Algorithms for nonlinear programming problems II

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Computational Aspects of Optimization

Algorithm convergence

Definition

Let $X \subseteq \mathbb{R}^p$, $Y \subseteq \mathbb{R}^q$ be nonempty closed sets. Let $F : X \to Y$ be a set-valued mapping. The map F is said to be **closed** at $x \in X$ if for any sequences $\{x^k\} \subset X$, and $\{y^k\}$ satisfying $x_k \to x$, $y^k \in F(x^k)$, $y^k \to y$ we have that $y \in F(x)$.
The map F is said to be closed on $Z \subseteq X$ if it is closed at each point in Z.

Algorithm convergence – Zangwill's theorem

Bazaraa et al. (2006), Theorem 7.2.3: Let

A1. $X \subseteq \mathbb{R}^p$ be a nonempty **closed** set

A2. $\hat{X} \subseteq X$ be a **nonempty solution set**

A3. $F : X \to X$ be a set-valued mapping **closed** over complement of $\hat{X}$

A4. Given $x^1 \in X$ the sequence $\{x^k\}$ is generated iteratively as follows: If $x^k \in \hat{X}$, then STOP; otherwise, let $x^{k+1} \in F(x^k)$ and repeat.

A5. the sequence $x^1, x^2, \ldots$ be contained in a **compact** subset of X

A6. there exist a **continuous function**[1] α such that $\alpha(y) < \alpha(x)$ if $x \notin \hat{X}$ and $y \in F(x)$

Then either the algorithm stops in a finite number of steps with a point in $\hat{X}$ or it generates an infinite sequence $\{x^k\}$ such that all accumulation points belong to $\hat{X}$ and $\alpha(x^k) \to \alpha(x)$ for some $x \in \hat{X}$.

[1] descent function: $\alpha(x) = f(x)$ or $\alpha(x) = \|\nabla f(x)\|$

Algorithm convergence – Newton method

Let $\overline{x}$ be an optimal solution, set

$$\begin{aligned} F(x) &= x - \left(\nabla^2 f(x)\right)^{-1} \nabla f(x), \\ \alpha(x) &= \|x - \overline{x}\|. \end{aligned}$$

More details: Bazaraa et al. (2006), Theorem 8.6.5

Algorithm classification

- **Order of derivatives**[2]**:** derivative-free, first order (gradient), second-order (Newton)
- **Feasibility of the constructed points**: interior and exterior point methods
- **Deterministic/randomized**
- **Local**/global

[2] If possible, deliver the derivatives.

Barrier method

$f : \mathbb{R}^n \to \mathbb{R}$, $g_j : \mathbb{R}^n \to \mathbb{R}$

$$\min_x f(x) \text{ s.t. } g_j(x) \le 0,\ j = 1, \dots, m.$$

(Extension including equality constraints is straightforward.)

Assumption

$$\{x : \ g(x) < 0\} \neq \emptyset.$$

Barrier method

Barrier functions: continuous with the following properties

$$B(y) \geq 0, y < 0, \lim_{y \to 0_-} B(y) = \infty.$$

Examples

$$\frac{-1}{y}, \; -\log \min\{1, -y\}.$$

Maybe the most popular barrier function:

$$B(y) = -\log -y$$

Set

$$\tilde{B}(x) = \sum_{j=1}^{m} B\big(g_j(x)\big),$$

and solve

$$\min f(x) + \mu \tilde{B}(x), \tag{1}$$

where $\mu > 0$ is a parameter.

Polynomial-time interior point methods for LP

$$\min\ c^T x - \mu \sum_{j=1}^{n} \log x_j \qquad \text{s.t. } Ax = b. \tag{2}$$

KKT-conditions ...

Interior point methods

$f : \mathbb{R}^n \to \mathbb{R}$, $g_j : \mathbb{R}^n \to \mathbb{R}$

$$\min_x f(x) \text{ s.t. } g_j(x) \le 0,\ j = 1, \dots, m.$$

(Extension including equality constraints is straightforward.)

