Lagrangian duality

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Computational Aspects of Optimization

Nonlinear Programming Problem (NLP)

Primal problem (P):

$$(P) = \min_{x \in X} f(x) \text{ s.t. } g_j(x) \le 0,\ j = 1, \dots, m,$$
$$h_i(x) = 0,\ i = 1, \dots, l.$$

Lagrangian function, $u \in \mathbb{R}^m_+$, $v \in \mathbb{R}^l$:

$$L(x, u, v) = f(x) + \sum_{j=1}^{m} u_j g_j(x) + \sum_{i=1}^{l} v_i h_i(x).$$

Dual problem

Dual function:

$$\theta(u, v) = \inf_{x \in X} L(x, u, v). \tag{1}$$

Dual problem (D):

$$(D) = \sup_{u \ge 0, v} \theta(u, v). \tag{2}$$

Weak Duality Theorem

Theorem

Let x be feasible for problem (P) and (u, v) be feasible for problem (D). Then

$$\theta(u, v) \le f(x).$$

Proof.

$$\theta(u, v) = \inf_{y} L(y, u, v) \le L(x, u, v) \le f(x),$$

where the last inequality follows from feasibility of x and (u, v), when $u_j g_j(x) \le 0$ and $v_i h_i(x) = 0$.

Weak Duality Theorem – Consequences

1. We obtain
$$(P) \geq (D).$$
2. If for some primal feasible $\overline{x}$ and dual feasible $(\overline{u}, \overline{v})$ holds
$$f(\overline{x}) = \theta(\overline{u}, \overline{v}),$$
then $\overline{x}$ is optimal solution of (P) and $(\overline{u}, \overline{v})$ is optimal solution of (D).
3. If $(P) = -\infty$ (unbounded primal problem), then $\theta(u, v) = -\infty$ for all $(u, v) \in \mathbb{R}^m_+ \times \mathbb{R}^l$.
4. If $(D) = \infty$, then (P) is infeasible.

Strong Duality Theorem

Theorem

Let

- X *be a nonempty convex set*
- f, g_j *be* convex
- h_i *be* affine
- Slater condition *be satisfied, i.e. there is* $\hat{x} \in X$ *such that* $g_j(\hat{x}) < 0, \forall j$ *and* $h_i(\hat{x}) = 0, \forall i$, *and* $0 \in \mathrm{int}\{(h_1(x), \dots, h_l(x)) : \ x \in X\} := h(X)$.

Then $(P) = (D)$.
Moreover, if (P) *is finite, then* sup *in (D) is achieved at* $(\overline{u}, \overline{v}) \in \mathbb{R}^m_+ \times \mathbb{R}^l$.
If inf *in (P) is achieved at* $\overline{x}$, *then* $\sum_{j=1}^m \overline{u}_j g_j(\overline{x}) = 0$.

A counterexample

Convexity alone is not sufficient. Consider

$$\begin{aligned} p^* = \min_{x,y} \ & e^{-x} \\ \text{s.t. } & x^2/y \leq 0, \\ & y > 0 \ \ (\text{or } y \geq \varepsilon). \end{aligned}$$

The optimal value is $p^* = 1$. The dual function is equal to

$$\theta(u) = \inf_{x,y>0} e^{-x} + ux^2/y = \begin{cases} 0 & u \geq 0, \\ -\infty & u < 0. \end{cases}$$

The dual problem is
$$d^* = \max_{u \geq 0} \theta(u)$$
with optimal value $d^* = 0$. Slater condition is not satisfied since $x = 0$ for any feasible (x, y), i.e. $x^2/y = 0$.

SDT proof

Bazaraa et al. (2006), Lemma 6.2.3:

Lemma

Let $X \subseteq \mathbb{R}^n$ *be a convex set,* $f, g_j : \mathbb{R}^n \to \mathbb{R}$ *be convex,* $h_i : \mathbb{R}^n \to \mathbb{R}$ *be affine. If System 1 has no solution, then System 2 has a solution* (u_0, u, v). *The converse holds true if* $u_0 > 0$.

System 1: $f(x) < 0$, $g_j(x) \leq 0$, $h_i(x) = 0$ *for some* $x \in X$.

System 2: $u_0 f(x) + \sum_{j=1}^m u_j g_j(x) + \sum_{i=1}^l v_i h_i(x) \geq 0$ *for all* $x \in X$, $(u_0, u) \geq 0$, $(u_0, u, v) \neq 0$.

