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Traction profiles of the Turek benchmark

and their relation to bifurcations for Navier Stokes fluids

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joint work with Jan Blechta, Sebastian Schwarzacher, Karel Tůma

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Equations and traction

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- Unsteady incompressible Navier-Stokes equations

$$
\begin{aligned}
\rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) &= \mathrm{div}\, \mathbb{T} \quad \text{in } \Omega, \\
\mathrm{div}\, \mathbf{v} &= 0 \quad \text{in } \Omega, \\
\mathbb{T} &= -p\mathbb{I} + \mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^T), \\
\mathbf{v} &= \mathbf{v}_D^i \quad \text{on } \Gamma_i \subset \partial\Omega, \\
\mathbb{T}\mathbf{n} &= 0 \quad \text{on } \partial\Omega \backslash \cup \Gamma_i.
\end{aligned}
$$

- Traction

$$
\mathbf{t} := \mathbb{T}\mathbf{n}.
$$

- Discretization: Taylor–Hood CG(2,1); Crank–Nicolson; monolithic Newton–LU (implemented in Firedrake)

Navier-Stokes theory – overview

- steady BVP non-unique in 2D, 3D (except low *Re*)
- IBVP unique in 2D (Ladyženskaja)
- uniqueness of time-periodic solutions in 2D, 3D
  - steady solution only unique time-per. solution in case of a single obstacle flow (low *Re*)
- questions:
  - what makes stable solution?
  - are long-time attractors unique?
  - are eigenvalues the key?

Pointwise traction – fluid bifurcation

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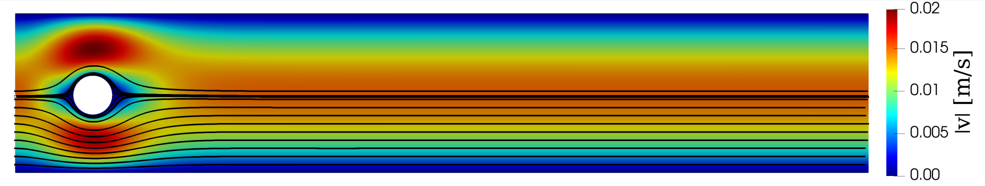
Stokes like

0 → *Re*



Pointwise traction – fluid bifurcation

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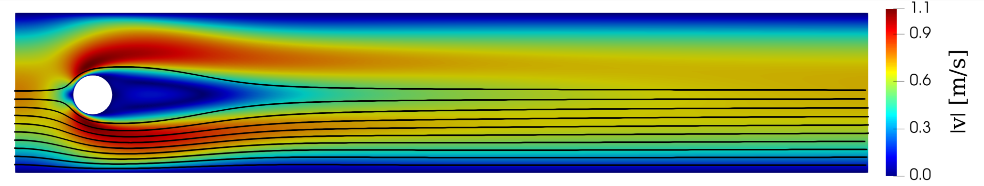
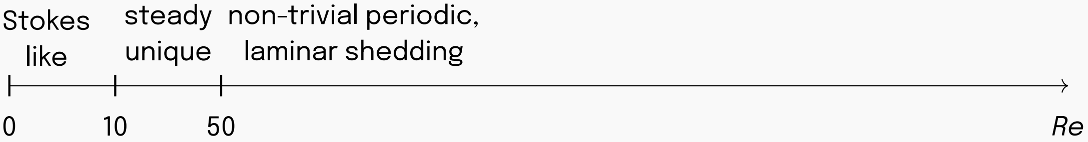
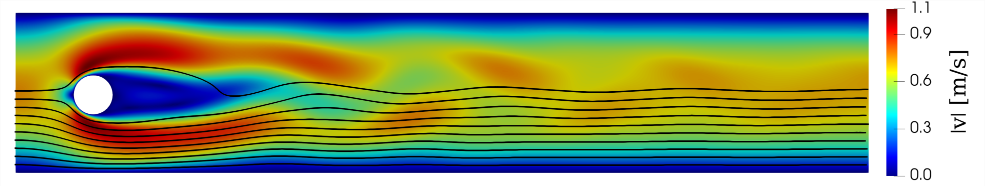
Stokes like | steady unique

0 | 10 | *Re*



Pointwise traction – fluid bifurcation





Pointwise traction – fluid bifurcation

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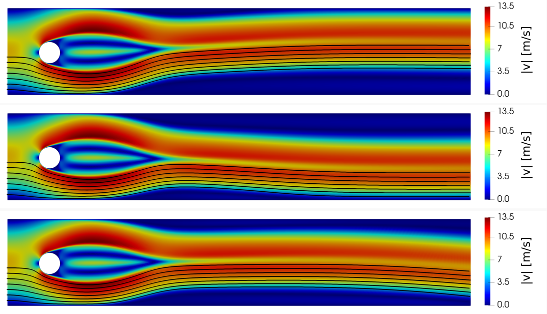
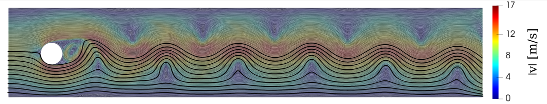
Stokes like | steady unique | non-trivial periodic, laminar shedding | steady non-unique, non-symmetric shedding

