned}$$

Then, we have the expense-loaded premium reserve

$${}_kV_x^{B} = {}_kV_x + {}_kV_x^{\alpha} + {}_kV_x^{\beta} + {}_kV_x^{\gamma},\ k = 0,\ldots,m+n-1.$$

□