Introduction to equilibrium problems and application to emission allowances markets

Michal Červinka*

In part based on the joint paper with
Jiří V. Outrata[1], Michael C. Ferris[2], Michal Outrata[3]

*,[1] Institute of Information Theory and Automation, Czech Academy of Sciences
[2] University of Wisconsin, Madison
*,[3] Charles University in Prague

Greenhouse gas emissions

- based on Kyoto Protocol: several emission allowances markets (Australia, China, EU, Japan, Korea, New Zealand and the U.S.)
- in EU region: EU commission approves national emission caps for regulated countries, each government then distributes a national permit allocation to regulated industries (power plants, combustion plants, oil refineries etc) - domestically issued allowances
- United Nations administrates carbon offset activities - installation of environmentally friendly facilities (renewable energy, energy efficiency improvements, reforestation, nature preservation etc) in developing countries without emission caps - carbon offset credits
- both allowances and offset credits give right to the holder to emit certain amount of carbon dioxide
- EU commission sets limits on imported offset credits with respect to the original endowment of allowances (around 13 percent)

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Given real-valued function f and a nonempty set A, consider

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & \\ & x \in A. \end{array} \tag{1}$$

let $\bar{x}$ be a local solution of (1). If $\bar{x}$ is an interior point of A and f is differentiable (in some sense) at that point, then Fermat rule specifies the necessary optimality condition:

$$f'(\bar{x}) = 0.$$

In not an interior point (but still assuming differentiability) the necessary condition still holds as variational inequality, which for convex A reads

$$\langle f'(\bar{x}), x - \bar{x} \rangle \geq 0 \;\text{ for any } x \in A.$$

Question: what to do in case of nondifferentiability of f; nonconvexity of f or A?
One can eventually rewrite the variational inequality to

$$0 \in \partial f(\bar{x}) + N_A(\bar{x}). \tag{2}$$

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To include the possibility that the function can attain $+\infty$ we often use the following construction: let $B \subset \mathbb{R}^n$ be nonempty and convex and $g : B \to \mathbb{R}$ be convex. Then the function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is defined by

$$f(x) = \begin{cases} g(x) & x \in B; \\ +\infty & x \notin B. \end{cases}$$

A function f is called proper if $f(x) < \infty$ for at least one $x \in \mathbb{R}^n$, i.e. (effective) domain is nonempty, and $f(x) > -\infty$ everywhere. However, it is quite pathological for a convex function to attain $-\infty$.

Convexity of function f is equivalent to convexity of its epigraph.

For convex $f : \mathbb{R}^n \to \bar{\mathbb{R}}$, good substitute for derivative is a subgradient. If $f(\bar{x}) < \infty$, a vector $v \in \mathbb{R}^n$ is a (convex) subgradient of f at $\bar{x}$ if

$$f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle \quad \text{for all } x.$$

We write $v \in \partial f(\bar{x})$, where $\partial f(\cdot)$ is a set-valued mapping called subdifferential of f. If f is differentiable at $\bar{x}$ then $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$.

A function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is called lower semi-continuous (lsc) at $\bar{x}$ if either $f(\bar{x}) = -\infty$ or for every $k < f(\bar{x})$ there exists a neigborhood U of $\bar{x}$ such that

$$k < f(x) \quad \text{for every } x \in U.$$

f is called lsc if it is lsc at every point. It is called upper semi-continuous (usc) if $-f$ is lsc. It is continuous (in the classical sense) if it is both lsc and usc. Equivalently, f is lsc if its epigraph is closed relatively to $\mathbb{R}^{n+1}$.

If $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is proper convex lsc function and interior of its domain is nonepty, then f is continuous on the interior of its domain.

Attainment of minimum:
If $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is proper and lsc then it is bounded from below on each bounded subset of $\mathbb{R}^n$. Thus, such f attains a minimum relative to any compact subset of $\mathbb{R}^n$ that meets its domain.