SDT proof

Let γ be a (finite) optimal value of (P) and consider the following system:

$$f(x) - \gamma < 0,\ g_j(x) \le 0, j = 1, \ldots, m,\ h_i(x) = 0, i = 1, \ldots, l,\ x \in X.$$

By the definition of γ the system has no solution. Hence, there exists $(u_0, u, v) \neq 0$ with $(u_0, u) \ge 0$ such that

$$u_0(f(x) - \gamma) + \sum_{j=1}^{m} u_j g_j(x) + \sum_{i=1}^{l} v_i h_i(x) \ge 0,\ \forall x \in X.$$

SDT proof

Suppose that $u_0 = 0$. By assumption there is an $\hat{x} \in X$ such that $g_j(\hat{x}) < 0, \forall j$ and $h_i(\hat{x}) = 0, \forall i$. Substituting into the inequality we obtain $\sum_{j=1}^{m} u_j g_j(\hat{x}) \ge 0$. Since $g_j(\hat{x}) < 0, \forall j$, we have $u_j = 0, \forall j$, and $u_0 = 0$. This implies that $\sum_{i=1}^{l} v_i h_i(x) \ge 0$ for all $x \in X$. Since $0 \in h(X)$, we can pick a $x \in X$ such that $h_i(x) = -\lambda v_i$, where $\lambda > 0$ (small). Therefore

$$\sum_{i=1}^{l} v_i h_i(x) = -\lambda \sum_{i=1}^{l} v_i^2 \ge 0,$$

which implies that $v_i = 0, \forall i$. But this is a contradiction with $(u_0, u, v) \neq 0$. Hence $u_0 > 0$...

SDT proof

Hence $u_0 > 0$. Thus, if we set $\tilde{u}_j = u_j / u_0$ and $\tilde{v}_i = v_i / u_0$, we get

$$f(x) + \sum_{j=1}^{m} \tilde{u}_j g_j(x) + \sum_{i=1}^{l} \tilde{v}_i h_i(x) \ge \gamma,\ \forall x \in X.$$

This shows that

$$\theta(\tilde{u}, \tilde{v}) = \inf_{x \in X} L(x, \tilde{u}, \tilde{v}) \ge \gamma.$$

Together with the Weak Duality Theorem we obtain that

$$\gamma = \theta(\tilde{u}, \tilde{v}) = \sup_{u \ge 0, v} \theta(u, v).$$

Example: Linear programming duality

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b, \\ & x \ge 0. \end{aligned}$$

Example: Linear programming

For $u \geq 0$

$$\begin{aligned} L(x,u,v) &= c^T x - u^T x + v^T (Ax - b) \\ &= c^T x - u^T x + v^T Ax - v^T b \\ &= (c^T - u^T + v^T A)x - v^T b. \end{aligned}$$

Then the dual function

$$\begin{aligned} \theta(u,v) &= \inf_x L(x,u,v) \\ &= -v^T b, \text{ if } c^T - u^T + v^T A = 0, \\ &= -\infty, \text{ if } c^T - u^T + v^T A \neq 0. \end{aligned}$$

Then the Lagrange dual problem is

$$\begin{aligned} \max\ & -b^T v \\ \text{s.t. } & c - u + A^T v = 0. \end{aligned}$$

Example: Linear programming

If we substitute $\tilde{v} = -v$ and realize that u can be seen as a vector of slack variables, we obtain

$$\begin{aligned} \max\ & b^T \tilde{v} \\ \text{s.t. } & A^T \tilde{v} \leq c, \end{aligned}$$

which is the standard LP dual.

Example: Ordinary least squares with equality constraints

$$\begin{aligned} \min\ & \|Ax - b\|_2^2 \\ \text{s.t. } & Fx = g. \end{aligned}$$

Langrangian lower bound is never worse than LP relaxation

Hooker (2009): Consider integer programming problem with complicated constraints $Ax \leq a$ and noncomplicated constraints $Bx \leq b$:

$$\begin{aligned} \min_x\ & c^T x \\ \text{s.t. } & Ax \leq a, \\ & Bx \leq b, \\ & x \in \mathbb{Z}^n_+. \end{aligned}$$

Langrangian lower bound is never worse than LP relaxation

Dual function obtained by relaxing the complicated constraints $Ax \le a$:

$$\begin{aligned}\theta(u) = \min_x\ & c^T x + u^T(Ax - a)\\ \text{s.t. } & Bx \le b,\\ & x \in \mathbb{Z}^n_+.\end{aligned}$$

Let $S = \{x \in \mathbb{Z}^n_+ :\ Bx \le b\}$, then the dual function can be rewritten as

$$\begin{aligned}\theta(u) = \min_x\ & c^T x + u^T(Ax - a)\\ \text{s.t. } & x \in \mathrm{conv}(S),\end{aligned}$$

where $\mathrm{conv}(S)$ can be described by (a large number of) linear inequalities.

