Comparison of different approaches to calculating pointwise traction in flow

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Introduction

For an incompressible Newtonian fluid flowing around an obstacle we are interested in the pointwise traction acting on it. To determine the local deformation of a solid obstacle, an accurate traction calculation is required. Besides the classical approach that concerns a direct calculation of the traction from the Cauchy stress tensor, we investigate the Poincaré-Steklov method based on calculating a dual problem and it seems to provide more accurate results. Indeed, we show a better convergence rate of the latter method with respect to the direct approach. The method is applied to the Turek benchmark, which considers a flow past a rigid cylinder. We also consider a rigid square prism as an obstacle. In this benchmark the total drag and lift acting on the cylinder is computed. We extended the benchmark and computed the point-wise traction for different mesh resolutions and Reynolds numbers.

Problem description

Steady incompressible Navier-Stokes equations with inflow and outflow

$$\rho\mathbf{v}\cdot\nabla\mathbf{v} = \operatorname{div}\mathbb{T} \quad \text{in } \Omega,$$
$$\operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega,$$
$$\mathbb{T} = -p\mathbb{I} + \mu(\nabla\mathbf{v} + \nabla\mathbf{v}^T),$$
$$\mathbf{v} = \mathbf{v}_D^i \quad \text{on } \Gamma_i \subset \partial\Omega,$$
$$\mathbb{T}\mathbf{n} = 0 \quad \text{on } \partial\Omega\setminus\cup\Gamma_i.$$

Analytical background

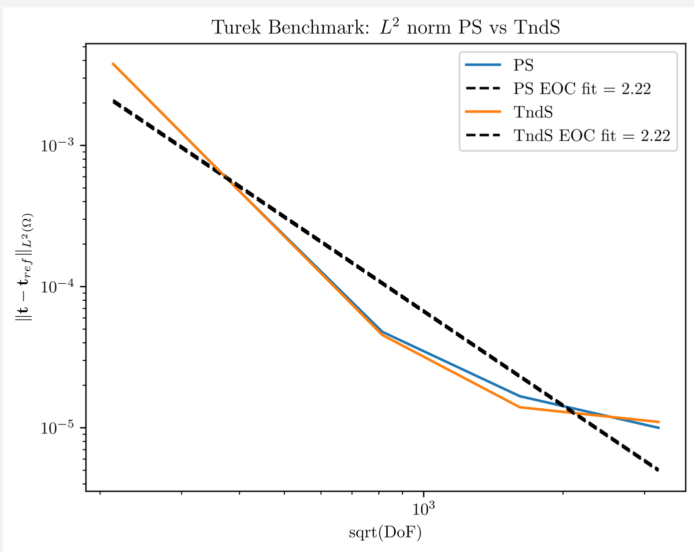
- Traction is defined on boundary using unit outward normal $\mathbf{n}$
$$\mathbf{t} := \mathbb{T}\mathbf{n}.$$
- Standard finite-dimensional estimate for Stokes eq. in bulk gives:
$$\|p - p_h\|_{L^2(\Omega)} + \|\nabla(\mathbf{v} - \mathbf{v}_h)\|_{L^2(\Omega)} \le Ch\|\nabla^2\mathbf{v}\|_{L^2(\Omega)}.$$
- The direct calculation of traction from computed $(\mathbf{v}, p)$ relies on the estimate on boundary that uses theory of traces
$$\|\nabla(\mathbf{v} - \mathbf{v}_h)\|_{L^2(\partial\Omega)} \le C\|v - v_h\|_{H^{3/2}(\Omega)} \le Ch^{\frac{1}{2}}\|\nabla^2\mathbf{v}\|_{L^2(\Omega)}.$$
- Disadvantages: loss of $1/2$ of convergence order, need of normal $\mathbf{n}$.
- In the Poincaré-Steklov computation, we view traction as a functional: Find $\mathbf{t}$ s.t. $\forall\varphi \in V \subset H^1(\Omega)$:
$$\int_\Gamma \mathbf{t}\cdot\varphi\, dS = -\int_\Omega \mathbb{T}\cdot\nabla\varphi\, dx + \int_{\partial\Omega\setminus\Gamma}(\mathbb{T}\mathbf{n})\cdot\varphi\, dS.$$
- Can be computed for Navier-Stokes eq., Γ is a boundary of interest, linear problem only.
- Advantages: no need for normal $\mathbf{n}$, retains former convergence order for the Stokes equation.
- Conjecture: preserves the former convergence order also for N-S eq.
- Proof: based on analogy between Stokes and Laplace eq. Assume we solved $-\Delta u = f$, $u|_{\partial\Omega} = 0$ and we are looking for t, which is an analogy of the traction, i.e., the flux through boundary:
$$\langle t, \varphi\rangle_{L^2(\partial\Omega)} = \langle\nabla u, \nabla\varphi\rangle_{L^2(\Omega)} + \langle f, \varphi\rangle_{L^2(\Omega)} \quad \forall\varphi \in V = H^1(\Omega).$$
- In finite-dimensional space:
$$\langle t_h, \varphi_h\rangle_{L^2(\partial\Omega)} = \langle\nabla u_h, \nabla\varphi_h\rangle_{L^2(\Omega)} + \langle f, \varphi_h\rangle_{L^2(\Omega)} \quad \forall\varphi_h \in V_h \subset V.$$
- Define harmonic extension Ψ of the traction t to the Ω: $-\Delta\Psi = 0$, $\Psi|_{\partial\Omega} = t$. It holds:
$$\langle t, t\rangle_{L^2(\partial\Omega)} = \langle\nabla u, \nabla\Psi\rangle_{L^2(\Omega)} + \langle f, \Psi\rangle_{L^2(\Omega)}.$$
- Assuming regularity $\|\nabla^2\Psi\|_{L^2(\Omega)} \le C$, using Galerkin orthogonality and Interpolation theorem ($k = 1$):
$$\begin{aligned}\langle t - t_h, t - t_h\rangle_{L^2(\partial\Omega)} &= \langle\nabla(u - u_h), \nabla(\Psi - \Psi_h)\rangle_{L^2(\Omega)}\\ &\le \|\nabla(u - \mathbb{P}_h u)\|_{L^2(\Omega)}\|\nabla(\Psi - \mathbb{P}_h\Psi)\|_{L^2(\Omega)}\\ &\le Ch^2\|\nabla^2 u\|^2_{L^2(\Omega)}.\end{aligned}$$
- Poincaré-Steklov approach improves the convergence rate $h^{\frac{1}{2}} \to h$.

