Chance Constrained Problems and Value at Risk
Homework 2

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Computational Aspects of Optimization

Chance constrained problems – single random constraint

Let $f, g(\cdot, \xi) : \mathbb{R}^n \to \mathbb{R}$ be real functions, $X \subseteq \mathbb{R}^n$, ξ be a real random vector, $\varepsilon \in (0, 1)$ small:

$$\begin{aligned} &\min_{x \in X} f(x) \\ &\quad \text{s.t.} \qquad P\left(g(x, \xi) \leq 0\right) \geq 1 - \varepsilon. \end{aligned}$$

INTERPRETATION: for a given $x \in X$, the probability of ξ for which the random constraint is fulfilled must be at least $1 - \varepsilon$:

$$P\left(g(x, \xi) \leq 0\right) = P\left(\{\xi : \; g(x, \xi) \leq 0\}\right).$$

Chance constrained problems – single random constraint

Let ξ have a finite discrete distribution with realizations $\xi^1, \dots, \xi^S$ and probabilities $p_s > 0$, $\sum_{s=1}^S p_s = 1$:

$$\begin{array}{rcl} \min_{x,y} f(x) & & \\ \text{s.t.} & & \\ \sum_{s=1}^S p_s y_s & \geq & 1 - \varepsilon, \\ g(x, \xi_s) & \leq & M(1 - y_s),\ s = 1, \dots, S \\ y_s & \in & \{0, 1\},\ s = 1, \dots, S, \\ x & \in & X, \end{array} \tag{1}$$

where $M \geq \max_{s=1,\dots,S} \sup_{x \in X} g(x, \xi_s)$.

Value at Risk (VaR)

Value at Risk (VaR) for a general **loss** random variable Z defined on probability space $(\Omega, \mathcal{A}, P)$, level $\alpha \in (0,1)$, usually 0.95, 0.99, 0.995:

$$VaR_\alpha(Z) = \min_z z \text{ s.t. } P(Z \le z) \ge \alpha.$$

Value at Risk (VaR)

Portfolio optimization problem:

$$\begin{aligned} & \min_{z,x} z \\ P\left(-\sum_{i=1}^{n} R_i x_i \le z\right) & \ge \alpha, \\ \sum_{i=1}^{n} \mathbb{E}[R_i] \cdot x_i & \ge r_{min}, \\ \sum_{i=1}^{n} x_i = 1,\ x_i & \ge 0, \end{aligned}$$

where R_i is random rate of return of $i-$th asset and minimal expected return r_{min} is selected in such way that the problem is feasible.

Homework 2

1. Rewrite the VaR minimization problem under a finite discrete distribution as a mixed-integer LP problem.
2. Use the same dataset as for the CVaR homework, i.e. at least 6 assets, but the number of scenarios is limited to 50 (if you have free GAMS, otherwise you can use all 100 returns).
3. Consider $\alpha = 0.95$ and run the problem for different 11 values $r_0 \in \{\min_i \overline{R}_i, \dots, \max_i \overline{R}_i\}$.
4. Plot the optimal values VaR_α against the corresponding values of r_0.

Joint chance constrained problems

Let $f, g_j(\cdot, \xi) : \mathbb{R}^n \to \mathbb{R}$ be real functions, ξ be a real random vector, $\varepsilon \in (0,1)$ small:

$$\min_{x \in X} f(x)$$
$$\text{s.t.} \quad P\left(g_1(x,\xi) \le 0, \dots, g_m(x,\xi) \le 0\right) \ge 1 - \varepsilon.$$

INTERPRETATION: for a given $x \in X$, the probability of ξ for which all (!) random constraints are fulfilled must be at least $1 - \varepsilon$.

Joint chance constrained problems

Let ξ have a finite discrete distribution with realizations $\xi^1, \ldots, \xi^S$ and probabilities $p_s > 0$, $\sum_{s=1}^S p_s = 1$:

$$\begin{aligned} \min_{x,y} f(x) & \\ \text{s.t.} & \\ \sum_{s=1}^S p_s y_s &\geq 1 - \varepsilon, \\ g_1(x, \xi_s) - M(1 - y_s) &\leq 0, \ s = 1, \ldots, S \\ &\vdots \\ g_m(x, \xi_s) - M(1 - y_s) &\leq 0, \ s = 1, \ldots, S, \\ y_s &\in \{0, 1\}, \ s = 1, \ldots, S, \\ x &\in X, \end{aligned} \tag{2}$$

where

$$M \geq \max_{j=1,\ldots,m} \max_{s=1,\ldots,S} \sup_{x \in X} g_j(x, \xi_s).$$

= A large mixed-integer (nonlinear) programming problem, Raike (1970).

Literature

- M. Branda, M.: Solving real-life portfolio problem using stochastic programming and Monte-Carlo techniques. Proceedings of the 28th International Conference on Mathematical Methods in Economics 2010, 67–72.
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