Optimization with application in finance – exercises

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HW: Examples 5.2 (part 1.), 5.7, 5.11.

5 Risk measures and mean–risk models

5.1 Markowitz model

Example 5.1 *Formulate the Markowitz model with the following properties*

1. *short sales not allowed,*
2. *short sales allowed,*
3. *short sales allowed with individual restrictions,*
4. *short sales allowed with margin requirements,*
5. *proportional and fixed transaction costs included.*

Add each restriction/property to the original model.

Solution: Let for the vector of random returns hold $\rho \sim (r, V)$, then the basic Markowitz model is:

$$\begin{aligned} \min_x \; & x^T V x \\ & r^T x \geq r_{min}, \\ & \mathbf{1}^T x = 1, \end{aligned}$$

where $\mathbf{1}$ is vector of ones and r_{min} denotes the minimal required expected return.

We add the following constraints to the original formulation:

1. short sales not allowed: $x_i \geq 0$,
2. short sales allowed: $x_i \in \mathbb{R}$,
3. short sales allowed with individual restrictions $u_i > 0$:

   $$x_i = x_i^+ - x_i^-, \; x_i^+ \geq 0, \; x_i^- \geq 0, \; x_i^- \leq u_i, \; \forall i,$$

   i.e. each short position is restricted.
4. short sales allowed with margin requirements, where the overall short position is bounded by 0.5 of the long position:

   $$x_i = x_i^+ - x_i^-, \; x_i^+ \geq 0, \; x_i^- \geq 0, \; \sum_{i=1}^n x_i^- \leq 0.5 \sum_{i=1}^n x_i^+, \; \forall i,$$

5. proportional p_i and fixed f_i transaction costs included:

$$l_i y_i \le x_i \le u_i y_i$$
$$\sum_{i=1}^{n}(\mu_i - p_i)x_i - \sum_{i=1}^{n}\frac{f_i}{B}y_i \ge r_{min},$$
$$y_i \in \{0,1\},$$

where l_i, u_i are lower and upper bounds on the weights, B denotes the initial budget. We can see that when the budget is large, the fixed transaction costs are negligible. Binary variables are used to identify which assets are bought.

Note that the properties introduced above are applicable in mean-risk models in general. □

Example 5.2 *Find the efficient solutions of the Markowitz model using the Karush–Kuhn–Tucker conditions where*

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} \sim \left(\begin{pmatrix} 12 \\ 14 \\ 12 \end{pmatrix}, \begin{pmatrix} 72 & 72 & -72 \\ 72 & 76 & -64 \\ -72 & -64 & 88 \end{pmatrix} \right),$$

with

1. *short sales allowed,*
2. *short sales not allowed.*

Use a suitable parametric programming reformulation.

5.2 Coherent risk measures

Let $V \subseteq \mathcal{L}_p(\Omega)$, usually $p \in \{1,2\}$, be a set of loss random variables.

Definition 5.3 *(Artzner et al. (1999))*
We say that $\mathcal{R} : V \to (-\infty, \infty]$ is a coherent risk measures if it satisfies:

(R1) *shift equivariance*[1]*: $\mathcal{R}(Z + c) = \mathcal{R}(Z) + c$ for all $Z \in V$ and constants c,*

(R2) *positive homogeneity*[2]*: $\mathcal{R}(0) = 0$, and $\mathcal{R}(\lambda Z) = \lambda\mathcal{R}(Z)$ for all $Z \in V$ and all $\lambda > 0$,*

(R3) *subadditivity*[3]*: $\mathcal{R}(Z_1 + Z_2) \le \mathcal{R}(Z_1) + \mathcal{R}(Z_2)$ for all $Z_1, Z_2 \in V$,*

(R4) *monotonicity*[4]*: $\mathcal{R}(Z_1) \le \mathcal{R}(Z_2)$ when $Z_1 \le Z_2$ a.s., $Z_1, Z_2 \in V$.*