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Recall:

$$\partial f(\bar{x}) := \{v \in \mathbb{R}^n | \langle v, x - \bar{x}\rangle \leq f(x) - f(\bar{x}) \ \forall x \in \mathbb{R}^n\}.$$

Its main purpose is to detect minimum points $\bar{x}$ for which, in the unconstrained case, $0 \in \partial f(\bar{x})$. In the constrained case, $\bar{x}$ is a minimum of f on set A iff $\bar{x}$ is an unconstrained minimum of $f + \delta_A$, i.e. iff $0 \in \partial(f + \delta_A)(\bar{x})$.

The function $\delta_A : \mathbb{R}^n \to \bar{\mathbb{R}}$, called the indicator function of A is defined by

$$\delta_A(x) := \begin{cases} 0 & x \in A; \\ +\infty & x \notin A. \end{cases}$$

δ_A is proper and convex whenever A is nonempty and convex.

If we could applying the chain rule

$$\partial(f_1 + f_2)(\bar{x}) \subset \partial f_1(\bar{x}) + \partial f_2(\bar{x})$$

and using

$$\partial \delta_A(\bar{x}) := N_A(\bar{x})$$

we arrive at the generalized equation (2) (the convex case).

If $f:\mathbb{R}^n\to\bar{\mathbb{R}}$ is proper convex function, then $\partial f(x)$ is nonempty and compact at any $x\in$int dom f.

Sum rule for convex subdifferential (Moreau-Rockafellar:
Let $f_1,f_2:\mathbb{R}^n\to\bar{\mathbb{R}}$ be proper convex functions. Then for every $x\in\mathbb{R}^n$

$$\partial(f_1+f_2)(x)\supset\partial f_1(x)+\partial f_2(x).$$

Further, assuming int dom $f_1\cap$ dom $f_2\neq\emptyset$, also

$$\partial(f_1+f_2)(x)\subset\partial f_1(x)+\partial f_2(x).$$

For $A\subset\mathbb{R}^n$, the (negative) polar cone of A is

$$A^\circ:=\{x^*\in\mathbb{R}^n|\,\langle x^*,x\rangle\leq 0\,\forall x\in A\}.$$

For A convex, the (convex) tangent cone $T_A(\bar{x})$ at $\bar{x}\in A$ is given by

$$T_A(\bar{x}):=cl\,(\mathbb{R}_+(A-\bar{x}))=\{x\in\mathbb{R}^n|\exists h_k\downarrow 0\,\exists x_k\to x:\bar{x}+h_kx_k\in A\,\forall k\},$$

and the (convex) normal cone $N_A(\bar{x})$ at $\bar{x}\in A$ by

$$N_A(\bar{x}):=(A-\bar{x})^\circ=\{v\in\mathbb{R}^n|\,\langle v,x-\bar{x}\rangle\leq 0\ \forall x\in A\}$$

. Also, $N_A(\bar{x})=(T_A(\bar{x}))^\circ=\partial\delta_A(\bar{x})$.

Recall optimization problem (1). Consider

$$A = \{x \in \mathbb{R}^n | g_i(x) \leq 0, i = 1, \ldots, m; x \in C\}.$$

If C is nonempty convex subset of dom $f \cap$ dom $g_1 \cap \cdots \cap$ dom g_m and f and $g_i, i = 1, \ldots, m$, are proper convex, then (1) is convex optimization problem.

Define the (enhanced) Lagrange function

$$L(x, \lambda, \mu_1, \ldots, \mu_m) := \lambda f(x) + \sum_{i=1}^{m} \mu_i g_i(x).$$

Under the Slater condition

$$\exists x_0 \in C : g_i(x_0) < 0 \;\; \forall i \in \{1, \ldots, m\},$$

the following is equivalent:

(L) there exists a vector of multipliers $(\bar{\lambda}, \bar{\mu}_1, \ldots, \bar{\mu}_m) \in \mathbb{R}^{m+1}_+ \setminus \{0\}$ such that $\bar{x}$ minimizes $L(\cdot, \bar{\lambda}, \bar{\mu}_1, \ldots, \bar{\mu}_m)$ on C and $\bar{\mu}_i g_i(\bar{x}) = 0, i = 1, \ldots, m$.