New slack decision variables $s \in \mathbb{R}^m$

$$\begin{aligned} \min_{x,s}\ & f(x) \\ \text{s.t. } & g_j(x) + s_j = 0,\ j = 1, \dots, m, \\ & s_j \ge 0. \end{aligned} \tag{3}$$

Interior point methods

Barrier problem

$$\begin{aligned} \min_{x,s}\ & f(x) - \mu \sum_{j=1}^{m} \log s_j \\ \text{s.t. } & g(x) + s = 0. \end{aligned} \tag{4}$$

The barrier term prevents the components of s from becoming too close to zero.

Lagrangian function

$$L(x,s,z) = f(x) - \mu \sum_{j=1}^{m} \log s_j - \sum_{j=1}^{m} z_j (g_j(x) + s_j).$$

Interior point methods

KKT conditions for barrier problem (matrix notation)

$$\begin{aligned}
\nabla f(x) - \nabla g^T(x) z &= 0,\\
-\mu S^{-1} e - z &= 0,\\
g(x) + s &= 0,
\end{aligned}$$

$S = \mathrm{diag}\{s_1, \ldots, s_m\}$, $Z = \mathrm{diag}\{z_1, \ldots, z_m\}$, $\nabla g(x)$ is the Jacobian matrix (components of function in rows?)

Multiply the second equality by S

$$\begin{aligned}
\nabla f(x) - \nabla g^T(x) z &= 0,\\
-SZe &= \mu e,\\
g(x) + s &= 0,
\end{aligned}$$

$=$ Nonlinear system of equalities $\rightarrow$ Newton's method

Newton's method

$$\nabla f(x+v) = \nabla f(x) + \nabla^2 f(x) v = 0$$

with the solution (under $\nabla^2 f(x) \succ 0$)

$$v = -\left(\nabla^2 f(x)\right)^{-1} \nabla f(x)$$

Interior point methods

Use Newton's method to obtain a step $(\Delta x, \Delta s, \Delta z)$

$$\begin{pmatrix} H(x,z) & 0 & -\nabla g^T \\ 0 & -Z & -S \\ \nabla g & I & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta s \\ \Delta z \end{pmatrix} = \begin{pmatrix} -\nabla f(x) + \nabla g^T(x) z \\ SZe + \mu e \\ -g(x) - s \end{pmatrix}$$

$H(x,z) = \nabla^2 f(x) - \sum_{j=1}^m z_j \nabla^2 g_j(x)$, $\nabla^2 f$ denotes the Hessian matrix.

Interior point methods

Stopping criterion:

$$E = \max \left\{ \left\| \nabla f(x) - \nabla g^T(x) z \right\|, \| SZe + \mu e \|, \| g(x) + s \| \right\} \leq \varepsilon,$$

$\varepsilon > 0$ small.

Interior point methods

ALGORITHM:

0. Choose x^0 and $s^0 > 0$, and compute initial values for the multipliers $z^0 > 0$. Select an initial barrier parameter $\mu^0 > 0$ and parameter $\sigma \in (0, 1)$, set $k = 1$.
1. Repeat until a stopping test for the nonlinear program (19.1) is satisfied:
    - Solve the nonlinear system of equalities using Newton's method and obtain (x^k, s^k, z^k).
    - Decrease barrier parameter $\mu^{k+1} = \sigma\mu^k$, set $k = k + 1$.

Interior point methods

Convergence of the method (Nocedal and Wright 2006, Theorem 19.1): continuously differentiable f, g_j, LICQ at any limit point, then the limits are stationary points of the original problem

Interior point method – Example

$$\begin{aligned} \min_{x}\ & (x_1-2)^4+(x_1-2x_2)^2 \\ \text{s.t. } & x_1^2-x_2\le 0. \end{aligned} \tag{5}$$

$$\begin{aligned} \min_{x,s}\ & (x_1-2)^4+(x_1-2x_2)^2 \\ \text{s.t. } & x_1^2-x_2+s=0, \\ & s\ge 0. \end{aligned} \tag{6}$$

$$\begin{aligned} \min_{x,s}\ & (x_1-2)^4+(x_1-2x_2)^2-\mu\log s \\ \text{s.t. } & x_1^2-x_2+s=0. \end{aligned} \tag{7}$$

Interior point method – Example

Lagrange function

$$L(x_1, x_2, s, z) = (x_1 - 2)^4 + (x_1 - 2x_2)^2 - \mu \log s - z(x_1^2 - x_2 + s).$$

Optimality conditions together with feasibility

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= 4(x_1 - 2)^3 + 2(x_1 - 2x_2) - 2zx_1 = 0,\\
\frac{\partial L}{\partial x_2} &= -4(x_1 - 2x_2) + z = 0,\\
\frac{\partial L}{\partial s} &= -\frac{\mu}{s} - z = 0,\\
\frac{\partial L}{\partial z} &= x_1^2 - x_2 + s = 0.
\end{aligned} \tag{8}$$

We have obtained 4 equations with 4 variables ...