Axioms (R2) & (R3) imply convexity: for arbitrary $\lambda \in (0,1)$ and $Z_1, Z_2 \in V$:

$$\mathcal{R}(\lambda Z_1 + (1-\lambda)Y) \le \mathcal{R}(\lambda Z_1) + \mathcal{R}((1-\lambda)Z_2) \le \lambda\mathcal{R}(Z_1) + (1-\lambda)\mathcal{R}(Z_2).$$

[1] Adding sure loss increases risk
[2] Increasing our position $\lambda-$times increases the risk proportionally
[3] Holding two assets together is never more risky than holding them separately $\leftrightarrow$ diversification
[4] Higher loss (almost sure), higher risk

5.3 Value at Risk

Value at Risk is defined at the quantile of the loss distribution, i.e. the losses lower or equal to VaR appear with a high probability α and the losses higher than VaR appear with low probability $1-\alpha$. Usually we consider α equal to 0.95, 0.99, 0.995.

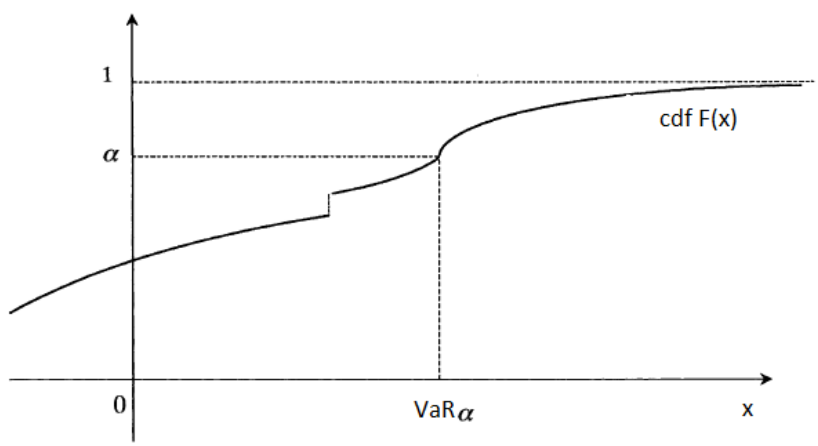
Definition 5.4 ***Value at Risk*** *(VaR) for a general loss random variable Z defined on probability space $(\Omega, \mathcal{A}, P)$, level $\alpha \in (0,1)$:*

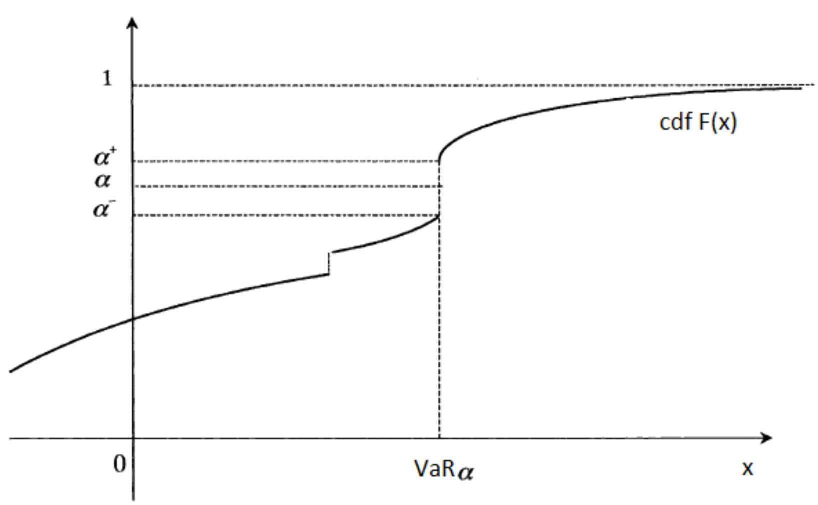
$$VaR_\alpha(Z) = q_\alpha(Z) = \min_z z \text{ s.t. } P(Z \leq z) \geq \alpha.$$