(SP) there exists $\bar{\mu} \in \mathbb{R}^m_+$ such that $(\bar{x}, \bar{\mu})$ is a saddle point of L for $\bar{\lambda} = 1$ with respect to $C \times \mathbb{R}^m_+$, i.e.,

$$L(\bar{x}, 1, u) \leq L(\bar{x}, 1, \bar{u}) \leq L(x, 1, \bar{u}) \;\; \forall (x, \mu) \in C \times \mathbb{R}^M_+.$$

(Min) $\bar{x}$ is a global solution of (1).

The previous are called global Karush-Kuhn-Tucker conditions. The following are local optimality conditions for (1)

(F-J) there exists a vector of multipliers $(\bar{\lambda}, \bar{\mu}_1, \ldots, \bar{\mu}_m) \in \mathbb{R}^{m+1}_+ \setminus \{0\}$ such that

$$0 \in \bar{\lambda}\partial f(\bar{x}) + \bar{\mu}_1 \partial g_1(\bar{x}) + \cdots + \bar{\mu}_m \partial g_m(\bar{x}) + N_C(\bar{x}),$$
$$0 = \bar{\mu}_i g_i(\bar{x}) \;\; i = 1, \ldots, m.$$

(KKT) there exists a vector $(\bar{\mu}_1, \ldots, \bar{\mu}_m) \in \mathbb{R}^m_+$ such that

$$0 \in \partial f(\bar{x}) + \bar{\mu}_1 \partial g_1(\bar{x}) + \cdots + \bar{\mu}_m \partial g_m(\bar{x}) + N_C(\bar{x}),$$
$$0 = \bar{\mu}_i g_i(\bar{x}) \;\; i = 1, \ldots, m.$$

Under Slater condition, both Fritz-John conditions and (local) KKT conditions are equivalent to $\bar{x} \in A$ being local minimum of (1).

Conditions ensuing $\bar{\lambda} \neq 0$ are called regularity conditions. Since these conditions are generally conditions on constraint functions, these conditions are also called constraint qualifications (CQs).

E.g., in case of $C = \mathbb{R}^n$, and all functions $g_i, i = 1, \dots, m$ continuously differentiable, the following are weaker CQs than Slater:

LICQ (linear independence CQ) vectors $\nabla g_j(\bar{x})$ are linearly independent for $j \in \{1, \dots, m\}$ such that $g_j(\bar{x}) = 0$.

MFCQ (Mangasarian-Fromowitz CQ) there exists $y \in \mathbb{R}^m$ such that

$$\langle \nabla g_j(\bar{x}), y \rangle < 0 \;\; \forall j : g_j(\bar{x}) = 0.$$

Slater implies LICQ, which in turn implies MFCQ. MFCQ can be also reformulated as positive linear independence of gradients of active constraints.

There is a broad range of even weaker CQs, like CR CQ, CPLD CQ, calmness CQ, Abadie CQ, Guignard CQ. The Giugnard CQ (involving polar to linearization cone to the constraint set) is known to be the (technically) weakest condition to ensure $\bar{\lambda} \neq 0$.

Consider A defined by

$$A = \{x \in C | g_i(x) \leq 0, i = 1, \ldots, m\}.$$

Then under LICQ

$$N_A(\bar{x}) = \left\{ \sum_{j:g_j(\bar{x})=0} \mu_j \nabla g_j(\bar{x}) + z \,\middle|\, \mu_j \geq 0, j : g_j(x) = 0; z \in N_C(\bar{x}) \right\}.$$