Interior point method – Example

Slight modification

$$\begin{aligned}
4(x_1-2)^3+2(x_1-2x_2)-2zx_1 &= 0, && (9)\\
-4(x_1-2x_2)+z &= 0, && (10)\\
-sz-\mu &= 0, && (11)\\
x_1^2-x_2+s &= 0. && (12)
\end{aligned}$$

Necessary derivatives

$$\begin{aligned}
H(x_1,x_2,z) &= \begin{pmatrix} 12(x_1-2)^2+2-2z & -4 \\ -4 & 8 \end{pmatrix} && (13)\\
\nabla g(x) &= \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} && (14)
\end{aligned}$$

Interior point method – Example

System of linear equations for Newton's step

$$\begin{pmatrix} 12(x_1-2)^2+2-2z & -4 & 0 & -2x_1 \\ -4 & 8 & 0 & 1 \\ 0 & 0 & -z & -s \\ 2x_1 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta s \\ \Delta z \end{pmatrix} =$$

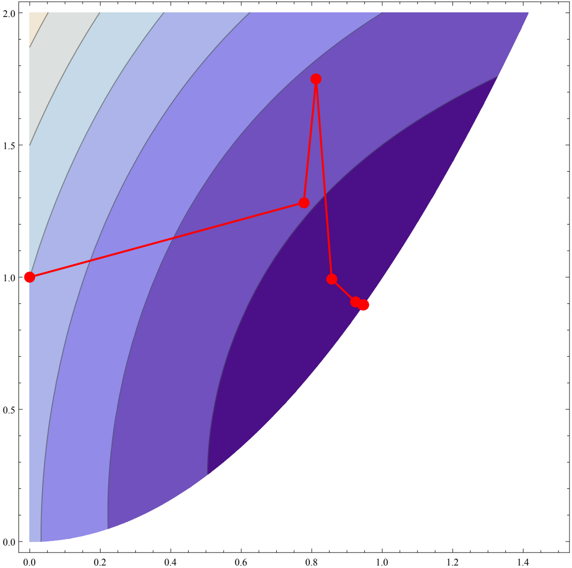
$$= \begin{pmatrix} -4(x_1-2)^3 - 2(x_1-2x_2) + 2zx_1 \\ 4(x_1-2x_2) - z \\ sz + \mu \\ -x_1^2 + x_2 - s \end{pmatrix}$$

Interior point method – Example

Starting point $x^0 = (0, 1)$, $z^0 = 1$, $s^0 = 1$, $\mu > 0$, then the step ...

$$\begin{pmatrix} 48 & -4 & 0 & 0 \\ -4 & 8 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta s \\ \Delta z \end{pmatrix} = \begin{pmatrix} 36 \\ -9 \\ 1+\mu \\ 0 \end{pmatrix}$$

Interior point method – Example



Sequential Quadratic Programming

Nocedal and Wright (2006): Let $f, h_i : \mathbb{R}^n \to \mathbb{R}$ be smooth functions,

$$\begin{aligned} &\min_x \; f(x) \\ &\text{s.t. } h_i(x) = 0, \;\; i = 1, \dots, l. \end{aligned} \tag{15}$$

Lagrange function

$$L(x, v) = f(x) + \sum_{i=1}^{l} v_i h_i(x)$$

and KKT optimality conditions

$$\begin{aligned} \nabla_x L(x, v) &= \nabla_x f(x) + A(x)^T v = 0, \\ h(x) &= 0, \end{aligned} \tag{16}$$

where $h(x)^T = (h_1(x), \dots, h_l(x))$ and
$A(x)^T = [\nabla h_1(x), \nabla h_2(x), \dots, \nabla h_l(x)]$ denotes the Jacobian matrix.