The optimal value of the dual problem

$$z_{LD} = \max_{u \ge 0} \theta(u)$$

is therefore equal to (it follows from LP duality)

$$\begin{aligned}z_{LD} = \min_x\ & c^T x\\ \text{s.t. } & Ax \le a,\\ & x \in \mathrm{conv}(S).\end{aligned}$$

Let $P = \{x \in \mathbb{R}^n_+ :\ Bx \le b\}$, i.e. $\mathrm{conv}(S) \subseteq P$, where the LP relaxation is

$$\begin{aligned}z_{LP} = \min_x\ & c^T x\\ \text{s.t. } & Ax \le a,\\ & x \in P,\end{aligned}$$

i.e. $z_{LP} \le z_{LD}$.

Generalized Benders Decomposition

Geoffrion (1972), Floudas (2009):

$$\begin{aligned}\min_{x,y}\ & f(x,y)\\ \text{s.t. } & g_j(x,y) \le 0,\ j = 1,\dots,m,\\ & x \in X, y \in Y.\end{aligned}$$

The problem can be rewritten as

$$\begin{aligned}\min_y \inf_x\ & f(x,y)\\ \text{s.t. } & g_j(x,y) \le 0,\ j = 1,\dots,m,\\ & x \in X, y \in Y.\end{aligned}$$

Generalized Benders Decomposition

Assumptions:

- $X \subseteq \mathbb{R}^n$ is a nonempty **compact convex** set, $Y \subseteq \mathbb{R}^s$, e.g. $Y = \{0,1\}^s$.
- $f(\cdot,y), g_j(\cdot,y) : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$ are **continuous convex** for each $y \in Y$.
- For each $y \in Y \cap V$, where

  $$V = \{y : g_j(\cdot,y) \le 0, \forall_j \text{ for some } x \in X\},$$

  the resulting problem is unbounded or is feasible and the Lagrange multipliers exist (under Slater CQ).

(Less stringent assumptions are available, see Floudas (2009).)

Generalized Benders Decomposition

Master problem

$$\begin{aligned} &\min\ v(y) \\ &\text{s.t. } y \in Y \cap V, \end{aligned}$$

where the **primal** (slave) **problem** is

$$\begin{aligned} v(y) = \inf_x\ & f(x,y) \\ \text{s.t. } & g_j(x,y) \le 0,\ j = 1,\dots,m, \\ & x \in X. \end{aligned}$$

We assume that $v(y)$ can be computed easily ...

Generalized Benders Decomposition

Feasibility Lagrange function: if the primal problem is infeasible for a given $y \in Y$, then consider

$$\overline{L}(x,y,u) = \sum_{j=1}^m u_j g_j(x,y),$$

where $u \in \Lambda = \{u \in \mathbb{R}^m_+ :\ \sum_{j=1}^m u_j = 1\}$. We obtain $y \in V$ if and only if

$$\sup_{u\in\Lambda} \inf_{x\in X} \overline{L}(x,y,u) \le 0.$$

... based on Lagrangian duality for the problem

$$\begin{aligned} \min_x\ & \sum_{i=1}^n 0x_i \\ \text{s.t. } & g_j(x,y) \le 0,\ j = 1,\dots,m, \\ & x \in X. \end{aligned}$$

Generalized Benders Decomposition

Optimality Lagrange function: if the primal problem is feasible for a fixed $y \in Y$, then (under Slater CQ) we can use the Lagrange function

$$L(x,y,u) = f(x,y) + \sum_{j=1}^m u_j g_j(x,y),$$

and the strong duality, i.e. for each $y \in Y \cap V$ we have

$$\begin{aligned} v(y) = & \inf_{x\in X} f(x,y) \text{ s.t. } g_j(x,y) \le 0,\ j = 1,\dots,m, \\ & = (SD) = \\ & = \sup_{u\ge 0} \inf_{x\in X} L(x,y,u). \end{aligned}$$

Generalized Benders Decomposition

Combining the feasibility and optimality Lagrange functions, we obtain an equivalent problem

$$\begin{aligned} \min_{y,\mu}\ & \mu \\ \text{s.t. } & \mu \ge \sup_{u\ge 0} \inf_{x\in X} L(x,y,u), \\ & 0 \ge \sup_{u\in\Lambda} \inf_{x\in X} \overline{L}(x,y,u), \\ & y \in Y, \end{aligned}$$

or

$$\begin{aligned} \min_{y,\mu}\ & \mu \\ \text{s.t. } & \mu \ge \inf_{x\in X} L(x,y,u), \forall u \ge 0, \\ & 0 \ge \inf_{x\in X} \overline{L}(x,y,u), \forall u \in \Lambda, \\ & y \in Y. \end{aligned}$$