This follows from the fact that for $C = C_1 \times \cdots \times C_n$, where each C_j is a closed interval in $\mathbb{R}$,

$$N_C(\bar{x}) = N_{C_1}(\bar{x}_1) \times \cdots \times N_{C_n}(\bar{x}_n),$$

where

$$N_{C_j}(\bar{x}_j) = \begin{cases} [0, \infty) & \bar{x}_j \text{ is the right endpoint of } C_j \\ (-\infty, 0] & \bar{x}_j \text{ is the left endpoint of } C_j \\ \{0\} & \bar{x}_j \text{ is the interior of } C_j \\ \emptyset & \bar{x}_j \notin C_j \\ (-\infty, \infty) & C_j \text{ is singleton set} \end{cases}$$

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The generalized equation (2) fits into a broader picture in cases when ∂f (or ∇f) is replaced by any mapping $F : D \to \mathbb{R}^n$. This format has applications beyond characterization of a minimum relative to a set, namely in description of equilibrium.

For a convex set $D \subset \mathbb{R}^n$ and any mapping $F : D \to \mathbb{R}^n$ the generalized equation

$$0 \in F(x) + N_D(x),$$

variational condition for D and F, can be equivalently written in the form of variational inequality problem $VI(D, F)$: find $x \in D$ such that

$$\langle F(x), u - x \rangle \geq 0 \;\; \forall u \in D.$$

Its special case, where $D = \mathbb{R}^n_+$ is known as the complementarity problem

$$x \geq 0, F(x) \geq 0, \langle F(x), x \rangle = 0.$$

It is sometimes summarized vectorially by notation $0 \leq x \perp F(x) \geq 0$. For $F = \nabla f$ the complementarity conditions are KKT conditions of optimization problem (1) for $A = \mathbb{R}^n_+$.

Existence of solutions of VIs
Let $D \subset \mathbb{R}^n$ be compact and convex set and $F : D \to \mathbb{R}^n$ be continuous mapping. Then $VI(D, F)$ has (at least one) a solution.

Results of this type (existence of equilibria) is (often) based on fixed point theorem. E.g., Brouwer fixed point theorem:
Every continuous function from convex compact set C of Euclidean space to C itself has a fixed point.

In cases when D is not compact but only closed, one can add a condition of coercivity of F to ensure existence of solution to VI.
Mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is coercive if

$$\frac{\langle F(x), x \rangle}{||x||} \to +\infty \text{ as } ||x|| \to \infty.$$

Function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is coercive whenever $f(x) \to +\infty$ as $||x|| \to \infty$.

Problem formulation

Consider an oligopolistic market with m producers, where each of them solves the profit-maximization problem

$$\begin{array}{ll} \text{minimize} & c_i(y_i) + \pi x_i - p(T) y_i \\ \text{subject to} & \\ & (y_i, x_i) \in (A_i \times \mathbb{R}) \cap \mathcal{B}_i. \end{array} \tag{3}$$

- y_i is the amount of commodity produced by the ith firm,
- x_i is the amount of purchased (sold) rare resource,
- π is the price of the rare resource,
- $c_i[\mathbb{R}_+ \to \mathbb{R}_+]$ specifies the production costs,
- $p[\text{ int } \mathbb{R}_+ \to \mathbb{R}_+]$ is the inverse demand function,
- $T = \sum_{i=1}^{m} y_i$ signifies the overall production,
- $A_i = [a_i, b_i]$ specifies the production bounds and
$$\mathcal{B}_i = \{(y_i, x_i) | q_i(y_i) \le x_i + e_i\}$$
- e_i is the initial endowment with the rare resource
- $q_i[\mathbb{R}_+ \to \mathbb{R}_+]$ is a (technological) function assigning each production value the corresponding (needed) amount of rare resource.

Assumptions:

A1: All functions c_i can be extended to open intervals containing the sets A_i. These extensions are convex and twice continuously differentiable.

A2: p is strictly convex and twice continuously differentiable on $\operatorname{int} \mathbb{R}_+$.

A3: $\alpha p(\alpha)$ is a concave function of α.