Sequential Quadratic Programming

We have system of $n + l$ equations in the $n + l$ unknowns x and v:

$$\nabla F(x, v) = \begin{bmatrix} \nabla_x f(x) + A(x)^T v \\ h(x) \end{bmatrix} = 0. \tag{17}$$

ASS. (LICQ) Jacobian matrix $A(x)$ has full row rank.
The Jacobian is given by

$$\nabla^2 F(x, v) = \begin{bmatrix} \nabla^2_{xx} L(x, v) & A(x)^T \\ A(x) & 0 \end{bmatrix}. \tag{18}$$

We can use the **Newton algorithm** ..

Sequential Quadratic Programming

Setting $f_k = f(x^k)$, $\nabla^2_{xx} L_k = \nabla^2_{xx} L(x^k, v^k)$, $A_k = A(x^k)$, $h_k = h(x^k)$, we obtain the Newton step by solving the system

$$\begin{bmatrix} \nabla^2_{xx} L_k & A_k^T \\ A_k & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_v \end{bmatrix} = \begin{bmatrix} -\nabla f_k - A_k^T v_k \\ -h_k \end{bmatrix} \tag{19}$$

Then we set $x^{k+1} = x^k + p_x$ and $v^{k+1} = v^k + p_v$.

Sequential Quadratic Programming

ASS. (SOSC) For all $d \in \{\tilde{d} \neq 0 : \; A(x)\tilde{d} = 0\}$, it holds

$$d^T \nabla^2_{xx} L(x, v) d > 0.$$

Sequential Quadratic Programming

Important alternative way to see the Newton iterations: Consider the **quadratic program**

$$\begin{aligned} \min_p\ & f_k + p^T \nabla f_k + \frac{1}{2} p^T \nabla^2_{xx} L_k p \\ \text{s.t. } & h_k + A_k p = 0. \end{aligned} \tag{20}$$

KKT optimality conditions

$$\begin{aligned} \nabla^2_{xx} L_k p + \nabla f_k + A_k^T \tilde{v} &= 0 \\ h_k + A_k p &= 0, \end{aligned} \tag{21}$$

$$\begin{bmatrix} \nabla^2_{xx} L_k & A_k^T \\ A_k & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_{\tilde{v}} \end{bmatrix} = \begin{bmatrix} -\nabla f_k \\ -h_k \end{bmatrix} \tag{22}$$

which is the same as the Newton system if we add $A_k^T v_k$ to the first equation. Then we set $x^{k+1} = x^k + p_x$ and $v^{k+1} = p_{\tilde{v}}$.

Sequential Quadratic Programming

Algorithm: Start with an initial solution (x^0, v^0) and iterate until a convergence criterion is met:

1. Evaluate $f_k = f(x^k)$, $h_k = h(x^k)$, $A_k = A(x^k)$, $\nabla^2_{xx} L_k = \nabla^2_{xx} L(x^k, v^k)$.
2. Solve the Newton equations OR the quadratic problem to obtain new (x^{k+1}, v^{k+1}).

If possible, deliver explicit formulas for first and second order derivatives.

Sequential Quadratic Programming

$f, g_j, h_i : \mathbb{R}^n \to \mathbb{R}$ are smooth. We use a **quadratic approximation** of the objective function and linearize the constraints, $p \in \mathbb{R}^p$

$$\begin{aligned} \min_p \ & f(x^k) + p^T \nabla_x f(x^k) + \frac{1}{2} p^T \nabla^2_{xx} L(x^k, u^k, v^k) p \\ \text{s.t. } & g_j(x^k) + p^T \nabla_x g_j(x^k) \le 0, \ j = 1, \ldots, m, \\ & h_i(x^k) + p^T \nabla_x h_i(x^k) = 0, \ i = 1, \ldots, l. \end{aligned} \tag{23}$$

Use an algorithm for quadratic programming to solve the problem and set $x^{k+1} = x^k + p_k$, where u^{k+1}, v^{k+1} are Lagrange multipliers of the quadratic problem which are used to compute new $\nabla^2_{xx} L$.

Convergence: Nocedal and Wright (2006), Theorem 18.1

Literature

- Bazaraa, M.S., Sherali, H.D., and Shetty, C.M. (2006). **Nonlinear programming: theory and algorithms**, Wiley, Singapore, 3rd edition.
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- P. Kall, J. Mayer: **Stochastic Linear Programming: Models, Theory, and Computation**. Springer, 2005.
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