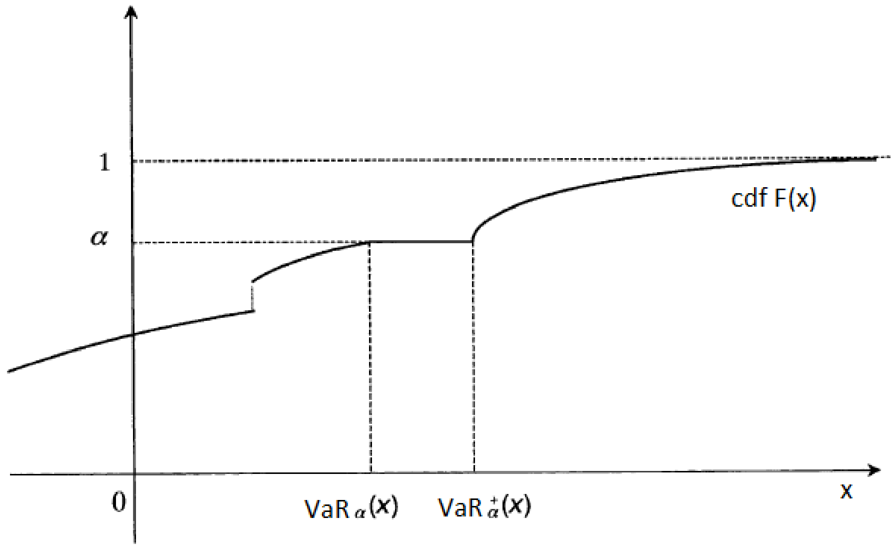
Upper Value at Risk *(upper-VaR)*

$$VaR_\alpha^+(Z) = q^\alpha(Z) = \inf_z z \text{ s.t. } P(Z \leq z) > \alpha.$$

Please look on the pictures by Rockafellar and Uryasev (2002) which explain the definition of VaR_α and VaR_α^+:







Example 5.5 *Show that Value at Risk fulfills (R1) shift equivariance, (R2) positive homogeneity, and (R4) monotonicity.*

Solution:
(R2) positive homogeneity: for $\lambda > 0$

$$\begin{aligned} VaR_\alpha(\lambda Z) &= \min_z z \text{ s.t. } P(\lambda Z \leq z) \geq \alpha \\ &= \min_z z \text{ s.t. } P\left(Z \leq \frac{z}{\lambda}\right) \geq \alpha \\ &= \min_{\tilde{z}} \lambda \tilde{z} \text{ s.t. } P\left(Z \leq \tilde{z}\right) \geq \alpha \\ &= \lambda \min_{\tilde{z}} \tilde{z} \text{ s.t. } P\left(Z \leq \tilde{z}\right) \geq \alpha \\ &= \lambda VaR_\alpha(Z). \end{aligned}$$

(R1) shift equivariance: for arbitrary c

$$\begin{aligned} VaR_\alpha(Z + c) &= \min_z z \text{ s.t. } P(Z + c \leq z) \geq \alpha \\ &= \min_{\tilde{z}} \tilde{z} + c \text{ s.t. } P\left(Z \leq \tilde{z}\right) \geq \alpha \\ &= VaR_\alpha(Z) + c. \end{aligned}$$

(R4) monotonicity: It is obvious. □

VaR under discrete distribution: Let Z be concentrated in finitely many points $z^{[1]} < z^{[2]} < \cdots < z^{[N]}$ with probabilities $P\big(Z = z^{[k]}\big) = p^{[k]} > 0$, $\sum_{k=1}^{N} p^{[k]} = 1$.
Find index k_α such that

$$\sum_{k=1}^{k_\alpha - 1} p^{[k]} < \alpha \leq \sum_{k=1}^{k_\alpha} p^{[k]}.$$

Then we have

$$VaR_\alpha(x) = z^{[k_\alpha]}. \tag{1}$$

In general, Value at Risk **does not fulfill** (R3) subadditivity.

