Optimization with application in finance – exercises

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HW: Example 6.2.

6 Stochastic dominance

Consider a random vector of returns (R_1, R_2) with the marginal **cumulative distribution functions**

$$F_{R_i}(x) = P(R_i \le x),$$

and the **twice cumulative probability distribution functions**

$$F^{(2)}_{R_i}(x) = \int_{-\infty}^{x} F_{R_i}(\nu)\, d\nu = \mathbb{E}[x - R_i]^+.$$

Then

1. FSD dominance: $R_1 \prec_{FSD} R_2$ if and only if $F_{R_1}(x) \ge F_{R_2}(x)$ for all x with at least one inequality strict.
2. SSD dominance: $R_1 \prec_{SSD} R_2$ if and only if $F^{(2)}_{R_1}(x) \ge F^{(2)}_{R_2}(x)$ for all x with at least one inequality strict.

Consider a random vector of returns (R_1, R_2) under a discrete distribution with **equiprobable realizations** (r_{1s}, r_{2s}), $s = 1, \dots, S$.

0. Sort the realizations for each asset, i.e.

$$r_{1[1]} < r_{1[2]} < \dots < r_{1[S]}, \quad r_{2[1]} < r_{2[2]} < \dots < r_{2[S]}.$$

1. FSD dominance: $R_1 \prec_{FSD} R_2$ iff $r_{1[s]} \le r_{2[s]}$ for $s = 1, \dots, S$ with at least one inequality strict.
2. SSD dominance: $R_1 \prec_{SSD} R_2$ iff $\sum_{s=1}^{t} r_{1[s]} \le \sum_{s=1}^{t} r_{2[s]}$ for $t = 1, \dots, S$ with at least one inequality strict.

Example 6.1 *Consider three assets (in rows) with the following realizations of discrete random returns (in columns)*

	1	*2*	*3*
R_1	*2*	*1*	*5*
R_2	*0*	*6*	*4*
R_3	*1*	*3*	*5*

1. *Identify FSD and SSD efficient assets if the realizations are equiprobable, i.e.* $p_1 = p_2 = p_3 = 1/3$.
2. *Add a nontrivial portfolio and identify FSD and SSD efficient portfolios again.*

3. Change the probabilities of realizations (columns) to $p_1 = 0.5$ and $p_2 = p_3 = 0.25$.

Solution: 1. Since the realizations are equiprobable, we can apply the approach reviewed above:

	1	2	3	[1]	[2]	[3]	[1]	[1+2]	[1+2+3]
R_1	2	1	5	1	2	5	1	3	8
R_2	0	6	4	0	4	6	0	4	10
R_3	1	3	5	1	3	5	1	4	9

We can see that the only SD dominance is $R_1 \prec_{FSD} R_3$, and $R_1 \prec_{SSD} R_3$, therefore we obtain

$$FSD - eff. = SSD - eff. = \{R_2, R_3\}.$$

2. We consider R_1, R_2, R_3 as trivial portfolios built out of one asset and add $R_4 = 0.5\,R_1 + 0.5\,R_2$. Then we have

	1	2	3	[1]	[2]	[3]	[1]	[1+2]	[1+2+3]
R_1	2	1	5	1	2	5	1	3	8
R_2	0	6	4	0	4	6	0	4	10
R_3	1	3	5	1	3	5	1	4	9
R_4	1	3.5	4.5	1	3.5	4.5	1	4.5	9

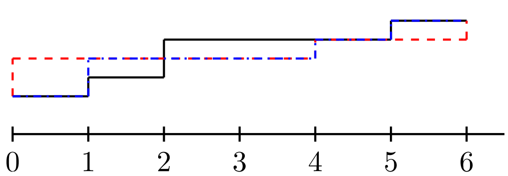
We obtained a new dominance $R_3 \prec_{SSD} R_4$, i.e.

$$\{R_2, R_3, R_4\} = FSD - eff. \supset SSD - eff. = \{R_2, R_4\}.$$

3. If we change the probabilities to $p_1 = 0.5$ and $p_2 = p_3 = 0.25$, the situation is different. To investigate the FSD relations, we must compare the cumulative distribution functions in all realizations of the random returns:

x	0	1	2	3	4	5	6
$P(R_1 \le x)$	0	0.25	0.75	0.75	0.75	1	1
$P(R_2 \le x)$	0.5	0.5	0.5	0.5	0.75	0.75	1
$P(R_3 \le x)$	0	0.5	0.5	0.75	0.75	1	1

We can also use the picture of CDFs



We can conclude that there is no dominance relation, i.e. all assets are FSD efficient. To investigate SSD efficiency, we must evaluate the twice cumulative probability distribution function

$$F^{(2)}_{R_i}(x) = \int_{-\infty}^{x} F_{R_i}(\nu)\, d\nu = \mathbb{E}[x - R_i]^+.$$

For our assets, we obtain

$$\begin{aligned}
F^{(2)}_{R_1}(x) &= 0.25[x-1]^+ + 0.5[x-2]^+ + 0.25[x-5]^+,\\
F^{(2)}_{R_2}(x) &= 0.5[x-0]^+ + 0.25[x-4]^+ + 0.25[x-6]^+,\\
F^{(2)}_{R_3}(x) &= 0.5[x-1]^+ + 0.25[x-3]^+ + 0.25[x-5]^+,
\end{aligned}$$

which are piecewise linear functions, i.e. we must compare their values in all considered realizations:

x	0	1	2	3	4	5	6
$F^{(2)}_{R_1}(x)$	0	0	0.25	1.0	1.75	2.5	3.5
$F^{(2)}_{R_2}(x)$	0	0.5	1.0	1.5	2.0	2.75	3.5
$F^{(2)}_{R_3}(x)$	0	0	0.5	1.0	1.75	2.5	3.5

Remind that $R_i \succ_{SSD} R_j$ if and only if $F^{(2)}_{R_i}(x) \le F^{(2)}_{R_j}(x)$ for all real x with at least one inequality strict. In our case

$$\begin{aligned}
&F^{(2)}_{R_1}(x) \le F^{(2)}_{R_2}(x) \text{ and } F^{(2)}_{R_1}(1) < F^{(2)}_{R_2}(1), \text{ i.e. } R_1 \succ_{SSD} R_2,\\
&F^{(2)}_{R_3}(x) \le F^{(2)}_{R_2}(x) \text{ and } F^{(2)}_{R_3}(1) < F^{(2)}_{R_2}(1), \text{ i.e. } R_3 \succ_{SSD} R_2,\\
&F^{(2)}_{R_1}(x) \le F^{(2)}_{R_3}(x) \text{ and } F^{(2)}_{R_1}(2) < F^{(2)}_{R_3}(2), \text{ i.e. } R_1 \succ_{SSD} R_3,
\end{aligned}$$

Therefore, we have obained

$$\{R_1, R_2, R_3\} = FSD - eff. \supset SSD - eff. = \{R_1\}.$$

Realize also that

$$\mathbb{E}[R_1] = \mathbb{E}[R_2] = \mathbb{E}[R_3] = \frac{5}{6}.$$

□

Example 6.2 *Consider a person with an income of 100 CZK which is deciding whether to accept a 10 CZK bet on the toss of a fair coin. Compare these two possibilities with respect to the first and second order stochastic dominance.*

Solution (outline): Consider two random variables: one degenerate with realization 100 with probability 1, and second one with realizations 90 and 110 with probabilities 1/2. Compare the cdf and the twice-cdf.