

Rough path theory in Besov-Orlicz spaces

Stochastic processes, for example (fractional) Brownian motion, that can be added to differential equations to account for non-systematic error or uncertainty in the model, typically have nowhere differentiable sample paths. Thus, a notion of an integral (and therefore that of a solution to the equation) has to be constructed in a different way than path by path, for example, as a limit of a Riemann-type sum in the L^2 space. This is the case for both the Itô and Stratonovich integrals. Considering B to be the standard Brownian motion, the solution map

$$S : B(\omega) \mapsto Y(\omega) \quad \text{where } Y \text{ solves the Itô SDE } dY_t = f(Y_t)dB_t,$$

known as the Itô map, is measurable but lacks continuity regardless of the norm used to equip the space of realisations of B . Rough path theory is, in a certain sense, a way to overcome this problem. It provides the following insight: the Itô solution map can be factorised into a measurable map Ψ and a continuous solution map $\hat{S}$ as

$$B(\omega) \xrightarrow{\Psi} (B, \mathbb{B})(\omega) \xrightarrow{\hat{S}} Y(\omega),$$

i.e. the procedure is broken down into two steps. In the first, probabilistic, step one constructs the iterated integral $\mathbb{B}$ of the path B (for example in the Itô or Stratonovich sense). This step, encoded in the map Ψ , is what can be called a rough path lift. The second step, encoded in the map $\hat{S}$, is analytical. It takes both the path B and its (iterated) integral $\mathbb{B}$ as input and solves the now deterministic equation for that input.

The standard rough path theory is built for drivers with Hölder continuous paths and therefore it gets Hölder continuity of the paths of the solution in exchange. However, there are more suitable function spaces for all the canonical examples of drivers in stochastic differential equations (for example exponential Besov-Orlicz spaces). It turns out that if we restrict ourselves to such spaces, the solution lives in these, smaller, spaces as well.