A4: $\forall$ i one has $0 \leq a_i < b_i$ and $\exists$ i_0 : $a_{i_0} > 0$.

A5: All functions q_i fulfill $q_i(0) = 0$ and can be extended to open intervals containing A_i. These extensions are convex, increasing and twice continuously differentiable.

A6: $\forall$ i one has $q_i(a_i) \leq e_i$ and $\exists$ i_0 : $q_{i_0}(a_{i_0}) < e_{i_0}$.

A7: $\pi \geq 0$.

By virtue of the above assumptions

$$(y_i, x_i) \in (A_i \times \mathbb{R}) \cap \mathcal{B}_i \Rightarrow x_i \geq -e_i.$$

In what follows

$$J_i(\pi, y_1, y_2, \ldots, y_m, x_i) := c_i(y_i) + \pi x_i - p(T)y_i.$$

and $x := (x_1, x_2, \ldots, x_m)$ and $y := (y_1, y_2, \ldots, y_m)$ are the cumulative vectors of the strategies $x_i, y_i, i = 1, 2, \ldots, m$.

Cournot-Nash equilibrium

Definition 1.

The strategy pair $(\bar{y}, \bar{x})$ is a Cournot-Nash equilibrium in the considered market for a given $\pi \geq 0$ provided for all i one has

$$J_i(\pi, \bar{y}, \bar{x}_i) = \min_{(y_i, x_i) \in (A_i \times \mathbb{R}) \cap \mathcal{B}_i} J_i(\pi, \bar{y}_i, \bar{y}_2, \ldots, \bar{y}_{i-1}, y_i, \bar{y}_{i+1}, \ldots, \bar{y}_m, x_i)$$

Let Ξ be the overall available amount of the rare resource so that

$$\Xi \geq \sum_{i=1}^{m} e_i.$$

Consequently, the excess demand amounts to

$$\sum_{i=1}^{m} (e_i + x_i) - \Xi.$$

Cournot-Nash-Walras equilibrium

Definition 2.(Flåm)

The triple $(\bar{\pi}, \bar{y}, \bar{x})$ is a Cournot-Nash-Walras (CNW) equilibrium in the considered market provided

(i) $(\bar{y}, \bar{x})$ is a Cournot-Nash equilibrium for $\pi = \bar{\pi}$, and

(ii) one has

$$\bar{\pi} \geq 0,\ \Xi - \sum_{i=1}^{m}(e_i + \bar{x}_i) \geq 0,\ \left\langle \bar{\pi}, (\Xi - \sum_{i=1}^{m}(e_i + \bar{x}_i)) \right\rangle = 0.$$

- The conditions in (ii) characterize a Walras equilibrium with respect to the rare resource which determines a price $\bar{\pi}$ under which the (secondary) market with the rare resource is cleared.
- From the point of view of the producers the computation of $\bar{\pi}$ is a dynamical process starting after the initial allocation has been conducted.
- From the point of view of the authority, controlling the rare resource, however, the whole problem can be solved in one step. The results provide the authority with information about the influence of the initial allocation on the CNW equilibrium.

Existence of CNW equilibria

Lemma 1.

There is a positive real L such that in all CNW equilibria one has $\pi < L$.

Elimination of variable x: We replace the inequality $q(y_i) \leq x_i + e_i$ by equality so that (3) becomes

$$\begin{array}{ll} \text{minimize} & c_i(y_i) + \pi(q_i(y_i) - e_i) - p(T)y_i \\ \text{subject to} & \\ & y_i \in A_i. \end{array} \tag{4}$$

Problems (4) for $i = 1, \ldots, m$ generate likewise a Cournot-Nash equilibrium in the standard way. It can be characterized by the GE

$$0 \in \begin{bmatrix} \nabla c_1(y_1) - y_1 \nabla p(T) - p(T) + \pi \nabla q_1(y_1) \\ \vdots \\ \nabla c_m(y_m) - y_m \nabla p(T) - p(T) + \pi \nabla q_m(y_m) \end{bmatrix} + \sum_{i=l}^{m} N_{A_i}(y_i). \tag{5}$$

Lemma 2.