Example 5.6 *Consider two independent one-year bonds with nominal value 1 CZK and the same parameters. No loss with probability 96%, loss 0.7 with probability 4%. Compute $VaR_{0.95}(Z_1)$, $VaR_{0.95}(Z_2)$, $VaR_{0.95}(Z_1 + Z_2)$.*

Solution: Value at Risk on the level 95% is equal to 0. If you buy both bonds, then we have the following losses and probabilities

- 0 with probability 92.16% (= 0.96 * 0.96)
- 0.7 with prob. 7.68% (= 2 * 0.96 * 0.04)
- 1.4 with prob. 0.16% (=0.04 * 0.04)

Thus Value at Risk of $Z_1 + Z_2$ is 0.7, i.e.

$$\mathrm{VaR}_{0.95}(Z_1 + Z_2) > \mathrm{VaR}_{0.95}(Z_1) + \mathrm{VaR}_{0.95}(Z_2).$$

Value at Risk is not subadditive. Even for independent losses (risks) it holds

$$\mathrm{VaR}_{0.95}(Z_1 + Z_2) \neq \mathrm{VaR}_{0.95}(Z_1) + \mathrm{VaR}_{0.95}(Z_2).$$

□

Example 5.7 *Consider two independent loss random variables Z_1, Z_2 with the following discrete distributions:*

s	z_{1s}	$P(Z_1 = z_{1s})$	z_{2s}	$P(Z_2 = z_{2s})$
1	*0*	*0.93*	*0*	*0.96*
2	*1*	*0.04*	*0.5*	*0.005*
3	*2*	*0.03*	*2.5*	*0.035*

Compute $VaR_{0.95}(Z_1)$, $VaR_{0.95}(Z_2)$, $VaR_{0.95}(Z_1+Z_2)$. (You can use a software tool.)

5.4 Conditional Value at Risk

For $Z \in \mathcal{L}_1(\Omega)$ with cdf $G(x) = P(Z \leq x)$, **Conditional Value at Risk** (CVaR) is defined as the mean of losses in the α-tail distribution with the distribution function:

$$G_\alpha(x) = \begin{cases} \frac{G(x)-\alpha}{1-\alpha}, & \text{if } x \geq \mathrm{VaR}_\alpha(Z), \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\mathrm{CVaR}_\alpha(Z) = \mathbb{E}_{G_\alpha}[Z].$$

Denote by $[x]^+ = \max\{x, 0\}$ the positive part of x. CVaR can be expressed using the following **minimization formula**, cf. Rockafellar and Uryasev (2000, 2002):

$$\mathrm{CVaR}_\alpha(Z) = \min_{\xi\in\mathbb{R}} \left\{ \xi + \frac{1}{1-\alpha}\mathbb{E}\left[Z-\xi\right]^+ \right\} \tag{2}$$

with the minimum attained at any α-th quantile, i.e. we have

$$\mathrm{CVaR}_\alpha(Z) = \mathrm{VaR}_\alpha(Z) + \frac{1}{1-\alpha}\mathbb{E}\left[Z - \mathrm{VaR}_\alpha(Z)\right]^+. \tag{3}$$

Before showing that CVaR is coherent, we prove an auxiliary lemma.

Lemma 5.8 *Let $f(\xi, z)$ be jointly convex real function and define*

$$g(z) = \min_{\xi\in\mathbb{R}} f(\xi, z).$$

Then $g(z)$ is convex on its domain $\mathrm{dom}\, g = \{z : g(z) \in \mathbb{R}\}$.

Proof: Take $\lambda \in (0, 1)$ and two arbitrary $z_1, z_2 \in \mathrm{dom}\, g$. Then there are $\xi_1, \xi_2 \in \mathbb{R}$ such that

$$g(z_1) = f(\xi_1, z_1),\ \ g(z_2) = f(\xi_2, z_2).$$

We have

$$\begin{aligned}
\lambda\, g(z_1) + (1-\lambda)\, g(z_2) &= \lambda\, f(\xi_1, z_1) + (1-\lambda)\, f(\xi_2, z_2) \\
&\geq f\big(\lambda\,\xi_1 + (1-\lambda)\,\xi_2,\ \lambda\, z_1 + (1-\lambda)\, z_2\big) \\
&\geq g\big(\lambda\, z_1 + (1-\lambda)\, z_2\big),
\end{aligned}$$

where the first inequality follows from the joint convexity of f and the second one from that g is minimal over all $z \in \mathbb{R}$. $\square$

Example 5.9 *Show that CVaR is a coherent risk measure.*

Proof:

(R1) shift equivariance: We can use formula (3) together with the properties of VaR showed in Example 5.5. We have

$$\begin{aligned}
\mathrm{CVaR}_\alpha(Z+c) &= \mathrm{VaR}_\alpha(Z+c) + \frac{1}{1-\alpha}\mathbb{E}\left[Z + c - \mathrm{VaR}_\alpha(Z+c)\right]^+ \\
&= \mathrm{VaR}_\alpha(Z) + c + \frac{1}{1-\alpha}\mathbb{E}\left[Z + c - \mathrm{VaR}_\alpha(Z) - c\right]^+ \\
&= c + \mathrm{VaR}_\alpha(Z) + \frac{1}{1-\alpha}\mathbb{E}\left[Z - \mathrm{VaR}_\alpha(Z)\right]^+ \\
&= c + \mathrm{CVaR}_\alpha(Z).
\end{aligned} \tag{4}$$

(R2) positive homogeneity: Similar approach can be used here, for $\lambda > 0$

$$\begin{aligned}
\mathrm{CVaR}_\alpha(\lambda Z) &= \mathrm{VaR}_\alpha(\lambda Z) + \frac{1}{1-\alpha}\mathbb{E}\left[\lambda Z - \mathrm{VaR}_\alpha(\lambda Z)\right]^+ \\
&= \lambda \mathrm{VaR}_\alpha(Z) + \frac{1}{1-\alpha}\mathbb{E}\left[\lambda Z - \lambda \mathrm{VaR}_\alpha(Z)\right]^+ \\
&= \lambda \mathrm{VaR}_\alpha(Z) + \frac{\lambda}{1-\alpha}\mathbb{E}\left[Z - \mathrm{VaR}_\alpha(Z)\right]^+ \\
&= \lambda \mathrm{CVaR}_\alpha(Z).
\end{aligned} \tag{5}$$

(R3) subadditivity: First, we show that CVaR is convex. This together with positive homogeneity implies subadditivity. We can use Lemma 5.8 and the auxiliary function from the minimization formula (2), and set

$$f(\xi, Z) := \xi + \frac{1}{1-\alpha}\mathbb{E}\left[Z - \xi\right]^+ .$$

This function is obviously convex[5] jointly in (ξ, Z). Lemma 5.8 implied that CVaR_α is convex.

(R4) monotonicity: It is obvious.

□

CVaR under discrete distribution: Let Z be concentrated in finitely many points $z^{[1]} < z^{[2]} < \cdots < z^{[N]}$ with probabilities $P\big(Z = z^{[k]}\big) = p^{[k]} > 0$, $\sum_{k=1}^N p^{[k]} = 1$.
Find index k_α such that

$$\sum_{k=1}^{k_\alpha - 1} p^{[k]} < \alpha \leq \sum_{k=1}^{k_\alpha} p^{[k]}.$$

Then we have

$$VaR_\alpha(x) = z^{[k_\alpha]} \tag{6}$$

and if $\alpha > 1 - p^{[N]}$, then

$$VaR_\alpha(x) = CVaR_\alpha(x) = z^{[N]}, \tag{7}$$

else

$$CVaR_\alpha(x) = \frac{1}{1-\alpha}\left[\Big(\sum_{k=1}^{k_\alpha} p^{[k]} - \alpha\Big) z^{[k_\alpha]} + \sum_{k=k_\alpha+1}^{N} p^{[k]} z^{[k]}\right]. \tag{8}$$

Sometimes, we are faced with some misunderstandings with upper and lower CVaR (called also Conditional Tail Expectations) which are defined as:

$$\begin{aligned}
\mathrm{CVaR}_\alpha^+(Z) &= \mathrm{CTE}^\alpha(Z) = \mathbb{E}[Z | Z > \mathrm{VaR}_\alpha(Z)], \\
\mathrm{CVaR}_\alpha^-(Z) &= \mathrm{CTE}_\alpha(Z) = \mathbb{E}[Z | Z \geq \mathrm{VaR}_\alpha(Z)].
\end{aligned}$$

[5]It is a sum of two convex functions where the second function is a composition of linear function $Z - \xi$ and convex function $[\cdot]_+$ multiplied by positive constant $\frac{1}{1-\alpha}$.

