LEE-CARTER

LG1

SURVIVAL function:

$$S_x(t) = P(T_x > t) = {}_tp_x$$

MORTALITY RATE

$$m_x = \frac{\int_0^1 S_0(x+u)\,\mu_{x+u}\,du}{\int_0^1 S_0(x+u)\,du}, \quad \text{where}$$

① $$\int_0^1 S_0(x+u)\,\mu_{x+u}\,du = \int_0^1 {}_{x+u}p_0\,\mu_{x+u}\,du =$$

$$= \cancel{P(T_0 \leq x+1) - P(T_0 \leq x)}$$

$$= P(T_0 < x+1 \mid T_0 > x) = P(T_x < 1) =$$

$$= S_0(x) - S_0(x+1) = {}_xp_0 - {}_{x+1}p_0 = {}_xp_0 \cdot \underbrace{(1-p_x)}_{=q_x}$$

② $$\int_0^1 S_0(x+u)\,du \approx \frac{S_0(x) + S_0(x+1)}{2} =$$

$$= \frac{{}_xp_0 \cdot (1 + p_x)}{2} = \frac{{}_xp_0 \cdot (1 + \overbrace{1 - 1 + p_x}^{-q_x})}{2} =$$

$$= \frac{{}_xp_0 \cdot (2 - q_x)}{2}$$

① & ②
$$\Rightarrow \quad m_x \cong \frac{2q_x}{2 - q_x} \quad !!!$$

$$\ln m_{x,t} = a_x + b_x \cdot k_t + \varepsilon_{x,t}$$

$m_{x,t}$ ... m. r. at age x during year t

a_x, b_x ... age specific constants

k_t ... unobservable time specific index

$\varepsilon_{x,t}$ ... white noise $\sim 0, \sigma_\varepsilon^2$

(& homoscedasticity)

Lee - Carter (1992)

→ NORMALIZATION : $\sum_{x=x_1}^{x_n} b_x = 1$

$$\sum_{t=t_1}^{t_n} k_t = 0$$

$x \in \{x_1, \dots, x_n\}$ ... not necessarily à 1 y

$t \in \{t_1, \dots, t_n\}$ ... —"— equidistant (à 1 y)

BUT WE ASSUME THAT !

Parameters estimation

1) Ordinary Least Squares

$$O_{LS}(a, b, k) = \sum_x \sum_t \left(\ln m_{x,t} - a_x - b_x k_t\right)^2$$

$\rightsquigarrow$ MIN

LC 2

$$\hat{a}_x = \frac{1}{t_n - t_1 + 1} \sum_{t=t_1}^{t_n} \ln \hat{m}_{x,t}$$

$$\to \quad z_{x,t} := \ln m_{x,t} - \hat{a}_x \, , \quad \forall_x \forall_t$$

$$\to \quad \tilde{O}_{LS}(b, k) = \sum_x \sum_t \left(z_{x,t} - b_x k_t \right)^2$$

$$A := \{ z_{x,t} \}_{x,t} \quad m \times n \, .$$

<u>Singular value decomposition</u> (SVD):

$$\underset{m \times m}{U} \; \underset{m \times n}{\Sigma} \; \underset{n \times n}{V^T} = A$$

- eigenvectors "u" of $A A^T$ $(m \times m)$
- — // — "v" of $A^T A$ $(n \times n)$
- eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$, $S = \cancel{diag}$ $\text{diag}\{\sqrt{\lambda_1} \dots \sqrt{\lambda_n}\}$

(*) $\to \quad \hat{b}_x = \dfrac{u_1}{\sum_{j=1}^{x_m - x_1 + 1} u_{1j}} \; (\neq 0)$ $\qquad \Sigma = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$

$$\hat{k} = \sqrt{\lambda_1} \cdot v_1 \cdot \left(\sum_{j=1}^{x_m - x_1 + 1} u_{1j} \right)$$ & additional normalization

(*) the best rank 1 approx of.

~~2)~~ Newton - Raphson alg.

3) <u>Maximum - likelihood method</u>

$\approx$ "generalized GLM"

$$D_{x,t} \sim Po\left(E_{x,t} \exp\{a_x + b_x \cdot k_t\}\right)$$

offset

$$\exp\left(\underline{\ln E_{x,t}}\right)$$

$$m_{x,t} = \frac{D_{x,t}}{E_{x,t}} \begin{array}{l} \text{.. no. of deaths at age } x \\ \text{.. no of living at age } x \end{array}$$

A Newton - Raphson & normalization

GENERALIZATION: Renshaw - Haberman (2006): a cohort effect (unobservable) $+ c_x i_{t-x}$

• age of birth

<u>Prediction</u> $x \in \{x_1, \dots, x_m\}$, but $\boxed{t > t_n}$

$\to k_t = ?$ $\boxed{\text{ARIMA}}$

$t = 2000$ napr.

$x = 20 \to 1980$

$= 30 \to 1970$

$= 40 \to 1960$