Zero-sum games of two players

Martin Branda

Charles University in Prague
Faculty of Mathematics and Physics
Department of Probability and Mathematical Statistics

COMPUTATIONAL ASPECTS OF OPTIMIZATION

Zero-sum games of two players

Definition

A triplet $\{X, Y, K\}$ is called a **game of two rational players with zero sum**, if

1. X is a set of strategies of Player 1 (P1),
2. Y is a set of strategies of Player 2 (P2),
3. $K : X \times Y \to \mathbb{R}$ is a payoff function of player 1, i.e. if P1 plays $x \in X$ and P2 plays $y \in Y$, then P1 gets $K(x, y)$ and P2 gets $-K(x, y)$.

Definition

For the zero-sum games $\{X, Y, K\}$ we define

- **upper value** of the game $\mathrm{uv}^* = \inf_{y \in Y} \sup_{x \in X} K(x, y)$,
- **lower value** of the game $\mathrm{lv}^* = \sup_{x \in X} \inf_{y \in Y} K(x, y)$,
- **upper price** of the game $\mathrm{up} = \min_{y \in Y} \sup_{x \in X} K(x, y)$,
- **lower price** of the game $\mathrm{lp} = \max_{x \in X} \inf_{y \in Y} K(x, y)$.

If the lower and upper prices exist and it holds $\mathrm{up} = \mathrm{lp}$, then we say that the game has the **price** $p = \mathrm{up} = \mathrm{lp}$.

Upper value can be seen as the lowest payoff of P1, if P1 knows strategy of P2 before his/her move.

Zero-sum games of two players

Definition

We say that

- $\hat{x} \in X$ is an optimal strategy of P1, if $K(\hat{x}, y) \geq \mathrm{lv}^*$ for all $y \in Y$.
- $\hat{y} \in Y$ is an optimal strategy of P2, if $K(x, \hat{y}) \leq \mathrm{uv}^*$ for all $x \in X$.

Zero-sum games of two players

Proposition

For each zero-sum game $\{X, Y, K\}$ the upper and lower value exits and it holds

$$\mathrm{lv}^* \leq \mathrm{uv}^*.$$

For each $\tilde{x} \in X$ and $\tilde{y} \in Y$ it holds

$$\begin{aligned} \inf_{y \in Y} K(\tilde{x}, y) &\leq K(\tilde{x}, \tilde{y}), \\ \sup_{x \in X} \inf_{y \in Y} K(x, y) &\leq \sup_{x \in X} K(x, \tilde{y}), \\ \mathrm{lv}^* = \sup_{x \in X} \inf_{y \in Y} K(x, y) &\leq \inf_{y \in Y} \sup_{x \in X} K(x, y) = \mathrm{uv}^*. \end{aligned} \tag{1}$$

Proposition

For each zero-sum game $\{X, Y, K\}$ is holds that

- *There is at least one optimal strategy of P1, if and only if the lower price exists.*
- *There is at least one optimal strategy of P2, if and only if the upper price exists.*

"$\Rightarrow$": Let $\hat{x} \in X$ be an optimal strategy of P1, i.e. $K(\hat{x}, y) \geq \mathrm{lv}^*$ for all $y \in Y$. Then

$$\mathrm{lv}^* \leq \inf_{y \in Y} K(\hat{x}, y) \leq \sup_{x \in X} \inf_{y \in Y} K(x, y) = \mathrm{lv}^*. \tag{2}$$

Thus

$$\mathrm{lv}^* = \inf_{y \in Y} K(\hat{x}, y) = \max_{x \in X} \inf_{y \in Y} K(x, y) = \mathrm{lp}. \tag{3}$$

Zero-sum games of two players

Proposition

Let $\{X, Y, K\}$ *be a zero-sum game with* X, Y *compact and* K *continuous. Then the upper and lower prices exist.*

Zero-sum games of two players

Theorem

A zero-sum game $\{X, Y, K\}$ has a price if and only if the payoff function has a saddle point, i.e. there is a pair[a] $(\hat{x}, \hat{y})$ *such that*

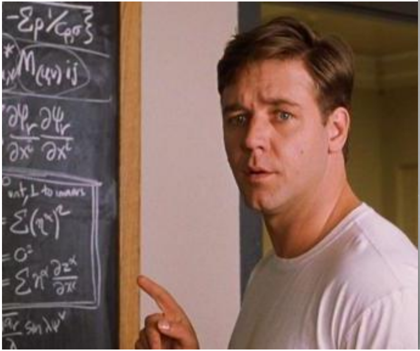
$$K(x, \hat{y}) \leq K(\hat{x}, \hat{y}) \leq K(\hat{x}, y)$$

for all $x \in X$ and $y \in Y$. Then $\hat{x}$ is an optimal strategy for P1, $\hat{y}$ is an optimal strategy for P2, and $p = K(\hat{x}, \hat{y})$ is the price of the game.