Numerical implementation

- FEM library Firedrake.
- Pressure robust method: Scott-Vogeliuis pair (CG2 velocity, DG1 pressure), triangles, barycentric split.
- Newton solver, sparse LU solver — MUMPS.
- Up to 55M DoFs on computational node with 512 GB RAM.
- Poincaré-Steklov problem is ill-posed.
- Regularization: Find $\mathbf{t} \in V$ such that
$$\int_\Gamma \mathbf{t}\cdot\varphi + \operatorname{id}(\mathbf{t}, \varphi) = F(\varphi) \qquad \text{for all } \varphi \in V.$$
- In code we add ones instead of zeros on the diagonal.
- Sparse LU regular factorization for linear Poincaré-Steklov problem.
- Question: Right discrete space for traction in PS problem. Now CG1.
- Evaluation of $\|\mathbf{t} - \mathbf{t}_{\mathrm{ref}}\|_{L^2(\Omega)}$, where the reference is obtained on the finest grid, is not straightforward due to projection / interpolation.

1) Comparison of traction computation approaches on Turek benchmark

- Considering laminar ($Re = 20$) Turek benchmark with cylinder and square obstacles.
- Very fast convergence in L^1 norm on the cylinder — drag and lift. But the correct norm for traction is L^2.
- With smooth boundary, such as a cylinder, we observe same order of convergence for direct (TndS) and Poincaré-Steklov (PS) approaches.
- The solution on the square obstacle is less regular, and hence we observe a loss of convergence order. This is partially saved by PS approach.
- Results are pointwise divergence free: $\int_\Omega \operatorname{div}\mathbf{v}\, dx \approx 10^{-16}$.