0 — 10 — 50 — 300 — Re

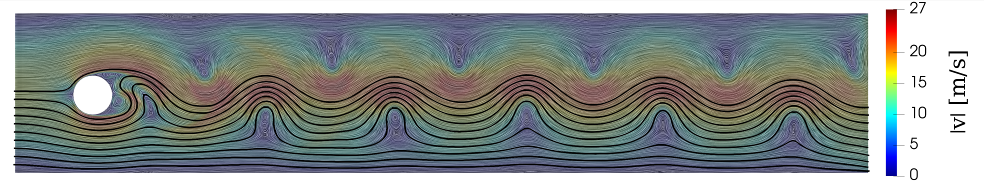




Pointwise traction – fluid bifurcation

Stokes like | steady unique | non-trivial periodic, laminar shedding | steady non-unique, non-symmetric shedding | vortex shedding

0 — 10 — 50 — 300 — 950 — *Re*



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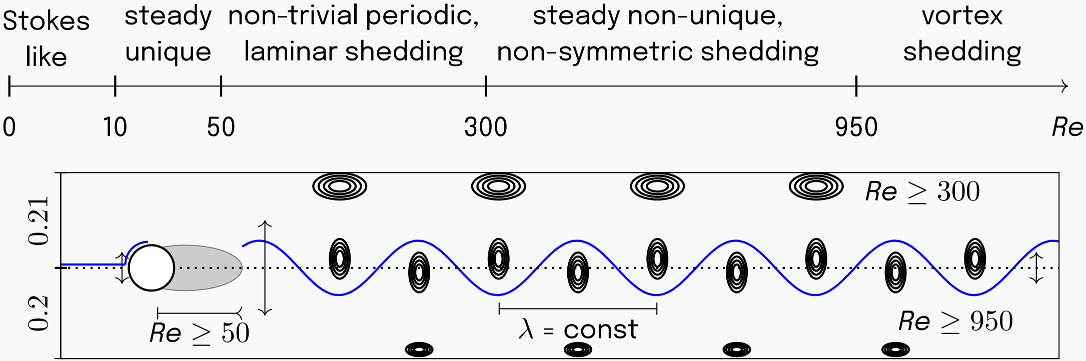
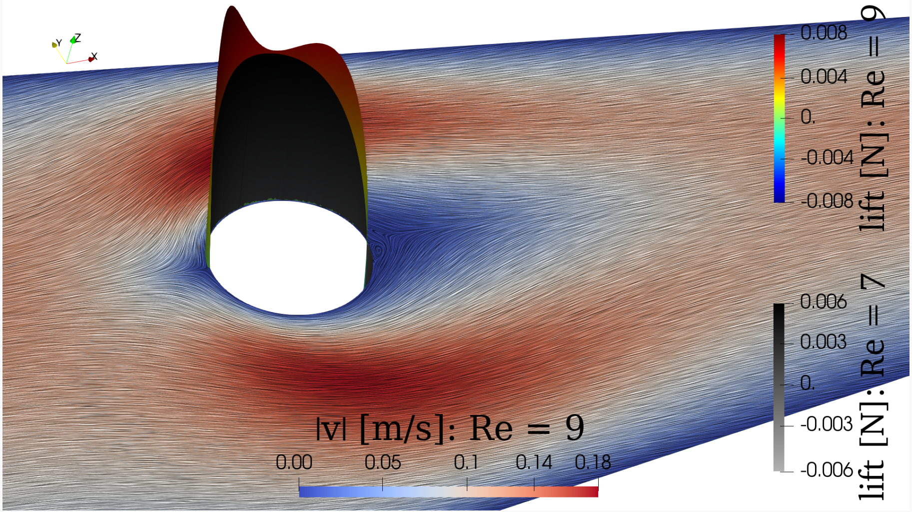
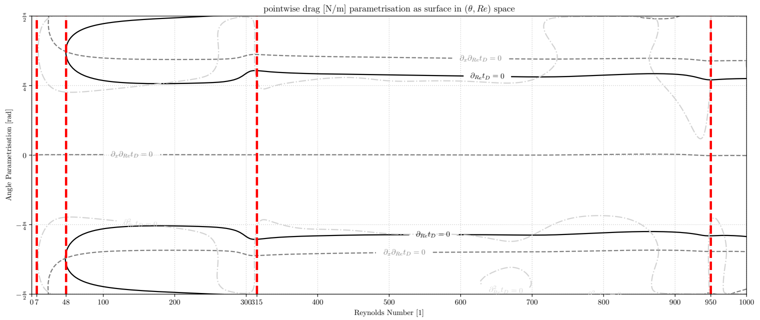
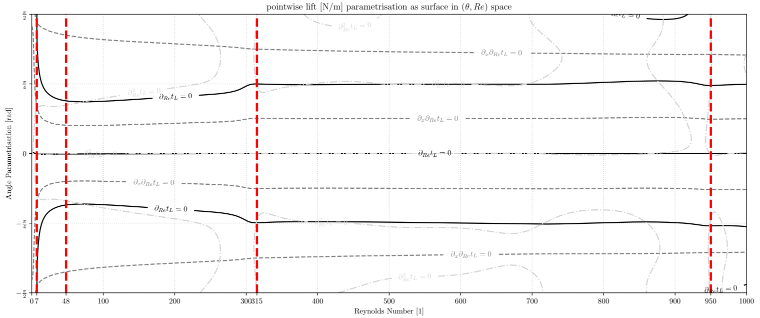
Pointwise traction – fluid bifurcation