[a] Such pair can be seen as a Nash equilibrium for two player games.

"$\Rightarrow$": $K(x, \hat{y}) \leq p \leq K(\hat{x}, y)$.

John Forbes Nash (1928–2015)




A Beautiful Mind (2001)

Minimax theorem

Theorem

Let $\{X, Y, K\}$ be a zero-sum game where X, Y are nonempty convex compact sets and $K(x, y)$ is continuous, concave in x and convex in y. Then, there exists the price of the game, i.e.

$$\min_{y \in Y} \max_{x \in X} K(x, y) = \max_{x \in X} \min_{y \in Y} K(x, y).$$

Applicable also out of the game theory, e.g. in robustness.

Generalizations: Rockafellar (1970)

Definition

We say that $\{X, Y, A\}$ is a **matrix game** if it a zero sum game (of two players), $A \in \mathbb{R}^{n \times m}$ is a matrix, and

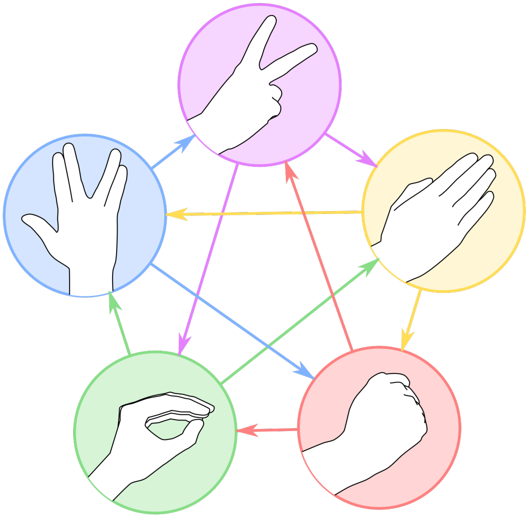
$$\begin{aligned} K(x, y) &= x^T A y, \\ X &= \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0 \right\}, \\ Y &= \left\{ y \in \mathbb{R}^m : \sum_{j=1}^{m} y_j = 1,\ y_j \geq 0 \right\}. \end{aligned} \tag{4}$$

Rock–paper–scissors

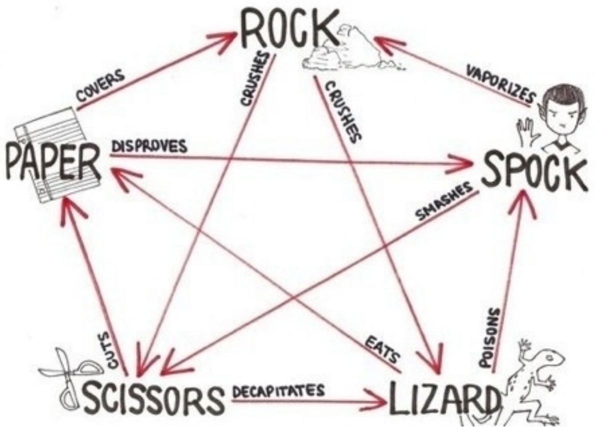
R–P–S

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tag{5}$$

Rock–paper–scissors–lizard–Spock



Rock–paper–scissors–lizard–Spock



Definition

For a matrix game $\{X, Y, A\}$, we define a matrix game with **pure strategies** $\{\overline{X}, \overline{Y}, A\}$, where

$$\begin{aligned} \overline{X} &= \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1,\ x_i \in \{0,1\} \right\}, \\ \overline{Y} &= \left\{ y \in \mathbb{R}^m : \sum_{j=1}^{m} y_j = 1,\ y_j \in \{0,1\} \right\}. \end{aligned} \tag{6}$$

We say that $\{X, Y, A\}$ has a **price in pure strategies** if both players have optimal pure strategies.

Matrix games

Proposition

Each matrix game has a price and both players have optimal strategies.