It can be shown that these measures are not always equal to CVaR.

Example 5.10 *Consider one-year bonds with nominal value 1 CZK and the parameters*

- *no loss with probability 96%,*
- *loss 0.7 with probability 4%.*

Compute and compare $\mathrm{CVaR}_{0.95}$, $\mathrm{CVaR}^-_{0.95}$, $\mathrm{CVaR}^+_{0.95}$.

Solution: From Example 5.6 we know that Value at Risk on the level 95% is equal to 0. Then

$$\begin{aligned}
\mathrm{CVaR}^+_{0.95}(Z) &= \mathbb{E}[Z|Z > \mathrm{VaR}_{0.95}(Z)] \\
&= \frac{1}{0.04}(0.04 \cdot 0.7) = 0.7, \\
\mathrm{CVaR}^-_{0.95}(Z) &= \mathbb{E}[Z|Z \geq \mathrm{VaR}_{0.95}(Z)] \\
&= \frac{1}{0.96 + 0.04}(0.96 \cdot 0 + 0.04 \cdot 0.7) = 0.028.
\end{aligned}$$

Using the formula (8) for CVaR under finite discrete distribution, we obtain

$$\mathrm{CVaR}_{0.95}(Z) = \frac{1}{1 - 0.95}\Big((0.96 - 0.95) \cdot 0 + 0.04 \cdot 0.7\Big) = 0.56$$

Obviously, it holds

$$\mathrm{CVaR}^-_{0.95}(Z) < \mathrm{CVaR}_{0.95}(Z) < \mathrm{CVaR}^+_{0.95}(Z).$$

□

Example 5.11 *Consider two independent loss random variables* Z_1, Z_2 *with the discrete distributions from Example 5.7. Compute* $\mathrm{CVaR}_{0.95}(Z_1 + Z_2)$, $\mathrm{CVaR}^-_{0.95}(Z_1 + Z_2)$, *and* $\mathrm{CVaR}^+_{0.95}(Z_1 + Z_2)$. *(You can use a software tool.)*

Example 5.12 *(VaR and CVaR under normal distribution)*
Let $Z \sim N(\mu, \sigma^2)$, *then*

$$\begin{aligned}
\mathrm{VaR}_\alpha(Z) &= \mu + z_\alpha \sigma, &(9)\\
\mathrm{CVaR}_\alpha(Z) &= \mu + \eta_\alpha \sigma, &(10)
\end{aligned}$$

where $z_\alpha = \Phi^{-1}(\alpha)$ *is a quantile of a standard normal distribution (with pdf* ϕ *and cdf* Φ*) and*

$$\eta_\alpha = \frac{\int_{\Phi^{-1}(\alpha)}^{\infty} t\phi(t)\,dt}{1 - \alpha}.$$

Solution: For Value at Risk

$$P(Z \leq VaR_\alpha) = P\left(\frac{Z-\mu}{\sigma} \leq \frac{VaR_\alpha - \mu}{\sigma}\right) = \Phi\left(\frac{VaR_\alpha - \mu}{\sigma}\right) = \alpha.$$

For Conditional Value at Risk

$$\begin{aligned}
\mathrm{CVaR}_\alpha(Z) &= \frac{1}{1-\alpha}\int_{\mu+\Phi^{-1}(\alpha)\sigma}^{\infty} \frac{z}{\sigma}\,\phi\left(\frac{z-\mu}{\sigma}\right) dz \\
&= \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^{\infty} \frac{\mu+t\sigma}{\sigma}\phi\,(t)\,\sigma dt \\
&= \frac{1}{1-\alpha}\left(\mu\int_{\Phi^{-1}(\alpha)}^{\infty}\phi\,(t)\,dt + \sigma\int_{\Phi^{-1}(\alpha)}^{\infty} t\phi\,(t)\,dt\right),
\end{aligned}$$

where

$$\int_{\Phi^{-1}(\alpha)}^{\infty}\phi\,(t)\,dt = 1-\alpha.$$

□

5.5 References

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- Markowitz, H. M. (1952). **Portfolio selection**. *The Journal of Finance* 7, No. 1, 77-91.

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