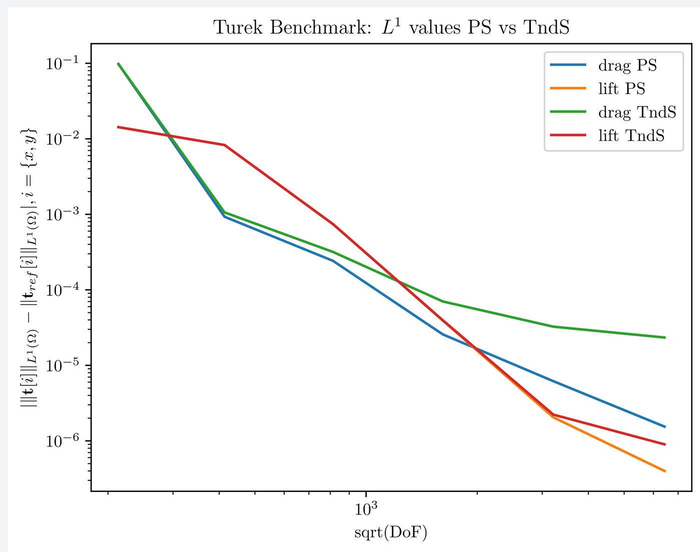
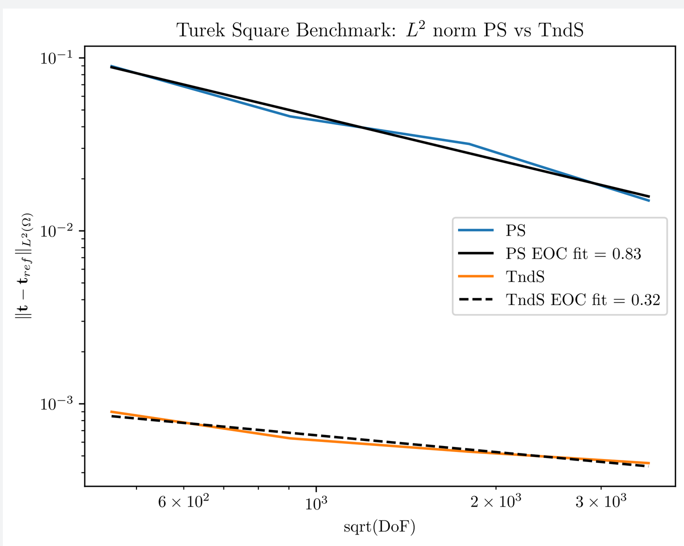


Figure: Reference obtained with 52M DoFs and from Turek benchmark. EOCs are drag = {3.03, 2.24}, lift = {3.37, 3.18} for methods {PS, TndS}.



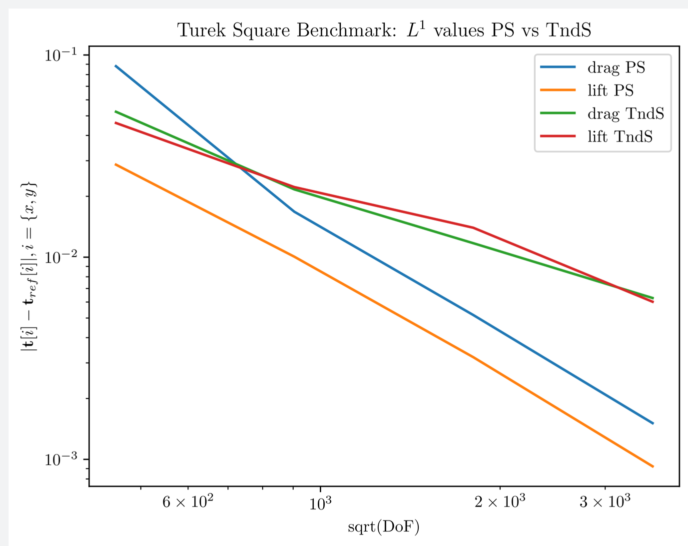
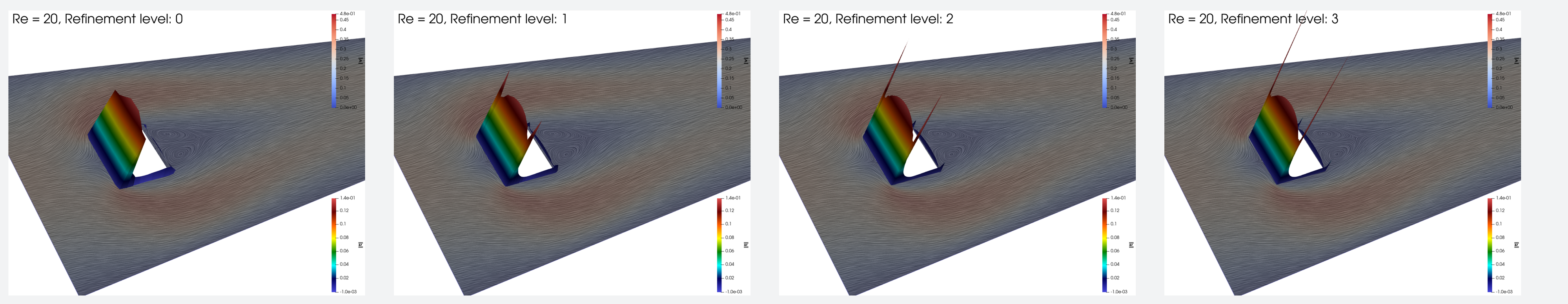


Figure: Reference obtained with 41M DoFs. EOCs are drag = {1.94, 1.01}, lift = {1.66, 0.95} for methods {PS, TndS}. PS by ≈ 1/2 convergence order better in L^2 norm. The worst absolute error in L^2 norm could be due to the wrong projection / interpolation on the reference mesh.