Proposition

Matrix game $\{X, Y, A\}$ *has a price in pure strategies if and only if* $\{\overline{X}, \overline{Y}, A\}$ *has a price.*

Proposition

Let $\{X, Y, A\}$ be a matrix game and $\hat{x} \in X$ and $\hat{y} \in Y$ with price p. Then

1. *$\hat{x}$ is an optimal strategy of P1 if and only if $\hat{x}^T A \geq (p, \ldots, p)$,*
2. *$\hat{y}$ is an optimal strategy of P2 if and only if $A\hat{y} \leq (p, \ldots, p)^T$.*

$\hat{x}^T A \geq (p, \ldots, p) \;\Leftrightarrow\; \hat{x}^T A y \geq p, \forall y \in Y$.
("$\Rightarrow$" $\cdot y$ & $\sum_i y_i = 1$, "$\Leftarrow$" $y = \mathrm{e}_i$)

Proposition

(Complementarity conditions) Let $\{X, Y, A\}$ *be a matrix game with price* p *and let* $\hat{x} \in X$ *and* $\hat{y} \in Y$ *be optimal strategies. Then*

1. *if* $\hat{x}_i > 0$, *then* $(A\hat{y})_i = p$
2. *if* $\hat{y}_j > 0$, *then* $(\hat{x}^T A)_j = p$.

Matrix games – Example

Consider

$$A = \begin{pmatrix} 5 & 1 \\ 0 & 7 \end{pmatrix}$$

$5x_1 \geq p,\ x_1 + 7x_2 \geq p,\ x_1 + x_2 = 1,\ x_1 \geq 0,\ x_2 \geq 0$

$$\max_{x \in X} \min\{5x_1, x_1 + 7x_2\} = p$$

and using $x_1 + x_2 = 1$

$$\max_{x_1 \geq 0} \min\{5x_1, 7 - 6x_1\} = p$$

Maximum is attained at $\hat{x}_1 = 7/11$, $\hat{x}_2 = 4/11$ with the price $p = 35/11$. Using complementarity conditions, we obtain $\hat{y}_1 = 6/11$, $\hat{y}_2 = 5/11$.

Matrix games

Let $a, b \in \mathbb{R}^n$. We say that a strictly dominates b (b is strictly dominated by a), if $a_i > b_i$ for all $i = 1, \ldots, n$.

Proposition

Let $\{X, Y, A\}$ be a matrix game.

1. *If a row $A_{k,\cdot}$ is strictly dominated by a convex combination of other rows, then each optimal strategy of P1 fulfills $\hat{x}_k = 0$.*
2. *If a column $A_{\cdot,k}$ strictly dominates a convex combination of other columns, then each optimal strategy of P2 fulfills $\hat{y}_k = 0$.*

Matrix games

$$\begin{pmatrix} 3 & 2 & 4 & 0 \\ 3 & 4 & 2 & 3 \\ 6 & 5 & 5 & 1 \\ 1 & 4 & 0 & 7 \end{pmatrix}$$

Show that $(0, 0, 7/11, 4/11)^T$ is optimal strategy for P1, $(0, 0, 6/11, 5/11)^T$ for P2, and the price is $p = 35/11$.

Matrix games

Proposition

Matrix game $\{X, Y, A\}$ has a price p in pure strategies if and only if matrix A has a saddle point , i.e. there is a pair of indices $\{k, l\}$ such that

$$A_{kl} = \min\{A_{kj} : \ j = 1, \dots, m\} = \max\{A_{il} : \ i = 1, \dots, n\}.$$

(minimum in the row, maximum in the column)

e_k,e_l are optimal strategies of P1, P2

$\Leftrightarrow$

$$\begin{aligned} (\mathrm{e}_k^T A)_j = A_{kj} \geq p, \forall j, \\ (A\mathrm{e}_l)_i = A_{il} \leq p, \forall i, \end{aligned} \tag{7}$$

$\Leftrightarrow$

$$A_{kl} = \min\{A_{kj} : \ j = 1, \dots, m\} = \max\{A_{il} : \ i = 1, \dots, n\}.$$

Find the saddle point(s) ..

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix}, \tag{8}$$

Literature

- Lachout, P. (2011). **Matematické programování**. Skripta k (zaniklé) přednášce Optimalizace I (IN CZECH).
- Rockafellar, R.T. (1970). **Convex Analysis**. Princeton University Press, Princeton (N.Y.).
- Webb, J.N. Game Theory (2007). **Decisions, Interactions and Evolution**. Springer-Verlag, London.