Let $\bar{y}$ be a solution of (5). Then the pair $(\bar{y}, \bar{x})$ with $\bar{x}_i = q_i(\bar{y}_i) - e_i \; \forall i$ is a Cournot-Nash equilibrium generated by (3). Conversely, for each Cournot-Nash equilibrium generated by (3), the component $\bar{y}$ is a solution of (5) whenever $\pi > 0$.

Existence of CNW equilibria

Theorem 1.

Under the posed assumptions there exists a CNW equilibrium.

Sketch of the proof. Define the mapping $Q[\mathbb{R}^m \rightarrow \mathbb{R}]$ by

$$Q(y) := \sum_{i=1}^{n} q_i(y_i).$$

By virtue of Lemma 2 it suffices to show the existence of a pair $(\bar{\pi}, \bar{y})$ which solves the (aggregated) GE

$$\left.\begin{array}{l} 0 \in \Xi - Q(y) + N_{\mathbb{R}_+}(\pi) \\ 0 \in \begin{bmatrix} \nabla c_1(y_1) + \pi \nabla q_1(y_1) - y_1 \nabla p(T) - p(T) \\ \vdots \\ \nabla c_m(y_m) + \pi \nabla q_m(y_m) - y_m \nabla p(T) - p(T) \end{bmatrix} + \displaystyle\sum_{i=1}^{m} N_{A_i}(y_i) \end{array}\right\} \quad (6)$$

in variables (π, y). Thanks to Lemma 1, $\mathbb{R}_+$ in the first line of (6) can be replaced by a bounded interval $[0, L]$. So, it remains to apply the standard existence result for VIs with bounded constraint sets. $\square$

Numerical approaches

(1) Direct solution of the coupled optimization problems (1) together with the complementarity problem in Def.2 (ii). This structure is called MOPEC (multiple optimization problems with equilibrium constraints). It can be solved eg by the PATH solver.

(2) Direct solution of GE (4).

(3) Solution via the mathematical program with equilibrium constraints (MPEC)

$$\begin{array}{lll} \text{minimize} & \pi \cdot (\Xi - Q(y)) & \\ \text{subject to} & & \\ & y \in S(\pi) & \text{(equilibrium constraint)} \\ & \pi \geq 0 & \text{(control constraint)} \\ & \Xi - Q(y) \geq 0, & \text{(state constraint)} \end{array} \tag{7}$$

where π is the control variable, y is the state variable and S assigns each π the corresponding set of Cournot-Nash equilibria. By virtue of Lemma 1 any solution $(\bar{\pi}, \bar{y})$ of (7) generates a CNW equilibrium with $\bar{x}_i = q_i(\bar{y}_i) - e_i$ provided the optimal value of the objective in (7) is zero.

The objective of (7) corresponds to the primal gap function [FP].

Solution via MPEC

MPEC (7) can be solved

(a) via nonlinear programming solvers (KNITRO, BARON);
(b) via regularization (MATLAB codes of Ch. Kanzow and A. Schwartz);
(c) via the a variant of the Implicit programming approach (ImP) which is able to deal with state constraints.

The ImP approach in this case is based on

Lemma 3.

Mapping S is single-valued and locally Lipschitz over $\mathbb{R}_+$.

ImP approach

The application of ImP to (7) amounts to the solution of the penalized (augmented) program

$$\begin{array}{ll} \text{minimize} & \pi \cdot (\Xi - Q \circ S(\pi)) + R[(Q \circ S(\pi) - \Xi)_+] \\ \text{subject to} & \\ & \pi \geq 0 \end{array} \qquad (8)$$

in variable π, where $R > 0$ is a suitably chosen penalty parameter.

Theorem 2.