The support vector classifier

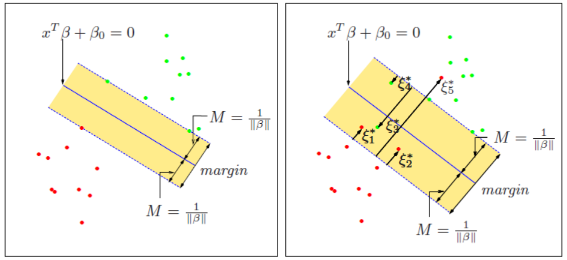
Hastie et al. (2009): Training data: N pairs (x_1, y_1), (x_2, y_2), ..., (x_N, y_N), $x_i \in \mathbb{R}^p$, $y_i \in \{-1, 1\}$ (classes).
A linear classification rule with $\|\beta\| = 1$

$$G(x) = \mathrm{sign}[x^T\beta + \beta_0].$$

Assume first that the data are separable. We would like to find **the biggest margin** between the training points for class 1 and -1:

$$\begin{aligned} &\max_{\beta_0,\beta} M \\ &\text{s.t. } y_i(x_i^T\beta + \beta_0) \ge M, \ i = 1, \dots, N, \\ &\quad \|\beta\| = 1. \end{aligned}$$

The support vector classifier



Hastie et al. (2009)

The support vector classifier

By setting $M = 1/\|\beta\|$:

$$\begin{aligned} &\min_{\beta_0,\beta} \|\beta\| \\ &\text{s.t. } y_i(x_i^T\beta + \beta_0) \ge 1, \ i = 1, \dots, N. \end{aligned}$$

If the classes overlap:

$$\begin{aligned} &\min_{\beta_0,\beta,\xi} \frac{1}{2}\|\beta\|^2 + C\sum_{i=1}^N \xi_i \\ &\text{s.t. } y_i(x_i^T\beta + \beta_0) \ge 1 - \xi_i, \ i = 1, \dots, N, \\ &\quad \xi_i \ge 0, \end{aligned}$$

where we penalize the overall overlap.

The support vector classifier

Lagrange function

$$\begin{aligned} L(\beta_0, \beta, \xi, \alpha, \mu) = &\frac{1}{2}\|\beta\|^2 + C\sum_{i=1}^N \xi_i - \sum_{i=1}^N \mu_i \xi_i \\ &- \sum_{i=1}^N \alpha_i (y_i(x_i^T\beta + \beta_0) - 1 + \xi_i), \ \alpha_i \ge 0, \mu_i \ge 0. \end{aligned}$$

The dual function

$$\theta(\alpha, \mu) = \inf_{\beta_0,\beta,\xi} L(\beta_0, \beta, \xi, \alpha, \mu).$$

The support vector classifier

$$L(\beta_0, \beta, \xi, \alpha, \mu) = \frac{1}{2}\|\beta\|^2 + C\sum_{i=1}^{N}\xi_i - \sum_{i=1}^{N}\mu_i\xi_i - \sum_{i=1}^{N}\alpha_i(y_i(x_i^T\beta + \beta_0) - 1 + \xi_i), \;\; \alpha_i \geq 0, \mu_i \geq 0$$

Use the derivatives to obtain the dual function:

$$\frac{\partial L}{\partial \beta_0} = \sum_{i=1}^{N}\alpha_i y_i = 0,$$

$$\frac{\partial L}{\partial \beta} = \beta - \sum_{i=1}^{N}\alpha_i y_i x_i = 0,$$

$$\frac{\partial L}{\partial \xi_i} = C - \alpha_i - \mu_i = 0.$$

The support vector classifier

We can express the dual function

$$\begin{aligned}\theta(\alpha,\mu) &= \frac{1}{2}\sum_{i=1}^{N}\sum_{i'=1}^{N}\alpha_i\alpha_{i'}y_iy_{i'}x_i^Tx_{i'} + C\sum_{i=1}^{N}\xi_i - \sum_{i=1}^{N}\sum_{i'=1}^{N}\alpha_i\alpha_{i'}y_iy_{i'}x_i^Tx_{i'} \\ &\quad -\beta_0\sum_{i=1}^{N}\alpha_iy_i + \sum_{i=1}^{N}\alpha_i - \sum_{i=1}^{N}\alpha_i\xi_i - \sum_{i=1}^{N}\mu_i\xi_i \\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{i'=1}^{N}\alpha_i\alpha_{i'}y_iy_{i'}x_i^Tx_{i'} + \sum_{i=1}^{N}\alpha_i,\end{aligned}$$

subject to $0 \leq \alpha_i \leq C$, $\sum_{i=1}^{N}\alpha_iy_i = 0$.

Literature

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