2) Visualisation of traction on square in Turek benchmark computed using Poincaré-Steklov approach

- Start computing with 4 points on side of the square and coarse mesh in bulk, and plot magnitude of traction on square.
- L^1 norms for direct and Poincaré-Steklov approaches coincide up to error on both shapes, however, L^2 norms are completely different numbers on square — possible reason are spikes in corners.

Fig: $\|\mathbf{t}\|_2$



3) Pointwise traction profiles for different Reynolds numbers computed using Poincaré-Steklov approach

- Profile of lift is dramatically changing with increasing Reynolds number and the fluid wants to deform the cylinder in non-obvious way.

Fig: Lift

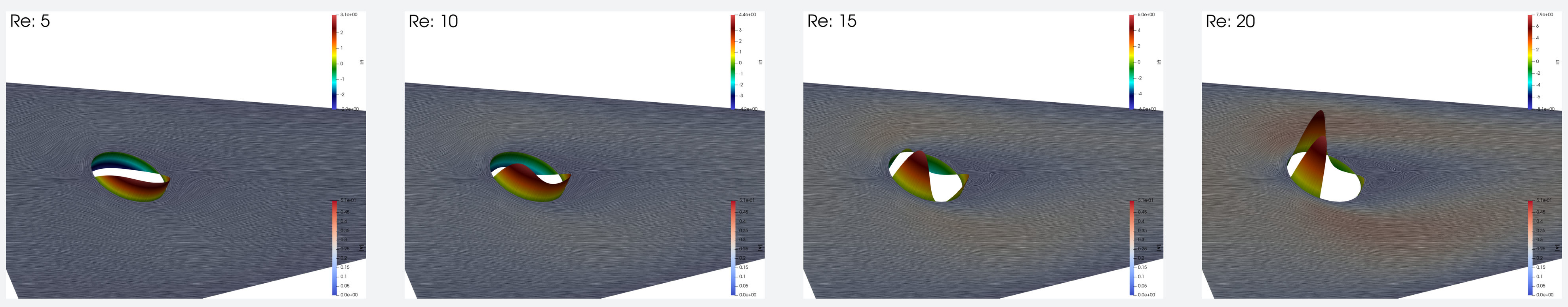
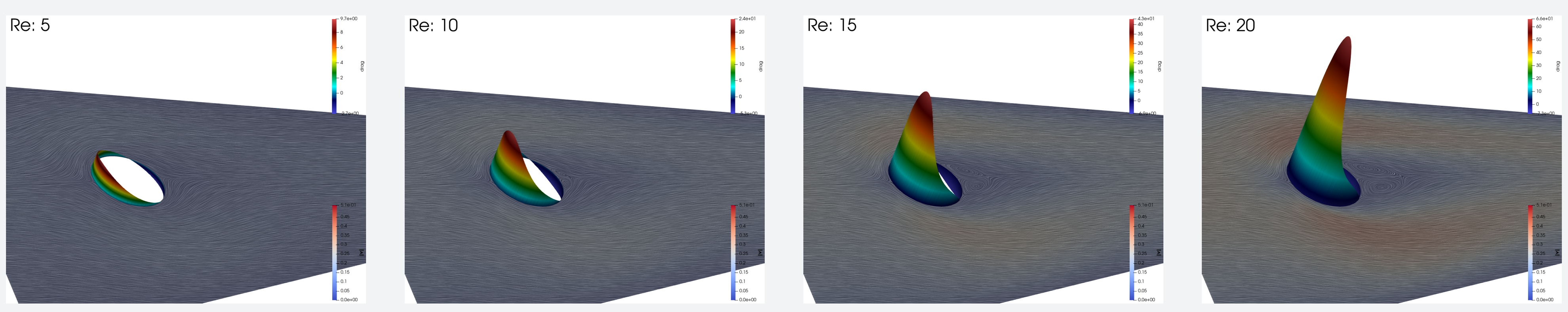


Fig: Drag



References & Acknowledgement

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