Let $(\bar{\pi}, \bar{y})$ be a solution of MPEC (7) and assume that the perturbation mapping

$$\mathcal{M}(z) = \{\pi \in \mathbb{R}_+ | \Xi - Q \circ S(\pi) \geq z\}$$

is calm at $(0, \bar{\pi})$. Then there is a positive real R such that $\bar{\pi}$ is a solution of the (augmented) program (8).

Remarks.

(1) The required calmness of $\mathcal{M}$ at $(0, \bar{\pi})$ can be ensured by non-restrictive conditions (MFCQ) in terms of problem data.

(2) To solve (8) numerically, e.g a bundle method for nonsmooth optimization can be used.

Test Example (based on example from [MSS])

Consider five firms supplying quantities $y_i \in \mathbb{R}_+, i = 1, \ldots, 5$, of some homogeneous product on the market with the inverse demand function

$$p(T) = 5000^{\frac{1}{\gamma}} T^{-\frac{1}{\gamma}},$$

where γ is a positive parameter termed demand elasticity. Let the functions q_i be linear in the form $q_i(y_i) = q_i y_i$. Let all the production cost functions be in the form

$$c_i(y_i) = c_i y_i + \frac{\beta_i}{1+\beta_i} K_i^{-\frac{1}{\beta_i}} (y_i)^{\frac{1+\beta_i}{\beta_i}},$$

where c_i, K_i and $\beta_i,\ i = 1, \ldots, 5$, are positive parameters.

Table : Parameter specification

	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
q_i	1.63	1.5	1.48	1.5	1.4
c_i	10	8	6	4	2
K_i	5	5	5	5	5
β_i	1.2	1.1	1.0	0.9	0.8

Further, let $\gamma = 1.3$ and $a_i = 0, b_i = 30$ and $e_i = 25$ for each $i = 1, \ldots, 5$. Assume further that there are no extra rare resources available ($\Xi = 125$) and $R = 50$.

In addition, we will consider the following modifications of the example above (case A):

- B Producer 1 has worse technology regarding rare resource such that $q_1 = 4$
- C upper bounds on production are increased to 35 and initial endowments with rare resource are increased to 45
- D initial endowments with rare resource are lowered to 5 and additional 100 units of rare resource are available

		Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
case A	$\pi = 6.375$					
	production	6.651	14.018	18.600	21.347	23.988
	profit	172.491	217.617	266.497	311.268	374.633
	purchased rare resource	-14.159	-3.973	2.528	7.021	8.584
case B	$\pi = 5.764$					
	production	0	16.215	20.608	23.132	25.342
	profit	144.097	220.921	274.314	321.432	383.849
	purchased rare resource	-25.000	-0.677	5.500	9.699	10.479
case C	$\pi = 0$					
	production	21.218	28.081	32.345	33.790	32.664
	profit	67.210	125.581	186.056	237.492	272.578
	purchased rare resource	-6.407	-2.878	2.870	5.685	0.729
case D	$\pi = 6.375$					
	production	6.651	14.018	18.600	21.347	23.988
	profit	44.983	90.109	138.989	183.760	247.124
	purchased rare resource	5.841	16.027	22.528	27.021	28.584

Social equilibrium problem

Further, we are able to establish that the mappings assigning CNW equilibria to the problem data (Ξ, γ, c, q) behave in the same stable way.
It follows that the authority (providing the allocation of rare resource) could, e.g., optimize Ξ in such a way that in the corresponding CNW, e.g., the overall production and the price π will be close to some desired values. One obtains the MPEC

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}(\Sigma_{i=1}^{m} y_i - T_d)^2 + \frac{1}{2}(\pi - \pi_d)^2 \\ \text{subject to} & \\ & (\pi, y) \text{ solves (4)}, \end{array} \tag{9}$$

where Ξ is now a control variable and can be subject to some constraints. (9) is in fact a social equilibrium problem and can, by virtue of the stability results, be solved via the ImP approach.

One can directly apply e.g. derivative-free method to solve (9), providing the derivative-free algorithm with objective values after solving (4) via one of the previously discussed approaches.

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