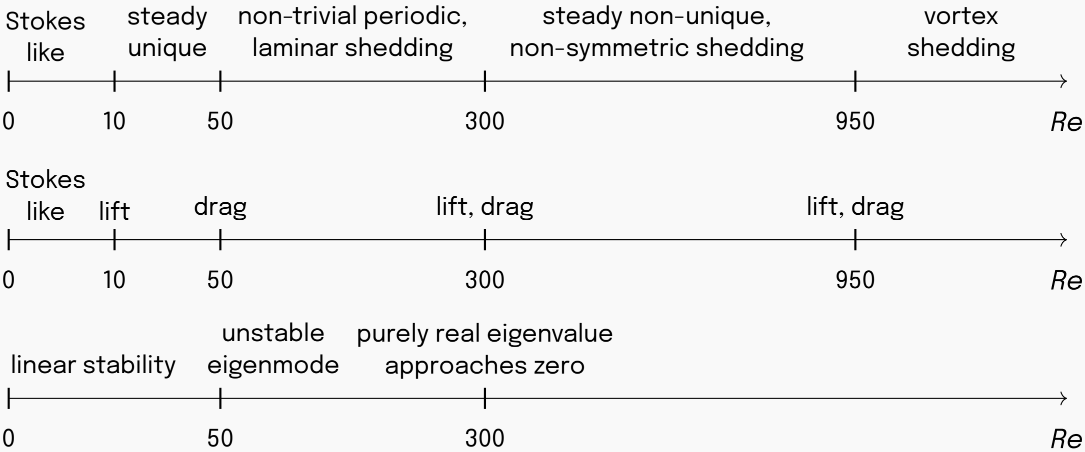
Pointwise traction of the STEADY solution




pointwise drag [N/m] parametrisation as surface in (θ, Re) space
Angle Parametrisation [rad]
Reynolds Number [1]
$\partial_x \partial_{Re} t_D = 0$
$\partial_{Re} t_D = 0$
0 7 48 100 200 300 315 400 500 600 700 800 900 950 1000


pointwise lift [N/m] parametrisation as surface in (θ, Re) space
Angle Parametrisation [rad]
Reynolds Number [1]
$\partial_x \partial_{Re} t_L = 0$
$\partial_{Re} t_L = 0$
$\partial^2_{Re} t_L = 0$
0 7 48 100 200 300 315 400 500 600 700 800 900 950 1000

Overview


Stokes like
steady unique
non-trivial periodic, laminar shedding
steady non-unique, non-symmetric shedding
vortex shedding
0
10
50
300
950
Re
Stokes like
lift
drag
lift, drag
lift, drag
0
10
50
300
950
Re
linear stability
unstable eigenmode
purely real eigenvalue approaches zero
0
50
300
Re

Summary

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Main message:

- solution bifurcations $\longleftrightarrow$ $\partial_{Re}\mathbf{t}_{\mathrm{st}}$ behavior
- upstream face of the obstacle – driver of all dynamics

We also observed:

- unique time-periodic global-in-time attractor
- selfsimilarities of regimes between transitions

References and acknowledgement

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2) David A. Ham et al. *Firedrake User Manual*. First edition. Imperial College London and University of Oxford and Baylor University and University of Washington. May 2023. DOI: 10.25561/104839.

3) P. Farrell, C. Beentjes, and Á. Birkisson, *The computation of disconnected bifurcation diagrams*, arXiv, Mar. 2016. doi:10.48550/arXiv.1603.00809.

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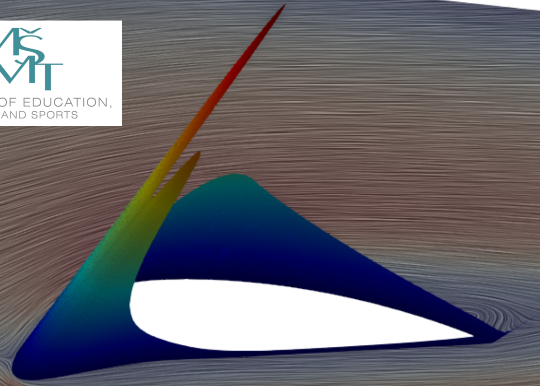

Questions?

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